INFORMATICA, 1997, Vol. 8, No. 2, 273–288

COMPUTING COEFFICIENTS OF FOURIER SERIES

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Abstract. In this paper, the following questions for computing coefficients of Fourier series are discussed: n-order Filon quadrature formula and its partial cases, some features of applying the Filon method in computing coefficients when the adaptive integration strategy is employed, the program implementation of 3-order and 5-order Filon quadrature formulas, using the adaptive integration strategy, and the experimental results of applying them in computing coefficients of Fourier series.

Key words: Fourier series, n-order Filon quadrature formula, adaptive integration.

1. Statement of the problem. The Fourier series for a periodical function $f(x)$ with a period $2l$ can be written

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right)$$

where $a_0 = \frac{1}{l} \int\limits_{-l}^{l} f(x)\, dx$,

$$a_n = \frac{1}{l} \int\limits_{-l}^{l} f(x) \cos \frac{n\pi}{l} x\, dx, \quad b_n = \frac{1}{l} \int\limits_{-l}^{l} f(x) \sin \frac{n\pi}{l} x\, dx.$$

Hence, for computing coefficients of Fourier series it is necessary to find values of the integrals $\int\limits_{-l}^{l} f(x) \cos kx\, dx$ and $\int\limits_{-l}^{l} f(x) \sin kx\, dx$ with some given precision ε.

In numeric integration, the videly used Newton-Cotes and Gaussian quadrature formulas for these integrals are not the best choice. For computation of these integrals, quadrature formulas taking into account the character of integrand function are needed. The first such quadrature formula was suggested by Filon in 1928 (Hamming, 1962; Tranter, 1956).

If we want to evaluate coefficients of Fourier series of many different frequencies, we can use DFT (Press *et al.,* 1992). This method, however, is not free from drawbacks.

- While computing Fourier coefficients at fixed frequencies, it is necessary before doing this to obtain them for the special values of frequencies $\omega_n = 2\pi n/N\Delta$, $n = 0, \dots, N-1$, where N is an integer power of 2, Δ is the length of the integration step, and afterwords to apply an interpolation.
- In the process of coefficient computation, this method involves an integration strategy based on the use of the same step in the whole integration interval. This results in the computation of a larger number of the function $f(x)$ values.
- It is difficull to evaluate the computation error.

A method for computing coefficients of Fourier series proposed in this paper has the following features which are not pertinent to the DFT method.

- The method allows to compute the coefficients of Fourier series at the fixed frequencies with a given precision.
- To obtain the result, an adaptive integration strategy is employed, which allows to compute an integral value with any desirable precision, using the smallest number of integrand function values.
- The method allows to create an universal procedure suitable for computing Fourier coefficients, using polynomial of any degree for approximation of the integrand function $f(x)$.

In this work, the following questions are considered:

1) n-order Filon quadrature formula and partial its cases for computing coefficients of Fourier series when the function $f(x)$ is being replaced by 3-order and 5-order Hermitian and Lagrangian polynomials;
2) some features of applying the Filon method for computing coefficients of Fourier series when the adaptive integration strategy is employed;
3) the experimental results of applying mentioned above quadrature formulas, using the adaptive integration strategy, for computing coefficients of Fourier series.

These questions were discussed in the reports presented at the conferences (Plukienė and Plukas, 1995; 1995b). This paper contains the generalized results of an investigation.

2. n-order Filon quadrature formula for coefficients of Fourier series. Assume we are asked to compute the value of the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\cos kx\,dx$. The idea of Filon method is very simple. The function $f(x)$ in the interval $[x_{i-1}, x_{i+1}]$ is replaced be the quadratic interpolating polynomial $y(x) = a + b(x - x_i) + c(x - x_i)^2$, going through the points $(x_{i-1}, f_{i-1}), (x_i, f_i)$, (x_{i+1}, f_{i+1}); here $x_i = (x_{i+1} + x_{i-1})/2$, and f_i is an abbreviation of $f(x_i)$. Then (cf. Tranter, 1956)

