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PIPELINED-BLOCK MODELS OF LINEAR DISCRETE-TIME SYSTEMS

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Abstract. For pipelining and block processing in linear time-varying (LTV), linear periodically time-varying (LPTV), and linear time-invariant (LTI) discrete-time systems, we suggest to use the general solution of state space equations. First, we develop three pipelined-block models for LTV, LPTV, and LTI discrete-time systems, and two pipelined-block structures. Afterwards, we analyse complete state controllability, complete output controllability, and complete observability of LTV and LTI pipelined-block discrete-time systems.

Key words: linear, time-varying, discrete systems, models, pipelining, block processing, controllability, observability.

1. Introduction. Due to rapid development of very large-scale integration (VLSI) technology, single chip digital signal processors have become a reality in recent years. However, the speed at which a given system can be implemented is limited by the operating speed of these processors which, in turn, limits the input sampling rate. For processing signals of high frequencies, it must be sampled at a high sample rate and, consequently, the digital signal processing algorithm must also be implemented for high throughput. One of the most common ways of achieving a high throughput implementation is by increasing the parallelism in the algorithm through a proper modification of the basic algorithm and employing many processing elements for a concurrent operation (Gnanasekaran, 1988).

Pipelining and block processing are two of several algorithmic transformation techniques that can be used to exploit the concurrency within a digital signal processing algorithm to improve its operating speed (Lucke and Parhi, 1994). Pipelining (Chung and Parhi, 1994; Parhi and Messerschmitt, 1989b;

Lucke and Parhi, 1994; Parhi and Messerschmitt, 1989c; Jump and Ahuja, 1978; Cappello and Steiglitz, 1983; Lim and Liu, 1992) increases the speed of a system at the expense of latency. Block processing (Parhi and Messerschmitt, 1989a; Burrus, 1971; Barnes and Shinnaka, 1980; Azimi-Sadjadi and King, 1986; Azimi-Sadjadi and Rostampour, 1989; Nikias, 1984) is a form of parallel processing which transforms a scalar system into a block system.

Block processing has been applied to numerous arreas in digital signal processing (Burrus, 1971; Meyer and Burrus, 1976; Lu, Lee, and Messerschmitt, 1985; Barnes and Shinnaka, 1980) and control (Khorasani and Azimi-Sadjadi, 1987). This is, to a great extent, due to its advantages and utilities in performing parallel processing with an increased throughput rate, increased computational efficiency, reduced roundoff error, and sensitivity performance (Burrus, 1971; Meyer and Burrus, 1976; Barnes and Shinnaka, 1980). Barnes and Shinnaka used the concept of block processing to arrive at a block state space formulation with states that propagate only at the edge of each block rather than at each sample. Parhi and Messerschmitt (1989b), introduced a look-ahead computation scheme for parallel implementation of the block state-space equation. They have also proposed an incremental block state-space structure which offers fewer computational complexities as compared to the standard and parallel block state space structures. Azimi-Sadjadi, Lu, and Nebot (1991) have derived two generalized sets of block Kalman filtering equations. Parallel and sequential implementation schemes have been considered and the corresponding block Kalman filter equations have been obtained.

In this paper, using the general solution of LTV discrete-time systems in state space, we have derived pipelined-block equations. In Section 2, we derived equations for pipelining and block processing of the LTV, LPTV, and LTI discrete-time systems. We proposed three pipelined-block models and two pipelined-block structures in state space. In Section 3, we analysed complete state and complete output controllability of the pipelined-block LTV discrete-time system. In Section 4, we analysed complete observability of the pipelined-block LTV discrete-time systems.

2. Pipelined-block models of LTV discrete-time systems in state space. We derive here equations for pipelining and block processing of LTV, LPTV, and LTI discrete-time systems in state space.

Consider a LTV discrete-time system described by

$$x(k+1) = A(k)x(k) + b(k)u(k), \tag{1a}$$

$$y(k) = c^T(k)x(k) + d(k)u(k), \quad k = 0, 1, 2, \dots, \tag{1b}$$

where the state $x(k)$ is $N \times 1$, the state update matrix $A(k)$ is $N \times N$, $b(k)$ and $c(k)$ are $N \times 1$, and $d(k)$, input sample $u(k)$ and output sample $y(k)$ are scalars, and N is the order of the system.

The solution of the dynamic equation (1a) is given by Chui and Chen (1991)

$$\begin{aligned} x(n) = & F(n,k)x(k) + \sum_{j=k}^{n-1} F(n, j+1)b(j)u(j), \\ & n = 0, 1, 2, \dots, \ k \leqslant n-1, \end{aligned} \tag{2}$$

where the following relationships

$$F(n,n) = I_N, \quad F(n+1,k) = A(n)F(n,k) \tag{3}$$

hold for the $N \times N$ state transition matrix $F(n,k)$.

From Eq. 3, we get

$$F(k+1,k) = A(k), \quad F(k+2,k) = A(k+1)A(k),$$

so

$$F(n,k) = \prod_{i=1}^{n-k} A(n-i), \quad \text{and} \quad F(n, j+1) = \prod_{i=1}^{n-j-1} A(n-i). \tag{4}$$

Substituting $x(n)$ from (2) into (1b), we have

$$\begin{aligned} y(n) = & c^T(n)F(n,k)x(k) + \sum_{j=k}^{n-1} c^T(n)F(n, j+1)b(j)u(j) \\ & + d(n)u(n), \quad n = 0, 1, 2 \dots . \end{aligned} \tag{5}$$

