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ADAPTIVE DISCRETE ALGORITHMS WITH IMPROVED TRANSIENT PERFORMANCES FROM FAST CONVERGENCE

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Abstract. This paper addresses the application of convergence rules of gradient-type discrete algorithms to discrete adaptive control algorithms for linear time-invariant systems, which are based on Lyapunov's – like functions, in order to improve the transient performances based on fast adaptation. In particular, the adaptation covergence is increased as a generalized or filtered error increases through the application of Armijo rule for regulating the decrease of each Lyapunov's-like function on which the particular adaptation algorithm is based. The proposed scheme can be implemented with minor modifications in systems subject to unmodelled dynamics if some weak knowledge on such a dynamics is available consisting of upper-bounds of the dimension and norm of the unmodelled parameter vector.

Key words: adaptation transients, adaptive algorithms, numerical methods for fast convergence.

1. Introduction. Many discrete adaptive algorithms have been derived during the last twenty years to be used in identification and adaptive control problems. The proof of stability and, more recently, the analysis of robustness has been a common pre-requirement in order to accept a given algorithm as a potential "a priori" useful one. The work on the topic has been exhaustively developed (see, for instance, Chalam (1987); Sastry and Bodson, (1989). A common gap in the theory has been the almost absence of a real investigation of the transient performances associated with the algorithms and the way of improving them. Some attempts were made in De la Sen (1984) concerning the on-line sub-optimization of the free parameters of the adaptive algorithms by using a optimization model parametrized according to "a priori" parametrical estimates, previous measures of the regressor and their predictions. The computing and time costs of such an optimization method can be adapted to the computing possibilities and real-time requirements by selecting the planning horizon sizes and the number of iterations. The use of adaptive sampling for discrete-time and hybrid controllers was proposed in De la Sen (1984), De la Sen (1985), De la Sen (1986). The sampling period was chosen with very simple up-dating rules while the plant becomes time-varying since sampling is not constant. The stability was guaranteed by making the sampling period to converge to a limit. Finally, the "a priori" parametrical estimation was modified "a posteriori" in Minambres and De la Sen (1986) according to the well-known Steffensen's method from Numerical Analysis to achieve a fast convergence to the limit of the parameter vector the method being only applicable if such a limit exists. The problem is recovered in this paper with an alternative solution in the sense that the adaptation rate is governed by the filtered/generalized adaptive scheme s error sequence and the adaptation rate of the Lyapunov's-like function used to prove stability is modified with that of such an error, the adaptation step size being selected by Armijo rule (Bertsekas, 1982). The scheme's stability is guaranteed by respecting the stability domains of the free-design parameters of the adaptation algorithm. The paper is organized as follows. Section 2 presents the main ideas on a basic algorithm developed in Lozano (1982) and then used in De la Sen (1984), De la Sen (1985), De la Sen (1986). The choice of this algorithm is due to the fact that it possesses two-free design parameters and a time-decreasing adaptation gain but the method can be applied to any updating algorithm having free-design parameters. Section 3 is devoted to the extension of the preceding ideas to more general algorithms and to the case of presence of unmodelled dynamics.

2. Basic scheme

2.1. Discrete plant and base algorithm. Consider a linear and time-invariant inversely stable plant whose filtered output $y_t^F = C_r(q^{-1})y_t = \theta^T\varphi(t)$ with $C_r(q^{-1})$ being a strictly Hurwitzian polynomial in the time-delay operator q^{-1} and $y_t,\ t \geqslant 0$ being the plant output sequence, θ is the plant parameter vector related to the filtered output, $\{\varphi_t,\ t \geqslant\}$ is the regressor vector consisting of a finite plant input/output sequence according to known upper-bounds of the plant poles and zeros and $d > 0$ is the known plant delay (or relative order). The adaptive control objective is that $y_t^F \to y_t^{MF}(t \to \infty)$ with bounded

$\{||v_t||,\ t \geqslant 0\}$ so that $y_t^{MF} = C_r(q^{-1})y_t^M$ and y_t^M being the filtered output of any explicit or implicit reference model. If θ is unknown, the next parameter estimation scheme was proposed in Lozano (1982); De la Sen (1984):

$$\begin{aligned} \widehat{\theta}_t &= \widehat{\theta}_{t-1} + \frac{F_t\varphi_{t-d}v}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} \\ F_{t+1} &= \lambda_t^{-1} F_t \left[I - \frac{\varphi_{t-d}\varphi_{t-d}^T}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} \right] \\ \widehat{\theta}_0 &= \theta_0; \quad F_0 = F_0^T > 0, \end{aligned} \tag{1}$$

where $\lambda_t,\ \varepsilon\,(0,1]$ and $c_t\,\varepsilon\,(0,\infty]$, all $t \geqslant 0$ are the forgetting factor and an updated gain which are free-design parameters within their admissibility constraints required for parameter convergence and closed-loop stability, Lozano (1982); De la Sen (1984); De la Sen (1985) and $v_t^0 = \widetilde{\theta}_{t-1}^T \varphi_{t-d} = y_t^F - \widehat{\theta}_{t-1}^T \varphi_{t-d}$, and $\widetilde{\theta}_t = \theta - \widehat{\theta}_t$ are the 'a priori' adaptation (or generalized) error and the parametrical error at the t-th sample, respectively. Direct calculations with Eqs. 1 yield the next relationships:

$$\begin{aligned} v_t^p &= \widetilde{\theta}_t^T \varphi_{t-d} = \varepsilon_t^F + (\widehat{\theta}_{t-d}^T - \widehat{\theta}_t^T)\varphi_{t-d} \\ &= \varepsilon_t^F - \sum_{i=t-d+1}^{t} \frac{\varphi_{i-d}^T F_i \varphi_{i-d} v_i^0}{c_i + \varphi_{i-d}^T F_i \varphi_{i-d}} = \rho_t v_t^0; \\ \rho_t &= c_t(c_t + \varphi_{t-d}^T F_t \varphi_{t-d}), \end{aligned} \tag{2}$$

where $\varepsilon_t^F = y_t^F - y_t^{MF} = C_r(q^{-1})\varepsilon_t = C_r(q^{-1})(y_t - y_t^M)$ is the filtered tracking error and v_t^p is the "a posteriori" adaptation error at the t-th sample. The stability of the algorithm (1) was investigated in Lozano (1982), De la Sen (1984) from the use of the Lyapunov's – like sequence $V_t = \widetilde{\theta}_t^T F_{t+1}^{-1} \widetilde{\theta}_t$ $(t \geqslant 0)$ whose gradient with respect to $\widetilde{\theta}_t$ is $\nabla V_t = F_{t+1}^{-1}\widetilde{\theta}_t$ $(t \geqslant 0)$. From (1), the parametrical error evolves as follows:

$$\begin{aligned} \widetilde{\theta}_t &= \widetilde{\theta}_{t-1} + \alpha_{t-1} d_{t-1}; \\ \alpha_{t-1} &= (c_t + \varphi_{t-d}^T F_t \varphi_{t-d})^{-1}; \\ d_{t-1} &= -F_t \varphi_{t-d} v_t^0 \end{aligned} \tag{3}$$

with α_t being a positive step size parameter and d_t is a descent direction since

$$d_{t-1}^T \nabla V_{t-1} = d_{t-1}^T F_t^{-1} \widetilde{\theta}_{t-1} = -(v_t^0)^2 \leqslant 0. \tag{4}$$

Note that $\nabla V_{t-1} = 0 \iff d_{t-1} = 0$ since $F_t^{-1}\tilde{\theta}_{t-1} = 0 \iff \tilde{\theta}_{t-1} = 0$ from the existence of F_t^{-1}. Note that (3), and equivalently (1), is an algorithm of generalized gradient method type (Bertsekas, 1982).

2.2. Selection of the free-design parameters according to Armijo rule. The convergence of $\{V_t,\ t \geqslant 0\}$ to a limit $V < \infty$ which ensures the scheme's stability can be governed and then made faster by increasing the modulus of the one-step increment $\Delta V_{t-1} = V_t - V_{t-1}$ $(t \geqslant 0)$ according to the "a priori" generalized error. One gets from (4) by using the Armijo rule:

$$|\Delta V_{t-1}| = -\Delta V_{t-1} = V_{t-1} - V_t \geqslant \sigma\beta^{m_{t-1}} s \left| d_{t-1}^T \nabla V_{t-1} \right| = \sigma\beta^{m_{t-1}} s v_t^{0^2}. \tag{5}$$

But, one gets directly from (1) – (2) that

$$\Delta V_{t-1} = V_t - V_{t-1} = \lambda_t \tilde{\theta}_t^T F_t^{-1} \tilde{\theta}_t + \frac{\lambda_t}{c_t} v_t^{p^2} - \tilde{\theta}_{t-1}^T F_t^{-1} \tilde{\theta}_{t-1}. \tag{6.a}$$

Since

$$\frac{\lambda_t \varphi_{t-d}^T \tilde{\theta}_{t-1} v_t^p}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} = \frac{\lambda_t v_t^p v_t^0}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} = \frac{\lambda_t c_t v_t^{0^2}}{(c_t + \varphi_{t-d}^T F_t \varphi_{t-d})^2},$$

$$\begin{aligned} \lambda_t \tilde{\theta}_t^T F_t^{-1} \tilde{\theta}_t &= \lambda_t \tilde{\theta}_{t-1}^T F_t^{-1} \left(\tilde{\theta}_{t-1} - \frac{F_t \varphi_{t-d} v_t^0}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} \right) - \frac{\lambda_t c_t v_t^{0^2}}{(c_t + \varphi_{t-d}^T F_t \varphi_{t-d})^2} \\ &= \lambda_t \tilde{\theta}_{t-1}^T F_t^{-1} \left(\tilde{\theta}_{t-1} - \frac{F_t \varphi_{t-d} v_t^0}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}} \right) - \frac{\lambda_t v_t^{p^2}}{c_t}. \end{aligned} \tag{6.b}$$

The substitution of (6.b) into (6.a) yields

$$-\Delta V_{t-1} = |\Delta V_{t-1}| = (1 - \lambda_t) V_{t-1} + \frac{\lambda_t v_t^{0^2}}{c_t + \varphi_{t-d}^T F_t \varphi_{t-d}}. \tag{7}$$

The Armijo rule is applied as follows. Fix scalars s, β and σ with $s > 0$, $\beta \varepsilon (0,1)$ and $\sigma \varepsilon (0, 1/2)$ with $\alpha_t = \beta^{m_t} s$ in (3) where m_k is the first nonnegative integer for which

$$|\Delta V_{t-1}| \geqslant \sigma\beta^{m_{t-1}} s \left| \nabla V_{t-1}^T d_{t-1} \right| = \sigma\alpha_{t-1} \left| \nabla V_{t-1}^T d_{t-1} \right| \tag{8}$$

which using (4) and (7) through (2) and (3) yields

$$\begin{aligned}|\Delta V_{t-1}| &= (1-\lambda_t)V_{t-1} + \lambda_t \beta^{m_{t-1}} s v_t^{0^2} \\ &\geqslant \sigma\beta^{m_{t-1}} s v_t^{0^2} \iff (\lambda_t - 1)V_{t-1} \leqslant (\lambda_t - \sigma)\beta^{m_{t-1}} s v_t^{0^2}, \quad (9)\end{aligned}$$

which is guaranteed if

$$\beta^{m_{t-1}} \geqslant \frac{\lambda_t - 1}{s(\lambda_t - \sigma)} \frac{V_{t-1}}{v_t^{0^2}} \quad (10)$$

in view of the stability constraints of the algorithm since $V_t \geqslant 0$ $(t \geqslant 0)$. After some direct calculations, Eq. 10 can be guaranteed by the choice

$$\lambda_t \geqslant \frac{\sigma s \beta^{m_t - 1} v^{0^2} - V_{t-1}}{s\beta^{m_t - 1} v_t^{0^2} - V_{t-1}} \quad (11)$$

which is guaranteed with stability of the scheme for any known bounded $\overline{V}_0 \geqslant V_0 \geqslant V_t$, all $t \geqslant 0$, if

$$1 \geqslant \lambda_t \geqslant \frac{\sigma s \beta^{m_{t-1}} v_t^{0^2}}{s\beta^{m_{t-1}} v_t^{0^2} - \overline{V}_0} \quad (12)$$

Two necessary conditions for coherency of (12) are

$$\begin{aligned}&(1) \quad s\beta^{m_{t-1}} v_t^{0^2} \geqslant \overline{V}_0, \\ &(2) \quad \sigma s \beta^{m_{t-1}} v_t^{0^2} \geqslant s\beta^{m_{t-1}} v_t^{0^2} - \overline{V}_0.\end{aligned}$$

Note that $\beta^{m_{t-1}} < 1$ so that $\ln(\beta^{m_{t-1}}) < 0$. Thus, Condition 2 is equivalent to $(1-\sigma)s\beta^{m_{t-1}} \geqslant \overline{V}_0$, or, $1/\beta^{m_{t-1}} \leqslant s(1-\sigma)\nu/\overline{V}_0 \implies m_{t-1} \leqslant \frac{1}{|\ln\beta|} \ln\left(\frac{s(1-\sigma)\nu}{\overline{V}_0}\right)$ for any sample t such that $v_t^{0^2} \leqslant \nu$, ν being a predefined positive constant with lower-bound $\frac{\overline{V}_0}{s(1-\sigma)}$ which guarantees that the upper-bound of m_{t-1} is nonnegative. Similarly, Condition 1 is equivalent to $m_{t-1} \leqslant \frac{1}{|\ln\beta|} \ln\left(\frac{s\nu}{\overline{V}_0}\right)$ for any t-sample such that $v_t^{0^2} \leqslant \nu$. Since $\sigma < 1$,

$$\ln\left(\frac{s(1-\sigma)\nu}{\overline{V}_0}\right) = \ln\left(\frac{s\nu}{\overline{V}_0}\right) - |\ln(1-\sigma)| \leqslant \ln\left(\frac{s\nu}{\overline{V}_0}\right),$$

and the solution for all t with $v_t^{0^2} \leqslant \nu$ satisfies for any $\nu \geqslant \dfrac{\overline{V}_0}{[s(1-\sigma)]}$

$$m_{t-1} \leqslant \frac{1}{|\ln\beta|}\ln\left(\frac{s(1-\sigma)\nu}{\overline{V}_0}\right). \tag{13}$$

When $v_t^0 < \nu$ and, since $v_t^0 \to 0$ as $t \to \infty$, Eq. 13, and thus Eq. 12, are impossible to accomplish with. Therefore, the global strategy is:

(a) For all $t \geqslant 0$ such that $v_t^{0^2} \geqslant \nu$, choose λ_t from (12) and then c_t such that $\operatorname{tr} F_{t+1}^{-1}$ is bounded, i.e., $\operatorname{tr} F_{t+1}^{-1} < \lambda_t M + \lambda_t \dfrac{\varphi_{t-d}^T \varphi_{t-d}}{c_t} \leqslant M = \operatorname{tr} F_0^{-1}$ or $\infty > c_t = \lambda_t \dfrac{\varphi_{t-d}^T \varphi_{t-d} + \sigma_0}{(1-\lambda_t)\operatorname{tr}(F_0^{-1})}$ leading to a bounded trace of F_{t+1}^{-1} with $\sigma_0 \,\varepsilon\, (0, \operatorname{tr} F_0^{-1})$ being a small positive constant.

(b) For all $t \geqslant 0$ such that $v_t^{0^2} < \nu$, fix $c_t \,\varepsilon\, (0, \infty)$ and then

$$\lambda_t = \frac{\operatorname{tr} F_0^{-1} - \sigma_0}{\operatorname{tr} F_0^{-1} + \dfrac{\varphi_{t-d}^T \varphi_{t-d}}{c_t}}.$$

The scheme's stability holds since $c_t \,\varepsilon\, (0, \infty)$ and

$$\lambda_t = \frac{\operatorname{tr} F_0^{-1} - \sigma_0}{\operatorname{tr} F_0^{-1} + \dfrac{\varphi_{t-d}^T \varphi_{t-d}}{c_t}},$$

(Lozano, 1982 ;De la Sen, 1984; De la Sen, 1985a; De la Sen, 1985b; De la Sen, 1986).

Proposition 1. *Fix $\nu > \overline{V}_0/[s(1-\sigma)]$ for some known $\overline{V}_0 \geqslant V_0$ (For instance, if $\theta \,\varepsilon\, \mathbf{B}(\theta_0, r)$ with $\mathbf{B}(\theta_0, r)$ being a closed ball centered at θ_0 and of radius r, with θ_0 and r known, then choose $\overline{V}_0 = \operatorname{Sup}(\|\theta - \widehat{\theta}\|_E^2)\|F_0^{-1}\|_E$) and fix also scalars $s > 0$, $\sigma \,\varepsilon\, (0, 1/2)$ and $\beta \,\varepsilon\, (0, 1)$. Define $T = \{t \in \mathbf{Z}_0^+ = \mathbf{Z}^+ \cup \{0\} : t \leqslant t_1, \text{some arbitrary finite} t_1\}$ and $\overline{T} = \mathbf{Z}_0^+ - T$. If the pair (λ_t, c_t) is chosen according to the rule:*

a)

$$\lambda_t \geqslant \frac{\sigma s \beta^{m_{t-1}} v_t^{0^2}}{s\beta^{m_{t-1}} v^{0^2} - \overline{V}_0};$$
$$m_{t_1} = \operatorname{Min}\left\{z \in \mathbf{Z}_0^+ : z = \frac{1}{|\ln\beta|}\ln\frac{(s(1-\sigma)\nu)}{\overline{V}_0}\right\}, \tag{14.a}$$

$$c_t = \frac{\lambda_t \varphi_{t-d}^T \varphi_{t-d} + \sigma_{01}}{(1-\lambda_t)\operatorname{tr}(F_0^{-1})}; \quad \sigma_{01} \geqslant 0, \tag{14.b}$$

if $t \varepsilon \mathbf{T}$.

b) Fix an arbitrary $c_t \varepsilon (0, \infty)$ and then fix

$$\lambda_t = c_t \frac{(\operatorname{tr} F_0^{-1} - \sigma_{02})}{c_t \operatorname{tr} F_0^{-1} + \varphi_{t-d}^T \varphi_{t-d}}; \quad \sigma_{02} \in [0, \operatorname{tr} F_0^{-1}), \tag{15}$$

if $t \varepsilon \overline{\mathbf{T}}$. Thus, $\Delta V_{t-1} \leqslant -\sigma s \beta^{m_{t-1}} v_t^{0^2} < 0$ for all $t \varepsilon \mathbf{T}$, and

$$\begin{aligned} |V_0 - V_\infty| &\geqslant \sum_{t \in T} \sigma s \beta^{m_{t-1}} v_t^{0^2} + \frac{\sigma}{s} \sum_{t \in \overline{T}} \frac{\lambda_t - 1}{\lambda_t - \sigma} V_{t-1} \\ &\geqslant \sigma s \nu \sum_{t \in T} \beta^{m_{t-1}} + \frac{\sigma}{s} \sum_{t \in \overline{T}} \frac{\lambda_t - 1}{\lambda_t - \sigma} V_{t-1} \quad \text{[Armijo rule]}, \end{aligned}$$

$\Delta V_{t-1} \leqslant 0$ if $t \varepsilon \overline{\mathbf{T}}$. Also, $\operatorname{tr} F_t^{-1} \leqslant \operatorname{tr} F_0^{-1} = M$ all $t \geqslant 0$ and $v_t^0 \to 0$, $v_t^p \to 0$, $\varepsilon_t^F \to 0$, $\varepsilon_t \to 0$ as $t \to \infty$ with uniformly bounded sequence $\{\|\varphi_t\|, t \geqslant 0\}$.

Proposition 1 establishes that the algorithm convergence over T is governed the Armijo rule with λ_t depending on the integers $m_{(\cdot)}$. However, if λ_t is made to converge to unity (or fixed to unity) over $\overline{\mathbf{T}}$. This is seen as follows. Use (10) into the last term of (9.a) to yield

$$|\Delta V_{t_1}| = (1 - \lambda_t) V_{t-1} + \lambda_t \beta^{m_{t-1}} s v_t^{0^2} \geqslant \frac{\sigma(\lambda_t - 1)}{s(\lambda_t - \sigma)} V_{t-1},$$

which is guaranteed if $\lambda_t \beta^{m_{t-1}} s v_t^{0^2} \geqslant \left(\lambda_t - 1 + \frac{\sigma(\lambda_t - 1)}{s(\lambda_t - \sigma)}\right) \overline{V}_0$, or

$$\frac{s^2 \lambda_t (\lambda_t - \sigma)}{(\lambda_t - 1)(\sigma + s(\lambda_t - \sigma))} \geqslant \frac{\overline{V}_0}{\beta^{m_{t-1}} v_t^{0^2}}. \tag{16}$$

Assume $\lambda_t = 1 - \rho v_t^{0^2}$, $t \in \overline{\mathbf{T}}$ some $\rho < v^{-1}$. Thus, (16) can be rewritten as

$$\frac{s^2 \lambda_t (\sigma - \lambda_t)}{\sigma + s(\lambda_t - \sigma)} \leqslant \frac{p \overline{V}_0}{\beta^{m_{t-1}}}, \tag{17}$$

which holds for the integer m_{t-1} such that

$$\frac{1}{\beta^{m_{t-1}}} \geqslant \frac{s^2 (1 - \rho v_t^{0^2})(\sigma + p v_t^{0^2} - 1)}{p \overline{V}_0 (\sigma + s(1 - \sigma - p v_t^{0^2}))}$$

or, equivalently, for $t \varepsilon \overline{\mathbf{T}}$ with

$$m_{t-1} = \mathrm{Min}\Big\{ z \varepsilon \mathbf{Z}_0^+ : z \leqslant \frac{1}{|\ln \beta|}\Big(\ln\Big[\frac{s^2}{p\overline{V}_0}\Big] + \ln \frac{(1 - p v_t^{0^2})(\sigma + p v_t^{0^2} - 1)}{\sigma + s(1 - \sigma - p v_t^{0^2})}\Big)\Big\}, \qquad (18)$$

whose right-hand-side is always positive for sufficiently small p since $v_t^{0^2}$ is bounded from the stability proof of Lozano (1982), the first relation in (9) still holds for a nonnegative integer m_{t-1} subject to (18) and the following result follows.

Proposition 2. *The Armijo rule is applied over all* $\mathbf{Z}_0^+$ *provided that* $p < v^{-1} < \frac{s(1-\sigma)}{\overline{V}_0}$ *is sufficiently small (what can be guaranteed for given* $\overline{V}_0$ *and* σ *for sufficiently small* s *) if* (λ_t, c_t) *are chosen as in* (16) *for* $t \varepsilon \mathbf{T}$ *and the choice of* (15) *is changed to* $\lambda_t = 1 - p v_t^{0^2}$ *and* c_t *is chosen according to* (14.b *for* $t \in \overline{\mathbf{T}}$. *The stability results of Proposition 1 still hold.*

Note that $\lambda_t \rightarrow 1$ as $t \rightarrow \infty$ under Proposition 2 over $\overline{\mathbf{T}}$ and the integers $\beta^{m_{t-1}}$ are not used directly to update λ_t but they are proved to exist guaranteeing the fulfillment of the first relationship of (9).

REMARK 1. An alternative way of selecting parameters over $\overline{\mathbf{T}}$ can be made by rearranging (16) to yield $s^2\lambda_t(\sigma - \lambda_t)\beta^{m_{t-1}} \leqslant p\overline{V}_0[\sigma + s(\lambda_t - \sigma)]$ what leads to

$$p(\lambda_t) = \lambda_t^2 + \frac{p\overline{V}_0 - s\sigma\beta^{m_{t-1}}}{d\beta^{m_{t-1}}}\lambda_t + \frac{p\overline{V}_0\sigma(1-s)}{s^2\beta^{m_{t-1}}} \geqslant 0, \qquad (19)$$

since $p(\lambda_t)$ is a convex function, the inequality (19) is

$$\lambda_t \geqslant \sigma + (s\beta^{m_{t-1}})^{-1}\Big[\sqrt{(p\overline{V}_0 - s\sigma\beta^{m_{t-1}})^2 + 4p\sigma(s-1)\overline{V}_0} - p\overline{V}_0\Big]. \quad (20)$$

If $s = 1$ and $p < v^{-1} < \frac{1-\sigma}{\overline{V}_0}$ then (20) reduces to $\lambda_t \geqslant 0$. As a result, there is a closed ball $\mathbf{B}(1, r_s)$ some $r_s > 0$ such that there are solutions $\lambda_t \varepsilon (0, 1]$ computed from (20) for all $s \varepsilon \mathbf{B}(1, r_s)$.

Thus, the following result follows.

Proposition 3. *The rule of choice of λ_t for $t \varepsilon \mathbf{\overline{T}}$ of Proposition 2 can be changed to* (20) *for* $p < v^{-1} < \frac{s(1-\sigma)}{\overline{V}_0}$ *and* $s \varepsilon \mathbf{B}(1, r_s)$ *some* $r_S > 0$ *such that* $\lambda_t \varepsilon (0, 1]$.

An alternative design can be implemented since

$$v_t^{0^2}/V_{t-1} = (\widetilde{\theta}_{t-1}^T \varphi_{t-d})^2 / \widetilde{\theta}_{t-1}^T F_t^{-1} \widetilde{theta}_{t-1} \leqslant \frac{\varphi_{t-d}^T \varphi_{t-d}}{\lambda_{\min}(F_t^{-1})} V_{t-1},$$

which is used for obtaining an upper-bound of the right-hand-side of (11) not being larger than unity by guaranteeing that

$$\frac{1}{\beta^{m_{t-1}}} \leqslant \frac{s v^{0^2}}{(1 + \varphi_{t-d}^T \varphi_{t-d} - \lambda_{\min}(F_t^{-1}))\overline{V}_0}.$$

This leads to the subsequent result:

Proposition 4. *Proposition 1 holds with the changes*

$$\lambda_t \geqslant \frac{[\sigma s \beta^{m_{t-1}} \varphi_{t-d}^T \varphi_{t-d} - \lambda_{\min}(F_t^{-1})]\overline{V}_0}{s \beta^{m_{t-1}} v_t^{0^2} - \overline{V}_0}, \quad t \in \mathbf{T},$$

with

$$m_{t-1} = \mathrm{Min}\left\{ z \,\varepsilon\, \mathbf{Z}_0^+ \,:\, z \leqslant \frac{1}{|\ln \beta|} \ln \frac{s v^2}{[1 + \varphi_{t-d}^T \varphi_{t-d} - \lambda_{\min}(F_t^{-1})]\overline{V}_0} \right\},$$

and

$$\lambda_{\min}(F_t^{-1}) < 1 + \varphi_{t-d}^T \varphi_{t-d}$$

being guaranteed if $\mathrm{tr}\, F_0^{-1} = M \leqslant 1$.

The condition $\mathrm{tr}\, F_0^{-1} \leqslant 1$ holds automatically in practice since $\mathrm{tr}\, F_0^{-1}$ is nominally fixed by the designer to $n10^{-\alpha}$, $\alpha \geqslant 6$, with n being the order of $F_{(\cdot)}$.

3. Modified algorithms

3.1 Basic ideas. A modified algorithm is now proposed by using the Lyapunov-like function:

$$\widetilde{V}_t = \widetilde{\theta}_t^T F_{t+1}^{-1} \widetilde{\theta}_t + q_{t+1} \varepsilon_{t+1}^{F^2} \tag{21}$$

with $F_0 = F_0^T > 0$ and $q_t > 0$ for all $t \geqslant 0$. Note that (21) weights both the parametrical and tracking errors. One gets from (21) the next gradient with respect to $\widetilde{\theta}_t$

$$\nabla V_{t-1} = F_t^{-1}\widetilde{\theta}_{t-1} + 2q_t\varphi_{t-1}\varepsilon_{t+d-1}^F, \tag{22}$$

since $\varepsilon_{t+d}^F = \widetilde{\theta}_t^T\varphi_t$. An alternative modified algorithm can be obtained by using the generalized error as follows

$$V_t' = \widetilde{\theta}_t^T F_{t+1}^{-1}\widetilde{\theta}_t + q_{t+1}{v_t^p}^2, \tag{23}$$

leading to

$$\nabla V_t' = F_{t+1}^{-1}\widetilde{\theta}_t + 2q_t v_t^p \varphi_{t-d}. \tag{24}$$

The algorithms derived from the above Lyapunov's-like functions could be induced in a more general one as proposed in the next section.

3.2. The general modified algorithm. Consider

$$V_t = \widetilde{\theta}_t^T F_{t+1}^{-1}\widetilde{\theta}_t + \sum_{i=0}^{\mu} q_{t+1-i}\, v_{t-i}^2(p), \quad t \geqslant 0, \tag{25}$$

with $q_{(\cdot)} > 0$ being scalars and $v_t(p) = \widetilde{\theta}_t^T\varphi_{t-p}$. Note that such a definition is applicable to all errors of Section 2 since

$$\begin{aligned} v_{t+1}^0 &= v_t(d-1) = \widetilde{\theta}_t^T\varphi_{t-d+1}; \\ v_t^p &= v_t(d) = \widetilde{\theta}_t^T\varphi_{t-d}; \\ \varepsilon_t^F &= v_{t-d}(0) = \widetilde{\theta}_{t-d}^T\varphi_{t-d}. \end{aligned} \tag{26}$$

The gradient of V_t with respect to $\widetilde{\theta}_t$ is from (25)

$$\begin{aligned} \nabla V_t &= F_{t+1}^{-1}\widetilde{\theta}_t + 2q_{t+1}v_t(p)\varphi_{t-p} \\ &= \left[F_{t+1}^{-1} + 2q_{t+1}\varphi_{t-p}\varphi_{t-p}^T\right]\widetilde{\theta}_t. \end{aligned} \tag{27}$$

Direct calculus yields

$$\begin{aligned} \Delta V_{t-1} = V_t - V_{t-1} &= \widetilde{\theta}_t^T F_{t+1}^{-1}\widetilde{\theta}_t + q_{t+1}v_t^2(p) + \sum_{i=1}^{\mu} q_{t+1-i}\, v_{t-i}^2(p) \\ &- \widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1} - q_t v_{t-1}^2(p) - \sum_{i=1}^{\mu} q_{t-i}\, v_{t-i-1}^2(p). \end{aligned} \tag{28}$$

The descendent gradient-type algorithm generalized from (1) is now

$$\widetilde{\theta}_t = \widetilde{\theta}_{t-1} + \alpha_{t-1}\overline{d}_{t-1};$$

$$\overline{d}_{t-1} = \begin{cases} d_{t-1} & \text{if } \nabla V_{t-1} \neq 0; \\ 0 & \text{otherwise;} \end{cases} \qquad F_{t+1}^{-1} = \lambda_t F_t^{-1} + \widetilde{F}_t^{-1}, \tag{29}$$

with F_0 being positive definite with $\alpha_{(\cdot)}, d_{(\cdot)}$ and $\widetilde{F}_{(\cdot)}$ to be defined later for all $t \in \mathbf{Z}_0^+$ such that $\nabla V_{t-1} = [F_t^{-1} + 2q_t\varphi_{t-p-1}\varphi_{t-p-1}^T]\widetilde{\theta}_{t-1} \neq 0$. The substitution of (2) into (28) yields

$$\begin{aligned}\Delta V_{t-1} =& q_{t+1}v_t^2(p) - q_{t-\mu}v_{t-\mu-1}^2(p) + \widetilde{\theta}_{t-1}^T\widetilde{F}_t^{-1}\widetilde{\theta}_{t-1}^{-1} \\ &+ \alpha_{t-1}\left[\alpha_{t-1}d_{t-1}^T + 2\widetilde{\theta}_{t-1}^T\right]F_{t+1}^{-1}d_{t-1} \\ &+ (\lambda_t - 1)\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1}.\end{aligned} \tag{30}$$

It is suitable that $V_t \leqslant V_{t-1} \leqslant V_0$, all $t \geqslant 0$ so that $-\Delta V_{t-1} = |\Delta V_{t-1}| \geqslant 0$ in (30), or,

$$\begin{aligned}&q_{t+1}v_t^2(p) + \widetilde{\theta}_{t-1}^T\widetilde{F}_t^{-1}\widetilde{\theta}_{t-1} + \alpha_{t-1}^2 d_{t-1}^T F_{t+1}^{-1}d_{t-1} \\ &\quad\leqslant q_{t-\mu}v_{t-\mu-1}^2(p) - 2\alpha_{t-1}\widetilde{\theta}_{t-1}^T F_{t+1}^{-1}d_{t-1} + (1-\lambda_t)\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1}\end{aligned} \tag{31}$$

Assume that $r_\theta \geqslant ||\theta||$ is known so that $|\widetilde{\theta}_{t-1}^T F_{t+1}^{-1}d_{t-1}| \leqslant ||F_{t+1}^{-1}d_{t-1}||(r_\theta + ||\widehat{\theta}_{t-1}||)$ and $\widetilde{\theta}_{t-1}^T\widetilde{F}_{t+1}^{-1}d_{t-1} \leqslant ||\widetilde{F}_t^{-1}||(r_\theta + ||\widehat{\theta}_{t-1}||)^2$ from the triangle's inequality. Thus, (31) holds if

$$\begin{aligned}&q_{t+1}v_t^2(p) + \alpha_{t-1}^2 d_{t-1}^T F_{t+1}^{-1}d_{t-1} \leqslant q_{t-\mu}v_{t-\mu-1}^2(p) \\ &\quad - (r_\theta + ||\widehat{\theta}_{t-1}||)\left[2\alpha_{t-1}||F_{t+1}^{-1}d_{t-1}|| + (r_\theta + ||\widehat{\theta}_{t-1}||)||\widetilde{F}_t^{-1}||\right] \\ &\quad + (1-\lambda_t)\widetilde{\theta}_{t-1}^T F_{t+1}^{-1}\widetilde{\theta}_{t-1}.\end{aligned} \tag{32}$$

Choose α_{t-1} as follows

$$\alpha_{t-1} = \frac{q_{t-\mu}v_{t-\mu-1}^2(p) - \left(r_\theta + ||\widehat{\theta}_{t-1}||\right)^2||\widetilde{F}_t^{-1}||}{2||F_{t+1}^{-1}d_{t-1}||\left(r_\theta + ||\widehat{\theta}_{t-1}||\right)}, \tag{33}$$

so that (32) becomes simplified as follows

$$q_{t+1}v_t^2(p) + \alpha_{t-1}^2 d_{t-1}^T F_{t+1}^{-1}d_{t-1} \leqslant (1-\lambda_t)\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1}. \tag{34}$$

On the other hand,

$$\nabla V_{t-1}^T d_{t-1} = -d_{t-1}^T \left[F_t^{-1} + 2q_t \varphi_{t-p-1}\varphi_{t-p-1}^T\right]\widetilde{\theta}_{t-1} \leqslant 0 \tag{35}$$

is guaranteed for $d_{t-1} = M_{t-1}(F_t^{-1} + 2q_t\varphi_{t-p-1}\varphi_{t-p-1}^T)\widetilde{\theta}_{t-1}$ with $M_{t-1} = M_{t-1}^T \geqslant 0$ being a symmetric matrix of bounded entries. In view of the choice of (33), the implementability of the adaptation rule (29) requires d_{t-1} to be measurable. Choose $M_{t-1} = F_t\varphi_{t-q}\varphi_{t-q}^T F_t \geqslant 0$, all integers $q \geqslant 0$ which lead to

$$\begin{aligned} d_{t-1} &= F_t\varphi_{t-q}\varphi_{t-q}^T\widetilde{\theta}_{t-1} + 2q_t F_t\varphi_{t-q}\varphi_{t-q}^T F_t\varphi_{t-p-1}\varphi_{t-p-1}^T\widetilde{\theta}_{t-1} && (36.a)\\ &= F_t\varphi_{t-1}v_{t-1}(q) + 2q_t\varphi_{t-q}\varphi_{t-q}^T F_t\varphi_{t-p-1}v_{t-1}(p) && (36.b)\end{aligned}$$

which is measurable from input-output data and updated values of $F_{(\cdot)}$ as well as it guarantees that (35) holds. In order to apply the Armijo rule, one has from (34) – (35)

$$\begin{aligned} |\Delta V_{t-1}| \geqslant &(1-\lambda_t)\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1} - \alpha_{t-1}^2 d_{t-1}^T F_{t+1}^{-1} d_{t-1} - q_{t+1}v_t^2(p)\\ \geqslant &\sigma\beta^{m_{t-1}}|\nabla V_{t-1}^T d_{t-1}| = \sigma\beta^{m_{t-1}} s\widetilde{\theta}_{t-1}^T\left[F_t^{-1} + 2q_t\varphi_{t-p-1}\varphi_{t-p-1}^T\right]\\ &\times F_t\varphi_{t-q}\varphi_{t-q}^T F_t\left[F_t^{-1} + 2q_t\varphi_{t-p-1}\varphi_{t-p-1}^T\right]\widetilde{\theta}_{t-1}. \end{aligned} \tag{37}$$

To simplify the notation, define the auxiliary matrix related to the expressions in (36.a) and (37)

$$\begin{aligned} A_{t-1} =&F_t\varphi_{t-q}\varphi_{t-q}^T\left[I + 2q_t F_t\varphi_{t-p-1}\varphi_{t-p-1}^T\right], && (38.a)\\ B_{t-1} =&\left[I + 2q_t\varphi_{t-p-1}\varphi_{t-p-1}^T F_t\right]\\ &\times \varphi_{t-q}\varphi_{t-q}^T\left[I + 2q_t F_t\varphi_{t-p-1}\varphi_{t-p-1}^T\right], && (38.a)\end{aligned}$$

so that $d_{t-1} = A_{t-1}\widetilde{\theta}_{t-1}$ and the rigth-hand-side of (37) is $\sigma\beta^{m-1}s\widetilde{\theta}_{t-1}^T B_{t-1} \times \widetilde{\theta}_{t-1}$. Thus, Eq. 37 can be compactly rewritten as

$$\begin{aligned} 1-\lambda_t \geqslant = \delta_t = \sigma\beta^{m_{t-1}} s\frac{\widetilde{\theta}_{t-1}^T B_{t-1}\widetilde{\theta}_{t-1}}{\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1}}\\ + \frac{1}{\widetilde{\theta}_{t-1}^T F_t^{-1}\widetilde{\theta}_{t-1}}\left(\alpha_{t-1}^2 d_{t-1}^T F_{t+1}^{-1} d_{t-1} + q_{t+1}v_t^2(p)\right). \end{aligned} \tag{39}$$

The fulfillment of (39) is guaranteed by looking for an upper-bound of $\overline{\delta}_t$ of δ_t as follows

$$\overline{\delta}_t = \lambda_{\max}(F_t)\Bigg[\sigma\beta^{m_{t-1}} s\lambda_{\max}(B_{t-1}) + \frac{\alpha_{t-1}^2 d_{t-1}^T d_{t-1}}{\lambda_{\min}(F_{t+1})} + q_{t+1}\|\varphi_{t-p}\|^2 \frac{\|\widetilde{\theta}_t\|_E^2}{\|\widetilde{\theta}_{t-1}\|_E^2}\Bigg]. \tag{40}$$

But $\|\widetilde{\theta}_t\|_E^2/\|\widetilde{\theta}_{t-1}\|_E^2 \leqslant 2(1+\alpha_{t-1})\lambda_{\max}(A_{t-1}^T A_{t-1})$ which used in (40) ensures $1-\lambda_t \geqslant \overline{\delta}_t \geqslant \delta_t$ if

$$\lambda_t \leqslant 1 - \lambda_{\max}(F_t)\Bigg[\sigma\beta^{m_{t-1}} s\lambda_{\max}(B_{t-1}) + \frac{\alpha_{t-1}^2 d_{t-1}^T d_{t-1}}{\lambda_{\min}(F_{t+1})} + 2q_{t+1}\|\varphi_{t-p}\|^2\big(1+\alpha_{t-1}\lambda_{\max}(A_{t-1}^T A_{t-1})\big)\Bigg]. \tag{41}$$

Note from (33) and (40) that

$$\frac{\alpha_{t-1}^2 d_{t-1}^T d_{t-1}}{\lambda_{\min}(F_{t+1})} \leqslant \frac{\big(q_{t-\mu}v_{t-\mu-1}^2(p) - (r_\theta + \|\widehat{\theta}_{t-1}\|)^2\|\widetilde{F}_t^{-1}\|\big)^2 \lambda_{\max}^2(F_{t+1})}{4(r_\theta + \|\widetilde{\theta}_{t-1}\|)^2 \lambda_{\min}(F_{t+1})} = \frac{1}{4}\frac{\lambda_{\max}^2(F_{t+1})\big(q_{t-\mu}v_{t-\mu-1}^2(p) - (r_\theta + \|\widehat{\theta}_{t-1}\|)^2\|\widetilde{F}_t^{-1}\|\big)^2}{(r_\theta + \|\widehat{\theta}_{t-1}\|)^2 \lambda_{\min}(F_{t+1})}.$$

A problem which could occur is that $F_{(\cdot)}$ could be a divergent matrix sequence, since λ_t in (41) can be negative and thus $\widetilde{F}_t^{-1} > -\lambda_t F_t^{-1}$ in order to achieve nonnegative definiteness of F_t, and then the problem is not well posed as $t \to \infty$ Therefore, the algorithm should be commuted to that of Section 2 after a finite time as time increases. The algorithm to be applied is as follows:

Step 0: Fix a finite $t_1 > 0$

Step 1: Apply the algorithm (29) which is equivalent to $\widehat{\theta}_t = \widehat{\theta}_{t-1} - \alpha_{t-1}\overline{d}_{t-1}$ over $t \in [0, t_1] \cap \mathbf{Z}$ subject to the choice of α_{t-1} and d_{t-1} in (33) and (36), respectively with λ_t subject to (41), with $s > 0$, $\sigma \in (0, 1/2)$, $\beta \in (0,1)$ and $\widetilde{F}_t^{-1} > -\lambda_t F_t^{-1}$.

Step 2: For each $t \geqslant t_1 + i$, $i = 1, 2, \ldots, \mu$, fix $q_t = 0$ and repeat Step 1.

Step 3: When all the μ last q_i have been zeroed and $\alpha_{t-1} = (c_t + \varphi_{t-d}^T \times F_t\varphi_{t-d})^{-1}$, choose $\widetilde{F}_t^{-1} = \frac{\lambda_t}{c_t}\varphi_{t-d}\varphi_{t-d}^T$ with the algorithm becoming the basic one (1) of Section 2 with the only change $\varphi_{t-d} \to \varphi_{t-p}$. Thus, apply any of Propositions 1 to 4 with the change $p \to q$ to implement the adaptive algorithm.

REMARK 2. Note from Step 2 of the above algorithm that a "quasi-Armijo" updating rule is applied since $m_{(\cdot)}$ is not necessarily the smaller nonnegative integer according to the theoretical Armijo rule since $\lambda_{(\cdot)}$ can be negative. Note also that the stability is guaranteed since we switch to the basic algorithm in finite time while respecting from that time the stability constraints of Lozano (1982) for the free-design parameters.

REMARK 3. The general algorithm of Section 3.2 can be extended slightly by considering any of the following Lyapunov-like functions

$$V_t = \sum_{i=0}^{v} \tilde{\theta}_{t-i}^T F_{t+1-i}^{-1} \tilde{\theta}_{t-i} + \sum_{i=0}^{\mu} q_{t-i} (\tilde{\theta}_{t-1-i}^T \varphi_{t-p-i})^2, \tag{43.a}$$

$$V_t = \sum_{i=0}^{v} \tilde{\theta}_{t-i}^T F_{t+1-i}^{-1} \tilde{\theta}_{t-i} + \sum_{i=0}^{\mu} q_{t-i} (\tilde{\theta}_{t-i}^T \varphi_{t-p-i})^2. \tag{43.b}$$

Note that (25) is a particular case of (43.b). Note that the real difference between (43.a) – (43.b) is that the last available regressor weighs the contribution of $\tilde{\theta}_t$ to V_t in (43.b) as it can be seen by expanding the corresponding second right-hand-side terms which yield when $\mu = v$:

$$V_t = \tilde{\theta}_t^T F_{t+1}^{-1} \tilde{\theta}_t + \sum_{i=0}^{\mu-1} \tilde{\theta}_{t-i-1}^T [F_{t-1}^{-1} + q_{t-i} \varphi_{t-p-i} \varphi_{t-p-i}^T] \tilde{\theta}_{t-i-1}$$

$$+ q_{t-\mu} \tilde{\theta}_{t-1-\mu}^T \varphi_{t-p-\mu} \varphi_{t-p-\mu}^T \tilde{\theta}_{t-1-\mu}, \tag{44a}$$

$$V_t = \sum_{i=0}^{\mu} \tilde{\theta}_{t-i}^T [F_{t+1-i}^{-1} + q_{t-i} \varphi_{t-p-i} \varphi_{t-p-i}^T] \tilde{\theta}_{t-i}, \tag{44b}$$

for (43.a) and (43.b), respectively.

3.3. Robustness issues in the presence of unmodelled dynamics

ASSUMPTION 1. It can exist linear unmodelled dynamics in the plant parametrized by an unknown constant $\omega = ||\theta'||$ with $\theta' = (\theta_u'^T, \theta_y'^T)^T \varepsilon \mathbf{R}^{p'+q'}$ of unknown dimensionalities p' and q' but with known upper-bounds $\theta'_* \geqslant ||\theta'||$, $\overline{p} \geqslant p'$ and $\overline{q} \geqslant q'$, $\overline{p}$ and $\overline{q}$ being nonnegative integers. Also, for the parameter vector of the modelled part, a nonnegative constant $\underline{\tilde{\theta}}_*$ is known such that either $||\widehat{\theta}_0|| \geqslant \underline{\widetilde{\theta}}_* + ||\theta||$ or $||\widehat{\theta}_0|| \geqslant ||\theta|| - \underline{\widetilde{\theta}}_*$ for the initial parameter estimates.

Under Assumption 1, the plant parameter vector is $\overline{\theta} = (\theta^T, \theta'^T)^T$. The parameter estimation is based on the modelled part of the plant so that the parametrical error is $\widetilde{\overline{\theta}}(t) = (\widetilde{\theta}^T(t), \theta'^T)^T$. Direct calculus with $\mu = v$ in (43.a) yields after the replacement $\widetilde{\theta}(t) \to \widetilde{\overline{\theta}}(t)$:

$$\Delta V_t = V_{t+1} - V_t = \Delta V_t^0 + g_t, \tag{45}$$

where

$$\begin{aligned}\Delta V_t^0 &= \widetilde{\theta}_t^T F_{t+1}^{-1}\widetilde{\theta}_t - \widetilde{\theta}_{t-\mu}^T F_{t+1-\mu}^{-1}\widetilde{\theta}_{t-\mu} + q_t\widetilde{\theta}_{t-1}^T\varphi_{t-p}\varphi_{t-p}^T\widetilde{\theta}_{t-1} \\ &\quad - q_{t-\mu}\big(\widetilde{\theta}_{t-p-\mu}^T\varphi_{t-p-\mu}\big)^2,\end{aligned} \tag{46.a}$$

$$\begin{aligned}g_t &= q_t\big(\theta'^T\varphi'_{t-p}\varphi'^T_{t-p}\theta' + 2\widetilde{\theta}_{t-1}^T\varphi_{t-p}\varphi'^T_{t-p}\theta'\big) \\ &\quad - q_{t-\mu}\big(\theta'^T\varphi'_{t-p-\mu}\varphi'^T_{t-p-\mu}\theta' + 2\widetilde{\theta}_{t-1}^T\varphi_{t-p-\mu}\varphi'^T_{t-p-\mu}\theta'\big),\end{aligned} \tag{46.b}$$

where V_t is defined in (43.a) with $\widetilde{\overline{\theta}}(t)$, $\Delta V_t^0 = V_{t+1}^0 - V_t^0$, with V_t^0 being now (44.a), is the increment of V_t from the t-th sampling instant by modifying $\widetilde{\theta}_t \to \widetilde{\theta}_{t+1}$ (i.e., the increment $\Delta V_t = V_{t+1} - V_t$ when $\theta' = 0$, and g_t is the contribution to ΔV_t due to θ' (i.e., $g_t = 0$ if $\theta' = 0$). The regressor φ'_t, associated with the unmodelled dynamics , is built at the current sample by adding new components of preceding inputs and outputs to φ_t according to the upper-bounds $\overline{p}$ and $\overline{q}$. If the Lyapunov's-like function (43.b) is used then (45) remains valid by replacing $\widetilde{\theta}_{t-1} \to \widetilde{\theta}_t$ in (46). The parameter-adaptive law is

$$\widetilde{\theta} = \widetilde{\theta}_{t-1} + \alpha_{t-1}d_{t-1} \tag{47}$$

with d_{t-1} to be determined such that $d_{t-1}^T\nabla V_{t-1} \leqslant 0$ for all $t \geqslant 0$ with $\nabla V_{t-1} = F_t^{-1}\widetilde{\theta}_{t-1}$ from inspection of (46) since g_{t-1} does not depend on $\widetilde{\theta}_{t-1}$. Note that ∇V_{t-1} takes the same form when (47) is derived from the Lyapunov's - like function (43.b). Choose

$$d_{t-1} = \varphi_{t-d}^T P\varphi_{t-d}F_t\varphi_{t-d}\varphi_{t-d}^T\widetilde{\theta}_{t-1}, \quad P = P^T > 0, \tag{48}$$

which implies that $d_{t-1}^T\nabla V_{t-1} = -\varphi_{t-d}^T P\varphi_{t-d}(\widetilde{\theta}_{t-1}^T\varphi_{t-d})^2 \leqslant 0$ and leads, by proceeding recursively in (47), to

$$\begin{aligned}&\widetilde{\theta}_{t-j} = \mathbf{A}_{t-j}(\mu)\widetilde{\theta}_{t-\mu}, \\ &F_{t-j}+1 = \lambda_t^{-1}(\mu)\mathbf{A}_{t-j}(\mu)F_{t-\mu}, \quad j = 1, 2, \ldots, \mu,\end{aligned} \tag{49}$$

by applying a similar adaptation rule for the adaptation gain, where

$$\begin{aligned} A_{t-j}(\mu) &= \prod_{i=j}^{\mu} \left[I - \alpha_{t-i}\overline{P}_{t-i}\right]; \\ \lambda_{t+j}(\mu) &= \prod_{i=j}^{\mu} \left[\lambda_j\right]; \\ \overline{P}_{t-i} &= \varphi_{t+1-d-i}^T P \varphi_{t+1-d-i} F_{t+1-i} \varphi_{t+1-d-i} \varphi_{t+1-d-i}^T, \end{aligned} \tag{49.c}$$

for $j = 0, 1, \ldots, m$, which substituted in (45) – (46) yields

$$\begin{aligned} \Delta V_t =& \widetilde{\theta}_{t-\mu}^T \left[\mathbf{A}_t^T(\mu) F_{t+1}^{-1} \mathbf{A}_t(\mu) - F_{t+1-\mu}^{-1} + q_t \mathbf{A}_{t-1}^T(\mu)\varphi_{t-p}\varphi_{t-p}^T \mathbf{A}_{t-1}(\mu) \right. \\ & \left. - q_{t-\mu}\varphi_{t-p-\mu}\varphi_{t-p-\mu}^T\right]\widetilde{\theta}_{t-\mu} \\ & + \theta'^T\left[q_t\varphi'_{t-p}\varphi'^T_{t-p} - q_{t-\mu}\varphi'_{t-p-\mu}\varphi'^T_{t-p-\mu}\right]\theta' \\ & + 2\widetilde{\theta}_{t-\mu}^T \mathbf{A}_{t-1}^T(\mu)\left[q_t\varphi_{t-p}\varphi'^T_{t-p} - q_{t-\mu}\varphi_{t-p-\mu}\varphi'^T_{t-p-\mu}\right]\theta' \\ =& -\left(\widetilde{\theta}_{t-\mu}^T, \theta'^T\right)Q(t)\left(\widetilde{\theta}_{t-\mu}^T, \theta'^T\right)^T; \\ Q(t) =& \text{Block}\left[Q_{ij};\ i,j = 1,2\right], \end{aligned} \tag{50}$$

where

$$\begin{aligned} Q_{11}(t) =& F_{t+1-\mu}^{-1} + q_{t-\mu}\varphi_{t-p-\mu}\varphi_{t-p-\mu}^T - \mathbf{A}_t^T(\mu)F_{t+1}^{-1}\mathbf{A}_t(\mu) \\ & - q_t\mathbf{A}_{t-1}^T(\mu)\varphi_{t-p}\varphi_{t-p}^T\mathbf{A}_{t-1}(\mu), & (51.a) \\ Q_{12}(t) =& Q_{21}^T(t) = \mathbf{A}_{t-1}^T(\mu)\left[q_{t-\mu}\varphi_{t-p-\mu}\varphi'^T_{t-p-\mu} - q_t\varphi_{t-p}\varphi'^T_{t-p}\right], & (51.b) \\ Q_{22}(t) =& q_{t-\mu}\varphi'_{t-p-\mu}\varphi'^T_{t-p-\mu} - q_t\varphi'_{t-p}\varphi'^T - t - p. & (51.c) \end{aligned}$$

