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A SET OF EXAMPLES OF GLOBAL AND DISCRETE OPTIMIZATION: APPLICATION OF BAYESIAN HEURISTIC APPROACH I

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Abstract. The following topics are important teaching operation research:

- games theory;
- decision theory;
- utility theory;
- queuing theory;
- scheduling theory;
- discrete optimization.

These topics are illustrated and the connection with global optimization is shown considering the following mathematical models:

- competition model with fixed resource prices, Nash equilibrium;
- competition model with free resource prices, Walras equilibrium;
- Inspector's problem, multi-stage game model;
- "Star War" problem, differential game model;
- "Portfolio" problem, resource investment model;
- exchange rate prediction, auto-regression-moving-average (ARMA) model;
- optimal scheduling, Bayesian heuristic model;
- "Bride's" problem, sequential statistical decisions model.

The first seven models are solved using a set of algorithms of continuous global and stochastic optimization. The global optimization software GM (see Mockus, 1996) is used. The underlying theory of this software and algorithms of solution are described in Mockus (1989, 1996). The last model is an example of stochastic dynamic programming.

For better understanding, all the models are formulated in simplest terms as "class-room" examples. However, each of these models can be regarded as simple representations of important families of real-life problems. Therefore the models and solution algorithms may be of interest for application experts, too.

The paper is split into two parts. In the part one the first five models are described. In the part two the rest three models and accompanying software are considered.

Key words: operations research, Bayesian, heuristic, optimization, global.

1. Optimization problems in competition models

1.1. Introduction. We consider optimization problems of a simple competitive model. There are several servers providing the same service. Each server fixes the price and the capacity of service. The servers capacity determines the average rate of served customers so defining the customers time losses while waiting in queue for the service. The customer goes to the server with less total service cost. The total cost includes the service price plus waiting losses. The customer goes away, if the total cost exceeds a certain critical level. Both the flow of customers and the service time are stochastic. There is no known analytical solution for this model. The results are obtained by Monte-Carlo simulation. The analytical solution of a simplified model is considered, too.

The model is used to illustrate the possibilities and limitations of the optimization theory and numerical techniques in competitive models. We consider optimization in two different mathematical frameworks: the fixed point and Lagrange multipliers. Two different economic and social objectives are considered: the equilibrium and the social cost minimization.

In the Nash case the servers rent their service equipment at fixed price per unit capacity[1]. We are looking for the equilibrium capacities and service prices. In the Walras case servers share resources which they own. Therefore we are looking not only for the equilibrium capacities and service prices but also for the prices of shared resources under the condition that a server cannot pay more than it gets.

The competitive model is applied as a test function for the Bayesian algorithms. However, the simple model may help to design more realistic ones describing the processes of competition better.

2. Competition model with fixed resource prices, Nash equilibrium

2.1. Optimization. The competitive model is applied as a test function for the Bayesian algorithms. However, a simple model may help to design more realistic ones describing the processes of competition better. Besides, one may use the competitive model for teaching Operations Research, too. Let us consider m servers providing the same service:

$$u_i = u_i(x_1, y_1, \ldots, x_m, y_m) = a_i y_i - x_i, \quad i = 1, \ldots, m, \tag{1}$$

[1]We call by capacity the average service rate in the case of non-stop operation.

where u_i is the profit, y_i is the service price, x_i is the server capacity, a_i is the rate of customers, and i is the server index. Assume, as a first approximation, that the a server capacity[2] is equal to the running cost x_i. The service cost

$$c_i = y_i + \gamma_i, \tag{2}$$

where γ_i is waiting cost. Assume that the waiting cost is equal to an average waiting time at the server i. A customer goes to the server i, if

$$c_i < c_j, \quad j = 1, \ldots, m, \ j \neq i, \quad c_i \leqslant c_0. \tag{3}$$

A customer goes away, if

$$\min_i c_i > c_0, \tag{4}$$

where c_0 is the critical cost. The rate a of incoming consumers is fixed:

$$a = \sum_{i=0}^{m} a_i, \tag{5}$$

where a_0 is the rate of lost customers.

Conditions (3) and (4) separate the flow of incoming customers into $m + 1$ flows thus making the problem very difficult for analytical solution. The separated flow is not simple even in the Poisson incoming flow case (Gnedenko, 1987). Thus we need Monte Carlo simulation, to define the average rates of customers $a_i, \ i = 0, 1, \ldots, m$, by conditions (3), (4), and the average profits $u_i, \ i = 1, \ldots, m$ by expression (1).

2.2. Search for Nash equilibrium. First we fix the the initial values, the "contract-vector" $(x_i^0, y_i^0, \ i = 1, \ldots, m)$. The transformed values, the "fraud-vector" $(x_i^1, y_i^1, \ i = 1, \ldots, m)$, is obtained by maximizing the profits of each server i, under the assumption that all the partners $j \neq i$ will honor the contract $(x_j^0, y_j^0, \ j = 1, \ldots, m, \ j \neq i)$

$$(x_i^1, y_i^1) = \arg\max_{x_i, y_i} u_i(x_i, y_i, x_j^0, y_j^0, \ j = 1, \ldots, m, \ j \neq i), \quad i = 1, \ldots, m. \tag{6}$$

[2] Average number of customers that may be served per unit time

Condition (6) transforms the vector $z^n = (x_i^n, y_i^n,\ i = 1,\dots,m) \in B \subset R^{2m},\ n = 0,1,2,\dots$ into the vector z^{n+1}. Denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0,1,2,\dots. \tag{7}$$

One may obtain the equilibrium at the fixed point z^n, where

$$z^n = T(z^n). \tag{8}$$

The fixed point z^n exists, if the feasible set B and the profit functions (1) are all convex (Michael, 1976). We obtain the equilibrium directly by iterations (7) , if the transformation T is contracting (Neuman and Morgenstern, 1953). If not, then we minimize the square deviation

$$\min_{z \in B} \| z - T(z) \|^2 . \tag{9}$$

The equilibrium is achieved, if the minimum (9) is zero. If the minimum (9) is positive then the equilibrium does not exist. One minimize (9) by the usual stochastic approximation techniques (Ermoljev and Wets, 1988), if square deviation (9) is unimodal. If not, then the Bayesian techniques of global stochastic optimization (see Mockus, 1989) are used.

Obviously an equilibrium will be stable if transformation (7) is locally contracting in the vicinity of fixed point (8). If not, then some stabilizing conditions should be introduced. The transformation $T(z)$ is referred to as locally contracting if there exists a constant $0 \leqslant \alpha < 1$ such that

$$||T(z^1) - T(z^2)|| \leqslant \alpha ||z^1 - z^2||, \tag{10}$$

for all $z^1, z^2 \in Z_\epsilon$, where Z_ϵ is an ϵ-vicinity of fixed point defined by the "natural" deviations from the equilibrium.