2.1. Pipelined-block Modell. Substituting $n = k + M$ into (2), where M is a pipelining level, we get

$$\begin{aligned} x(k+M) &= F(k+M,k)x(k) + \sum_{j=k}^{k+M-1} F(k+M, j+1)b(j)u(j) \\ &= F(k+M,k)x(k) + \sum_{j=0}^{M-1} F(k+M, k+j+1)b(k+j)u(k+j) \\ &= \prod_{i=1}^{M} A(k+M-i)x(k) + \sum_{j=1}^{M} \prod_{i=1}^{M-j} A(k+M-i)b(k+j-1)u(k+j-1), \end{aligned}$$

or, in a matrix form:

$$x(k+M) = \overline{A}(k)x(k) + \overline{B}(k)\overline{u}(k), \quad k = 0,1,2,\ldots, \tag{6}$$

where the $N \times N$ matrix

$$\overline{A}(k) = \prod_{i=1}^{M} A(k+M-i). \tag{7}$$

The $N \times M$ matrix $\overline{B}(k)$ is defined by

$$\overline{B}(k) = [B_1, \ldots, B_j, \ldots, B_M], \tag{8}$$

in which

$$B_j = \prod_{i=1}^{M-j} A(k+M-i)b(k+j-1), \quad j = 1,2,\ldots, M-1,$$
$$B_M = b(k+M-1),$$

and

$$\overline{u}(k) = \left[u(k), u(k+1), \ldots, u(k+M-1)\right]^T.$$

Assume $k = mM + l$, $m = 0,1,2,\ldots,$ $l = 0,1,\ldots,M-1$. From (6) and (1b), we obtain Model1 of the pipelined-block LTV discrete-time system:

$$x(mM+M+l) = \overline{A}(mM+l)x(mM+l) + \overline{B}(mM+l)\overline{u}(mM+l),$$
$$y(mM+l) = c^T(mM+l)x(mM+l) + d(mM+l)u(mM+l),$$

or, in a matrix form:

$$X(m+1) = \overline{A}(m)X(m) + \overline{B}(m)\overline{U}(m), \tag{9a}$$
$$Y(m) = \overline{C}(m)X(m) + \overline{D}(m)U(m), \quad m = 0,1,2,\ldots, \tag{9b}$$

where $\overline{A}(m), \overline{B}(m), \overline{C}(m)$, and $\overline{D}(m)$ are $NM \times NM$, $NM \times M^2$, $M \times NM$, and $M \times M$ diagonal matrices, respectively, defined by

$$\overline{A}(m) = \operatorname{diag}\left\{\overline{A}(mM), \ldots, \overline{A}(mM+l), \ldots, \overline{A}(mM+M-1)\right\},$$
$$\overline{B}(m) = \operatorname{diag}\left\{\overline{B}(mM), \ldots, \overline{B}(mM+l), \ldots, \overline{B}(mM+M-1)\right\},$$
$$\overline{C}(m) = \operatorname{diag}\left\{c^T(mM), \ldots, c^T(mM+l), \ldots, c^T(mM+M-1)\right\},$$
$$\overline{D}(m) = \operatorname{diag}\left\{d(mM), \ldots, d(mM+l), \ldots, d(mM+M-1)\right\},$$

in which

$$\overline{A}(mM+l) = \prod_{i=1}^{M} A(mM+l+M-i),$$
$$\overline{B}(mM+l) = \left[B_1,\ldots,B_j,\ldots,B_M\right], \quad l = 0,1,\ldots,M-1,$$

where

$$B_j = \prod_{i=1}^{M-j} A(mM+l+M-i)b(mM+l+j-1), \quad j = 1,2,\ldots,M-1,$$

$B_M = b(mM+l+M-1)$;

$X(m+1)$ and $X(m)$ are $NM \times 1$ vectors given by

$X(m+1) = [x(mM+M),\ldots x(mM+2M-1)]^T$,

$X(m) = \left[x(mM),\ldots,x(mM+M-1)\right]^T$;

$\overline{U}(m)$ is $M^2 \times 1$ vector given by

$\overline{U}(m) = \left[\overline{u}(mM),\overline{u}(mM+1),\ldots,\overline{u}(mM+M-1)\right]^T$;

$Y(m)$ and $U(m)$ are $M \times 1$ vectors given by

$Y(m) = \left[y(mM),\ldots,y(mM+M-1)\right]^T$,

$U(m) = \left[u(mM),\ldots,u(mM+M-1)\right]^T$.

For LPTV systems, if M is equal to periodicity, then

$$\overline{A}(mM+l) = \overline{A}(l) = \prod_{i=1}^{M} A(l+M-i), \quad l = 0,1,\ldots,M-1,$$
$$\overline{B}(mM+l) = \overline{B}(l) = \left[B_1,\ldots,B_j,\ldots B_m\right],$$

in which

$$B_j = \prod_{i=1}^{M-j} A(l+M-i)b(l+j-1), \quad j = 1,2,\ldots,M-1,$$

and $B_M = b(l+M-1)$;

$$c^T(mM+l) = c^T(l), \text{ and } d(mM+l) = d(l), \; l = 0,1,\ldots,M-1.$$

For LTI systems, $\overline{A}(mM+l) = \overline{A} = A^M$, $\overline{B}(mM+l) = \overline{B} = [A^{M-1}b, \ldots, b]$, $\overline{c^T}(mM+l) = c^T$, and $d(mM+l) = d$, $l = 0, 1, \ldots, M-1$.

Some remarks. Eqs. 9a and 9b of the pipelined-block model are of the same form as Eqs. 1a and 1b of the LTV system. Model1 is useful when we need to calculate all the states simultaneously. In Parhi and Messerschmitt (1989c), the intermediate states were lost due to the block-state update process. That is the reason why the multiplication complexity of Model1 is greater than in (Parhi and Messerschmitt, 1989c). To reduce the multiplication complexity, we can use the decomposition technique of the matrix $\overline{B}(k)$ (Parhi and Messerschmitt, 1989b). Fig. 2.1 shows the pipelined-block implementation of the Nth order LTV discrete-time system for pipelining level and block size of M (Model1).