$$
\begin{aligned}
R = &\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\cos kx\,dx \approx \int\limits_{x_{i-1}}^{x_{i+1}} y(x)\cos kx\,dx \\
= &\frac{1}{k}\Bigg(\left(f_{i+1} - \frac{2c}{k^2}\right)\sin kx_{i+1} - \left(f_{i-1} - \frac{2c}{k^2}\right)\sin kx_{i-1} \\
&+ \frac{1}{k}\left(\frac{3f_{i+1} - 4f_i + f_{i-1}}{2h}\cos kx_{i+1} - \frac{4f_i - f_{i+1} - 3f_{i-1}}{2h}\cos kx_{i-1}\right)\Bigg),
\end{aligned}
$$

where $h = \dfrac{x_{i+1} - x_{i-1}}{2}$, $c = \dfrac{f_{i+1} - 2f_i + f_{i-1}}{2h^2}$.

An analogous quadrature formula for the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\sin kx\,dx$ is also given in (Tranter, 1956).

Armed with this idea, we can give the n-order Filon quadrature formula for coefficients of Fourier series and develop an efficient algorithm of its calculation.

Suppose the n-order polynomial

$$y(x) = a_n(x - x_i)^n + a_{n-1}(x - x_i)^{n-1} + \ldots + a_1(x - x_i) + a_0 \quad (1)$$

aproximates function $f(x)$ defined in the interval $[x_{i-1}, x_{i+1}]$; here and in the text below $x_i = (x_{i+1} + x_{i-1})/2$. Then

$$\begin{aligned} R_C &= \int\limits_{x_{i-1}}^{x_{i+1}} f(x)\cos kx\,dx \approx \int\limits_{x_{i-1}}^{x_{i+1}} y(x)\cos kx\,dx = \sum_{j=0}^{n} a_j C_j, \\ R_S &= \int\limits_{x_{i-1}}^{x_{i+1}} f(x)\sin kx\,dx \approx \int\limits_{x_{i-1}}^{x_{i+1}} y(x)\sin kx\,dx = \sum_{j=0}^{n} a_j S_j, \end{aligned} \tag{2}$$

where

$$C_j = \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^j \cos kx\,dx, \quad S_j = \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^j \sin kx\,dx.$$

For computing these integrals it is resonable to use the recurence formulas. Obviously,

$$\begin{aligned} C_0 &= \int\limits_{x_{i-1}}^{x_{i+1}} \cos kx\,dx = \frac{1}{k}(\sin kx_{i+1} - \sin kx_{i-1}), \\ S_0 &= \int\limits_{x_{i-1}}^{x_{i+1}} \sin kx\,dx = -\frac{1}{k}(\cos kx_{i+1} - \cos kx_{i-1}). \end{aligned} \tag{3}$$

We can integrate C_j and S_j by parts

$$\begin{aligned} C_j &= \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^j \cos kx\,dx = \left.\frac{(x - x_i)^j \sin kx}{k}\right|_{x_{i-1}}^{x_{i+1}} \\ &\quad - \frac{j}{k} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^{j-1} \sin kx\,dx \\ &= \frac{h^j}{k}(\sin kx_{i+1} + (-1)^{j+1} \sin kx_{i-1}) - \frac{j}{k} S_{j-1}, \\ S_j &= \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^j \sin kx\,dx = -\left.\frac{(x - x_i)^j \cos kx}{k}\right|_{x_{i-1}}^{x_{i+1}} \\ &\quad + \frac{j}{k} \int\limits_{x_{i-1}}^{x_{i+1}} (x - x_i)^{j-1} \cos kx\,dx \\ &= \frac{j}{k} C_{j-1} - \frac{h^j}{k}(\cos kx_{i+1} + (-1)^{j+1} \cos kx_{i-1}), \end{aligned}$$

here and in the text below $h = (x_{i+1} - x_{i-1})/2$. By introducing additional variables, we can write

$$\begin{aligned} C_j &= (h^j t_S - jS_{j-1})/k, \\ S_j &= (jC_{j-1} - h^j t_C)/k,\ j \geqslant 1, \end{aligned} \tag{4}$$

where C_0 and S_0 are integrals expressed by (3), and

$$\begin{aligned} t_S &= \sin kx_{i+1} + (-1)^{j+1} \sin kx_{i-1}, \\ t_C &= \cos kx_{i+1} + (-1)^{j+1} \cos kx_{i-1}, \end{aligned}$$

By virtue of (4), for computing the quadrature formulas (2) it is convenient to apply the following algorithm stated using the Pascal language syntax.

3. An algorithm for computing the integrals $\int_{x_1}^{x_2} f(x)\cos kx\,dx$ and $\int_{x_1}^{x_2} f(x)\sin kx\,dx$ using n-order Filon quadrature formula.

Input: the interval of integration $[x1, x2]$, $(x1 \leqslant x2)$, the order n of polynomial (1), and the array $(a_0, a_1, \ldots, a_n)$ of the coefficients of (1).

Output: $rc = \sum_{j=0}^{n} a_j C_j$ and $rs = \sum_{j=0}^{n} a_j S_j$.

```
begin
    rc := 0; rs := 0; h := (x2 - x1)/2;
    c1 := cos(k * x1); c2 := cos(k * x2);
    s1 := sin(k * x1); s2 := sin(k * x2);
    tcp := c1 + c2; tcm := c2 - c1;
    tsp := s1 + s2; tsm := s2 - s1;
    cj := 0; sj := 0; hj := 1;
    for j := 0 to n do
        begin
            if j mod 2 = 0    then    begin  ts := tsm;
                                              tc := tcm;
                                      end
                              else    begin  ts := tsp;
                                              tc := tcp;
                                      end;
```

```
        ct := (hj * ts - j * sj)/k;
        st := (j * cj - hj * tc)/k;
        cj := ct; sj := st;
        rc := rc + a[j] * cj;
        rs := rs + a[j] * sj;
        hj := hj * h
        end;
end;
```

4. Specific realizations of the approximating function $y(x)$. We now discuss different realizations of the approximating function $y = f(x)$ and the remainder terms of the quadrature formulas obtained.

1. *Filon method.* As already mentioned above, in this case

$$y(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2$$

is the Lagrange interpolating polynomial obeying the equalities $y(x_j) = f_j$, $j = i-1, i, i+1$. This condition allows to write

$$\begin{aligned} a_0 &= f_i, \\ a_1 &= \frac{f_{i+1} - f_{i-1}}{2h}, \\ a_2 &= \frac{f_{i+1} - 2f_i + f_{i-1}}{2h^2}. \end{aligned} \tag{5}$$

As shown in (Plukienė and Plukas, 1995), the remainder term of the Fourier series coefficients obtained using Filon method satisfies the following inequality

$$|E_F| \leqslant \frac{2l}{45}\sqrt{\left(\frac{f^{(4)}(c_1)}{4}\right)^2 + \left(kf^{(3)}(c_2)\right)^2}\, h^4, \quad \text{here } c_1, c_2 \in [-l, l].$$

2. *Third order Filon quadrature formula.* In this case the function $y(x)$ is the third order Hermitian polynomial $y(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3$, satisfying the following conditions

$$\begin{aligned} &y(x_j) = f_j, \ j = i-1, i, i+1, \\ &y'(x_i) = f_i'. \end{aligned}$$

It follows from above that

$$
\begin{aligned}
a_0 &= f_i, \\
a_1 &= f_i', \\
a_2 &= \frac{f_{i+1} - 2f_i + f_{i-1}}{2h^2}, \\
a_3 &= \frac{f_{i+1} - 2f_i'h - f_{i-1}}{2h^3}.
\end{aligned}
\tag{6}
$$