By inspection of (51), $Q(t) = Q^T(t) \geqslant 0$ if $Q_{ii}(t) \geqslant 0$ $(i = 1, 2)$ and Block Diag$(Q_{11}, Q_{22}) \geqslant Q -$ Block Diag(Q_{11}, Q_{22}). The analysis is split into three parts, namely:

(a) $Q_{11}(t) \geqslant 0$. This condition is guaranteed from (51.c) if $0 \leqslant q_t \leqslant \left(\varphi'^T_{t-p-\mu}\varphi'_{t-p-\mu}/\varphi'^T_{t-p}\varphi'_{t-p}\right)q_{t-\mu}$;

(b) the use of the above constraint for q_t in (51.a) together with (49.b) yields

$$Q_{11}(t) = \begin{cases} \overline{Q}_{11}(t) & \text{if } q_t = 0 \quad \text{(this always occurs if } \varphi'^T_{t-p-\mu}\varphi'_{t-p-\mu} \neq 0 \text{ from (a),} \\ \overline{Q}'_{11}(t) & \text{if } \overline{q}_t \neq 0, \end{cases} \tag{52.a}$$

$$\begin{aligned}\overline{Q}_{11}(t) =& F_{t+1-\mu}^{-1} - \lambda_t(\mu)\mathbf{A}^T(\mu)F_{t+1}^{-1}\mathbf{A}_t(\mu)\\ =& \left[I - \lambda_t\lambda_{t-1}(\mu)\mathbf{A}_t^T(\mu)\right]F_{t+1-\mu}^{-1},\end{aligned} \tag{52.b}$$

$$\begin{aligned}\overline{Q}'_{11} =& q_t\Bigg[\frac{\varphi'^T_{t-p}\varphi'_{t-p}\varphi_{t-p-\mu}\varphi^T_{t-p-\mu}}{\varphi'^T_{t-p-\mu}\varphi'_{t-p-\mu}}\\ &- \mathbf{A}^T_{t-1}(\mu)\varphi_{t-p}\varphi^T_{t-p}\mathbf{A}_{t-1}(\mu)\Bigg].\end{aligned} \tag{52.c}$$

Note from the first identity in (52.b) that $Q_{11}(t) \geqslant 0$ if $F_{t+1-\mu}^{-1} \geqslant \lambda_t(\mu)\mathbf{A}^T(\mu)F_{t+1}^{-1}(\mu)\mathbf{A}_t(\mu)$ which is guaranteed from the second identity of (52.b) if $\lambda_t \leqslant \lambda_{t-1}^{-1}(\mu)\lambda_{\min}(\mathbf{A}_t^{-T}(\mu))$ and $Q(t) \geqslant 0$ if $\overline{Q}_{11}(t) \geqslant 0$ and

$$\begin{aligned}q_t \leqslant& \left(\varphi^T_{t-p}\mathbf{A}_{t-1}(\mu)\mathbf{A}^T_{t-1}(\mu)\varphi_{t-p}\right)^{-1}\left(\lambda_{\min}(F_{t+1-\mu}^{-1})\right.\\ &\left.- \lambda_t\lambda_{t-1}(\mu)\lambda_{\max}\left(\mathbf{A}_t(\mu)\right)\lambda_{\max}(F_{t+1-\mu}^{-1})\right),\end{aligned}$$

since $\lambda_{\max}\left(\mathbf{A}^T_{t-1}(\mu)\varphi_{t-p}\varphi^T_{t-p}\mathbf{A}_{t-1}(\mu)\right) = \varphi^T_{t-p}\mathbf{A}_{t-1}(\mu)\mathbf{A}^T_{t-1}(\mu)\varphi_{t-p}$.

(c) Note from (50) and (51.b) – (51.c) that $\theta'^T Q_{22}(t)\theta' + 2\widetilde{\theta}'^T_{t-1}Q_{12}(t)\theta' \geqslant 0$ if $\lambda_{\max}(A_{t-1}(\mu)) \leqslant \lambda_{\min}(Q_{22}(t))/\lambda_{\max}(Q_{22}(t))$ and then $|\Delta V_t| \geqslant \widetilde{\theta}^T_{t-\mu}Q_{11}(t)\widetilde{\theta}_{t-\mu}$ with $\Delta V_t \leqslant 0$ for all $t \geqslant 0$.

Now, choose with $\alpha = \dfrac{\rho_{t-1}\varphi^T_{t-d}F_t\varphi_{t-d}\varphi^T_{t-d}P\varphi_{t-d}}{c_{t-1} + \varphi^T_{t-d}F_t\varphi_{t-d}\varphi^T_{t-d}P\varphi_{t-d}}$ with $\rho_{(\cdot)}$ and $c_{(\cdot)}$ being some positive scalars. It follows from (50) that $\mathbf{A}_t(\mu)(I - \alpha_{t-1}\overline{P}_{t-1})$ $\mathbf{A}_{t-1}(\mu)$ has spectral radius less than $1/2$ if $||I - \alpha_{t-1}\overline{P}_{t-1}||$ and $||\mathbf{A}_0(\mu)||_2 \leqslant 1/2$ and $||I - \alpha_{t-1}\overline{P}_{t-1}||_2 < 1$ for all $t \geqslant 0$ to guarantee $||\mathbf{A}_t(\mu)||_2 < 1$ for all $t \geqslant 0$. Direct calculus yields that $||\mathbf{A}_t(\mu)|| < 1$ if $c_i = \rho_i = 0$, $i = 0, 1, \ldots, d-1$, and $c_{d-1} \leqslant (2\rho_{d-1} - 1)\varphi_0^T F_d\varphi_0\varphi_0^T P\varphi_0$; $c_t \leqslant (2\rho_{t-1} - 1)\varphi^T_{t-d+1}F_{t+1}\varphi_{t-d+1}\varphi^T_{t-d+1}P\varphi_{t-d+1}$, all $t \geqslant d$. The above partial results obtained in (a) to (c) can be summarized in the following proposition.

Proposition 5. *Assume that the updating algorithm* (49) *is implemented with*

$$\begin{aligned}&d_{t-1} = -\varphi^T_{t-d}P\varphi_{t-d}F_t\varphi_{t-d}\varphi^T_{t-d}\widetilde{\theta}_{t-1};\\ &\alpha_{t-1} = (\rho_{t-1}\varphi^T_{t-d}P\varphi_{t-d}\varphi^T_{t-d}F_t\varphi_{t-d})/(c_{t-1} + \varphi^T_{t-d}P\varphi_{t-d}\varphi^T_{t-d}F_t\varphi_{t-d}),\end{aligned}$$

with $P = P^T > 0$. Thus, V_t and $||\widehat{\theta}_t||$ are uniformly bounded and converge asymptotically to finite limits.

Note that $|d_{t-1}^T \nabla V_{t-1}| = \varphi_{t-d}^T P \varphi_{t-d}(\widetilde{\theta}_{t-1}^T \varphi_{t-d})^2 \leqslant 2\varphi_{t-d}^T P \varphi_{t-d}(|v_t^0|^2 + \theta_*'^2 \|\varphi_{t-d}'\|^2)$ since $(\widetilde{\theta}_{t-1}'^T \varphi_{t-d})^2 \leqslant 2(|v_t^{0^2}| + \theta_*'^2 \|\varphi_{t-d}'\|^2)$. Direct calculus with (50) yields

$$\begin{aligned}|\Delta V_t| =& (\widetilde{\theta}_{t-\mu}^T, \theta'^T) Q(t) (\widetilde{\theta}_{t-\mu}^T, \theta'^T)^T = \widetilde{\theta}_{t-\mu}^T Q_{11}(t) \widetilde{\theta}_{t-\mu} \\ &+ 2\widetilde{\theta}_{t-\mu}^T Q_{12}(t)\theta' + \theta'^T Q_{22}(t)\theta' \\ &\geqslant |\Delta V_t^0| + [\lambda_{\min}(Q_{22}(t))\theta_*' - 2\lambda_{\max}(Q_{22}(t))\widetilde{\theta}_*]\widetilde{\theta}_*', \end{aligned} \tag{53}$$

where

$$\begin{aligned}|\Delta V_t^0| =& \widetilde{\theta}_{t-\mu}^T Q_{11}(t)\widetilde{\theta}_{t-\mu} \geqslant \lambda_{\min}(\mathbf{A}_{t-\mu}^T(0) Q11(t) \mathbf{A}_{t-\mu}(0)) \|\widetilde{\theta}_0^T\|^2 \\ &\geqslant \lambda_{\min}^2(\mathbf{A}_{t-\mu}(0))\lambda_{\min}(Q_{11}(t))\underline{\widetilde{\theta}}_*^2, \end{aligned} \tag{54}$$

where $\underline{\widetilde{\theta}}_* \leqslant \|\widetilde{\theta}_0\|$ exists from the last part of Assumptions 1. Since, from (51), $\lambda_{\min}^2(\mathbf{A}_{t-\mu}) \geqslant a_{t-1}$; $\lambda_{\min}(Q_{11}(t)) \geqslant b_{t-1}$ with

$$\begin{aligned}a_{t-1} =& \left(1 - \alpha_{t-1}\varphi_{j-d}^T P \varphi_{j-d} \varphi_{j-d}^T F_j \varphi_{j-d}\right)^2 \underline{a}_{t-1}; \\ \underline{a}_{t-1} =& \prod_{i=0}^{t-1} \left(1 - \alpha_j \varphi_{j+1-d}^T P \varphi_{j+1-d} \varphi_{j+1-d}^T \right. \\ & \left. F_{j+1}\varphi_{j+1-d}\right)^2, \text{(55.a)} \\ b_{t-1} =& \frac{1}{\lambda_{\max}(F_{t+1-\mu})} - \lambda_t \lambda_{t-1} \frac{\lambda_{\max}(\mathbf{A}_t(\mu))}{\lambda_{\min}(F_{t+1-\mu}(\mu))} \\ &- q_t \varphi_{t-d}^T \mathbf{A}_{t-1}^T(\mu)\varphi_{t-d}. \end{aligned} \tag{55.b}$$

From (54) - (55), it follows that $\Delta V_t^0 \leqslant 0$ with

$$\begin{aligned}|\Delta V_t^0| \geqslant& \left[(1 - \alpha_{t-1}^0 \varphi_{j-d}^T P \varphi_{j-d}\varphi_{j-d}^T F_j \varphi_{j-d})^2 \underline{a}_{t-1}^2\right] \left[\frac{1}{\lambda_{\max}(F_{t+1-\mu})} \right. \\ &\left. - \lambda_t^0 \lambda_{t-1} \frac{\lambda_{\max}(\mathbf{A}_t(\mu))}{\lambda_{\min}(F_{t+1-\mu})}\right] - q_t \varphi_{t-d}^T \mathbf{A}_{t-1}^2(\mu)\varphi_{t-d} \\ \geqslant& 2\sigma_0 s_0 \beta^{m_{t-1}^0} \varphi_{t-d}^T P \varphi_{t-d}(|v_t^0|^2 + \theta_*'^2 \|\varphi_{t-d}'\|^2). \end{aligned} \tag{56}$$

Similarly, one has for $\alpha_t = \alpha_t^0 + \Delta\alpha_t^0$; $\lambda_t = \lambda_t^0 + \Delta\lambda_t^0$; $\sigma = \sigma_0 + \Delta\sigma_0$; $m_t = m_t^0 + \Delta m_t^0$ and $\Delta V_t > 0$ is guaranteed if (56) holds and, furthermore,

$$\begin{aligned}&\Delta\alpha\left[\Delta\alpha + 2(\alpha_0 - 1)\right]\underline{a}_{t-1} - \lambda_t^0 \Delta\lambda_t^0 \lambda_{t-1}(\mu) \frac{\lambda_{\max}(\mathbf{A}_t(\mu))}{\lambda_{\min}(F_{t+1-\mu}(\mu))} \\ &\geqslant 2\beta^{m_{t-1}^0}\left[\sigma_0 s_0(\beta^{\Delta m_t^0} - 1) + (s_0\Delta\sigma + \sigma_0\Delta s + \Delta\sigma\Delta s)\beta^{\Delta m_t^0}\right]\varphi_{t-d}^T P \varphi_{t-d} \\ &\times \left(|v_t^0|^2 + \theta_*'^2 \|\varphi_{t-d}'\|^2\right) + 2q_{t-\mu}\|\varphi_{t-p-\mu}'\|^2 \theta_*'(\theta_*' + \widetilde{\theta}_*) \end{aligned} \tag{57}$$

which holds since $\lambda_{\max}(Q_{22}(t)) \leqslant 2q_{t-\mu}\varphi'^T_{t-p-\mu}\varphi'_{t-p-\mu}$ from (51.c) and the choice of q_t in the part (a) of the proof. Note that (57) always holds if $q_{t-\mu} = \Delta\sigma = \Delta m_t^0 = 0$ for all $t \geqslant 0$. Thus, (57) holds if

$$0 \leqslant q_{t-\mu} \leqslant \max\left(0, q^1_{t-\mu}\right),$$
$$\Delta = \left(\Delta\alpha, \Delta\lambda:\ \alpha_t \leqslant \frac{1}{2}, \lambda_t < 1\right) \quad \text{for all} \quad t \geqslant 0,$$

$$\begin{aligned} q'_{t-\mu} = &\Big\{\mathrm{Max}\Big\{\Delta\alpha[\Delta\alpha + 2(\alpha_0 - 1)]\underline{\widetilde{\alpha}}_{t-1} \\ &- \lambda_t^0\Delta\lambda_t^0\lambda_{t-1}(\mu)\frac{\lambda_{\max}(\mathbf{A}_t(\mu))}{\lambda_{\max}(F_{t+1-\mu}(\mu))}\Big\} - 2\beta^{m^0_{t-1}}\left[\alpha_0 s_0\left(\beta^{\Delta m_t^0} - 1\right)\right. \\ &\left.+ \left(s_0\Delta\sigma + \sigma_0\Delta s + \Delta\sigma\Delta s\right)\beta^{\Delta m_t^0}\right]\Big\}\frac{1}{2\|\varphi'_{t-p+\mu}\|^2\theta'_*(\theta'_* + \widetilde{\theta}_*)}. \end{aligned} \tag{58}$$

Now, the following result follows.

Proposition 6. *Assume that Assumption 1 hold and choose the sequence $\{q_t,\ t \geqslant 0\}$ so that Condition (c) and (58) are satisfied simultaneously. Assume also that $(\lambda_t^0, \alpha_t^0)$ are chosen so that (56) holds for some $s_0 \,\varepsilon\, (0, 1/2)$, $s > 0$, m_t^0, being a positive integer. Then, Proposition 1 holds provided that q_t does not converge to zero as $t \to \infty$. If q_t converges to zero as $t \to \infty$ then all the results of Proposition 1 hold except for the uniform boundedness of $\{\|\varphi_t,\ t \geqslant 0\|\}$.*

The proof follows as in Proposition 1.

Note that the fact that $\widetilde{\theta}_*$ is a known upper-bound of $\widetilde{\theta}(t)$ is not a key fact to ensure the boundedness of V_t and then that of $\widetilde{\theta}(t)$ since $\widetilde{\theta}_*$ can be fixed sufficiently large and then proving the uniform boundedness of V_{t-1}. It follows that the parametrical error $\widetilde{\theta}(t)$ cannot diverge from the definition of V_t in (43.a) and the uniform boundedness of that function. Note also that Proposition 6 follows with minor modifications if the updating algorithm is derived from the Lyapunov's-like function (43.b).

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ADAPTYVŪS DISKRETINIAI ALGORITMAI SU PAGERINTOMIS PEREINAMOJO PROCESO CHARAKTERISTIKOMIS

Manuel De la SEN

Straipsnyje nagrinėjama gradientinių diskretinių algoritmų konvergavimo taisyklių panaudojimo galimybės pagerinti tiesinių invariantinių sistemų adaptyvaus valdymo algoritmų, sudarytų pagal Liapunovo tipo funkcijas, adaptacijos proceso pereinamasias charakteristikas. Tuo atveju, kai naudojama Armijo taisyklė, adaptacijos greitis didėja, jeigu didėja apibendrinta arba filtruota paklaida, reguliuojanti kiekvienos Liapunovo tipo funkcijos mažėjimą. Pasiūlytas algoritmas gali būti realizuotas, kai apie sistemos dinamiką žinome labai mažai, t.y., žinome parametrų vektoriaus dimensijos viršutinę ribą ir normą.