2.2. Pipelined-block Model2. Substituting $n = kM + M$, and $k = kM$ into Eq. 2, we have

$$\begin{aligned} x(kM+M) &= F(kM+M, kM)x(kM) \\ &\quad + \sum_{j=kM}^{kM+M-1} F(kM+M, j+1)b(j)u(j) \\ &= F(kM+M, kM)x(kM) \\ &\quad + \sum_{j=1}^{M} F(kM+M, kM+j)b(kM+j-1)u(kM+j-1), \end{aligned} \tag{10}$$

where

$$F(kM+M, kM) = \prod_{i=1}^{M} A(kM+M-i),$$

$$F(kM+M, kM+j) = \prod_{i=1}^{M-j} A(kM+M-i).$$

Then we get from Eq. 10 the state equation of the pipelined-block Model2 in matrix form:

$$\overline{x}(k+1) = \overline{A}(k)\overline{x}(k) + \overline{B}(k)\overline{u}(k), \quad k = 0, 1, 2, \ldots, \tag{11}$$

where the $N \times N$ matrix $\overline{A}(k)$ is defined by

$$\overline{A}(k) = \prod_{i=1}^{M} A(kM+M-i). \tag{12}$$

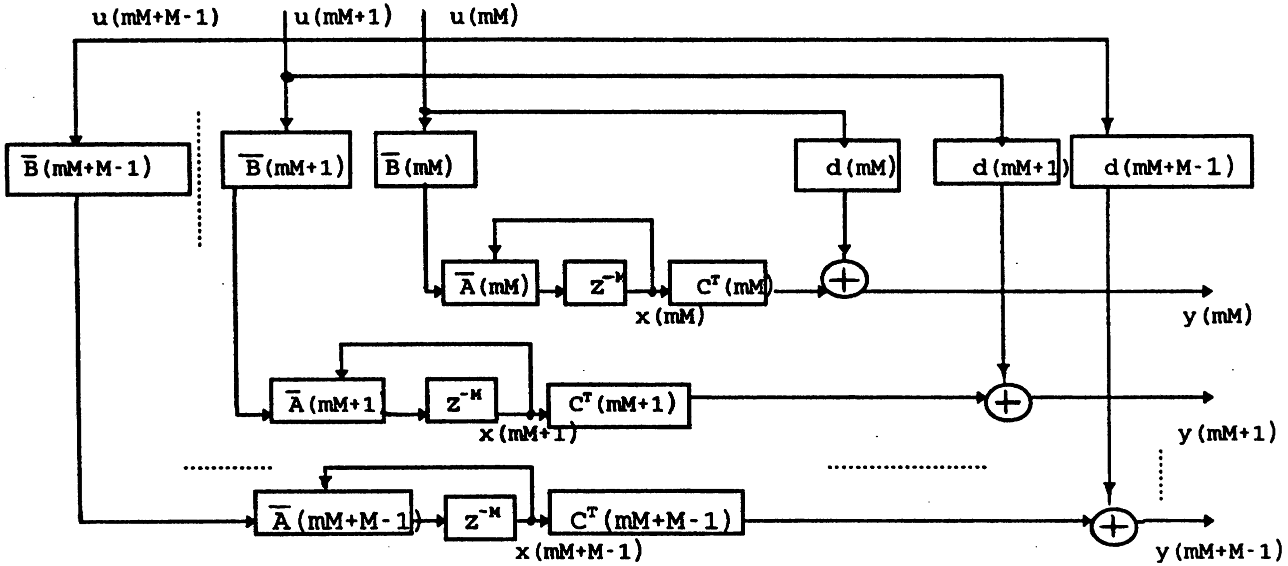


Fig. 2.1. Pipelined-block implementation of the Nth order LTV discrete-time system in state space for pipelining level and block size of M (Model1).

The $N \times M$ matrix $\overline{B}(k)$ is defined by

$$\overline{B}(k) = \big[B_1, \ldots, B_j, \ldots, B_M\big], \tag{13}$$

in which

$$\begin{aligned}
B_j &= \prod_{i=1}^{M-j} A(kM + M - i)b(kM + j - 1), \quad j = 1, 2, \ldots, M-1, \\
B_M &= b(kM + M - 1), \\
\overline{u}(k) &= \big[u(kM), u(kM+1), \ldots, u(kM + M - 1)\big]^T, \\
\overline{x}(k) &= x(kM), \quad \overline{x}(k+1) = x\big[(k+1)M\big].
\end{aligned}$$

Substituting $n = kM + i$, and $k = kM,\ k = 0, 1, 2, \ldots, i = 0, 1, \ldots, M-1$ into (5), we obtain

$$\begin{aligned}
y(kM + i) &= c^T(kM + i)F(kM + i, kM)x(kM) \\
&+ \sum_{j=kM}^{kM+i-1} c^T(kM + i)F(kM + i, j + 1)b(j)u(j) + d(kM + i)u(kM + i)
\end{aligned}$$

or

$$\begin{aligned}
y(kM + i - 1) &= c^T(kM + i - 1)F(kM + i - 1, kM)x(kM) \\
&+ \sum_{j=1}^{i} c^T(kM + i - 1)F(kM + i - 1, kM + j)b(kM + j - 1) \\
&\times u(kM + j - 1) + d(kM + i - 1)u(kM + i - 1), \\
&\qquad i = 1, 2, \ldots, M,
\end{aligned} \tag{14}$$

where

$$F(kM + i - 1, kM) = \prod_{j=1}^{i-1} A(kM + i - j - 1) = \prod_{j=2}^{i} A(kM + i - j),$$

and

$$F(kM + i - 1, kM + j) = \prod_{l=1}^{i-j-1} A(kM + i - l - 1) = \prod_{l=2}^{i-j} A(kM + i - l).$$

Then we get from Eq. 14 the output equation of the pipelined-block Model2 in a matrix form:

$$\overline{y}(k) = \overline{C}(k)\overline{x}(k) + \overline{D}(k)\overline{u}(k), \quad k = 0, 1, 2, \dots, \tag{15}$$

where the $M \times N$ matrix $\overline{C}(k)$ is defined by

$$\overline{C}(k) = [C_1, \dots, C_i, \dots, C_M]^T, \tag{16}$$

in which

$$C_1 = c^T(kM),$$
$$C_i = c^T(kM + i - 1)\prod_{j=2}^{i} A(kM + i - j), \quad i = 2, 3 \dots, M.$$

The $M \times M$ matrix $\overline{D}(k)$ is defined by

$$\overline{D}(k) = [d_{ij}], \tag{17}$$

in which

$$d_{ij} = 0, \quad \text{if} \quad i < j,$$
$$d_{ij} = d(kM + i - 1), \quad \text{if} \quad i = j,$$
$$d_{ij} = c^T(kM + i - 1)b(kM + j - 1), \quad \text{if} \quad i = j + 1,$$
$$d_{ij} = c^T(kM + i - 1)\prod_{l=2}^{i-j} A(kM + i - l)b(kM + j - 1), \text{ if } i > j + 1,$$
$$\overline{y}(k) = \left[y(kM), y(kM + 1), \dots, y(kM + M - 1)\right]^T.$$

For LPTV discrete-time systems, $A(k)$, $b(k)$, $c^T(k)$, and $d(k)$ are L-periodic, i.e., $A(k + L) = A(k)$, $b(k + L) = b(k)$, $c^T(k + L) = c^T(k)$, and $d(k + L) = d(k)$. In case the pipelining level M is equal to the periodicity L of the LPTV system, we get simpler expressions from Eqs. 12 and 13 for calculating matrices $\overline{A}(k) = \overline{A}$ and $\overline{B}(k) = \overline{B}$.

$$\overline{A} = \prod_{i=1}^{L} A(L - i), \tag{18}$$

$$\overline{B} = [B_1, \dots, B_j, \dots, B_L], \tag{19}$$

in which

$$B_j = \prod_{i=1}^{L-j} A(L-i)b(j-1), \quad j = 1, 2, \ldots, L-1,$$
$$B_L = b(L-1).$$

If $M = mL$, $m = 2, 3, \ldots$, in Eqs. 18 and 19, then some matrices A will be raised to the mth power, for example, $\overline{A} = A^m(L-1)A^m(L-2)\ldots A^m(0)$. If $M \neq mL$, then the matrices $\overline{A}$ and $\overline{B}$ are calculated using Eqs. 12 and 13, in which some matrices A and vectors b will have the same values.

In case the periodicity is equal to the block size of the LPTV discrete-time system, we obtain simpler expressions for calculating matrices $\overline{C}(k) = \overline{C}$, and $\overline{D}(k) = \overline{D}$. Then we get using Eqs. 16 and 17,

$$\overline{C} = [C_1, \ldots, C_i, \ldots, C_L]^T, \tag{20}$$

in which

$$C_1 = c^T(0),$$
$$C_i = c^T(i-1)\prod_{j=2}^{i} A(i-j), \quad i = 2, 3 \ldots, L,$$

and the $L \times L$ matrix

$$\overline{D} = [d_{ij}], \tag{21}$$

in which

$$d_{ij} = 0, \quad \text{if} \quad i < j,$$
$$d_{ij} = d(i-1), \quad \text{if} \quad i = j,$$
$$d_{ij} = c^T(i-1)b(j-1), \quad \text{if} \quad i = j+1,$$
$$d_{ij} = c^T(i-1)\prod_{l=2}^{i-j} A(i-l)b(j-1), \text{ if } i > j+1.$$

Eq. 11 with matrices (12) and (13) can be used for pipelining of the Nth order LTV discrete-time system and the Nth order LPTV discrete-time system with an arbitrary pipelining level M. Therefore, the pipelining level is independent of the periodicity L of the LPTV discrete-time system coefficients. In expressions (16) and (17), M is the block size, which is not the same (or may be the same) as the periodicity L of the LPTV discrete-time system.

For LTI discrete-time systems, $A(k) = A$, $b(k) = b$, $c^T(k) = c^T$, and $d(k) = d$. Hence, matrices (12), (13), (16), and (17) are of the forms, respectively

$$\overline{A}(k) = \overline{A} = A^M, \tag{22}$$

$$\overline{B}(k) = \overline{B} = [A^{M-1}b, \ldots, Ab, b], \tag{23}$$

$$\overline{C}(k) = \overline{C} = [c^T, c^T A, \ldots, c^T A^{M-1}]^T, \tag{24}$$

$$\overline{D}(k) = \overline{D} = [d_{ij}], \tag{25}$$

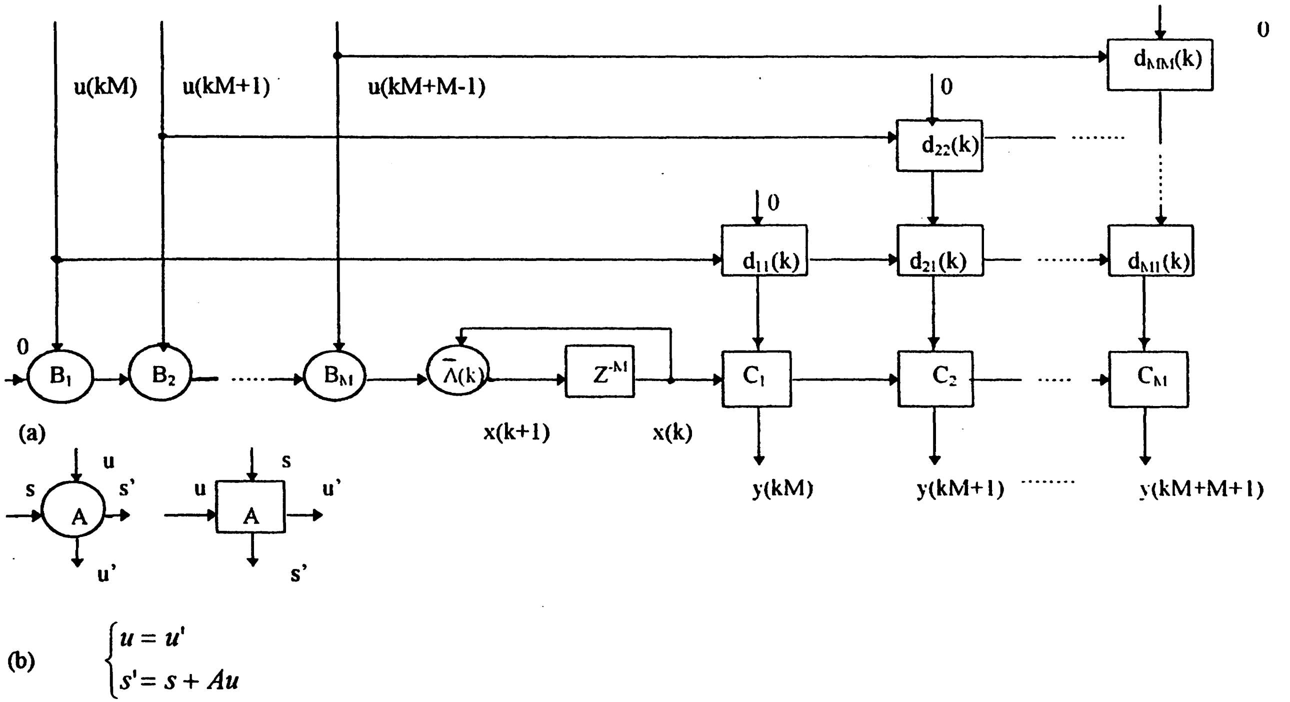
in which

$$\begin{aligned} d_{ij} &= 0, \quad \text{if} \quad i < j, \\ d_{ij} &= d, \quad \text{if} \quad i = j, \\ d_{ij} &= c^T b, \quad \text{if} \quad i = j + 1, \\ d_{ij} &= c^T A^{i-j-1} b, \ \text{if}\ i > j + 1. \end{aligned}$$

Fig. 2.2. shows the pipelined-block implementation of the Nth order LTV discrete-time system for the pipelining level and block size of M (Model2). In the pipelined-block discrete-time system, we use $\overline{x}(0)$ to compute the block of outputs $[y(0), \ldots, y(M-1)]$, and to update $\overline{x}(1) = x(M)$. In the next cycle, $\overline{x}(1)$ is used to compute the next block of outputs, and to update the state $\overline{x}(2) = x(2M)$, and so on.

2.3. Pipelined-block Model3. Substituting $n = kM + ML$, and $k = kM$ into Eq. 2, we get

$$\begin{aligned} x(kM + ML) = {} & F(kM + ML, kM)x(kM) \\ & + \sum_{j=kM}^{kM+ML-1} F(kM + ML, j + 1)b(j)u(j). \\ = {} & F(kM + ML, kM)x(kM) \\ & + \sum_{j=1}^{ML} F(kM + ML, kM + j)b(kM + j - 1)u(kM + j - 1), \end{aligned} \tag{26}$$



$$\begin{cases} u = u' \\ s' = s + Au \end{cases}$$

Fig. 2.2. a) Pipelined-block implementation of the *N*th order LTV discrete-time system in state space for pipelining level and block size of *M* (Model2); b) Definition of processing elements.

where the state transition matrix

$$F(kM+ML,kM)=\prod_{j=1}^{ML}A(kM+ML-j),$$

and

$$F(kM+ML,kM+j)=\prod_{i=1}^{ML-j}A(kM+ML-i).$$

Thus, we get from Eq. 26 the state equation of pipelined-block Model3 in a matrix form:

$$\overline{x}(k+L)=\overline{A}(k)\overline{x}(k)+\overline{B}(k)\overline{u}(k),\quad k=0,1,2,\ldots, \tag{27}$$

where the $N\times N$ matrix $\overline{A}(k)$ is defined by

$$\overline{A}(k)=\prod_{j=1}^{ML}A(kM+ML-j). \tag{28}$$

The $N\times ML$ matrix $\overline{B}(k)$ is defined by

$$\overline{B}(k)=[B_1,\ldots,B_j,\ldots,B_{ML}], \tag{29}$$

in which

$$B_j=\prod_{i=1}^{ML-j}A(kM+ML-i)b(kM+j-1),\quad j=1,2,\ldots,ML-1,$$
$$B_{ML}=b(kM+ML-1),$$

and

$$\overline{u}(k)=\left[u(kM),u(kM+1),\ldots,u(kM+M-1)\right]^T,$$
$$\overline{x}(k)=x(kM),\quad \overline{x}(k+L)=x\left[(k+L)M\right].$$

For the *pipelined-block LPTV discrete-time systems*

$$\overline{A}(k)=\prod_{j=1}^{ML}A(ML-j), \tag{30}$$

$$\overline{B}(k)=[B_1,\ldots,B_j,\ldots,B_{ML}], \tag{31}$$

in which

$$B_j = \prod_{i=1}^{ML-j} A(ML-i)b(j-1), \quad j = 1, 2, \ldots, ML-1,$$
$$B_{ML} = b(ML-1).$$

For the pipelined-block LTI discrete-time systems (see also Parhi and Messerschmitt, 1989c, p. 1130)

$$\overline{A}(k) = \overline{A} = A^{ML}, \tag{32}$$

and

$$\overline{B}(k) = \overline{B} = \left[A^{ML-1}b, \ldots, Ab, b\right]. \tag{33}$$

The output of the discrete-time system can be computed using Eq. 15 of Model2.

