

## NON-PERIODIC AND ADAPTIVE SAMPLING. A TUTORIAL REVIEW

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**Abstract.** The paper describes the use of adaptive and non-periodic sampling in different fields of System Theory and Control. The review is organized in a very comprehensive way and it presents results of the last thirty years about the problem of signal applications using as main tool adaptive sampling schemes including results is the improvement of the transient behaviors. Also, related results are presented about the use of non-periodic sampling in compensation as an alternative design to the well-known frequency domain methods and about the choice of the sampling points in order to improve the transmission of measuring and/or rounding errors towards the results when studying the properties of dynamic systems such as controllability, observability and identifiability.

**Key words:** non-periodic sampling, sampling systems, adaptive systems.

**1. Introduction.** Sampled-data control systems have received a great attention during the last years. The main reasons are:

(a) Sampling is applicable in problems in which, by nature, continuous control is not applicable (radar, sonar, etc.).

(b) Sampled-data systems present important advantages in designs subject to very strong specifications about transient behaviours. For instance, they allow the achievement of dead-beat responses in linear systems.

(c) Sampled-data systems are the natural alternative to continuous systems in many applications because of their computation power possibilities.

The use of discrete techniques with constant sampling periods is well known in classical and modern control fields like Optimization, Regulator Design and Identification. The reason of the facility of the treatment of discrete systems is that their input-output descriptions can be modeled by discrete transfer functions using  $z$ -transforms. The parameters of the associated difference equations are

time-varying and dependent on the parameters of the continuous plant (for instance, zeros, poles and static gains of the transfer functions which constitute the continuous system and on the sampled period). However, it is obvious that the possible freedom in the choice of the sampling rate is lost once that rate has been selected. In this context, non-periodic sampling allows an additional engineering tool involving an important freedom in the designs. In the past thirty years, non-periodic sampling has been motivated by its usefulness in several different control problems. Some of them are listed below:

1. The development of adaptive sampling laws. The advantages of their use arise from the fact that similar response characteristics as in the case of constant sampling can be achieved with less number of samples on a given time interval.

2. The application of adaptive sampling techniques to discrete and hybrid adaptive control, with sampling intervals being dependent on the tracking error, may improve the transient responses in terms of smaller deviations from the reference signals.

3. Fixed, dynamics or optimal variations of the sampling period may approximately compensate parametrical changes for processes in some cases without needing a new design of the nominal controllers.

4. The transmission of absolute/relative errors and/or rounding errors from the data to the results may be improved by using non-periodic sampling when studying typical properties of the dynamic systems such as observability, controllability or identifiability.

5. Non-periodic sampling naturally describes many biological processes and it is useful in some chemical industrial processes in which equilibria states suffer from variations. Also, multirate sampling is useful in complex systems involving many fast and slow variables and in vehicle control applications while a class of signal-dependent sampling referred to as pulse frequency modulation PFM has led to applications towards noise filtering.

The absence of a review in the literature or the theoretical and practical developments in this area has encouraged us to present this review work. Some related problems like random sampling in Communication Theory are also reviewed. The paper is organized as follows. Section 2 is devoted to an overview of existing particular and general adaptive sampling criteria together with some applications to adaptive discrete and hybrid control. In Section 3, the following related topics are dealt with: sensitivity compensating methods to local

parametrical process changes, combined optimal regulator and sampling sequence designs, multirate regulators and PFM design. In Section 4, some algebraic properties of dynamic systems under non-periodic sampling are described. Also, the optimal transmission of measuring/rounding errors by adequate selection of the sampling instants is dealt with. Section 5 presents alternative models describing non-periodic sampling systems. In Section 6, some industrial and biological applications of interest involving non-periodic sampling are given and, finally, conclusions end the paper. A list of references is also provided divided into sections as follows:

A – Adaptive sampling laws, adaptive sampling versus adaptive control. Refs. [1]–[25];

B – Analysis and modeling of non-periodic systems. Refs. [26]–[43];

C – Regulator design, sensitivity compensation of discrete systems, PFM systems. Refs. [44]–[70];

D – Non-periodic sampling and fundamental properties of dynamic systems. Refs. [71]–[81];

E – Biomedical and industry applications. Refs. [82]–[105];

F – Another recent bibliography. Refs. [106]–[117].

The various references are related to the topic they are more relevant although there are obvious links between topics so that many references have also interest in other topics apart from the one they are directly related with according to the list.

**2. Adaptive sampling.** The purposes of this section are: (1) to state a bridge between sampling and non-periodic and adaptive sampling; (2) to describe particular and general adaptive (or signal dependent) sampling laws and some comparisons of involved performances with the continuous and discrete (constant sampling) cases; (3) to introduce such laws to improve the transients in adaptive discrete and hybrid control as higher order level control in a hierarchical structure.

**2.1. Classical adaptive sampling.** Adaptive sampling consists, in general, of implementing state-dependent rules for the computation of the sampling interval in real time. Most of these rules were derived heuristically. The earlier works on the subject were reported by Dorf *et al.* [9] and Jury [16–17]. The first one may be an example of how heuristical sampling schemes are derived.

Its main characteristics are:

- the frequency varies linearly with the modulus of the closed-loop tracking/regulation error time-derivative;
- lower and upper-bounds for the sampling rate are prefixed by taking into account stability and bandwidth requirements;
- the sampling law is based on sampling efficiency analysis. This efficiency is evaluated by selecting the sampling period by taking the area between the system output and the zero-order-hold signal as a prescribed constant.

Mellado *et al.* [98] use a constant absolute difference amplitude criterion. In Bekey and Tomovic [2], [8], time-sensitivity approaches were taken so that sampling occurs at  $t = t_{n+1} = t_n + T_n$  ( $t_n$  being the previous sampling point), if  $|y(t) - y(t_n)|$  or its approximation  $|\Delta y(t_n)| = |y(t_n)T_n + \dot{y}(t_n)T_n^2/2 + \dots|$  is greater than or equal to a prescribed positive threshold "A". An equivalent criterion is used in [42] by using analogic technology. The criterion proposed in Ciscato and Mariani [3], considers the constant difference amplitude criterion for the time-integral error before computing the sampling period. Gupta [12] proposes an analytic criterion which minimizes the quadratic square mean of the difference between the plant output and the output of the zero-order-hold device over each sampling interval with respect to the sampling period. That sampling period is calculated by eliminating the trivial solution  $T_n = 0$  and substituting possible negative solutions by positive ones leading to the same value of the cost criterion. The second historical step in this subject was to establish comparisons between the different methods by deriving the existing sampling laws from unified criteria which also allows obtaining some new ones. Hsia [14] generalizes the area laws ([3], [9]) by implementating the criterion  $|\int_{t_n}^{t_n+T_n} (e(t) - e(t_n))dt| = A$  using different methods of derivation were given by Dormido *et al.* [8], Hsia [13], [15] and Tait [23] by generalizing three area-type criteria, [9], to polynomial functions of the sampling period or the analytic criterion, [12], to more general mean error powers so that unsuitable sampling periods were eliminated from the solutions. The following design specifications have been found.

- The adaptive sampling system must be asymptotically stable if the time continuous regulator is stable. This idea implicitly appears in most of the references. See, for instance, [2], [8], [9], [12–18], [22–24].
- The adaptive sampling law must be compatible with the suitable closed-

loop bandwidth ([8–9], [12], [23]). – Sampling must be held asymptotically (or more explicitly,  $T_k \geq \varepsilon$  for all integer  $k > 0$  and  $\lim_{k \rightarrow \infty} T_k \geq \varepsilon$ ). – The adaptive sampling system must be more efficient than the corresponding constant sampling scheme in terms of either less samples with similar (or better) performances over a given time interval or more closeness of the output to that of the continuous system. See, for instance, [9], [13–15], [22–23].

The above first and second items are obviously needed for elementary engineering purposes. The third one arises naturally if sampling is constant. However, in some state-dependent sampling schemes, like for instance, those related to the use of the amplitude difference criterion (see [2], [3], [8], [13], [15], [24]), it can occur that sampling does not hold (namely, the system operates in a continuous fashion) for weak signal levels if additional cautions are not taken. In general, it suffices the establishment of a “p priori” lower and upper-bounds on the sampling law to deal with the above three items. This is a general strategy in all references of section A related to this subject.

**General criteria for sampling.** Several general criteria were proposed, namely,

(a) the systematic design of sampling criteria was focused in [22] by defining a vector function  $Q(\cdot)$  of the state vector  $x(\cdot)$  so that the next sampling instant  $t_{n+1}$  occurs when the identity

$$\left| \int_{t_n}^{t_{n+1}} Q(x(t)) dt \right| = A \leftrightarrow \left| \int_{t_n}^{t_{n+1}} Q(x(t)) dt \right| = A \operatorname{sign} \left\{ \int_{t_n}^{t_{n+1}} x(t) dt \right\} \quad (2.1)$$

holds where  $A$  is a prefixed positive real constant.

(b) The area-type criteria were generalized in [8] to the general criterion

$$\left| \int_{t_n}^{t_{n+1}} (e(t) - e(t_n)) dt \right| = \varphi(T_n) = \sum_{-3}^3 a_j T_n^j \quad (a_j \text{ being real}) \quad (2.2)$$

All the coefficients in (2.2) are, in practice, zero except one or two to the designer’s choice. This allows the derivation of many previously existing sampling laws and the derivation of some new ones. For instance,  $a_0 \neq 0$ ,  $a_j = 0$ ,  $\forall j \neq 0$ , lead to the former Dorf *et al.* law. This is not surprising since different functional forms for the  $\varphi(\cdot)$  function can lead to equivalent sampling

laws. Two conditions were imposed "a priori" on (2.2) in order to consider a potential sampling law as coherent, namely, (1) the  $a$ -coefficients must be selected in such a way that  $\varphi(T_n) > 0$ ; (2)  $\varphi(\tau) < \varphi(T_n)$  all  $\tau \in [0, T_n)$ . The first condition arises from the physical fact that the left-hand side of (2.2) must be positive. The second one expresses that sampling must occur when  $\tau = T_n$ . The possible choices of the  $a$ -coefficients which do not fulfill those conditions were directly rejected for design purposes.

(c) The objective of the general criterion of Hsia [13–15] was to eliminate the zero trivial solutions for the sampling period of [12] as well as to generalize the objective function. Two main considerations were used, namely, (1) a cost functional due to sampling itself must be added to the objective so that the trivial solution be eliminated; (2) the cost per-sample per-unit of time must be inversely proportional to the current sampling period. Then the functional of (2.1) should be a monotonically decreasing positive function of the current sampling period. Suitable choices of the overall loss function are

$$J(T_n) = J_0 + J_1; \quad J_0 = 1/T_n^a \int_{t_n}^{t_n+T_n} [|e(t) - e(t_n)|]^b dt + J_1, \quad (2.3)$$

where  $a \geq 1$  and  $b \geq 2$  and the new cost associated with sampling is  $J_1 = Ae^{-BT_n}$  or  $A/T_n$  with  $A, B$  (real) and  $p$  (integer) are positive. In Table 2.1, some of the adaptive sampling laws of [15] as particular solutions of (2.3) are given. Several of them are also particular laws of the criteria of [8] and [23] and can be obtained from (2.3).

The approximation  $e(t) \simeq e(t_n) + \dot{e}(t_n)(t - t_n)$  is used in all the laws. The sampling constraint lower and upper bounds  $T_{\min} \leq T_n \leq T_{\max}$  depend on the bandwidth and stability requirements. If it is violated by the sampling law, the sampling rates  $T_{\min}$  or  $T_{\max}$  are used. Also, two possible first-order (Type 1) or second-order (Type 2) approximations are used for  $e^{-bT_n}$ . Four sampling laws numbers 1, 4, 5a and 6 in Table 2.1 were compared. The point of view used for the comparisons of efficiencies was keep the sample numbers the same as in the constant sampling case by an appropriate choice of the constant of the sampling law. Then, the output deviations from that of the continuous system (i.e., the reference response) were compared with respect to the cost criterion  $E = \int_0^{1.2} |y^*(t) - y(t)| dt$ , where  $y^*(t)$  is the step response of the continuous closed-loop system. The plant chosen for study was that of transfer function

**Table 2.1.** Sampling laws from the objective below

$$J = \frac{1}{(T_n)^a} \int_{t_n}^{t_n+T_n} [|e(t) - e(t_n)|]^b dt + Ae^{(-BT_n)}, \quad a, b > 0.$$

No.	Objective function parameters	Adaptive sampling law	Sampling law parameters	Type of approximation used for $e^{-BT_n}$
1*	$a = 1$ $b = 2$	$T_n = \frac{T_{\max}}{\alpha \dot{e}_n^2 + 1}$	$a = \frac{2}{3AB^2}$ $B = \frac{1}{T_{\max}}$	Type 1
2	$a = 0$ $b = 2$	$T_n = \frac{C_1}{ \dot{e}_n }$	$C_1 = \sqrt{AB}$	Type 2
3*	$a = -1$ $b = 2$	$T_n = \frac{C_2}{ \dot{e}_n^2 ^{1/3}}$	$C_2 = \frac{\sqrt{3AB}}{2}$	Type 2
4	$a = 1$ $b = 1$	$T_n = T_{\max} - K \dot{e}_n $	$K = \frac{1}{2AB^2}$ $B = \frac{1}{T_{\max}}$	Type 2
5a	$a = 0$	$T_n = \frac{T_{\max}}{\alpha_1 \dot{e}_n  + 1}$	$\alpha_1 = \frac{1}{AB^2}$ $B = \frac{1}{T_{\max}}$	Type 1
5b	$b = 1$	$T_n = \frac{C_3}{ \dot{e}_n }$	$C_3 = AB$	Type 2
6	$a = -1$ $b = 1$	$T_n = \frac{C_4}{\sqrt{ \dot{e}_n }}$	$C_4 = \sqrt{\frac{2}{3}AB}$	Type 1
* New results presented in [8, 15].				

$G(s) = 10(s + 10)/s^2$ . The efficiency with respect to a step response against periodic sampling is shown in Table 2.2.

**Table 2.2.** Efficiency of the adaptive sampling laws

Response error $E$		Sample numbers on $[0, 1.2]$		
Sampling schemes		30	20	15
Periodic sampling	$T(\text{sec.})$	0.04	0.06	0.08
$T_n = T$	$E$	0.05962	0.09704	0.14314
$T_n = \frac{0.1}{\alpha_1 \dot{e}_n^2 + 1}$	$\alpha_1$	2.5	0.05	0.08
	$E$	0.03594	0.055604	0.09876
$T_n = \frac{0.1}{\alpha_2  \dot{e}_n  + 1}$	$\alpha_2$	1.4	0.275	0.05
	$E$	0.03442	0.05504	0.11105
$T_n = \frac{C}{\sqrt{ \dot{e}_n }}$	$C$	0.03	0.08	0.15
	$E$	0.03756	0.05443	0.0915
$T_n = 0.1 - K \dot{e}_n $	$K$	0.065	0.01	0.003
	$E$	0.03584	0.05587	0.1162
$T_{\max} = 0.15, T_{\min} = 0.02$ (all constants are positive)				

**2.3. Transients in adaptive control.** It is well known that the sampling rate is a crucial parameter in the performance of a control scheme [16–17]. In the two above subsections, it has been emphasized that such a performance can be greatly improved with the use of signal-dependent (adaptive) sampling. Adaptive control is normally formalized under constant sampling rate because the discrete system is defined by constant parameters provided that the associated continuous system is time-invariant ([4], [19–20]). Note that the word “adaptive” in this context is referred to the real-time estimation of the plant and/or controller parameters and it has a radically different sense to that used for adaptive sampling in the previous subsection. It was obvious after examining the performances of plants of known parameters under adaptive sampling that it would be possible to improve the poor transients normally observed in adaptive



control schemes if sampling adaptation were used as a second adaptation level. This was addressed in [5–7] for discrete and hybrid adaptive control by using the previously existing adaptive sampling laws ([8], [13–15]). A better adaptation transient in terms of smaller overshoots and smaller output deviations with respect to the continuous system than the discrete scheme under constant sampling was observed as in the case of known plants (see [8], [13–15], [18], [21], [22–23]). In all the schemes the parameter-adaptive controller is of a standard (discrete [5] or hybrid [6–7]) type often based in a least squares parameter estimation. The elements included in the overall adaptive controller under adaptive sampling are:

- the reference model which gives the desired output;
- the sampling rate controller which physically implements an adaptive sampling law of the same type as in Section 2.2. The signal used for sampling adaptation is the tracking/regulation error (i.e., the error between the outputs of the reference model and the current adaptive system). Two characteristic typical actions of this controller are: (a) in discrete schemes, the sampling law acts on the period over which the plant input is applied so that it acts directly on the system dynamics and, in a natural fashion, on the adaptive controller updating; (b) in hybrid schemes, sampling acts on the parameter updating action of the adaptive controller because of the structures of such schemes in which the parameters are updated through the implementation of time integrals involving the adaptation error.

The approximation of the error time-derivatives has been proposed in [5–7] by using the difference equation approach which involves number (according to a desired prefixed order) of preceding ones. This strategy eases the sensor implementations. Some specific characteristics of the adaptive sampling scheme with respect to those reported in Sections 2.1–2.2 are:

- the admissibility domain for the sampling period variations is of a local nature around a nominal sampling period and has a radius being compatible with the above requirements. The nominal sampling period is chosen according to the needs of each application;
- the adaptive sampling is of a bang-bang (or linear within a bang-bang range) type according to the above admissible domain. This makes the sampling law to be less sensitive to the choice of its free constants. The choice of the bounds where the bang-bang or saturation laws operate becomes critical and an

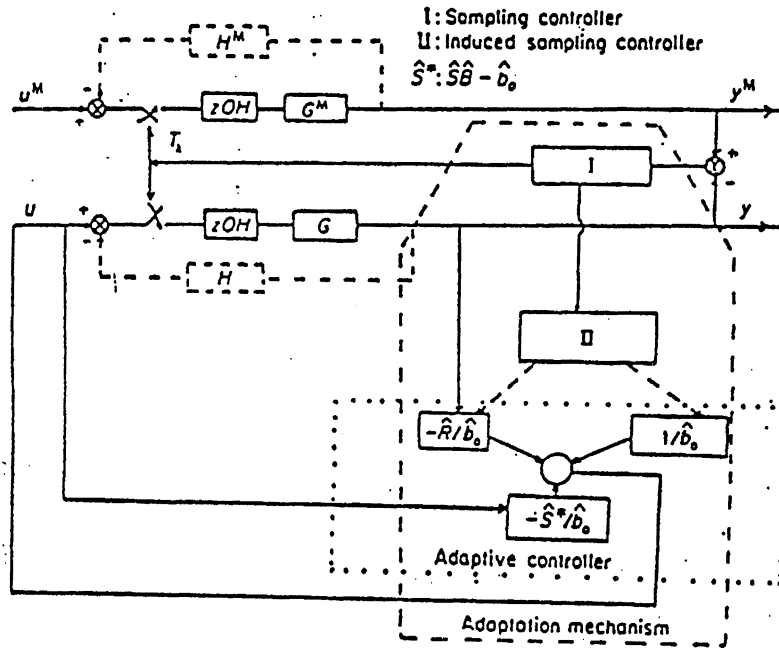
“a priori” knowledge on the plant and on the previously registered performances must be used in the involved design. This is necessary to update on-line the sampling rate while keeping the stability, bandwidth and scheme’s performance.

In purely discrete (non-hybrid) designs, the sampling period is made constant within its above admissibility domain either in the limit, as time increases, or after a finite number of samples being chosen accordingly to the registered transient performances. In this way, the discrete parameters become asymptotically constant and the classical updating schemes can be used. This updating process is not required in hybrid schemes in which, although the parameter updating process is discrete, the plant input is generated in a continuous fashion. The possibility of using standard parameter-adaptive algorithms is due to the fact that a discrete system involving non-periodic sampling can be modeled by a time-varying difference equation, [31 – 32]. In Section 5.1, such a model will be presented. *The fact that the description is parametric and involves a finite number of parameters allows the implementation of slight extensions of the classical adaptive controllers.* The schemes used for improvements of the adaptive transients in [5] use the general sampling criteria of [8], [13 – 15] and [23]. In hybrid schemes, [4], the sampling period is time-varying and its choice is crucial to guarantee the scheme’s stability. In [6 – 7], the sampling period is signal-dependent while used to improve the transient performances. In the scheme of [19], two constant sampling rates are used so that the input to the process is generated at a faster sampling rate than the associated one with the adaptive controller updating process. The scheme ensured global asymptotic stability and signal boundedness. That multirate control strategy was also used in other schemes (see, for instance, [10 – 11], [25], [27], [44], [40 – 41]). In [39], a similar multirate idea was proposed for digital filtering. In Fig. 2.1, the overall adaptive scheme is shown for a discrete system. Extensions to the hybrid case are direct (see [5 – 7], [31 – 32] and Section 5.2 below).

The problem is stated as follow. Let  $SI$  be the sequence of sampling points obtained from any adaptive sampling law. Thus, *a time-varying difference equation is given for the discrete plant under non-periodic sampling:*

$$A(q'(t))y(t) = B(q(t))u(t - d) \quad (2.4)$$

for all  $t \in SI$ ,  $d$  is a nonnegative integer representing the discrete delay and  $q(\cdot)$ ,  $q'(\cdot)$  are time-varying delay operators being associated with the real and



$$C_r(q^{-1}) = 1.$$

Fig. 2.1. Adaptive controller with adaptive sampling.

so-called induced sampling points. The time-varying nature of the coefficients of the  $A$  and  $B$  polynomials as well as the fact that the induced sampling instants defining  $A(\cdot)$  are, in general, different from the real sampling instants is inherent to the use of non-periodic sampling and will be described in Section 5.1.

Then, apply a parameter-adaptive algorithm of the same class as those normally used under constant sampling rate for discrete estimation of the controller parameters:

$$\hat{\theta}(t) = \hat{\theta}(t-1) + F(t)\phi(t-d)e(t), \quad \forall t \in SI, \quad (2.5)$$

where  $\phi(t)$  is the measurement or regression vector which contains input/output sequence as components.  $F(t)$  is a positive definite adaptation gain matrix (in

many schemes, it is constant and fixed to identity) and  $e(t)$  is the adaptation error which can depend on design parameters  $c(t)$  and  $\lambda(t)$  (forgetting factor). If  $c(t)$  is unity then the adaptive algorithm is one of the recursive least-squares type. The design parameters have to belong to certain domains in order to keep the algorithm's stability ([4–7], [10–11], [18–20]). In the particular algorithm of [6–7],  $c(\cdot)$  must be positive and the forgetting factor must be within the interval  $(0, 1]$  in order to guarantee the stability. The regulation/tracking objectives are to achieve that the regulation/tracking error  $e^F(t) = y^F(t) - y^{MF}(t) \rightarrow 0$ , as  $t \rightarrow \infty$ , where superscript  $M$  and  $F$  stand, respectively, for the uniformly bounded reference sequence and filtered signals through the use of a Hurwitz polynomial  $C(\cdot)$ . In many schemes, the direct outputs rather than their filtered versions were used. The design philosophy is the (perhaps filtered) adaptation error so as to improve the transient performances. For hybrid schemes, a standard change in (2.5) is addressed through the use of time-integrals over each current sampling interval for obtaining the current updated parameter vector.

**3. Compensation and optimal regulation.** Several topics about the non-periodic design of regulators are presented in this section, namely:

*Compensation techniques:* the sampling rate is either fixed or adaptively changed to compensate for parametrical variations of the continuous plant parameters.

*Optimal regulation:* the optimal design of sampling sequences is presented as a second optimization level in optimal regulation.

*Multirate regulation:* the optimal design of multiple sampling rates is focused with special emphasis in aeronautical applications.

Also, the PFM (pulse frequency modulated) regulators are reviewed as a class of signal-dependent sampling systems. In most of the theory and applications of the above underlined topics, the sampling periods are computed in an off-line fashion as auxiliary design tools to a classical compensation and regulation. Although the PFM regulation has been developed in the context of signal-dependent sampling, it is preferred its review in this section because of its clear classification in a regulation context in many of its associated developments. Its related biological and industry applications will be overviewed in Section 6.

**3.1. Compensation techniques.** It is well known that the methods involving

adaptive control designs and those involving parameter insensitive designs are useful in order to reduce the sensitivity of the dead-beat response to process parametrical deviations. In [51], general sensitivity models for both continuous and discrete models were given. Also, perturbation and sensitivity approaches in sampled-data control systems have been presented in many references (see [48], [52], [54] and [59]). In particular, the behavior of the stationary quadratic optimal regulator versus sampling rates is investigated in [48]. More related results have been reported in [53] and [58]. In this section, the reviewed results have as main objective the compensation of process deviations through the use of a sampling rate modification. The general design characteristics are the following: (a) they are an alternative to the design of classical compensators like, for instance, phase lead/lag networks; (b) they are easily applicable in cases of known local variations of the continuous plant parameter vector; (c) the design philosophy consists of achieving insensitive systems for combined variations of the continuous plant parameters (i.e., the parametrical disturbances) and the sampling period (i.e., the compensating parameter) while keeping invariant the nominal controller, if any.

At a first glance, the next conclusions follow from the above characteristics:

- The compensation is approximate for local variations. Thus, it is of the same applicability domain as the classical compensation based on frequency methods. Also, the new sampling periods are used to compensate known process variations.

- Since the nominal controller remains fixed, the computational effort being associated with the compensation is lower than in the typical adaptive situation where all the controller parameters are re-updated. The price to be paid is usually a poor performance with respect to the standard adaptive situation.

The design is addressed as follows. Consider a discrete system given by its state equation

$$x((k+1)T) = \phi(T)x(kT) + h(T)u(kT). \quad (3.1)$$

Then, the following sensitivity relation is obtained for local deviations  $\Delta p = p - p_0$  from the nominal parameter vector  $p_0$  which parameterizes the continuous-time process before the discretization is applied:

$$\frac{dx((k+1)T)}{dp_j} = \phi \frac{dx(kT)}{dp_j} + h \frac{du(kT)}{dp_j} + \frac{d\phi}{dp_j} x(kT) + \frac{dh}{dp_j} u(kT), \quad (3.2)$$

if the input depends on  $p$  via feedback. All the above derivatives are partial derivatives with respect to the continuous parameters evaluated in the nominal parameter vector. Note that (3.2) is a sensitivity trajectory subject to zero initial conditions.

The reviewed compensation methods are divided into two basic types, namely:

a) Those which use sensitivity methods in the frequency or, in general, in the transform domains. See [60].

(b) Those which establish sensitivity methods in the time domain. See [47], [54], [60]. This strategy can be combined with the use of extra compensating inputs or higher levels of regulation designs. See, for instance, [45].

In both situations, the sampling period, even when time-varying, must lie in the good stability/bandwidth region. Now, further brief discussions on the above strategies are given.

*Discussion of (a).* Assume that the system suffers from local variations of its static gain  $K_0$  only and that sufficient smoothness conditions are available to compute parametrical and sampling period  $T_0$ -sensitivities of the  $z$ -transform of the output so that

$$S_K^{Y(z)} \simeq \frac{\Delta^K Y(z) K_0}{\Delta K Y_0(z)}; \quad S_T^{Y(z)} \simeq \frac{\Delta^T Y(z) T_0}{\Delta T Y_0(z)}, \quad (3.3)$$

where subscripts 0 stand for the nominal values and  $Y(z)$  are output variations in the  $z$ -transform domain due to gain (disturbance) and sampling period (compensating) variations denoted respectively with superscripts  $K$  and  $T$ . With the sensitivities of (3.3), the first-order approximations of the output variations with respect of local variations of both parameters around their nominal values are computed. The sampling period compensating variation is designed so that the overall (first-order) output variation due to  $K$  and  $T$  is zero. Other variants were presented in [60] and [96] which are based in the use of a modified  $z$ -transform. In a series expansion of the output, each coefficient depends on a  $m$ -parameter with  $0 \leq m \leq 1$ . Each of the coefficients gives the value of the output for some sampling point within one of the intervals, the particular sample depending on  $m$ . Either the coefficients being relevant to the transient response such as the first  $d$  coefficients in a dead-beat response of  $d$ -order, or some weighted combination of output errors with respect the nominal situation, are designed as first-order insensitive with respect to combined parametrical and sampling

rate disturbances. This insensitive design is performed at the new sampling points, the sampling rate to be re-updated still being the compensating parameter against plant parametrical disturbances. For instance, in dead-beat designs, if the static gain varies in  $\Delta K > 0$ , the third term  $y_2$  ( $m = 2\Delta T/T_0$ ), for a second order system, is equalized to the reference since it gives the response between the second and third sampling points.

*Discussion of (b)–(c).* The application of the  $z$ -transform methods becomes difficult when other continuous plant parameters distinct or apart from the static gain such as poles or zeros vary. Therefore, another approach given in [60] relies in the application of the above methods with sensitivity approaches in the time-domain. In particular, a possibility proposed in that paper was the modification of the threshold of the constant amplitude criteria of [2], [8], [13–15] and [23]. See also [95–99]. The same strategy was later used to improve the steady-state response of systems subject to such sampling criteria by reducing the amplitude of the possibly existing limit cycles (see [42]). In particular, a least-squares weighted loss function was established for the error between the current plant output suffering from parametrical and (compensating) sampling period variations and the nominal output over an “a priori” prefixed number of samples. This compensation horizon was chosen as being relevant to the transient response and all the signals were approximated by using their first-order sensitivities. The resulting variation for the sampling period, with respect to its nominal value, is given by

$$\Delta T = - \frac{\sum_{k=1}^N \sum_{i=1}^r v_k^{p_i} \Delta p_i (v_k^{T'} - v_k^T)}{\sum_{k=1}^N (v_k^{T'} - v_k^T)^2}, \quad (3.4)$$

in the case when all the weighting coefficients are unity where the  $v$ -functions denote the global sensitivities with respect to the nominal and current sampling periods denoted by the superscripts  $T$  and  $T'$ , respectively, and the  $r$  continuous plant parameters superscripted by  $p(\cdot)$ .  $N$  is the size of the compensation horizon. The value of  $N$  must be carefully chosen depending on each problem at hand (namely, minimum system, dead-beat system, etc.), the plant variations (if  $p$  increases  $N$  should normally increase), and the computational capability. See [45], [60], [95–99]. This difficulty was overcome in part in [60] by

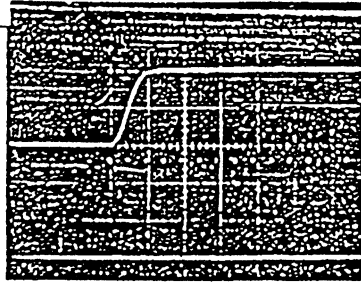
proposing an optimization scheme which separates the compensation horizon  $[1, N']$  from the optimization horizon  $[1, N]$  with  $N' \leq N$  so that the loss function used for obtaining (3.4) is slid in time, sample per sample, to perform the compensation at each time using the overall size of the optimization horizon to compute each current sampling period.

Some experimental analogic results which concern the plant of transfer function  $K_0(1 - e^{-T_0s})/s^2(s + 1)$  with unity feedback are shown in Fig. 3.1–3.2 which nominally led to a dead-beat response closed-loop system. Under gain perturbations, the response of systems subject to such sampling nominal digital controller was kept invariant while the sampling period was redesigned by using the sensitivities (3.3).

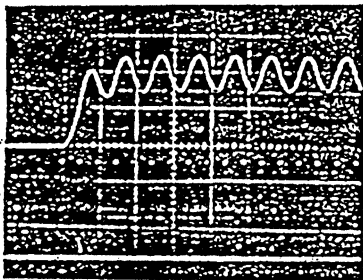
In oscillograms I and VI, the nominal outputs are shown. Oscillograms II, VII, IV and IX show the outputs of the system, which is disturbed in its gain, with relative variations of 20% (the first two oscillograms), and 40% and 60% (the two remaining ones). Oscillograms III, VII, V and X show the compensated system outputs by sampling period redesign. As far as the time-sensitivity compensation methods are concerned (cf. Eq. 3.4), their main drawback is that non-parametrical models are involved so that computational problems appear as time increases and, on the other hand, the planning horizon sizes are limited by the need of maintaining well-posed computations related to the involved sensitivity functions. Also, there are *no general criteria* to compare the efficiency of the above methods to the classical phase lead/lag compensation ones so that trial-error experiments are usually involved.

**3.2. Optimization and regulation.** The problem of choosing an optimal sampling interval or an optimal sampling sequence and that of approximating continuous optimal quadratic regulators by discrete ones received some attention in the past years. A general theory was given in [46]. A sensitivity study of optimal sampled-data regulators was supplied in [38–39], [48] and [59]. The optimal linear regulation with state-dependent or aperiodic sampling was studied in [50]. Also, the problem of optimal choices of the sampling period sequence as a higher optimization level in optimal deterministic and stochastic quadratic regulators was investigated in [45–46], [53], [55], [58] and [97]. In [45], two higher order optimization levels were proposed, the first one being valid to supply an optimal compensating input sequence for a quadratic cost taking into account parametrical continuous process changes while the second one is

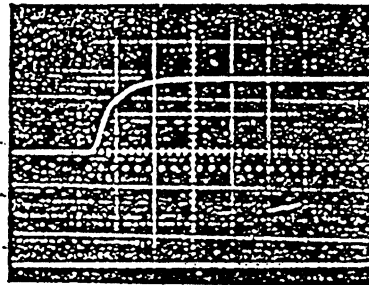




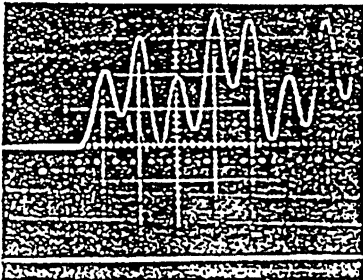
I.—Y: 5 V/cm; 0.2 seg/cm  
 Y\*: 2.5 V/cm; 0.2 seg/cm



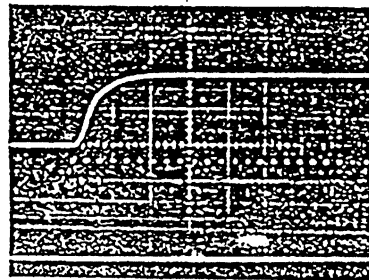
II.—Y: 5 V/cm; 0.2 seg/cm  
 Y\*: 2.5 V/cm; 0.2 seg/cm



III.—Y: 5 V/cm; 0.2 seg/cm  
 Y\*: 2.5 V/cm; 0.2 seg/cm

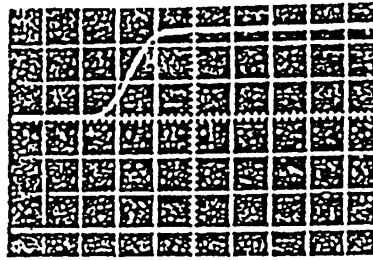


IV.—Y: 5 V/cm; 0.2 seg/cm  
 Y\*: 2.5 V/cm; 0.2 seg/cm

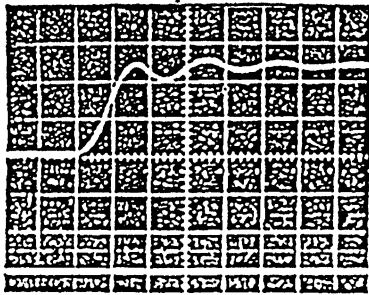


V.—Y: 5 V/cm; 0.2 seg/cm  
 Y\*: 2.5 V/cm; 0.2 seg/cm

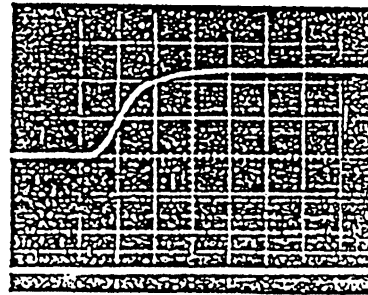
Fig. 3.1. Numerical values:  $K_0 = 100$ ,  $T_0 = 0.1$ .



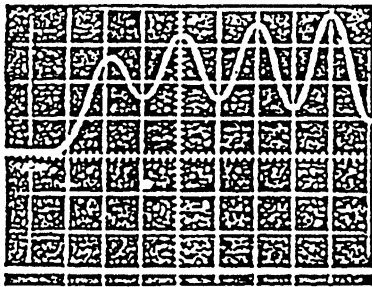
VI.—Y: 5 V/cm; 1 seg/cm  
Y\*: 2.5 V/cm; 1 seg/cm



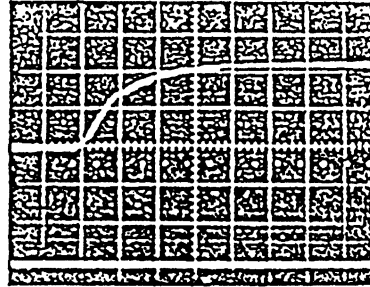
VII.—Y: 5 V/cm; 1 seg/cm  
Y\*: 2.5 V/cm; 1 seg/cm



VIII.—Y: 5 V/cm; 1 seg/cm  
Y\*: 2.5 V/cm; 1 seg/cm



IX.—Y: 5 V/cm; 1 seg/cm  
Y\*: 2.5 V/cm; 1 seg/cm



X.—Y: 5 V/cm; 1 seg/cm  
Y\*: 2.5 V/cm; 1 seg/cm

Fig. 3.2. Numerical values:  $K_0 = 10$ ,  $T_0 = 1$ .

a constrained sampling optimization procedure which is very similar to that used in [53]. In particular, in Melzer and Kuo [59], the first and second order sensitivities of the Riccati and feedback matrices associated with an optimal regulator are calculated. The sensitivities are calculated around a zero sampling invariant plant and a quadratic performance index are, respectively, converted into a difference equation and an associate discrete optimization index. This optimization index weights both the state and the input of the system and is used to compute the optimal input together with the optimal sampling period if desired. It is found (see [59]) that  $dK(T)/dT = 0$  and  $dJ(T)/dT = 0$  at  $T = 0$ , namely, the stationary Riccati solution  $K(T)$  and the cost function  $J(T)$  are both insensitive to the variations of the sampling period around the zero sampling period. On the other hand, the second-order sampling period sensitivity of the Riccati matrix sequence is given by

$$\left. \frac{d^2 K(T)}{dT^2} \right|_{T=0} = [A^T - K(0)BR^{-1}B^T] \frac{S}{6} - [A - BR^{-1}B^T K(0)], \quad (3.5)$$

with  $S$  being the positive semidefinite solution of the Lyapunov function. The optimal (constant) sampling period is computed in [59] from the minimization of the maximum eigenvalue of  $K(T)$  since it is proved that  $J(T) \leq (1/2)\lambda_{\max}(J(T))x^T(0)x(0)$ . In that paper, it was also proved that the minimum value of  $J(T)$  is not necessarily monotonically non-decreasing. This arises naturally from the general sampled-data theory (see, for instance, [58]) since discrete systems lose their controllability with respect to their continuous counterparts when  $T = n\pi/\omega_k$ ,  $\omega_k = \text{Im}(\lambda_k) = \text{eigenvalues of the dynamics matrix of the continuous system with } n \text{ being any positive integer}$ . In [38], the above method has been improved by computing the first and second-order sensitivities to the zero sampling period of the precompensator and feedback matrices of the discretized system. It was argued that for  $T \in [0.3, 0.7]$ , it is sufficient, in practice, the consideration of two or three terms in the Taylor's series expansion around a zero sampling period to achieve very close responses between the approximated discrete system and the original continuous one. The following design rules for the sampling period have been proposed in [58]:

*Rule 1.* The loss function grows asymptotically as  $T^{2\rho-1}$  where  $\rho$  is the highest multiplicity of the zero roots. Then, several candidate values for  $T$  must be tested accordingly.

*Rule 2.* If there are complex conjugate values of the matrix of dynamics of the continuous system then the location of the imaginary parts of the complex eigenvalues determine the sequence of values of the sampling period  $T$  for which the discrete system may not be controllable. The design is feasible if  $T \in (0, \bar{T})$  where the system is controllable. If the current  $T$  is close to  $\bar{T}$ , further investigation is necessary for the appropriate sampling rate design.

*Rule 3.* If the above mentioned matrix of dynamics has complex eigenvalues, a maximum sampling interval  $T_{\max}$  is determined by  $\pi/\omega_{\max} = \pi/(\text{Im}[\lambda_{\max}])$ . The best domain for  $T$  is found to be  $T \in (0, aT_{\max})$ ,  $0 < a \leq 1$ .

The following results are derived in [53] and [97] for the case of constrained sampling times:

*Result 1.* If a linear sampling constraint  $Ly(t_i) = LCx(t_i) = \Delta_i$  for  $0 < a \leq t_{i+1} - t_i \leq b_i$ ; ( $a_1 \neq b_i$ ) holds, then there exists a solution for the deterministic optimal linear regulator with state-dependent sampling ( $C$  being the output matrix).

*Result 2.* An optimal closed-loop control law exists for the optimal stochastic regulator with constrained sampling times and initial conditions given by  $E x_0 = \xi_0$ ;  $E(x_0 - \xi_0)(x_0 - \xi_0)^T = LC$  for the deterministic cost functional  $E(J(T))$ ,  $J(\cdot)$  being quadratic in the state and input as discussed before for the deterministic case.

In the related optimal designs, the Riccati matrix sequence of the optimal regulator is dependent on a set  $T_i, 1 \leq i \leq N$  of sampling intervals with  $T(\cdot) \in [a(\cdot), b(\cdot)]$ . They are computed from a numerical search algorithm of Fletcher-Powell of conjugate gradient type which ensures the convergence of the sampling period sequence. The following penalty function is used to constraint the sampling periods

$$p(T) = \delta \left[ \sum_{i=0}^{N-1} [\min(0, T_i - b_i)](T_i - b_i) + [\min(0, a_i - T_i)](a_i - T_i) \right]. \quad (3.6)$$

Some real constant  $\delta > 0$ . For the first-order system which is a simplified model for satellite altitude control:  $\dot{x} = u$ ,  $x(t_0) = 1$ ,  $x(t_1) = \Delta_1 = 0.5$ ,  $x(t_2) = \Delta_2 = (0.0)$  with the constraints  $0.01 < t_{i+1} - t_i = T_i < 100.0$ ,  $i = 0, 1$  and  $u(t) = u_0(t_0 < t_1/T)$ ,  $u_1(t_1 \leq t < t_2)$ , it is found that convergence occurs with five iterations. The following values are obtained  $\tilde{T}^* = (0.467, 1.220)^T$ ,  $J = 0.7351$ ,  $x = (1.0, 0.5, 0.1)^T$ ,  $U = (1.09, 4.08)^T$ . Additional results for this example are shown in Fig. 3.3.

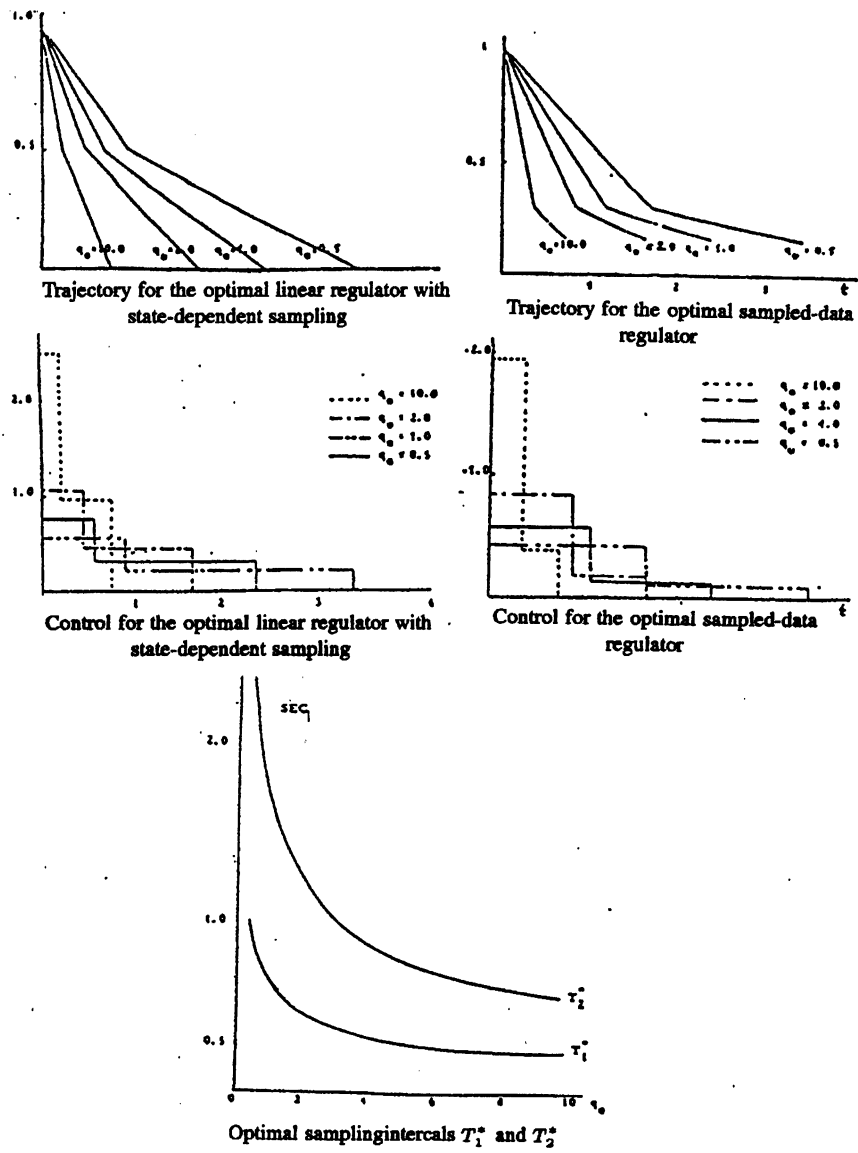


Fig. 3.3. Optimal sampled-data regulator.

In the above design, note that, in spite of their structure, the regulators take into account the original continuous nature of the system through the weighting matrices in the performance index. Note also that, because of the randomness of the initial state assumed in the design, the stochastic optimal regulator leads to an optimal control which is, in practice, independent of the initial conditions of the plant.

In the approach taken in [45], the optimization was directly focused on the discrete equations by using a multiobjective optimal control involving two quadratic performance indices. The first one involves the state and the input of the current plant while the second one involves the state error between the current plant and the nominal one together with a term involving a second compensating input. As a result, two coupled Riccati equations appear in the solution. This coupling is solvable under weak nonsingularity conditions and both inputs can be generated by state feedback. The approaches of [53] and [97] are more sensitive to changes in the sampling period than that of [45] since the sampling sequence appear explicitly in the weighting matrices because the index is obtained from an initial optimization index for the continuous system. In works by McDermott and Mellichamp (see, for instance, [20]), it has been found that the closed-loop performances are better (and the robustness becomes improved) for fast sampling, in spite of the dominant pole to be close to unity if the sampling rate yields an optimal closed-loop pole in the range [0.2–0.7]. This is motivated by the fact that the sensitivity to the sampling rate diminishes in these circumstances. Very close problems and solutions to the above ones are presented in optimization of digital encoders (see, for instance, [56]).

**3.3. Multirate regulators.** Multirate regulators have received much attention. A good description of those regulators can be found in Amit [1], Araki and Yamamoto [27] and Glasson [47–48]. Their main characteristic is that they use several sampling rates which are, in general, fixed. The practical need for the use of multiple sampling rates in control systems arises from the finite computational capabilities of the digital computers used in the implementations. The literature on the subject is abundant, see, for instance, [1], [10–11], [25–29], [36–37], [39–41], [43–44], [89]. An input-output analysis is presented in [37]. The filtering problem using multirate sampling is dealt with in [30] and [39]. A model of relevance for analysis which describes multirate systems by using an extended system with periodic regulator gains is described in [10]

and related papers. A combination of adaptive and multirate sampling techniques has been reported in a set of papers (see, for instance, [40–41] and [89]). In particular, it has been seen in [10] and [89] that multirate sampling can be of usefulness in aeronautical applications in both the case of known plant parameters and its adaptive version. More specifically, two combined maneuvers like, for instance, ailerons/rudder operations, can improve filtering against unsuitable disturbances if appropriate sampling frequencies, related to those of the unsuitable disturbances, are used. According to the general results about transients of [6–8], [10], [40] and [89], the following ideas have to be used when implementing a multirate regulator:

– *Slow sampling rate diminishes the effects of the unmodelled dynamics but may not work in the presence of output-additive sinusoidal perturbations. It can act as a filter to high frequencies of undesired disturbances but can also lead to poor transient responses.*

– *Slow sampling rate can lead to large steady-state errors for low plant static gains if a sufficient number of integrators (according to the reference input) is not incorporated.*

– *High sampling rate leads to better transient responses.*

In Fig. 3.4, the multirate regulator structure used in [10–11] is shown for two sampling rates, where  $u$  and  $u'$  denote slow and fast changing inputs. A recomputation of  $u$  is accomplished by adding a quantity  $v$  to the holding circuit when  $k = il$  (see the model in Section 5.2 below for details). The purpose of the crossfeed is to compensate for excitations of the fast modes of the plant caused by  $u$  on cycles between slow control updates. The periodic gains  $C_{jk}$ ,  $C_{jil}$  and  $C_{sfk}$  are obtained by propagating the Riccati equation of the optimal regulator from infinity to the (periodic) steady-state backwards in time (Barry [44], