INFORMATICA, 1996, Vol. 7, No. 4, 411–418

ANALYTIC SOLUTION OF HOMOGENEOUS POINT-DELAY SYSTEMS. THE FIRST ORDER CASE

Carlos F. ALASTRUEY and Manuel de la SEN

University of the Basque Country, Department of Electricity and Electronics
P.O. Box 644 – 48080 Bilbao, Spain
E-mail: wepalmec@lg.ehu.es

Abstract. New formulae providing exact explicit solutions for a class of scalar delayed systems are given.

Key words: point-delayed system, stability.

1. Introduction. The stability of a linear delay-differential system with a point delay in its state has been studied in different works (Hmamed, 1985; Hmamed, 1986a; Hmamed 1986b; Mori, 1986; Bourlès, 1987; Mori and Kokame, 1989; De la Sen, 1992a; De la Sen, 1992b). However, the main difficulty when dealing with delayed systems in the time-domain is the non availability of exact explicit solutions for any $t > 0$ and the absence of direct stability tests of easy testing for such systems. If an exact explicit solution could be obtained for a delayed system for any $t > 0$, the study of its stability would become direct. In this note a simple general method to obtain such a solution is proposed, and an exact explicit solution for any t is provided for the linear scalar invariant case with one point-delay.

In many works exact formulae for solving delayed systems have been given. The most general of those results can be found in De la Sen (1988). However, in the literature none of the exact given solutions for systems with state-delays is explicit. This means that the exact solutions always include a subsidiary delayed differential equation that is not directly solvable to compute the fundamental matrix (De la Sen, 1988; Burton, 1985). Let us consider a first-order homogeneous and invariant linear differential system with one point delay in its state, as shown in the next equation:

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h), \tag{1}$$

with initial conditions $x(t) \equiv 0 \ \forall t < 0$; $x(0) = \varphi(0) = x_0$, where $a_0, a_1 \in R^*$ and $h \in R^+$. For t such that $0 \leqslant t < h$ system (1) is reduced to the non-delayed system $\dot{x}(t) = a_0 x(t)$ with initial condition $x(0) = x_0$ which can be solved as an ordinary first-order differential equation with a known forcing term in the case when $\varphi(t)$ is nonzero on $[-h, 0)$, (Burton, 1985). Since the global stability is independent of the chosen initial conditions, it is sufficient the choice of $\varphi(t) = 0$ on $[-h, 0)$ with $\varphi(0) = x_0$ to investigate the stability of (1). Again, note that for t such that $h < t \leqslant 2h$ Eq. 1 becomes

$$\dot{x}(t) = a_0(t)x(t) + a_1(t)\phi_1(t-h). \tag{2}$$

Once again, it could be possible to solve (2) as an ordinary first-order differential equation with a known forcing term, and find an exact explicit solution $\phi_2(t)$ of $x(t)$ for t such that $h < t \leqslant 2h$. By following successively this simple method, it would be possible to find the exact explicit solution for any interval. However, the problem is to find a general formula for the n term, and then to solve exactly and explicitly the delayed differential equation and then cheking directly its stability by using the above solution. This goal is pursued in the next section for a class of first-order linear delayed systems by following the so-called method of steps (Hale, 1977).

2. Study of a class of first-order homogeneous linear systems with one point-delay. Let us consider a first-order homogeneous and invariant linear differential system with one point delay in its state, as shown in Eq. 1, subject to initial conditions $x(t) \equiv 0 \ \forall t < 0$; $x(0) = \varphi(0) = x_0$, where $a_0, a_1 \in R^*$ and $h \in R^+$. For t such that $0 \leqslant t < h$ system (1) is reduced to the non-delayed system $\dot{x}(t) = a_0 x(t)$ with initial condition $x(0) = x_0$ and with exact explicit solution $\phi_1(t) = x_0 \mathbf{e}^{a_0 t}$. For successive intervals is possible to apply the method described in Section 1. The exact explicit solutions for the second and third intervals are the following ones:

For t such that $h \leqslant t < 2h$,

$$x(t) = \phi_2(t) = x_0 \mathbf{e}^{a_0 t}\left[1 - \frac{a_1 h}{\mathbf{e}^{a_0 h}} + \frac{a_1 t}{\mathbf{e}^{a_0 h}}\right]. \tag{3}$$

For t such that $2h \leqslant t < 3h$,

$$x(t) = \phi_3(t) = x_0 \mathbf{e}^{a_0 t}\left[1 - \frac{a_1 h}{\mathbf{e}^{a_0 h}} + \frac{2a_1^2 h^2}{\mathbf{e}^{2a_0 h}} + \frac{a_1 t}{\mathbf{e}^{a_0 h}} - \frac{2a_1^2 ht}{\mathbf{e}^{2a_0 h}} + \frac{a_1^2 T^2}{2\mathbf{e}^{2a_0 h}}\right]. \tag{4}$$