2.2.1. Simplified illustration. To illustrate the idea of equilibrium we consider very simple deterministic model. We express the waiting time as

$$\gamma_i = a_i / x_i, \quad i = 1,2. \tag{11}$$

We may compare the simplified expression (11) with the well-known expression of average waiting time in the Poisson case, see Gnedenko and Kovalenko (1987)

$$\gamma_i = \frac{a_i}{x_i} \frac{1}{x_i - a_i}. \tag{12}$$

Assume the steady-state conditions[3]

$$a_i/x_i + y_i = q. \tag{13}$$

Here q is a steady-state factor. From expression (4)

$$q \leqslant c. \tag{14}$$

From steady-state conditions (13)

$$a_i = (q - y_i)x_i \tag{15}$$

and

$$u_i = (q - y_i)x_i y_i - x_i. \tag{16}$$

Maximizing the profit

$$\max_{x_i y_i q} x_i \left((q - y_i)y_i - 1\right), \quad 0 \leqslant x_i \leqslant a, \quad q \leqslant c, \tag{17}$$

we obtain the optimal values

$$x_i = a, \quad y_i = q/2, \quad i = 1, 2, \quad q = c. \tag{18}$$

From expressions (17) and (18) the maximal profit

$$u_i = a\left((c/2)^2 - 1\right). \tag{19}$$

We achieve a positive profit equilibrium, if $c > 2$. These results may be helpful understanding the model (1).

2.2.2. Monte-Carlo simulation. Assume that the n-th customer estimates the average waiting time at the server i as the relation

$$\gamma_i(n) = n_i/x_i, \quad n = 1, \ldots, N, \quad i = 1, \ldots, m, \tag{20}$$

[3] This example is merely an illustration, stability of equilibrium conditions is not considered here.

Table 1. Simulation results

Alg.	Prices y_i and rates x_i						Profits u_i			Object.
No	x_1	y_1	x_2	y_2	x_3	y_3	u_1	u_2	u_3	min f
1	0.04	0.95	2.45	8.30	1.31	1.25	0.03	2.87	1.50	1.78
2	3.24	8.81	0.13	0.09	0.45	1.32	9.48	0.02	0.77	0.45
3	0.63	6.87	0.63	1.88	3.13	9.38	2.73	2.15	4.67	0.88
4	0.01	0.95	2.45	8.30	1.31	1.25	0.01	3.46	1.50	1.48
5	2.23	3.56	0.73	0.38	3.52	2.99	0.12	0.09	0.13	1.72

Algorithm Numbers				
1	2	3	4	5
MIG1	BAYES1	LPMIN	EXKOR	GLOPT

where n_i is queue length at the server i when the n-th customer arrives. Then from expressions (2) and (20) the service costs

$$c_i = y_i + n_i/x_i, \quad n = 1, \ldots, N, \quad i = 1, \ldots, m. \tag{21}$$

Table 1 indicates the possibilities and limitations of direct Monte-Carlo simulation of transformation (7) using different algorithms of global optimization. The simulation parameters are:

$$m = 3, \ N = 500, \ a = 2, \ c = 12.$$

The constraints are:

$$0.0001 \leqslant x_i \leqslant 10, \ 0.0001 \leqslant y_i \leqslant 10, \ i = 1, 2.$$

The positive result is that we obtained relatively small deviation from the equilibrium (0.45 using BAYES1 and 0.88 using LPMIN). However, we need much greater accuracy to answer a number of questions, for example:

- why using algorithm 2 we obtained the profit u_1 which is much greater as compared with u_2 and u_3 in the symmetric conditions?
- why using different algorithms we obtained so different results?
- is the equilibrium solution unique?
- is the algorithm accuracy sufficient?

The visual inspection shows that the profit functions u_i look like convex. That indicates the existence of the equilibrium. However one needs a large amount of computing to obtain the answers to these and related questions. That is outside the objective of this paper because we consider the competitive and the other models mainly as test functions to compare different algorithms including the Bayesian ones. Table 1 shows that in this special case the best results we obtained by the Bayesian methods and the second best by the uniform search algorithms during comparable time (see lines 2 and 3 correspondingly).

2.3. "Social" model. Define the "social" cost of service as follows

$$\sum_i (x_i + a_i \gamma_i). \tag{22}$$

Expression (22) defines a sum of running and waiting costs. For example, both the running and waiting costs may be defined as a time lost by the members of society while running the servers and waiting in the queues. The prices y are not present in social cost expression (22), since the service rates x_i are not limited.

In the simplified case (11), the optimal service rates

$$x_i = a_i, \quad i = 1, 2. \tag{23}$$

Here the prices y are eliminated, since the of service rates are not limited.

2.4. Lagrange multipliers. Consider now the Lagrangian model. We limit the total service rate of both servers by b

$$\sum_i x_i \leqslant b, \tag{24}$$

fix the customer rates a_i, $i = 1, 2$, and minimize the service losses

$$\min_x \sum_i a_i \gamma_i. \tag{25}$$

Problem (25), (24) can be solved by Lagrange multipliers, assuming the convexity of γ_i

$$\max_{y \geqslant 0} \min_{x \geqslant 0} \left(\sum_i a_i \gamma_i + y\Big(\sum_i x_i - b \Big) \right). \tag{26}$$

Here the Lagrange multiplier y means the "price" which the server pays to the supplier of service resources x_i. First let us fix y and minimize (26) by x. Thus the optimal rate $x_i = x_i(y)$ is defined as a function of price y. Next we maximize (26) by $y \geqslant 0$ to obtain the equilibrium (max-min) price $y = y_0$. Now each server can define the optimal service rate $x_i = x_i(y_0)$ by minimizing the social service cost

$$\min_{x_i \geqslant 0}(y_0 x_i + a_i \gamma_i), \quad i = 1, 2. \tag{27}$$

Apparently this model is not quite competitive, since the customer rate is fixed for each server. One defines equilibrium between the supplier and the servers. Here we assume a competition not between servers, like in (1), but between the supplier and the servers. The servers are of "non-profit" type. They minimize the social service cost including the customer waiting losses γ_i plus the price $y_0 x_i$ paid by the server to obtain the resource x_i.

