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ON ESTIMATES OF THE LOSS PROBABILITY FOR AN M/M/n QUEUEING SYSTEM WITH CHANNELS OF DIFFERENT SERVICE PRODUCTIVITY

Stasys PUŠKORIUS

Lithuanian Military Academy
Šilo 5a, 2055 Vilnius, Lithuania

Saulius MINKEVIČIUS

Institute of Mathematics and Informatics
Akademijos 4, 2600 Vilnius, Lithuania
E-mail: mathematica.ktl.mii.lt

Abstract. The queueing system theory is well developed. Such an important problem as the efficient of customer service in efficiency a multichannel queueing system with different productivity of service channels is well developed, too. Exact formulas are obtained from which the loss probability can be computed (if the input stream of customers distributed as Poisson and service time of the customer is the exponential service time). However, these formulas are very complex. So, in this paper, two theorems are proved, in which upper and lower estimates of the loss probability are presented. These estimates are simple formulas that don't become more complex with the growing number of service channels in the queueing system.

Key words: M/M/n queueing system, output stream of customers, the loss probability, upper and lower estimates of the loss probability.

1. Introduction. In applied problems of probability theory it is important to estimate the upper and the lower bound for different probability characteristics of systems, especially if the obtained formulas are very complex, cumbersome and not suitable an practical use. Mikhailov (1994) and Zubkov (1994) give some examples and reviews of this research trend in the probability theory.

Similar problems are in the applied theory of queueing systems. Shakhabazov (1969) obtained a formula for the loss probability of a multichannel queueing system. Models of multichannel queueing systems are well studied and widely applied. A review of these results is in Saati, (1965); Borovkov (1980); Ivchenko *et al.* (1982); Gnedenko *et al.* (1987).

The problems of estimating the upper and the lower bound of probability characteristics in queueing systems are very important in applied problems of operations research. A review of operations research results is in Ventsel (1972); Handbook of operations research (1978); Taha (1985); Saati *et al.* (1991).

In this paper, the authors are trying to estimate the loss probability in multichannel queueing systems with channels of different service productivity. The obtained estimates make it easier to analyse the influence of different channels of a queueing system on the whole queueing system.

2. Problem formulation. In the estimatation of the output stream capacity of a queueing system, there frequently occurs a situation, such that the service time of the queueing system is an exponentially distributed random variable, and the parameters of channels are different for each channel of the queueing system. Such a situation is due to some reasons related with different material and technical equipment of each channel, as well as with different qualification of specialists working with the multichannel queueing system, etc. This important problem was solved by Shakhabazov (1962) with the following predictions.

There is a multichannel queueing system with n channels and a Poisson input of customers with parameters λ. The service time of the j-th server is a random and exponentially distributed variable with the mean value m_j.

Then the loss probability can be obtained by the formula in Shakhabazov (1962):

$$P^{-1} = 1 + \sum_{j=1}^{n} j! \sum_{C_j} \frac{1}{\alpha_1 \alpha_2 \ldots \alpha_j}, \quad \alpha_j = \frac{\lambda}{\mu_j}, \quad \mu_j = m_j^{-1}, \qquad (1)$$

where μ_j is productivity of the j-th channel; c_j is any combination of j numbers from the series of numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ ($\sum\limits_{C_j}$ denotes that summation is taken over all the combinations).

In case there are some channels with the same service of productivity in the queueing system, it is reasonable to bring formula (1) to the form

$$P^{-1} = 1 + \sum_{j=1}^{n} j! \sum_{X_i = j} \frac{C_{n_1}^{x_1} C_{n_2}^{x_2} \ldots c_{n_r}^{x_r}}{\alpha_1^{x_1} \alpha_2^{x_2} \ldots \alpha_r^{x_r}}, \quad X_i = \sum_{l=1}^{i} x_l, \qquad (2)$$

where $n = \sum_{i=1}^{r} n_i$ is the whole number of server channels in the queueing system;

n_i is the number of server channels in the i-th group of servers with the same average service time;

r is the number of different groups of servers with the same service productivity;

x_i is the current value of the number of channels in the i-th group $x_i = 0, 1, \ldots, n_i$;

$C_{n_i}^{x_i}$ is the number of combinations from n_i to the number of service channels x_i.

The practical use of formulas (1) and (2) is connected with cumbersome computing. It is difficult to tabulate these formula, using personal computers since they depend on many variables, they are incapable in many practical cases. So it is necessary to develop an approximate formula for estimating the loss probability (that would essentially simplify computations).

Such approximate formulas are obtained on the basis of two theorems.