3. Controllability of pipelined-block LTV discrete-time systems. Consider the pipelined-block LTV discrete-time system described by

$$\overline{x}(k+1) = \overline{A}(k)\overline{x}(k) + \overline{B}(k)\overline{u}(k), \tag{34a}$$
$$\overline{y}(k) = \overline{C}(k)\overline{x}(k) + \overline{D}(k)\overline{u}(k), \quad k = 0, 1, 2, \ldots, \tag{34b}$$

where $\overline{A}(k), \overline{B}(k), \overline{C}(k)$, and $\overline{D}(k)$ are defined in expressions (12), (13), (16), and (17), respectively.

DEFINITION 3.1. Complete state controllability. A LTV discrete-time system described by (34a) is said to be complete state controllable if, for any initial time k_0, there exists a realizable control sequence $\overline{u}(k)$, $k = k_0, k_0 + 1, \ldots, n-1$, which will drive the state in a finite time interval from every initial state $\overline{x}(k_0)$ to a certain final state $\overline{x}(n)$.

Theorem 3.1. *Pipelined-block LTV discrete-time system* (34a) *is complete state controllable, if the rank* $Q = N$, *where the* $N \times (n-k_0)M$ *matrix*

$$Q = \left[Q_{k_0}, \ldots, Q_k, \ldots, Q_{n-1}\right], \tag{35}$$

in which

$$Q_k = \Phi(n, k+1)\overline{B}(k), \quad k = k_0, k_0+1, \ldots, n-1,$$
$$\Phi(n, k+1) = \prod_{j=1}^{n-k-1} \overline{A}(n-j),$$
$$\overline{A}(n-j) = \prod_{i=1}^{M} A\left[(n-j)M + M - i\right].$$

Proof. The solution of equation (34a) is given by

$$\overline{x}(n) = \Phi(n, k_0)\overline{x}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\overline{B}(k)\overline{u}(k), \tag{36}$$

or

$$\overline{x}(n) - \Phi(n, k_0)\overline{x}(k_0) = \sum_{k=k_0}^{n-1} \Phi(n, k+1)\overline{B}(k)\overline{u}(k). \tag{37}$$

Eq. 37 in a matrix form is as follows

$$X(n, k_0) = QU, \tag{38}$$

where the $N \times 1$ vector

$$X(n, k_0) = \overline{x}(n) - \Phi(n, k_0)\overline{x}(k_0),$$

Q is the $N \times (n - k_0)M$ matrix as in (35), and the $(n - k_0)M \times 1$ vector

$$U = \left[\overline{u}(k_0), \overline{u}(k_0 + 1), \dots, \overline{u}(n-1)\right]^T,$$

in which

$$\overline{u}(k_0) = \left[u(k_0 M), \dots, u(k_0 M + M - 1)\right]^T,$$
$$\overline{u}(k_0 + 1) = \left[u((k_0 + 1)M), \dots, u((k_0 + 1)M + M - 1)\right]^T.$$

In the case of complete state controllability, every initial state $\overline{x}(k_0)$ for any initial time k_0 must be transformed by a realizable control sequence $\overline{u}(k),\ k = k_0, k_0+1, \dots, n-1$, into the final state $\overline{x}(n)$. So we have such a problem: in an N-dimensional state space we must find a vector U such that it satisfies Eq. 38 in case the matrix Q and the vector $X(n, k_0)$ are known. System of equations (37) will have a solution, if equations are linear independent. A necessary and sufficient condition of independence is the rank $Q = N$.

Another way to prove that the matrix Q has the rank N is based on the fact that the $N \times N$ grammian

$$QQ^T = \sum_{k=k_0}^{n-1} Q_k Q_k^T$$

is nonsingular. The rank Q is equal to N, so the rank Q^T is also equal to N. That is why QQ^T must be of the rank N or be nonsingular.

Theorem 3.2. *System* (34a) *is complete state controllable if the grammian*

$$QQ^T = \sum_{k=k_0}^{n-1} \Phi(n, k+1)\overline{B}(k)\overline{B}^T(k)\Phi^T(n, k+1)$$

is nonsingular.

DEFINITION 3.2. Complete output controllability. A LTV discrete-time system described by (34a) and (34b) is said to be complete output controllable, if for any initial time k_0 there exists a control sequence $\overline{u}(k),\ k = k_0, k_0 + 1, \ldots, n-1$ that will drive the output in the finite time interval from any initial state to a certain final output value $\overline{y}(n)$.

Theorem 3.3. *A pipelined-block LTV discrete-time system* (34a) – (34b) *is complete output controllable if the rank* $T = M$, *where the* $M \times M(n - k_0 + 1)$ *matrix*

$$T = \left[T_{k_0}, \ldots, T_k, \ldots, T_n\right], \tag{39}$$

where

$$T_k = \begin{cases} \overline{C}(n)\Phi(n, k+1)\overline{B}(k), & \text{if } k = k_0, \ldots, n-1, \\ \overline{D}(n), & \text{if } k = n. \end{cases}$$

Proof. Substituting transition state equation (36) into output equation (34b), we get

$$\begin{aligned} \overline{y}(n) =& \overline{C}(n)\Phi(n, k_0)\overline{x}(k_0) \\ &+ \sum_{k=k_0}^{n-1} \overline{C}(n)\Phi(n, k+1)\overline{B}(k)\overline{u}(k) + \overline{D}(n)\overline{u}(n), \quad n = 0, 1, 2, \ldots, \end{aligned}$$

or

$$\begin{aligned} Y(n, k_0) =& \overline{C}(n)\Phi(n, k_0+1)\overline{B}(k_0)\overline{u}(k_0) \\ &+ \overline{C}(n)\Phi(n, k_0+2)\overline{B}(k_0+1)\overline{u}(k_0+1) + \ldots \\ &+ \overline{C}(n)\Phi(n, n)\overline{B}(n-1)\overline{u}(n-1) + \overline{D}(n)\overline{u}(n), \end{aligned} \tag{40}$$

where

$$Y(n, k_0) = \overline{y}(n) - \overline{C}(n)\Phi(n, k_0)\overline{x}(k_0).$$

Eq. 40 in matrix form is

$$Y(n, k_0) = TU = \sum_{k=k_0}^{n} T_k \overline{u}(k), \tag{41}$$

where the $M(n - k_0 + 1) \times 1$ vector U and $M \times M(n - k_0 + 1)$ matrix T are:

$$U = \left[\overline{u}(k_0), \overline{u}(k_0 + 1), \ldots, \overline{u}(n)\right]^T,$$
$$T = \left[T_{k_0}, \ldots, T_k, \ldots, T_n\right], \quad k = k_0, k_0 + 1, \ldots, n,$$

in which

$$T_k = \begin{cases} \overline{C}(n)\Phi(n, k + 1)\overline{B}(k), & \text{if } k = k_0, \ldots, n - 1, \\ \overline{D}(n), & \text{if } k = n. \end{cases}$$

Expression (41) consists of M linear equations. So these equations will have a solution according the control sequence, if for any initial state $\overline{x}(k_0)$ and a certain final output $\overline{y}(n)$, the rank of matrix T will be equal to M.

Theorem 3.4. *Pipelined-block LTV discrete-time system* (34a) – (34b) *is complete output controllable, if the $M \times M$ grammian*

$$TT^T = \sum_{k=k_0}^{n} T_k T_k^T$$

is nonsingular.

Proof. The rank T is equal to M, so the rank T^T is also equal to M. That is why TT^T must be of the rank M or be nonsingular.

4. Observability of pipelined-block LTV discrete-time systems

DEFINITION 4.1. Complete observability. A LTV discrete-time system described by (34a) and (34b) is said to be complete observable, if for some k_0 any state $\overline{x}(k_0)$ can be obtained from a finite number of output $\overline{y}(k)$ and input, $\overline{u}(k)$, $k_0 \leqslant k < n$ variables.

Theorem 4.1. *A pipelined-block LTV discrete-time system described by* (34a) – (34b) *is complete observable, if the rank $L = N$, where the $N \times M(n - k_0)$ matrix*

$$L(k_0, n - 1) = \left[\overline{C}^T(k_0), \Phi^T(k_0 + 1, k_0)\overline{C}^T(k_0 + 1), \ldots, \Phi^T(n - 1, k_0)\overline{C}^T(n - 1)\right], \tag{42}$$

or the $N \times N$ matrix

$$L(k_0, n-1)L^T(k_0, n-1) = \sum_{k=k_0}^{n-1} \Phi^T(k, k_0)\overline{C}^T(k)\overline{C}(k)\Phi(k, k_0) \qquad (43)$$

is nonsingular.

Proof. Transition state equation (36), after setting $n = k$, is of the form:

$$\overline{x}(k) = \Phi(k, k_0)\overline{x}(k_0) + \sum_{k=k_0}^{k-1} \Phi(k, k+1)\overline{B}(k)\overline{u}(k). \qquad (44)$$

Substituting (44) into (34b), we have a system of equations

$$\begin{aligned}\overline{y}(k) =& \overline{C}(k)\Phi(k, k_0)\overline{x}(k_0) + \sum_{k=k_0}^{k-1} \overline{C}(k)\Phi(k, k+1)\overline{B}(k)\overline{u}(k)\\ &+ \overline{D}(k)\overline{u}(k), \quad k = k_0, k_0+1, \ldots, n-1,\end{aligned}$$

or in matrix form:

$$\begin{bmatrix} \overline{y}(k_0) \\ \overline{y}(k_0+1) \\ \vdots \\ \overline{y}(n-1) \end{bmatrix} = \begin{bmatrix} \overline{C}(k_0) \\ \overline{C}(k_0+1)\Phi(k_0+1, k_0) \\ \vdots \\ \overline{C}(n-1)\Phi(n-1, k_0) \end{bmatrix} \overline{x}(k_0)$$

$$+ \begin{bmatrix} \overline{D}(k_0) & 0 & 0 & 0 \\ \overline{C}(k_0+1)\overline{B}(k_0) & \overline{D}(k_0+1) & 0 & 0 \\ \overline{C}(k_0+2)\Phi(k_0+2, k_0+1)\overline{B}(k_0) & \overline{C}(k_0+2)\overline{B}(k_0+1) & \overline{D}(k_0+2) & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \overline{C}(n-1)\Phi(n-1, k_0+1)\overline{B}(k_0) & \overline{C}(n-1)\Phi(n-1, k_0+2)\overline{B}(k_0+1) & \ldots & \overline{D}(n-1) \end{bmatrix} \begin{bmatrix} \overline{u}(k_0) \\ \overline{u}(k_0+1) \\ \overline{u}(k_0+2) \\ \vdots \\ \overline{u}(n-1) \end{bmatrix} \qquad (45)$$

In order to get $\overline{x}(k_0)$ from (45), where $\overline{y}(k)$ and $\overline{u}(k)$, $k = k_0, k_0+1, \dots, n-1$, are known, it is obvious that the $M(n-k_0) \times N$ matrix

$$L^T(k_0, n-1) = \begin{bmatrix} \overline{C}(k_0) \\ \overline{C}(k_0+1)\Phi(k_0+1, k_0) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \overline{C}(n-1)\Phi(n-1, k_0) \end{bmatrix}$$

must be of the rank N.