Define $b = a_1 e^{-a_0 h}$. After some simple algebraic manipulations, it is possible to rewrite (3) – (4) as follows.

$$\phi_1(t) = x_0 e^{a_0 t}, \tag{5}$$

$$\phi_2(t) = \phi_1(t)\big[1 + b(t-h)\big], \tag{6}$$

$$\phi_3(t) = \phi_1(t)\Big[1 + b\Big[(t-h) + \frac{b}{2}(t-2h)^2\Big]\Big]. \tag{7}$$

It is seen below that the exact explicit solution for the n-th interval is

$$\phi_n(t) = x_0 e^{a_0 t}\left(1 + \sum_{i=1}^{n-1}\left((t-ih)^i \prod_{j=1}^{i}\left(\frac{a_1 e^{-a_0 h}}{j}\right)\right)\right). \tag{8}$$

This solution is formalized in the following main result whose proof is given by using complete induction in Appendix A.

Theorem (Main result). *Consider system (1). An exact explicit unique solution for that system on $[0,\infty)$, subject to the initial conditions $\varphi(t) = 0,\ t \in [-h,0)$, and $\varphi(0) = x(0) = x_0$, is built in closed-form by using truncated functions as follows*

$$\Omega(t) = \sum_{i=1}^{\infty} \phi_{iT}(t), \quad \textit{all} \quad t \geqslant 0 \tag{9}$$

subject to the initial conditions $\varphi(t) = 0,\ t \in [-h,0)$, and $\varphi(0) = x(0) = x_0$, where

$$\phi_{1T}(t) = x_0 e^{a_0 t}\big(U(t) - U(t-h)\big), \tag{10a}$$

$$\phi_{iT}(t) = x_0 e^{a_0 t}\left(1 + \sum_{k=1}^{i-1}\left((t-kh)^k \prod_{j=1}^{k}\left(\frac{a_1 e^{-a_0 h}}{j}\right)\right)\right) \times \big(U(t-(i-1)h) - U(t-ih)\big)$$
$$(\Rightarrow \phi_{iT}(t) = 0 \quad \textit{for} \quad t < (i-1)h \quad \textit{or} \quad t \geqslant ih) \quad \textit{for} \quad i \geqslant 2, \tag{10b}$$

where $U(t)$ is the unity step function at $t = 0$.

Conditions for stability of system (1) could be directly obtained by taking the limit when n tends to infinity in Eq. 8.

Numerical example. Let us consider the following scalar delayed system

$$\dot{x}(t) = -6x(t) + x(t-2), \tag{11}$$

with initial conditions $x(t) \equiv 0 \ \forall t < 0$; $x(0) = \varphi(0) = 3$. For t such that $0 \leqslant t < 2$ system (11) is reduced to the non-delayed system $\dot{x}(t) = -6x(t)$ with initial condition $x(0) = 3$ and with exact explicit solution $\phi_1(t) = 3e^{-6t}$. For t such that $2 \leqslant t < 4$,

$$x(t) = \phi_2(t) = 3e^{-6t}\left[1 - \frac{2}{e^{-12}} + \frac{t}{e^{-12}}\right], \tag{12}$$

and note that $\phi_1(2) = \phi_2(2)$. According to the Theorem, the exact explicit solution for the n-th interval is

$$\phi_n(t) = 3e^{-6t}\left(1 + \sum_{i=1}^{n-1}\left((t-i2)^i \prod_{j=1}^{i}\left(\frac{e^{12}}{j}\right)\right)\right). \tag{13}$$

3. Conclusions. The simple method presented in this note has been used to find an explicit exact solution for a class of delay-differential systems with one point delay which does not depend on a time-delayed fundamental matrix.

APPENDIX A. Proof of the main result by complete induction. For $n = 1$ ($\Rightarrow \ 0 \leqslant t < h$), the proposed solution (9) and system (1) become, respectively, to

$$\Omega(t) = \phi_{1T}(t) = x_0 e^{a_0 t}\big(U(t) - U(t-h)\big) = x_0 e^{a_0} t = \phi_1(t), \tag{A1}$$

$$\dot{x}(t) = a_0 x(t), \ x(0) = x_0. \tag{A2}$$

For $n = 2$ ($\Rightarrow \ h \leqslant t < 2h$), the proposed solution (9) and system (1) become, respectively, to

$$\Omega(t) = \phi_{1T}(t) + \phi_{2T}(t) = x_0 e^{a_0 t}\left[1 + a_1 e^{-a_0 h}(t-h)\right], \tag{A3}$$

$$\dot{x}(t) = a_0 x(t) + a_1\phi_1(t-h) = a_0 x(t) + a_1 x_0 e^{a_0(t-h)}. \tag{A4}$$

By direct substitution it can be seen that (A1) is a solution for (A2) and (A3) is a solution for (A4). Therefore, the theorem is demonstrated for $n = 1, 2$. Let us suppose that the proposed solution (9) is valid for both $n-2$ and for $n-1$, this is, for $(n-3)h \leqslant t < (n-2)h$ and for $(n-2)h \leqslant t < (n-1)h$. Thus

$$\phi_{n-2}(t) = x_0 \mathrm{e}^{a_0 t}\left(1 + \sum_{i=1}^{n-3}\left((t-ih)^i \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right) \tag{A5}$$

is a solution for system (1) for any t such that $(n-3)h \leqslant t < (n-2)h$ and

$$\phi_{n-1}(t) = x_0 \mathrm{e}^{a_0 t}\left(1 + \sum_{i=1}^{n-2}\left((t-ih)^i \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right) \tag{A6}$$

is a solution for system (1) for any t such that $(n-2)h \leqslant t < (n-1)h$. In particular, $\phi_{n-1}(t)$ must satisfy the state equation (1) for such interval:

$$\dot{\phi}_{n-1}(t) = a_0\phi_{n-1}(t) + a_1\phi_{n-2}(t-h). \tag{A7}$$

Note that, by differentiating $\phi_{n-1}(t)$, one obtains

$$\dot{\phi}_{n-1}(t) = a_0\phi_{n-1}(t) + x_0 \mathrm{e}^{a_0 t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1} \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right]. \tag{A8}$$

Substitute (A8) in (A7) to yield

$$\begin{aligned} a_1\phi_{n-2}(t-h) &= x_0 \mathrm{e}^{a_0 t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1} \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right] \\ &\Rightarrow a_1 x_0 \mathrm{e}^{a_0(t-h)}\left[1 + \sum_{i=1}^{n-3}\left((t-(i+1)h)^i \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right] \\ &= x_0 \mathrm{e}^{a_0 t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1} \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right]. \end{aligned} \tag{A9}$$

The proposed solution (9) for the n-th interval, $(n-1)h \leqslant t < nh$, becomes

$$\Omega(t) = \phi_n(t) = x_0 \mathrm{e}^{a_0 t}\left(1 + \sum_{i=1}^{n-1}\left((t-ih)^i \prod_{j=1}^{i}\left(\frac{a_1 \mathrm{e}^{-a_0 h}}{j}\right)\right)\right) \tag{A10}$$

and therefore it must be demonstrated that (A10) satisfies the state equation as follows

$$\dot{\phi}_n(t) = a_0\phi_n(t) + a_1\phi_{n-1}(t-h). \tag{A11}$$

Note that the right-hand side of (A11) can be rewritten as

$$a_0\phi_n(t) + a_1\phi_{n-1}(t-h) = a_0\phi_n(t) + a_1 x_0 \mathbf{e}^{a_0(t-h)}\left[1+\sum_{i=1}^{n-2}\left((t-(i+1)h)^i\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]. \quad \text{(A12)}$$

By adding and substracting the left-hand and the right-hand sides of identity (A9) to (A12) one obtains the next chain of identities

$$\begin{aligned}
&a_0\phi_n(t) + a_1\phi_{n-1}(t-h)\\
&= a_0\phi_n(t) + a_1 x_0 \mathbf{e}^{a_0(t-h)}\left[1+\sum_{i=1}^{n-2}\left((t-(i+1)h)^i\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&\quad - a_1 x_0 \mathbf{e}^{a_0(t-h)}\left[1+\sum_{i=1}^{n-3}\left((t-(i+1)h)^i\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&\quad + x_0\mathbf{e}^{a_0t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1}\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&= a_0\phi_n(t) + a_1 x_0 \mathbf{e}^{a_0(t-h)}(t-(n-2+1)h)^{n-2}\prod_{j=1}^{n-2}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\\
&\quad + x_0\mathbf{e}^{a_0t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1}\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&= a_0\phi_n(t) + x_0\mathbf{e}^{a_0t}(n-1)(t-(n-1)h)^{n-2}\left(\frac{a_1\mathbf{e}^{-a_0h}}{n-1}\right)\prod_{j=1}^{n-2}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\\
&\quad + x_0\mathbf{e}^{a_0t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1}\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&= a_0\phi_n(t) + x_0\mathbf{e}^{a_0t}(n-1(t-(n-1)h)^{n-2}\prod_{j=1}^{n-1}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\\
&\quad + \quad x_0\mathbf{e}^{a_0t}\left[\sum_{i=1}^{n-2}\left(i(t-ih)^{i-1}\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&= a_0\phi_n(t) + x_0\mathbf{e}^{a_0t}\left[\sum_{i=1}^{n-1}\left((i-1)(t-ih)^{i-1}\prod_{j=1}^{i}\left(\frac{a_1\mathbf{e}^{-a_0h}}{j}\right)\right)\right]\\
&= \dot{\phi}_n(t),
\end{aligned} \quad \text{(A13)}$$

and hence Eq. 9 satisfies (1) for any positive integer n and it is then a solution for (1) subject to initial conditions $x(t) \equiv 0 \ \forall t < 0;\ x(0)\varphi(0) = x_0$. The unicity follows from the fact that the right-hand side of (1) is globally Lipschitz (Burton, 1985). Note also that the existing unicity of the solution is maintained if $\varphi\colon [-h, 0]$ is any absolutely continuous function with possible isolated bounded steps (Burton, 1985).

REFERENCES

Burton, T.A. (1985). *Stability and Periodic Solutions of Ordinary and Functional Differential Equations.* Academic Press, New York.

Bourlès, H. (1987). α-Stability of systems governed by a functional differential equation – extension of results concerning linear delay systems. *Int. J. of Control,* **45**(6), 2233–2234.

Hale, J.K. (1977). *Theory of Functional Differential Equations.* Springer Verlag, New York.

Hmamed, A. (1985). Comments on 'On an estimate of the decay rate for stable linear delay systems'. *Int. J. of Control,* **42**(2), 539–540.

Hmamed, A. (1986a). On the stability of time-delay systems: new results. *Int. J. of Control,* **43**(1), 321–324.

Hmamed, A. (1986b). Stability conditions of delay-differential systems. *Int. J. of Control,* **43**(2), 455–463.

Mori, T. (1986). Further comments on 'Comments on "On an estimate of the decay rate for stable linear delay systems" '. *Int. J. of Control,* **43**(5), 1613–1614.

Mori, T., and H. Kokame (1989). Stability of $\dot{x}(t)=Ax(t)+Bx(t-\tau)$. *IEEE Trans. Autom. Control,* **34**(4), 460–462.

De la Sen, M. (1988). Fundamental properties of linear control systems with after-effect – I. *Math. Comput. Modelling,* **10**(7), 473–489.

De la Sen, M. (1992a). New results in stability of a class of hereditary linear systems. *Int. J. Systems Sci.,* **23**(6), 915–933.

De la Sen, M. (1992b). Relations between the stabilization properties of two classes of hereditary linear systems with commensurate delays. *Int. J. Systems Sci.,* **23**(10), 1667–1691.

Received June 1996

C.F. Alastruey was born in the Basque Country in 1963. In 1981 he became Bahá'í. He received the degree of Physicist (are: Electronics and Control) from the University of the Basque Country in 1988, and the degree of Doctor in Physics from the same university in 1993. He is professor in the Department of Electricity and Electronics in the University of the Basque Country. His research interests are mathematical and computing modelling of functional differential equations, control and identification of delay-differential systems and control of industrial processes.

M. de la Sen was born in Arrigoriaga (Bizkaia) in the Spanish Basque Country. He obtained the M. Sc. degree in Applied Physics by the University in 1979 and the degree of Docteur d'Etat-ès-Sciences Physiques (specialitè Automatique et Traitement du Signal) by the University of Grenoble (France) in 1987 with "mention très honorable". He has been Associate Professor of Applied Physics and Associate Professor of Systems and control Engineering at the University of the Basque Country while teaching courses on Ordinary and Partial-Derivative Differential Equations and Complex-Variable for Physicists as well as courses on Nonlinear Control Systems and Adaptive control. He is Currently full Professor of Systems and Control Engineering at the University of the Basque Country. He interest research fields are: Differential Functional Equations of Systems with Delays, Algebraic Systems Theory and Adaptive Systems. He was published a number of papers in those areas and has been visiting Professor at the Universities of Grenoble (France), Newcastle in New South Wales (Australia) and Australian National University at Canberra.

PIRMOSIOS EILĖS HOMOGENINIŲ SISTEMŲ SU VĖLINIMU ANALITINIS SPRENDIMAS

Carlos F. ALASTRUEY ir Manuel de la SEN

Straipsnyje nagrinėjami pirmosios eilės invariantinių tiesinių diferencialinių skaliarinių sistemų su būsenos vėlinimu tikslūs sprendiniai. Suformuluota teorema ir pateiktas jos įrodymas. Nagrinėjamas vienas skaitmeninis pavyzdys.