In the simplified case (11),

$$\max_{y \geqslant 0} \min_{x \geqslant 0} \left(\sum_i a_i^2/x_i + y\Big(\sum_i x_i - b\Big)\right). \tag{28}$$

First we fix y and obtain optimal $x_i = x_i(y)$ as a function of y

$$x_i(y) = a_i/\sqrt{y}. \tag{29}$$

Next we maximize by y

$$\max_{y \geqslant 0} \left(\sqrt{y}\sum_i a_i + y\Big(1/\sqrt{y}\sum_i a_i - b\Big)\right) = \max_{y \geqslant 0}(2a\sqrt{y} - by) \tag{30}$$

and obtain the optimal price

$$y = y_0 = (a/b)^2. \tag{31}$$

This and expression (29) imply

$$x_i = a_i/ab. \tag{32}$$

All the solutions of simplified models are illustrative. However, they may be used as a first approximation considering more complicated models, correctly representing the stochastic service and processes.

2.5. Stable coalition (Core of game). If $m > 0$ then the formation and stability of coalition $S \subset M$ is important. Denote by $S = i$ a coalition of an individual server i. Denote by $S = (i_1, i_2)$ a coalition of two servers i_1 and i_2, and so on. Denote by $S = (i_1, \ldots, i_m) = M$ the coalition of all the m servers. A coalition S of servers means equal service prices $y_i = y(S)$, equal service rates $x_i = x(S)$, and equal share of profit $u_i(S) = 1/|S|\ u(S)$ for all $i \in s$. Here $|S|$ denotes the number of servers in the coalition S and $u(S) = \sum_{i \in S} u_i$ assuming that all the remaining servers form the opposite coalition $M \setminus S$ and that the profits u_i corresponds to the equilibrium service prices and rates $y(S), x(S), y(S), y(M \setminus S)$. This way we consider the m-server system as a set of two-server systems, where the first server is coalition S and the second one is coalition $M \setminus S$. The coalition $S \subset M$ is stable if there are no dominant coalition $S \subset M$ such that

$$u_i(S) > u_i(S), \quad \text{for all } i \in S. \tag{33}$$

The definition of stable coalition is related to the definition of game core **C**, see Rosenmuller (1981). It is well known that if

$$u(M) > \sum_{i \in M} u_i(M), \tag{34}$$

$$u(S) + u(M \setminus S) = u(m), \tag{35}$$

then the game core is empty $\mathbf{C} = \varnothing$ and thus there is no stable coalition S. The server system is clearly not a constant-sum game. Thus one of two "non-existence" conditions (35) is not satisfied meaning that a stable coalition S may exist. If a stable coalition S exists one may determine it testing if there exists a dominant coalition S satisfying condition (33).

3. Competition model with free resource prices, Walras equilibrium. In the previous Nash model the cost of service capacity unit is known and the individual server i controls the service capacity x_i and the price y_i charged for the service. In the Walras model the service capacity w_i of the server i depends on the resource vector $x_i = (x_{ij},\ i, j = 1, \ldots, m)$ defining the share of resources. The server i controls the price of its own resource p_i. The server also controls the resource vector x_i. The notion of "credit" is introduced defining the credit v_i as the difference between what the server i pays for resources obtained from partner-servers j and what it gets selling them its own resource. The server also controls the price y_i of its outside services, as in the Nash model. Now we shall describe the Walras model in the terms similar to those of the Nash model (see expression (1)).

Suppose there are m servers providing the same service. In this case we may write:

$$\begin{aligned} u_i =& u_i(x_i, y_i, p_i,\ x_j, y_j, p_j,\ j = 1, \ldots, m,\ j \neq i) \\ =& a_i y_i - (1 + \alpha_i) v_i, \quad i = 1, \ldots, m, \end{aligned} \tag{36}$$

where u_i is the profit, y_i is the service price, x_i is the resource vector determining its capacity w_i, v_i is the credit of the server i, $1 + \alpha_i$ defines the credit "price"[4] , a_i is the rate of customers, and i is the server index. The bank interest is defined as

$$\alpha_i = \begin{cases} \alpha_1, & \text{if } v_i \geqslant 0, \\ \alpha_2, & \text{if } v_i < 0, \end{cases} \tag{37}$$

where $\alpha_1 > \alpha_2$. Expression (37) means that one pays more for the bank credit comparing to what one gets for his deposit (denoted as a "negative credit" $v_i < 0$). Suppose that each server i owns a single resource b_i. Assume that a service capacity w_i is an increasing function of the resource vector $x_i = (x_{ij}, j = 1, \ldots, m)$.

$$w_i = \phi_i(x_i). \tag{38}$$

The resource component x_{ij} denotes the amount of resource b_j used by server i. The amount of resources is limited by the budget condition

$$\sum_{i=1}^{m} p_j x_{ij} = p_i b_i + v_i, \quad i = 1, \ldots, m. \tag{39}$$

Assuming the lower and upper limits $a_{p_i}, a_{x_{ij}}, a_{y_i}, b_{p_i}, b_{x_{ij}}, b_{y_i}$, $i, j = 1, \ldots, m$, we obtain the inequalities

$$a_{p_i} \leqslant p_i \leqslant b_{p_i}, \quad a_{x_{ij}} \leqslant x_{ij} \leqslant b_{x_{ij}}, \quad a_{y_i} \leqslant y_i \leqslant b_{y_i}.$$

It is natural to assume the following upper resource limit $b_{x_{ij}} = b_j$.

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[4]Here α_i is a bank interest.

The service cost, the waiting cost and the customer behavior remains the same as in the Nash model. Namely, the service cost

$$c_i = y_i + \gamma_i, \tag{41}$$

where γ_i is waiting cost. Assume that the waiting cost is equal to an average waiting time (see expression (20) at the server i. A customer goes to the server i, if

$$c_i < c_j, \quad j = 1, \ldots, m, \ j \neq i, \ c_i \leqslant c_0. \tag{42}$$

A customer goes away, if

$$\min_i c_i > c_0, \tag{43}$$

where c_0 is the critical cost. The rate a of incoming consumers flow is fixed:

$$a = \sum_{i=0}^{m} a_i, \tag{44}$$

where a_0 is the rate of lost customers.

3.1. Search for Walras equilibrium. We fix a contract-vector $(x_i^0, y_i^0, p_i^0, i = 1, \ldots, m)$. Then the fraud-vector $(x_i^1, y_i^1, p_i^1, \ i = 1, \ldots, m)$, is obtained by maximizing the profits of each server i and assuming that all the partners $j \neq i$ will honor the contract $(x_i^0, y_i^0, p_i^0, \ i = 1, \ldots, m)$

$$\begin{aligned} &(x_i^1, y_i^1, p_i^1) \\ &= \arg \max_{x_i, y_i, p_i} u_i(x_i, y_i, p_i, \ x_j^0, y_j^0, p_j^0, \ j = 1, \ldots, m, \ j \neq i), \end{aligned} \tag{45}$$

satisfying the budget condition

$$\sum_{j=1}^{m} p_j x_{ij} = p_i b_i + v_i, \tag{46}$$

and the constraints

$$\begin{aligned} &a_{p_i} \leqslant p_i \leqslant b_{p_i}, \\ &a_{x_{ij}} \leqslant x_{ij} \leqslant b_{x_{ij}}, \\ &a_{y_i} \leqslant y_i \leqslant b_{y_i}, \quad j = 1, \ldots, m, \ i = 1, \ldots, m. \end{aligned} \tag{47}$$

Condition (45) transforms the vector z^n, $n = 0, 1, 2, \dots$ into the vector z^{n+1}, where $z^n = (x^n, y^n, p^n)$, $x^n = (x_1^n, \dots, x_m^n)$, $y^n = (y_1^n, \dots, y_m^n)$, and $p^n = (p_1^n, \dots, p_m^n)$. Denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0, 1, 2, \dots. \tag{48}$$

Here the vector $z = (x_i, y_i, p_i, v_i,\ i = 1, \dots, m) \in B \subset R^{m^2+2m}$. We obtain the equilibrium at the fixed point z^n, where

$$z^n = T(z^n). \tag{49}$$

We may obtain the equilibrium directly by iterations (48), if the transformation T is contracting (Neuman and Morgenstern, 1953). If not, then we minimize the square deviation

$$\min_{z \in B} \| z - T(z) \|^2 . \tag{50}$$

The equilibrium is achieved, if the minimum (50) is zero.

3.2. Walras equilibrium regarding bank as server. The difference from the previous Walras model is that the bank providing funds for the other servers is considered as an additional server with zero index $i = 0$. In this case adding the bank profit expression is added to the equalities (36)

$$\begin{aligned} u_0 =& u_0(p_0,\ x_j^0, y_j^0, p_j^0,\ j = 1, \dots, m) = \sum_{i=1}^{m} p_0(i) v_i, \\ u_i =& u_i\left(p_0^0(i),\ x_i, y_i, p_i, v_i,\ x_j^0, y_j^0, p_j^0,\ j = 1, \dots, m,\ j \neq i\right) \\ =& a_i y_i - p_0^0(i) v_i, \quad i = 1, \dots, m, \end{aligned} \tag{51}$$

where u_i is the profit, y_i is the service price, x_i is the resource vector determining the service capacity, v_i is the credit of the server i, $p_0(i) = 1 + \alpha_i$ defines a bank interest, a_i is the rate of customers, and i is the server index. Note that the credit parameter $p_0(i)$ is equal to $1 + \alpha_1$ or to $1 - \alpha_2$ depending on the sign of v_i, see expression (37).

The service cost, the waiting cost and the customer behavior remains the same as in the previous model.

3.3. Search for Walras equilibrium regarding bank as a server. Let us to fix a contract-vector $(x_i^0, y_i^0, p_i,\ i = 1, \dots, m,\ \alpha_1^0, \alpha_2^0)$. Then the fraud-vector $(x_i^1, y_i^1, p_i^1,\ i = 1, \dots, m,\ \alpha_1^1, \alpha_2^1)$ is obtained by maximizing the profits of each server $i = 0, 1, \ldots, m$, under the assumption that all the partners $j \neq i$ will honor the contract $(x_i^0, y_i^0, p_i^0,\ i = 1, \ldots, m,\ \alpha_1^0, \alpha_2^0)$. Then, from expression (51)

$$(\alpha_1^1, \alpha_2^1) = \arg\max_{\alpha_1, \alpha_2} \sum_{i=1}^{m} p_0(i) v_i,$$
$$a_{\alpha_1} \leqslant \alpha_1 \leqslant b_{\alpha_1}, \quad a_{\alpha_2} \leqslant \alpha_2 \leqslant b_{\alpha_2}; \tag{52}$$

$$(x_i^1, y_i^1, p_i^1) = \arg\max_{x_i, y_i, p_i} \left(a_i y_i - p_0(i) v_i\right), \quad i = 1, \ldots, m; \tag{53}$$

$$\sum_{j=1}^{m} p_j x_{ij} = p_i b_i + v_i; \tag{54}$$

$$a_{p_i} \leqslant p_i \leqslant b_{p_i}, \quad i = 0, \ldots, m,$$
$$a_{x_{ij}} \leqslant x_{ij} \leqslant b_{x_{ij}}, \quad a_{y_i} \leqslant y_i \leqslant b_{y_i}, \quad j = 1, \ldots, m,\ i = 1, \ldots, m. \tag{55}$$

Condition (53) transforms the vector z^n, $n = 0, 1, 2, \ldots$ into the vector z^{n+1}, where $z^n = (x^n, y^n, p^n, \alpha_1^n, \alpha_2^n)$, $x^n = (x_1^n, \ldots, x_m^n)$, $y^n = (y_1^n, \ldots, y_m^n)$, $p^n = (p_1^n, \ldots, p_m^n)$. Denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0, 1, 2, \ldots. \tag{56}$$

Here the vector $z = (x_i, y_i,\ i = 1, \ldots, m,\ p_i\ i = 1, \ldots, m,\ \alpha_1, \alpha_2) \in B \subset R^{m^2+2m+2}$. We obtain the equilibrium at the fixed point z^n, where

$$z^n = T(z^n). \tag{57}$$

We obtain the equilibrium directly by iterations (56), if the transformation T is contracting (Neuman and Morgenstern, 1953). If not, then we minimize the square deviation

$$\min_{z \in B} \| z - T(z) \|^2 . \tag{58}$$

The equilibrium is achieved, if the minimum (58) is zero.

4. Inspection model, multi-stage game

4.1. Linear case. Denote by $x = (x_1, \ldots, x_m)$, $x_i \geqslant 0$, $\sum_i x_i = 1$ the inspection vector and by $y = (y_1, \ldots, y_m)$, $y_j \geqslant 0$, $\sum_j y_j = 1$ the violation vector. Here x_i denotes the probability of the area i to be inspected and y_j

means the probability of violation in the area j. Denote by $u(i,j)$ the inspection utility function when the object i is inspected and the object j is violated. Denote by $v(i,j)$ the violation utility function when the object i is inspected and the object j is violated. Denote by $U(x,y)$ and $V(x,y)$ the expected values of the inspection and violation utility functions using inspection and violation vectors x, y defining the probabilities of inspection and violation. For example,

$$u(i,j) = \begin{cases} p_i g_i q_i, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{59}$$

and

$$v(i,j) = \begin{cases} -q_j p_i g_j + (1-p_i) q_j g_j, & \text{if } i = j, \\ q_j g_j, & \text{otherwise}, \end{cases} \tag{60}$$

Here p_i is the probability of detecting the violation if it happens in the area i, q_i the probability of successful violation[5] if it occurs the area i, and g_i is the utility of successful violation in the area i. The average utility functions at fixed inspection and violation vectors x and y

$$U(x,y) = \sum_{i,j} x_i u(i,j) y_i, \tag{61}$$

and

$$V(x,y) = \sum_{i,j} x_i v(i,j) y_i. \tag{62}$$

4.2. Search for equilibrium. We fix the contract-vector $x^0 = (x_i^0, y_i^0,\ i = 1,\ldots,m)$. Then the fraud-vector $x^1 = (x_i^1, y_i^1,\ i = 1,\ldots,m)$, is obtained by maximizing the expected utilities $U(x,y)$ and $V(x,y)$ separately, under the assumption that the "partner" will honor the contract $(x_i^0, y_i^0,\ i = 1,\ldots,m)$

$$x^1 = \arg\max_x U(x,y^0), \quad U = \max_x U(x,y^0), \tag{63}$$

$$y^1 = \arg\max_y V(x^0,y), \quad V = \max_y V(x^0,y). \tag{64}$$

These expressions define two linear programming problems that may be solved for any fixed contract vectors $x^0 = (x_i^0, y_i^0,\ i = 1,\ldots,m)$. However, using the standard simplex algorithm, an optimal base solution of linear programming problem representing the "best" vertex on the simplex is obtained,

[5]For example, killing the prey.

as usual. It follows from expressions (61), (62), (63) and (64) that the best vertex is defined by conditions

$$x_k^1 = \begin{cases} 1, & \text{if } \sum_j u(i,j)y_j > \sum_j u(k,j)y_j, \quad k \neq i, \\ 0, & \text{otherwise,} \end{cases} \tag{65}$$

and

$$y_l^1 = \begin{cases} 1, & \text{if } \sum_i v(i,j)x_i > \sum_i v(i,l)x_i, \quad l \neq j, \\ 0, & \text{otherwise.} \end{cases} \tag{66}$$

These conditions define a set of pure strategies. However, mixed strategies are needed to obtain the equilibrium (see Owen, 1968). A mixed strategy may be determined using a continuous set of all the solutions not just the base ones. The following expressions define such a set

$$\sum_{j=1}^{m} u(i,j)y_j^0 = U, \quad i = 1, \ldots, m; \tag{67}$$

$$\sum_{i=1}^{m} v(i,j)x_i^0 = V, \quad j = 1, \ldots, m; \tag{68}$$

$$\sum_{j=1}^{m} y_j^0 = 1; \tag{69}$$

$$\sum_{i=1}^{m} x_i^0 = 1; \tag{70}$$

$$x_i^0 \geqslant 0, \quad y_j^0 \geqslant 0, \qquad i,j = 1, \ldots, m. \tag{71}$$

The expressions (67), (69) define values of y_j^0 providing the multiple "continuous" maximum of $U(x, y^0)$ in a sense that any vector $x^1, \sum_i x_i^1 = 1,\ x_i \geqslant 0$, maximizes the average utility at given $y = y^0,\ y^0 = (y_1^0, \ldots, y_m^0)$:

$$x^1 = \arg\max_x U(x, y^0), \quad U = \max_x U(x, y^0), \tag{72}$$

and, correspondingly,

$$y^1 = \arg\max_y V(x^0, y), \quad V = \max_y V(x^0, y). \tag{73}$$

Since any probability distribution x^1 satisfies the maximum condition we may obtain the equilibrium just by setting

$$x^1 = x^0, \tag{74}$$

and

$$y^1 = y^0, \tag{75}$$

respectively.

4.3. Zero-sum case. It is well known (see Owen, 1968) that in the zero-sum case where $v_{ij} = -u_{ij}$ the equilibrium may be defined as two two linear programming problems. One may reduce the utilities (59) and (60) to the zero-sum case by subtracting the penalty terms $(1-p_i)q_jg_j$ and q_ig_i from the inspectors utilities. In such a case

$$u(i,j) = \begin{cases} p_ig_iq_i - (1-p_i)q_jg_j, & \text{if } i = j, \\ -q_ig_i, & \text{otherwise,} \end{cases} \tag{76}$$

and

$$v(i,j) = \begin{cases} -q_jp_ig_j + (1-p_i)q_jg_j, & \text{if } i = j, \\ q_jg_j, & \text{otherwise.} \end{cases} \tag{77}$$

It follows fom expressions (76) and (77) that $v_{ij} = -u_{ij}$. Then we may obtain the equilibrium by solving the following two linear programming problems (see Owen, 1968): the direct problem

$$\max_x U,$$
$$\sum_{i=1}^m x_iu_{ij} \geqslant U, \quad j = 1,\ldots,m, \quad \sum_{i=1}^m x_i = 1, \; x_i \geqslant 0, \tag{78}$$

and the dual one

$$\min_y V$$
$$-\sum_{j=1}^m y_ju_{ij} \leqslant V, \quad i = 1,\ldots,m, \quad \sum_{i=1}^m y_i = 1, \; y_i \geqslant 0. \tag{79}$$

The important difference between the zero-sum expressions (78), (79) of linear programming and the system of equalities and inequalities (67)–(71) is that in the zero-sum case the mini-max condition holds

$$\min_y V = \max_x U. \tag{80}$$

This condition makes the zero-sum case a convenient tool for testing the results of non-zero-sum cases where the mini-max condition is not true.

4.4. Nonlinear case. Suppose that the probability distributions x and y are not controlled directly by the inspector and/or the violator and depend on some control vectors $s \in A \subset R^k$ and $t \in B \subset R^l$

$$x = x(s), \quad y = y(t). \tag{81}$$

Then the average utility functions at fixed controls s and t

$$U\left(x(s), y(t)\right) = \sum_{i,j} x_i(s) u(i,j) y_i(t), \tag{82}$$

and

$$V\left(x(s), y(t)\right) = \sum_{i,j} x_i(s) v(i,j) y_i(t). \tag{83}$$

We fix the contract-vector $z^0 = (s_i^0, t_j^0,\ i = 1, \ldots, k,\ j = 1, \ldots, l)$. Then the fraud-vector $z^1 = (s_i^1, t_j^1,\ i = 1, \ldots, k,\ j = 1, \ldots, l)$ is obtained by maximizing the expected utilities $U(x(s), y(t))$ and $V(x(s), y(t))$ separately, under the assumption that the "partner" will honor the contract z^0

$$s^1 = \arg\max_s\ U\left(x(s), y(t^0)\right), \tag{84}$$

$$t^1 = \arg\max_t\ V\left(x(s^0), y(t)\right). \tag{85}$$

Conditions (84), (85) transforms the vector z^n, $n = 0, 1, 2, \ldots$ into the vector z^{n+1}, where $z^n = (s^n, t^n)$, $s^n = (s_1^n, \ldots, s_k^n)$, and $t^n = (t_1^n, \ldots, t_l^n)$. Denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0, 1, 2, \ldots. \tag{86}$$

Here the vector $z = (x_i, y_i,\ i = 1, \ldots, m) \in B \subset R^{k+l}$. We obtain the equilibrium at the fixed point z^n, where

$$z^n = T(z^n). \tag{87}$$

We obtain the equilibrium directly by iterations (86), if the transformation T is contracting (Neuman and Morgenstern, 1953). If not, then we minimize the square deviation

$$\min_{z \in B} \| z - T(z) \|^2. \tag{88}$$

The equilibrium is achieved, if the minimum (88) is zero.

A reduction to the linear case (61), (62) is an another way for solving nonlinear inspector game (82), (83) using the equations relating the probabilities x, y and the control parameters s, t

$$x_i = x_i(s_1, \ldots, s_k), \quad y_j = y_j(t_1, \ldots, t_l), \quad i, j = 1, \ldots, m. \tag{89}$$

There are two solution stages:

- the equilibrium values of x^0 and y^0 are found by expressions (67)–(71);
- the equilibrium values of s^0, t^0 are obtained by solving system (89) at given x^0, y^0.

One may see that the linearization way is possible if the solution of system (89) is not too difficult. Otherwise the direct optimization of control parameters s, t using expressions (84), (85), (88) is preferable.

5. "Star Wars" problem, differential game model

5.1. Introduction. Consider two objects in space trying to destroy each other. Assume, as a first approximation, that there are three control parameters: the initial points z_0, w_0, the rates a, b and the firing times t_1, t_2 of both objects. Suppose that these parameters are set before the start. Denote the control parameters by vectors $x = (x_i, \ i = 1, \ldots, m)$ and $y = (y_i, \ i = 1, \ldots, m)$ correspondingly. In the illustrative example $x_1 = z_0, x_2 = a, x_3 = t_1$ and $y_1 = w_0, y_2 = b, y_3 = t_2$. The trajectories of the objects are described in the "hight-time space" by the equations

$$dz(t)/dt = az(t), \tag{90}$$
$$dw(\tau)/d\tau = bw(\tau), \quad \tau = 2 - t. \tag{91}$$

Then the trajectories are

$$z(t) = z_0 e^{at} \tag{92}$$
$$w(\tau) = w_0 e^{b\tau}. \tag{93}$$

Denote by $d(t)$ the distance between the objects at the moment t

$$d(t) = \|(w(\tau), \tau) - (z(t), t)\|. \tag{94}$$

Denote by $p(t)$ the hitting probability

$$p(t) = 1 - (d(t)/D)^{\alpha} . \tag{95}$$

Here $D \geqslant \max_t d(t)$ and $\alpha > 0$.

The expected utility function of the first object

$$U(x,y) = \begin{cases} p(x_3) - (1 - p(x_3))\, p(y_3), & \text{if } x_3 < y_3, \\ -p(y_3) + (1 - p(y_3))\, p(x_3), & \text{if } x_3 > y_3, \\ p(x_3) - p(y_3)0, & \text{if } x_3 = y_3. \end{cases} \tag{96}$$

The expected utility function of the second object

$$V(x,y) = \begin{cases} p(y_3) - (1 - p(y_3))\, p(x_3), & \text{if } x_3 < y_3, \\ -p(x_3) + (1 - p(x_3))p(y_3), & \text{if } x_3 > y_3, \\ p(y_3) - p(x_3), & \text{if } x_3 = y_3. \end{cases} \tag{97}$$

The expected utility functions (96) and (97) follows from hitting probability expression (95) assuming that utility of hitting the hostile object is plus one and the utility to be hit is minus one.

5.2. Convex version. The expected utilities (96) and (97) are not convex functions of the variables $p(x_3)$ and $p(y_3)$. The convex version may be obtained assuming that the object "hears" the enemy fire and keeps moving until $d(t) = 0$ (correspondingly $p(t) = 1$). In such a case modifying expressions (96) and (97) the expected utility function of the first object

$$U(x,y) = \begin{cases} p(x_3) - (1 - p(x_3)) , & \text{if } x_3 < y_3, \\ -p(y_3) + (1 - p(y_3)) , & \text{if } x_3 > y_3, \\ p(x_3) - p(y_3), & \text{if } x_3 = y_3. \end{cases} \tag{98}$$

The expected utility function of the second object

$$V(x,y) = \begin{cases} p(y_3) - (1 - p(y_3)) , & \text{if } x_3 < y_3, \\ -p(x_3) + (1 - p(x_3)) , & \text{if } x_3 > y_3, \\ p(y_3) - p(x_3), & \text{if } x_3 = y_3. \end{cases} \tag{99}$$

It is easy to see that expected utilities (96) and (97) are convex functions of probabilities $p(x_3)$ and $p(y_3)$ at the equilibrium point

$$p(x_3) = 0.5,\quad p(y_3) = 0.5. \tag{100}$$

Note that convexity of expected utilities (96) and (97) as functions of probabilities $p(x_3)$ and $p(y_3)$ does not provide their convexity as functions of firing times x_3 and y_3 and other parameters. One may exploit conditions (100) in two ways:

- testing the validity of solution (104);
- reducing the dimension of search for equilibrium using expressions (101) by defining the firing times x_3 and y_3 directly from equalities (100) at fixed parameters x_1, x_2, y_1, y_2.

5.3. Search for equilibrium. We fix the contract-vector $(x^0, y^0) = (z_0^0, a^0, t_1^0, w_0^0, b^0, t_2^0)$. Then the fraud-vector $(x^1, y^1) = (z_0^1, a^1, t_1^1, w_0^1, b^1, t_2^1)$ is obtained by maximizing the expected utilities $U(x, y)$ and $V(x, y)$ separately, under the assumption that the "partner" will honor the contract (x^0, y^0)

$$x^1 = \arg\max_x U(x, y^0),\quad y^1 = \arg\max_y V(x^0, y). \tag{101}$$

Condition (101) transforms the vector z^n, $n = 0, 1, 2, \ldots$ into the vector z^{n+1}, where $z^n = (x^n, y^n)$. Denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0, 1, 2, \ldots. \tag{102}$$

Here the vector $z \in B \subset R^{2m}$. In the example $m = 3$. The equilibrium is at the fixed point z^n, where

$$z^n = T(z^n). \tag{103}$$

One may obtain the equilibrium directly by iterations (102) , if the transformation T is contracting (Neuman and Morgenstern, 1953). If not, then one minimizes the square deviation

$$\min_{z \in B} \| z - T(z) \|^2 . \tag{104}$$

The equilibrium is achieved, if the minimum (104) is zero.

5.4. One-dimensional example. In one-dimensional space from expressions (90)–(97) the trajectories

$$z(t) = t, \tag{105}$$

$$w(\tau) = 2 - \tau, \tag{106}$$

the hiting probability

$$p(t) = 1 - d(t)/D. \tag{107}$$

Here $D \geqslant \max_t d(t)$. The control parameters are the firing times $x = t_1$ and $y = t_2$ of the first and the second object respectively. Assuming, that $D = 2$ the expected utility function of the first object from expression (98)

$$U(x,y) = \begin{cases} x - (1-x), & \text{if } x < y, \\ -y + (1-y), & \text{if } x > y, \\ x - y, & \text{if } x = y. \end{cases} \tag{108}$$

The expected utility function of the second object from expression (99)

$$V(x,y) = \begin{cases} y - (1-y), & \text{if } x < y, \\ -x + (1-x), & \text{if } x > y, \\ y - x, & \text{if } y = x. \end{cases} \tag{109}$$

The equilibrium is reached at the firing moment $x = y = 1/2$. The one-dimensional example may be used testing the results of the two-dimensional case (104).

6. "Portfolio" problem, resource investment model

6.1. Utility-theoretical investment model. The portfolio problem is to maximize the average utility of wealth obtained by the optimal distribution of available capital between different objects with uncertain parameters. Denote the part of the capital invested into the object i by x_i and the corresponding wealth returned with interest $\alpha_i > 0$ by $y_i = c_i x_i$, where the return $c_i = 1 + \alpha_i$. Denote by $p_i = 1 - q_i$ the reliability, where q_i is the insolvency probability. Denote by $u(y)$ the utility function of the wealth y and by $U(y)$ the expected utility function which we should maximize. This means

$$\max_x U(y), \tag{110}$$

$$\sum_{i=1}^{n} x_i = 1, \quad x_i \geqslant 0. \tag{111}$$

Here the objective $U(y)$ depends on n variables x_i and is defined as follows:

$$U(y) = \mathbf{E}u(y) = \int_0^{\infty} u(y)p(y)dy, \tag{112}$$

where $p(y)$ is probability density of wealth y.

6.2. Average utility. One may define probabilities $p(y^j)$ of discrete values of wealth y^j, $j = 1, 2, \ldots$, by the exact expressions, for example,

$$\begin{aligned}
p(y^1) &= p_1 \prod_{i \neq 1} q_i, \\
p(y^2) &= p_2 \prod_{i \neq 2} q_i, \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
p(y^n) &= p_n \prod_{i \neq n} q_i, \\
p(y^{n+1}) &= p_1 p_2 \prod_{i \neq 1, i \neq 2} q_i, \\
p(y^{n+2}) &= p_1 p_3 \prod_{i \neq 1, i \neq 3} q_i, \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{113}$$

Then from expressions (124) and (113) follows that

$$U(y) = \sum_{k=1}^{M} u(y^k)p(y^k), \tag{114}$$

where M is the number of different values of wealth y.

Using Monte Carlo approach we may determine $U(y)$ approximately:

$$U_K(y) = 1/K \sum_{k=1}^{K} u(y^k). \tag{115}$$

Here

$$y^k = \sum_{i=1}^{n} y_i^k, \tag{116}$$

where

$$y_i^k = \begin{cases} c_i x_i, & \text{if } \eta_i^k \in [0, p_i], \\ 0, & \text{otherwise,} \end{cases} \tag{117}$$

In these expressions K is the number of Monte Carlo samples and η_i^k is a random number uniformly distributed on the unit interval.

It is easy to see that

$$U(y) = \lim_{K \to \infty} U_K(y). \tag{118}$$

6.3. Optimal portfolio, special cases. The optimal portfolio depends on the utility function $u(y)$. Consider, for example, the optimal portfolio for three different utility functions The first utility function is linear

$$u(y) = cy. \tag{119}$$

The second one is of the "non-risky" type

$$u(y) = \begin{cases} 0, & \text{if } 0 \leqslant y < a, \\ 1, & \text{if } a \leqslant y \leqslant c. \end{cases} \tag{120}$$

The third example is a "risky" utility function

$$u(y) = \begin{cases} 0, & \text{if } 0 \leqslant y < c, \\ 1, & \text{if } y = c. \end{cases} \tag{121}$$

Here a is a risk threshold and $c = \max_i c_i x_i$ denotes the maximal return of invested capital, see expression (111). It easy to show that in the linear case (119) the optimal portfolio is to invest all the capital in the object with highest product $p_i c_i$. If $a = 1/m \min_i c_i x_i$ then in the not-risky case (120) an optimal decision is $x_i^* = 1/m,\ i = 1, \dots, m$. Here one divides the capital equally between all the objects[6]. In the risky case (121) one should invest all the capital in the object with highest return, thus $x_i = 1$ if $c_i = \max_j c_j$.

6.4. Defining utility function. The utility function $u(y)$ is different for each individual person or organization. It can be defined by some lottery $L(A, B, p) = \{pA + (1 - p)B\}$, where p is the probability to win the best

[6] The optimal decision $x=1/m$ is not a unique, any decision satisfying the inequality $c_i x_i \geqslant a,\ i=1,\dots,m$, minimizes the expected utility function $U(y)$.

event A, and $(1-p)$ is the probability to get the worst one. Denote by C the "ticket price" of this lottery. Denote by $p(C)$ the "hesitation" probability, when one cannot decide what is better: to keep the money C in a safe or to risk this money hoping to win A with probability $p(C)$. Denote the "hesitation" lottery as

$$L(A,B,C,p(C)) = [C \approx \{p(C)A + (1-p(C))\,B\}]. \qquad (122)$$

Then, from Rosenmuller (1981) it follows that assuming the utilities $u(A)=1$ and $u(B)=0$ the utility $u(C)=p(C)$.

Suppose, for example, that event C when $y=1$ means keeping a unit capital in the safe (no risk, no profit). Assume that the event A when $y=2$ means doubling the unit capital and the event B when $y=0$ means losing this capital. Denote by $p(1)$ the hesitation probability. Then $u(1) = u(0) + p(1)(u(2) - u(0))$. In the case $u(0)=0,\ u(2)=1$ the utility of the unit capital $u(1)=p(1)$. Thus we obtained utilities at the three points: $y=0,\ y=1,\ y=2$. To define a reasonable approximation of the utility function $u(y)$ one needs at least two additional points. For example, the points of half and one-and-half of the unit capital $y=0.5,\ y=1.5$. We may define the corresponding utilities by the hesitation probabilities $p(0.5)$ and $p(1.5)$ obtained by the two hesitation lotteries

$$\begin{aligned} &L(1.0, 0.0, 0.5, p(0.5)) \\ &\quad = [(y=0.5) \approx \{p(0.5)(y=1) + (1-p(0.5))\,(y=0)\}], \end{aligned}$$

and

$$\begin{aligned} &L(2.0, 1.0, 1.5, p(1.5)) \\ &\quad = [(y=1.0) \approx \{p(1.5)(y=2.0) + (1-p(1.5))\,(y=1)\}]. \end{aligned}$$

In this case following Rosenmuller (1981) we obtain utility values $u(0) = 0,\ u(0.5) = p(0.5)u(1),\ u(1) = p(1),\ u(1.5) = u(1) + p(1.5)(u(2) - u(1)),\ u(2)=1$. The remaining utility values may be defined by a simple expression of linear interpolation

$$\begin{aligned} u(y) =& u(y_i) + p(y_i)\,(u(y_{i+1} - u(y_i)), \\ & y_i \leqslant y < y_{i+1}, \quad i = 0,1,\ldots,4. \end{aligned} \qquad (123)$$

6.5. Potential portfolio model. Consider the exponential utility function

$$u(y) = 1 - e^{-ay}, \qquad (124)$$

where $a \geqslant 0$ is risk aversion constant and the wealth $y = \sum_{i=1}^{n} x_i \xi_i$. Assume that the return vector $\xi = (\xi_i,\ i = 1, \ldots, n)$ is a Gaussian one (c, Σ), where $c = (c_i,\ i = 1, \ldots, n)$ and $\Sigma = (\sigma_{ij},\ i, j = 1, \ldots, n)$. In this case (see Freund, 1956) the optimal portfolio may be obtained by maximizing the average utility function

$$U(y) = c'x - a/2\ x'\Sigma x, \quad \sum_{i=1}^{n} x_i = 1,\ x_i \geqslant 0. \tag{125}$$

Here c' and x' are transposed vectors c and x, and $c_i \geqslant 0,\ i = 1, \ldots, n$.

Assuming that $\sigma_{ij} = \sigma_i,\ i = j$ and $\sigma_{ij} = 0,\ i \neq j$ we express the Lagrangian as

$$L(x, \lambda) = \sum_{i=1}^{n} c_i x_i - a/2 \sum_{i} \sigma_i x_i^2 - \lambda\left(\sum_{i=1}^{n} x_i - 1\right)x. \tag{126}$$

If $a > 0$ then maximizing Lagrangian (126) we define the optimal portfolio

$$x_i^* = \frac{1}{a}\left(\frac{c_i}{\sigma_i} - \frac{1}{\sigma_i}\lambda^*\right), \tag{127}$$

where

$$\lambda^* = \frac{\sum_{i=1}^{n} \frac{c_i}{\sigma_i} - a}{\sum_{i=1}^{n} \frac{1}{\sigma_i}}. \tag{128}$$

If one does not care of risk, meaning that $a = 0$, then the optimal porfolio may be obtained directly maximizing expression (125):

$$x_i^* = \begin{cases} 1, & \text{if } c_i = \max_j c_j, \\ 0, & \text{otherwise.} \end{cases} \tag{129}$$

If one feels a strong risk aversion, meaning that $a \to \infty$ then from expression (127) the optimal porfolio is

$$x_i^* = \frac{1}{\sigma_i \sum_{j=1}^{n} \sigma_j}. \tag{130}$$

We see that in the case when the risk aversion constant tends to infinity the optimal investment is equal distribution of capital between all the objects if variances $\sigma_i = \sigma,\ i = 1, \ldots, n$. If the risk aversion is zero then the optimal investment is to put all the capital into the object with largest return c_i. Thus in the case of "risky" exponential utility we obtained asymptotically the same optimal investment as in the case of "risky" piece-vise utility functions (see expressions (120) and (121)). In the case of "non-risky" functions the optimal portfolio remains the same only if variances $\sigma_i = \sigma$ are equal. That shows the important difference between the two portfolio problems defined by expressions (110) and (125). In the first case the utility function is based on the assumptions of utility theory and the returned wealth is determined by the reliability of different objects, what is natural for banks. In the second case the utility function is assumed to be exponential and the return wealth is defined as a Gaussian random variable.

The advantage of exponential case (125) is the possibility of analytical solution. The disadvantage is that there is only one "control" parameter a to "adapt" the utility function to the individual user. Besides, the Gaussian returned wealth model is not convenient describing the reliability of investment objects.

In the case of piece-wise linear utility function (see expression (123)) there are many control parameters, for example, $p(0.5),\ p(1),\ p(1.5)$ which are defined following the procedure of the "hesitation lotteries" which follows from the well-known results of utility theory. A number of different utility functions has been considered, see, for example, Ziemba *et al.*, 1974. We prefer the "hesitation lottery" approach since it is based on the simple and clear assumptions of utility theory, see Rosenmuller, 1981.

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GLOBALINIO IR DISKRETINIO OPTIMIZAVIMŲ PAVYZDŽIŲ RINKINYS: BAJESO HEURISTINIŲ METODŲ TAIKYMAS. I

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Dėstant operacijų tyrimą svarbu susipažinti su lošimų, naudingumo, eilių, tvarkaraščių bei nuoseklių sprendimų teorijomis. Straipsnyje šios teorijos iliustruojamos bei jų ryšys su globaliniu optimizavimu parodomas nagrinėjant aštuonis pavyzdžius. Visi pavyzdžiai formuluojami lengvai suvokiamais įvairių specialybių studentams terminais, tačiau kiekvienas iš jų atstovauja svarbioms uždavinių šeimoms. Todėl aprašomi modeliai bei jų optimizavimo algoritmai gali būti įdomūs ir patyrusiems atitinkamų sričių ekspertams.

Straipsnis padalintas į dvi dalis. Pirmoje dalyje aprašomi penki modeliai, antroje – likę trys, o taip pat bendra programinė įranga.