If the matrix $L(k_0, n-1)$ is of the rank N, then the $N \times N$ matrix $L(k_0, n-1)L^T(k_0, n-1)$ must also be of the rank N, whence condition (43) follows.

5. Controllability of pipelined-block LTI discrete -time systems

Theorem 5.1. *A pipelined-block LTI discrete-time system described by*

$$\overline{x}(k+1) = \overline{A}\overline{x}(k) + \overline{B}\overline{u}(k), \quad k = 0, 1, 2, \dots, \tag{46}$$

where $\overline{A}$ and $\overline{B}$ are defined in (22) and (23), respectively, is complete state controllable, if the rank $Q = N$, where the $N \times nM$ matrix

$$Q = [\overline{B} \quad \overline{A}\,\overline{B} \quad \overline{A}^2\overline{B} \dots \overline{A}^{n-1}\overline{B}].$$

Proof directly follows from the complete state controllability (see Theorem 3.1) of LTV discrete-time systems.

Theorem 5.2. *A pipelined-block LTI discrete-time system (46) is complete state controllable, if the $N \times N$ grammian*

$$QQ^T = \sum_{k=0}^{n-1} \overline{A}^{n-k-1}\overline{B}\,\overline{B}^T(\overline{A}^{n-k-1})^T$$

is nonsingular for $n \geqslant N$.

Proof directly follows from the complete state controllability (see Theorem 3.2) of pipelined-block LTV discrete-time systems.

Theorem 5.3. *A pipelined-block LTI discrete-time system described by*

$$\overline{x}(k+1) = \overline{A}\overline{x}(k) + \overline{B}\overline{u}(k), \tag{47a}$$

$$y(k) = \overline{C}\overline{x}(k) + \overline{D}\overline{u}(k), \quad k = 0, 1, 2, \dots, \tag{47b}$$

where $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$ are defined in (22), (23), (24), and (25), respectively, is complete output controllable, if the rank $T = M$, where the $M \times M(n+1)$ matrix

$$T = [\overline{C}\,\overline{A}^{n-1}\overline{B}, \ldots, \overline{C}\,\overline{A}\,\overline{B},\ \overline{C}\,\overline{B},\ \overline{D}].$$

Proof directly follows from the complete output controllability condition of LTV discrete-time systems. Using Eq. 39, we get

$$T_k = \begin{cases} \overline{C}\,\overline{A}^{n-k-1}\overline{B}, & k = 0, 1, \ldots, n-1, \\ \overline{D}, & k = n. \end{cases}$$

6. Observability of pipelined-block LTI discrete-time systems

Theorem 6.1. *A pipelined-block LTI discrete-time system* (47a) – (47b) *is complete observable, if the rank $L = N$, where the $N \times Mn$ matrix*

$$L = [\overline{C}^T, \overline{A}^T\overline{C}^T, \ldots, (\overline{A}^T)^{n-1}\overline{C}^T].$$

Proof directly follows from the complete observability condition (42) of pipelined-block LTV discrete-time systems. In the general case, information on $\overline{y}(k)$ and $\overline{u}(k)$ is needed only for $k = 0, 1, \ldots, n-1$. If a LTI discrete-time system is observable in some interval $[k_0, n]$, then it will be observable in any other interval. So for LTI discrete-time systems a complete observability implies global observability.

Theorem 6.2. *A pipelined-block LTI discrete-time system described by* (47a) – (47b) *is complete observable, if the $N \times N$ grammian*

$$V = \sum_{k=0}^{n-1} (\overline{A}^{n-k-1})^T \overline{C}^T \overline{C}\,\overline{A}^{n-k-1}$$

is nonsingular.

Proof directly follows from the complete observability condition (43) of the pipelined-block LTV discrete-time system.

7. Conclusions. It has been shown that any form of pipelined-block LTV, LPTV, or LTI discrete-time systems can be obtained using the general solution of state space equations. At first, we have derived pipelined-block equations for LTV discrete-time systems. LPTV and LTI discrete systems are partial cases of the LTV discrete-time systems, so we get pipelined-block equations for LPTV, and LTI discrete-time systems from pipelined-block equations of LTV discrete-time systems. In Model1 we compute all the states, so the multiplication complexity of Model1 is greater than the multiplication complexity of Model2.

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TIESINIŲ DISKRETINIŲ SISTEMŲ KONVEJERINIAI-BLOKINIAI MODELIAI

Kazys KAZLAUSKAS

Straipsnyje parodyta, kaip iš būsenų lygčių bendrojo sprendinio galima gauti įvairius tiesinių diskretinių kintamų, periodinių ir pastovių parametrų sistemų konvejerinius-blokinius modelius. Nagrinėjami trys konvejeriniai-blokiniai modeliai ir dviejų modelių struktūros. Analizuojamos tiesinių diskretinių kintamų ir pastovių parametrų konvejerinio-blokinio modelio būsenų ir išėjimo signalo valdomumo ir stebimumo sąlygos.