Theorem 1. *There exists a lower estimate for the loss probability in a multichannel queueing system consisting of channels of different service productivity.*

$$P \geqslant \frac{\alpha_a^n}{n!}\left(\sum_{i=0}^{n}\frac{\alpha_a^i}{i!}\right)^{-1} = P_a, \tag{3}$$

$$\alpha_a = \frac{\lambda}{\mu_a}, \tag{4}$$

$$\mu_a = \frac{1}{n}\sum_{i=1}^{n}\mu_i n_i. \tag{5}$$

Proof. Let us consider a situation, when all the service channels of the queueing system are of different service productivity. Then $n_i = 1,\ r = n$, and formula (2) has the form

$$P^{-1} = 1 + \sum_{j=1}^{n} j! \sum_{X_i=j} \frac{1}{\alpha_1^{x_1}\alpha_2^{x_2}\ldots\alpha_n^{x_n}}, \tag{6}$$

where $x_i = 0, 1$.

Thus, inserting expression P from (6) into inequality (3), we see that to prove this theorem it is necessary and sufficient to prove that the inequality

$$C_n^k \mu_a^k \geqslant \sum_{C_n^k} \mu_1 \mu_2 \dots \mu_n \tag{7}$$

is fulfilled for any $k = 1, 2, \dots, n$, where $\sum\limits_{C_n^k}$ means that in this inequality, summation is taken over all the combinations $\mu_1, \mu_2, \dots, \mu_n$, from n to k channels (i.e., the number of combinations in the product $\mu_1, \mu_2, \dots, \mu_n$ is equal to k).

To prove inequality (7) we construct a Lagrange function

$$\Phi(\mu_1, \dots, \mu_n) = \frac{C_n^k}{n^k}\left(\sum_{i=1}^{n} \mu_i\right)^k - \sum_{C_n^k} \mu_1 \dots \mu_n + \lambda\left(\sum_{i=1}^{n} \mu_i - n\mu\right), \tag{8}$$

where $\sum\limits_{i=1}^{n} \mu_i = n\mu$ is the equation of connection, and μ is the average channel productivity in the queueing system.

We investigate the function Φ for the extremum.

Note that a necessary condition for the extremum at some point M is that all partial derivatives of function (8) be equal to derivatives zero.

So

$$\frac{\partial \Phi}{\partial \mu_i} = \frac{C_n^k}{n^k} k \left(\sum_{i=1}^{n} \mu_i\right)^{k-1} - \sum_{C_{n-1}^{k-1}} \mu_1 \dots \mu_{i-1}\mu_{i+1} \dots \mu_n + \lambda = 0,$$

$$\sum_{i=1}^{n} \mu_i = n\mu, \quad i = 1, 2, \dots, n. \tag{9}$$

In order to solve a system of equations (9) we calculate the last n-th equation in (9) from the first equation in (9). We get

$$\sum_{C_{n-1}^{k-1}} (\mu_1 - \mu_n)\mu_2 \dots \mu_{n-1} = 0.$$

Hence, it is obvious that $\mu_1 = \mu_n$. Anologous it is possible to prove that for every $\mu_i = \mu_n$, $i = 1, \dots, n-1$. Hence, by making use of the connection equation we find that $\lambda = 0$, and the coordinates of the equilibrium point $M(\mu_i = \mu,\ i = 1, 2, \dots, n)$

To verify the sufficient condition for the extremum to exist, at the point M we define a second differential as

$$d^2\Phi = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi}{\partial \mu_i \partial \mu_j} d\mu_i d\mu_j,$$

where

$$\frac{\partial^2 \Phi}{\partial \mu_i \partial \mu_j} = \frac{C_n^k}{n^2} k(k-1)\mu^{k-2} - C_{n-2}^{k-2}\mu^{k-2},\ i \neq j$$

and

$$\frac{\partial^2 \Phi}{\partial \mu_i^2} = \frac{C_n^k}{n^2} k(k-1)\mu^{k-2},\ i = j,\ i,j = 1,2,\ldots,n$$

at this equilibrium point. The second differential is equal to

$$d^2\Phi = a \sum_{i=1}^{n} d\mu_i^2 - \frac{a}{n-1} \sum_{i,j=1}^{n} d\mu_i d\mu_j, \tag{10}$$

where

$$a = \frac{C_n^k}{n^2} k(k-1)\mu^{k-2}, \quad k \geqslant 2.$$

We may transform formula (10) into the form

$$d^2\Phi = \frac{a}{2(n-1)} \sum_{i,j=1}^{n} \left(d\mu_i - d\mu_j \right)^2.$$

Since $a > 0$, we have $d^2\Phi > 0$ and the point M is the point of local minimum. The Lagrange function Φ is equal to zero at the point M. Consequently, when $\mu_i \neq \mu_j,\ i = 1,2,\ldots,n$, the function Φ is positive, what implies that inequality (3) is valid.

Thus, we found the corresponding lower bound for the loss probability. In order to find the upper bound for this loss probability we need to prove the following theorem.

Theorem 2. *There exists an upper estimate for the loss probability in a multichannel queueing system with channels of different service productivity*

$$P \leqslant \frac{\alpha_q^n}{n!} \left(\sum_{i=0}^{n} \frac{\alpha_q^i}{i!} \right)^{-1} = P_q, \tag{11}$$

where

$$\alpha_q = \frac{\lambda}{\mu_q}; \tag{12}$$

$$\mu_q = \prod_{i=1}^{n} \mu_i^{1/n}. \tag{13}$$

Proof. After inserting expression P from (6) into inequality (11) it is evident that to prove this theorem it is necessary and sufficient to show that the inequality

$$\sum_{C_n^k} \mu_1 \ldots \mu_n \geqslant C_n^k \prod_{j=1}^{n} \mu_j^{k/n} \tag{14}$$

holds.

Similarly as in the proof of Theorem 1 we introduce a Lagrange function

$$\Phi_1(\mu_1, \ldots, \mu_n) = \sum_{C_n^k} \mu_1 \ldots \mu_n - C_n^k \prod_{j=1}^{n} \mu_j^{k/n} + \lambda\left(\prod_{i=1}^{n} \mu_i^{1/n} - \mu\right)$$

and investigate it for the extremum.

From the system of equations

$$\frac{\partial \Phi_1}{\partial \mu_i} = \sum_{C_{n-1}^{k-1}} \mu_1 \ldots \mu_{i-1}\mu_{i+1} \ldots \mu_n - C_n^k \frac{k}{n\mu_i}\left(\prod_{j=1}^{n} \mu_j^{k/n}\right)$$
$$+ \lambda \frac{1}{n\mu_i}\left(\prod_{j=1}^{n} \mu_j^{1/n}\right) = 0,$$
$$\prod_{i=1}^{n} \mu_i^{1/n} = \mu, \quad i = 1, 2, \ldots, n.$$

we can find that $\lambda = 0$, and the coordinates of the equilibrum point M_1 ($\mu_i = \mu, \quad i = 1, 2, \ldots, n$).

Now we compute a second differential of function Φ at the point M_1.

So

$$d^2\Phi_1 = \sum_{i,j=1}^{n} \frac{\partial^2 \Phi_1}{\partial \mu_i \partial \mu_j} d\mu_i d\mu_j,$$

where

$$\frac{\partial^2\Phi_1}{\partial\mu_i^2} = C_n^k\frac{k(k-1)}{n^2}\mu^{k-2} = a_1, \quad i = j,$$
$$\frac{\partial^2\Phi_1}{\partial\mu_i\partial\mu_j} = C_n^k\frac{k(k-1)}{n(n-1)}\mu^{k-2} - C_n^k\frac{k^2}{n^2}\mu^{k-2} = \frac{a_1}{n-1},$$
$$i \neq j, \ i,j = 1,2,\dots,n, \ k \geqslant 2.$$

at this point. Then the second differential can be expressed as

$$d^2\Phi_1 = a_1\sum_{i=1}^{n} d\mu_i^2 - \frac{a_1}{n-1}\sum_{i,j=1}^{n} d\mu_i d\mu_j$$
$$= \frac{a_1}{2(n-1)}\sum_{i,j=1}^{n}(d\mu_i - d\mu_j)^2.$$

Since $a_1 > 0$, we have that $d^2\Phi_1 > 0$ is positive and the point M_1 is the point of local minimum.

Consequently, when $\mu_i \neq \mu$, $i = 1,2,\dots,n$, the function Φ_1 is positive, and it means that inequality (14) holds.

It is necessary to note that the validity of inequalities (7) and (14) can also be proved by more general inequalities (see Markus *et al.* (1972; p.145)).

Thus we found the upper bound of the loss probability. Values of the function $F = \frac{\alpha^n}{n!}\left(\sum\limits_{i=0}^{n}\frac{\alpha^i}{i!}\right)^{-1}$ can be obtained from special tables (see, for example, Novikov *et al.* (1969)). Therefore, to obtain the upper and the lower estimate of the loss probability it suffices to insert α_a instead of α, in one case (see formulas (4,15)), and to take α_q, in another case (see formulas (12,13)). We also consider the loss probability as the average value, defined by the formula

$$\widetilde{P} = \frac{P_a + P_q}{2}.$$

Note that the absolute error of definition of the loss probability does not exceed the value $\Delta = \frac{P_q - P_a}{2}$, and a relative error does not exceed the value $\delta = \frac{P_q - P_a}{P_q + P_a}100\%$.

The error of definition of the loss probability can also be expressed by the initial variables.

So

$$\Delta = \frac{\Delta_1 + \Delta_2}{2},$$

where

$$\Delta_1 = \left\{ \sum_{k=1}^{n} (b_k - a_k) \right\} \left(\sum_{k=1}^{n} b_k \right)^{-1} \left(\sum_{k=1}^{n} a_k \right)^{-1},$$

$$\Delta_2 = \left\{ \sum_{k=1}^{n} (a_k - c_k) \right\} \left(\sum_{k=1}^{n} a_k \right)^{-1} \left(\sum_{k=1}^{n} c_k \right)^{-1},$$

$$a_k = \frac{k!}{\lambda^k} \left(\sum_{C_n^k} \mu_1 \ldots \mu_n \right), \quad b_k = \frac{n!}{(n-k)!} \frac{\mu_a^k}{\lambda^k},$$

$$c_k = \frac{n!}{(n-k)!} \left(\lambda^k \right)^{-1} \left(\prod_{j=1}^{n} \mu_j^{k/n} \right).$$

Obviously, when computing the errors of definition of the loss probability, it is more convenient to use the formulas expressed by P_a and P_q.

Numerical example. In the next table we present the values of the upper and lower bound for the loss probability and the error of definition of this probability for some initial data. δ_1' is defined here as the error of definition of the loss probability relative to its real value.

Table 1. Computing results

Indices	λ = customer· minutes^{-1}, if μ_1 = 1minutes^{-1}				λ = customer· minutes^{-1}, if μ_1 = 0.5minutes^{-1}			
	0.5	2	4	8	0.5	2	4	8
P_a	0.0403	0.276	0.492	0.692	0.0541	0.330	0.549	0.735
P_q	0.0438	0.292	0.511	0.707	0.0769	0.400	0.615	0.780
$\tilde{P}$	0.0421	0.284	0.502	0.700	0.0655	0.365	0.582	0.758
P	0.0435	0.286	0.500	0.696	0.0714	0.364	0.571	0.744
δ_1 %	4.2	2.8	1.9	1.1	17.4	9.6	5.7	3
δ_1' %	3.2	0.7	0.3	0.6	8.3	0.3	1.9	1.9

REMARK. Computations are performed for $n = 2$, $n_1 = 1$, $n_2 = 1$, $\mu_2 = 2(\text{minutes}^{-1})$.

3. Conclusions. Consequently, the theorems proved in this paper allow us to determine the approximate loss probability by simple formulas using tabulated functions in a multichannel queueing system with a Poisson input stream. Apart from that, the obtained estimates make it easier to analyse the influence of productivity of different service channels on the whole productivity of a multichannel system.

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S. Puškorius was born in 1935, received a doctoral degree in cybernetics in 1974 and Doctor Habilius degree of Military Sciences in 1984. Now he is a professor and deputy rector at the Lithuanian Military Academy. His research interests include operations research, especially queueing systems.

S. Minkevičius graduated from the Department of Mathematics of Vilnius State University, in 1984 and received a doctoral degree from the Belorussian State University, Minsk in 1994. Now he is a researcher at the Department of Statistical Modelling, Institute of Mathematics and Informatics, Vilnius, Lithuania. His research interests include nets of stochastic neurons and queueing networks in heavy traffic.

PRARADIMO TIKIMYBĖS DAUGIAKANALINĖJE MASINIO APTARNAVIMO SISTEMOJE SU SKIRTINGO NAŠUMO KANALAIS ĮVERTINIMAS

Stasys PUŠKORIUS ir Saulius MINKEVIČIUS

Nagrinėjama n-kanalinė masinio aptarnavimo sistema su praradimais ir su skirtingo našumo kanalais, esant Puasono paraiškų srautui ir eksponentiniam paraiškos aptarnavimo laikui kanaluose. Praradimo tikimybė įvertinama iš apačios ir iš viršaus paprastų formulių pagalba. Gauti įvertinimai supaprastina atskirų kanalų įtakos analizę visai masinio aptarnavimo sistemai.