The remainder term of this third order Hermitian polynomial is $\frac{f^{(3)}(c)}{3!}\Omega_1(x)$, where $c \in [x_{i-1}, x_{i+1}]$, and $\Omega_1(x) = (x - x_{i-1})(x - x_i)^2(x - x_{i+1})$ (Kvedaras and Sapagovas, 1974). So the remainder term of quadrature formula (6) for the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x) \cos kx \, dx$ is expressed as follows

$$
E = \int\limits_{x_{i-1}}^{x_{i+1}} \frac{f^{(3)}(c)}{3!}\Omega_1(x) \cos kx \, dx.
$$

Observing that $\Omega_1(x) \leqslant 0$ when $x \in [x_{i-1}, x_{i+1}]$ and letting the function $f^{(3)}(x)$ be bounded in the interval $[x_{i-1}, x_{i+1}]$, we can give evaluation of the remainder term. Thus

$$
|E| \leqslant \frac{2}{45} M_3 h^5, \quad \text{here } M_3 = \max_{x \in [x_{i-1}, x_{i+1}]} |f^{(3)}(x)|.
$$

It is easy to see that the remainder term E_F of the Fourier series coefficients obtained using the quadrature formula (6) satisfies the following inequality

$$
|E_F| \leqslant \frac{2l}{45} M_3^* h^4, \quad \text{here } M_3^* = \max_{x \in [-l, l]} |f^{(3)}(x)|.
$$

3. *The 5th order Filon quadrature formula.* In this case the function $y(x)$ is 5th order Hermitian polynomial

$$
y(x) = a_0 + a_1(x - x_i) + a_2(x - x_i)^2 + a_3(x - x_i)^3 + a_4(x - x_i)^4 + a_5(x - x_i)^5,
$$

satisfying the following conditions

$$
\begin{aligned}
y(x_j) &= f_j, \\
y'(x_j) &= f_j', \; j = i - 1, i, i + 1.
\end{aligned}
$$

These conditions imply that

$$\begin{aligned} a_0 &= f_i, & a_3 &= \frac{5t_2 - t_3}{4h^3}, \\ a_1 &= f_i', & a_4 &= \frac{t_4 - 2t_1}{4h^4}, \\ a_2 &= \frac{4t_1 - t_4}{4h^2}, & a_5 &= \frac{t_3 - 3t_2}{4h^5}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} t_1 &= f_{i+1} - 2f_i + f_{i-1}, \\ t_2 &= f_{i+1} - 2f_i'h - f_{i-1}, \\ t_3 &= (f_{i+1}' - 2f_i' + f_{i-1}')h, \\ t_4 &= (f_{i+1}' - f_{i-1}')h. \end{aligned}$$

Since the remainder term of the 5th order Hermitian polynomian is $\frac{f^{(5)}(c)}{5!}\Omega_2(x)$, where $c \in [x_{i-1}, x_{i+1}]$, and $\Omega_2(x) = (x - x_{i-1})^2(x - x_i)^2(x - x_{i+1})^2$ (Kvedaras and Sapagovas, 1974), it follows that the remainder term of the quadrature formula (7) for the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\cos kx\,dx$ can be obtained using the following formula

$$E = \int\limits_{x_{i-1}}^{x_{i+1}} \frac{f^{(5)}(c)}{5!}\Omega_2(x)\cos kx\,dx. \tag{8}$$

Observing that $\Omega_2(x) \geqslant 0$ when $x \in [x_{i-1}, x_{i+1}]$ and letting the function $f^{(5)}(x)$ be bounded in the interval $[x_{i-1}, x_{i+1}]$, we can give evaluation of the remainder term. Hence

$$|E| \leqslant \frac{2}{1575}M_5h^7, \quad \text{here } M_5 = \max_{x\in[x_{i-1},x_{i+1}]} |f^{(5)}(x)|.$$

Obviously, the remainder term E_F of the Fourier series coefficients computed using the quadrature formula (7) satisfies the following inequality

$$|E_F| \leqslant \frac{2l}{1575}M_5^*h^6, \quad \text{here } M_5^* = \max_{x\in[-l,l]} |f^{(5)}(x)|.$$

In practise, the remainder terms of quadrature formulas are evaluated aplying the Richardson-Romberg method (Forsythe, Malcolm and Moler, 1980). Suppose the remainder term of quadrature formula is proportional to h^p . For example, for quadrqture formulas (5) and (6) $p = 4$, while for (7) $p = 6$. Then the absolute value of E can be expressed

$$|E| = \left|\frac{R_{h/2} - R_h}{2^p - 1}\right|,$$

where R_h and $R_{h/2}$ are the values of the integral obtained using the integration steps h and $h/2$, respectively.

5. An adaptive integration. Suppose we are asked to find the value $R = \int_a^b y(x)\,dx$ with some given precision ε. We can use a quadrature formula for integration the remainder term of which is proportional to h^p, where h is an integration step. An adaptive integration strategy can be formulated as follows.

1. Set the length of the integration interval $H = b - a$.
2. Apply quadrature formula in the interval $[a, b]$ to obtain the interval value R_h, here h stands for the integration step.
3. Using integration step $h/2$, compute $R_{h/2} := R^l_{h/2} + R^r_{h/2}$, where $R^l_{h/2}$ and $R^r_{h/2}$ are the values of the integral R in the left and right half-intervals of the integration interval, respectively.
4. If $\left|\dfrac{R_{h/2} - R_h}{2^p - 1}\right| > \dfrac{H}{b-a}\varepsilon$, then the required precision of integration in the current intrerval is not attained. In this case, keep the following values: the integral value $R^r_{h/2}$ on the right half-interval, x-coordinates of integration and coresponding values of the integrand function. Take integration interval := left half-interval, $R_h := R^l_{h/2}$, $H := H/2$ and return to 3.

   If $\left|\dfrac{R_{h/2} - R_h}{2^p - 1}\right| \leqslant \dfrac{H}{b-a}\varepsilon$, then the required precision of integration in the current interval (of length H)is already $R_H := R_{h/2} + \dfrac{R_{h/2} - R_h}{2^p - 1}$.

   Take the right half-interval (if any) letting $R_H := R^r_{h/2}$, extract x-coordinates of integration over the right half-interval and corresponding values of the integrand function stored earlier, integration interval := right half-interval, compute the length H of the integration interval and return to 3.

If the list of right half-integrals is empty, then stop. The integral value R is equal to the sum of all R_H with precision ε.

We apply this strategy of adaptive integration for computing coefficients of Forier series, using quadrature formulas given above.

For Filon quadrature formula, there exists such a mesh x_{i-1}, x_i, x_{i+1} that $R_h = R_{h/2}$ independently of the integrand function $f(x)$. Consequently, the estimate of the remainder term based on the Richardson-Romberg method is erroneous. As shown in (Plukienė and Plukas, 1995), such a degenerate mesh for the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\cos kx\,dx$ is $x_{i-1} = (2l - t + 1)\frac{\pi}{2k}$, $h = \frac{t\pi}{k}$, while for the integral $\int\limits_{x_{i-1}}^{x_{i+1}} f(x)\sin kx\,dx$ such the analogous mesh is $x_{i-1} = (2l - t)\frac{\pi}{2k}$, $h = \frac{t\pi}{k}$, where $l = 0, \pm 1, \pm 2, \ldots$, $t = 1, 2, 3, \ldots$. In (Plukienė and Plukas, 1995), a refinement of the adaptive integration strategy is discussed which allows ascertain and adjust a degenerate mesh. The problem of the degenerate mesh ceases whenever the quadrature formulas (6) and (7) are used.

6. Program implementation. For computing coefficients of Fourier series, the PASCAL-language procedures *Fourier, Fourier3 and Fourier5*, using the formulas (5), (6) and (7) respectively, were written. While creating these procedures, the structure of procedure $rc8$ (Forsythe, Malcolm and Moler, 1980) was taken into account. In this procedure, an adaptive integration strategy with 8th order Newton-Cotes quadrature formula employed is realized.

The results of experimentation with the procedures *Fourier, Fourier3 and Fourier5* are given bellow.

We applied our procedures for computing coefficients of Fourier series for the functions $f(x) = e^x$ and $f(x) = \frac{1}{1+x^2}$. The precission was chosen equal to 10^{-8}. At the top of each table the function and the length of the half-period are shown. The first column of each table indicates coefficient of the series (be giving its index n). The rest columns display the number of function values needed for computing coefficients of Fourier series with a specified precision.

In the case of computing Fourier coefficients by the DFT method, the integrand function $f(x)$ is approximated by the Lagrangian interpolating polynomial.

Table 1. $f(x) = e^x$, $l = 2$

n	Fourier	Fourier3	Fourier5
1	375	344	58
5	539	424	58
10	654	408	50
15	731	392	34
20	766	352	26
25	791	256	26
30	830	336	18
35	831	304	10
40	850	360	10
45	863	392	10
50	886	368	10
55	903	250	10
60	930	329	10
70	982	129	10
80	1030	241	10
90	962	241	10
100	1086	153	10
150	1038	121	10

Table 2. $f(x) = e^x$, $l = 10$

n	Fourier	Fourier3	Fourier5
10	1890	872	154
20	2762	976	138
30	3542	1032	106
40	4182	1104	130
50	4706	1056	146
60	4710	1200	130
70	4714	1232	138
80	–	1288	106
90	–	1298	114
100	–	1104	106

Dashes in the second column mean that the required precision of 10^{-8} was not attained after using 5000 values of $f(x)$.

Table 3. $f(x) = \dfrac{1}{1+x^2}$, $l = 2$

n	Fourier	Fourier3	Fourier5
10	654	505	114
20	694	465	98
30	710	409	50
40	734	393	10
50	738	393	10
60	782	337	10
100	1010	313	10

Table 4. $f(x) = \dfrac{1}{1+x^2}$, $l = 10$

n	Fourier	Fourier3	Fourier5
10	1284	1456	274
20	1434	1424	274
30	1646	1408	242
50	1974	1264	210
70	2130	1264	194
90	2150	1184	194
100	2278	1074	10

The algorithm given earlier for computing the integrals $\int\limits_c^d f(x)\cos kx\,dx$ and $\int\limits_c^d f(x)\sin kx\,dx$ allows to compare experimentally the accuracy of the quadrature formulas based on the Lagrangian and Hermitian polynomials.

The coefficients of the cubic polynomial

$$y = a_3(x-z)^3 + a_2(x-z)^2 + a_1(x-z) + a_0,$$

defined by four neighboring points (x_k, y_k), $k = 0, \dots, 3$, laying at the same distance one from another are given by the following expressions

$$a_0 = \frac{9(y_1+y_2)-(y_0+y_3)}{16}, \quad a_1 = \frac{27(y_2-y_1)-(y_3-y_0)}{16h},$$

$$a_2 = \frac{9(y_0 - y_1 - y_2 + y_3)}{16h^2}, \quad a_3 = \frac{9(y_3 - y_0 - 3(y_2 - y_1))}{16h^3},$$

where $z = (x_0 + x_3)/2$, and $h = (x_3 - x_0)/2$.

Analogously, coefficients of the 5-order polynomial

$$y = a_5(x-z)^5 + a_4(x-z)^4 + a_3(x-z)^3 + a_2(x-z)^2 + a_1(x-z) + a_0,$$

defined by points (x_k, y_k), $k = 0, \dots, 5$, laying at the same distance one from another are expressed by the following formulas

$$\begin{aligned}
a_4 &= \frac{625}{384h^4}\left(\frac{1}{2}(y_0 + y_5) - \frac{3}{2}(y_1 + y_4) + y_2 + y_3\right), \\
a_2 &= -\frac{25}{192h^2}\left(\frac{5}{2}(y_0 + y_5) - \frac{39}{2}(y_1 + y_4) + 17(y_2 + y_3)\right), \\
a_0 &= \frac{1}{2}(y_0 + y_5) - a_4h^4 - a_2h^2, \\
a_5 &= \frac{625}{384h^5}\left(\frac{1}{2}(y_5 - y_0) - \frac{5}{2}(y_4 - y_1) + 5(y_3 - y_2)\right), \\
a_3 &= -\frac{25}{192h^3}\left(\frac{5}{2}(y_5 - y_0) - \frac{195}{6}(y_4 - y_1) + 85(y_3 - y_2)\right), \\
a_1 &= \frac{1}{h}\left(\frac{1}{2}(y_5 - y_0) - a_5h^5 - a_3h^3\right),
\end{aligned} \tag{9}$$

where $z = (x_0 + x_5)/2$, and $h = (x_5 - x_0)/2$.

Applying the above mentioned algorithm to the examples considered earlier and choosing the integration strategy based on the same step in the whole interval, the following results were obtained.

- Quadrature formulas using 3-order Lagrangian and Hermitian polynomials essentialy are the same accuracy, that is in both cases the coefficients of the same order of accuracy are obtained provided the same number of integrand function values is used.
- The quadrature formula (7) is considerably more accurate then the quadrature formula (9). To obtain the same occuracy, the quadrature formula based on the 5-order Hermitian polynomial uses on the average ten times less integrand function values than the formula based on 5-order Lagrangian polynomial.

7. Conclusions

1. In this work, we investigated the n-order Filon quadrature formula for coefficients of Fourier series which allows in aproximate the function $f(x)$ by the Hermitian or Lagrangian interpolating polynomial of the desired order.
2. Although computing of the 5th order Hermitian interpolating polynomial requires more values of $f(x)$ and $f'(x)$ at each iteration than for polynomials of lower order, using this polynomial for computing coefficients of Fourier series is considerably more efficient than in the case where lower order polynomials are employed. This can be explained by the fact that the polynomial approximates the function $f(x)$ in each interval $[x_{i-1}, x_{i+1}]$ with a greater accuracy than the lower order polynomials.

REFERENCES

Hamming, R.W. (1962). *Numerical Methods for Scientists and Engineers.* McGraw-Hill, New-York.

Tranter, K.J. (1956). *Integral Transformations in Mathematical Physics.* GITTL, Moscow (in Russian).

Press, W.H., S.A. Teukolsky, W.T. Vetterling and B.P. Flannery (1992). *Numerical Recipes in C.* Cambridge University Press.

Forsythe, G.E., M.A. Malcolm and C.B. Moler (1980). *Computer Methods for Mathematical Computations.* Moscow (in Russian).

Plukienė, D., and K. Plukas (1995). Adaptive integration using Filon method. Mathematics and mathematical simulation. In *Proceed. of the Conference "Lithuanian Science and Industry",* Kaunas University of Technology. pp. 54–58 (in Lithuanian).

Plukienė, D., and K. Plukas (1995). A refinement of Filon method. In *Abstracts of XXXVI Conference of the Lith. Math. Society,* Vilnius. pp. 161–162 (in Lithuanian).

Kvedaras, B., and M. Sapagovas (1974). *Numerical Methods.* Mintis, Vilnius (in Lithuanian).

Received January 1997

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FURJE EILUTĖS KOEFICIENTŲ APSKAIČIAVIMAS

Kostas Plukas ir Danutė Plukienė

Darbe nagrinėjami tokie Furje eilutės koeficientų apskaičiavimo klausimai: n-tosios eilės Filono kvadratūrinė formulė ir jos atskiri atvejai, kai periodinė funkcija keičiama 3-osios ir 5-osios eilės Ermito ir Lagranžo interpoliaciniais polinomais; Furje eilutės koeficientų apskaičiavimo Filono metodu ypatumai, kai naudojama adaptyviojo integravimo strategija; minėtų kvadratūrinių formulių programinė realizacija, panaudojant adaptyviojo integravimo strategiją ir jų eksperimentinis tyrimas, skaičiuojant Furje eilutės koeficientus.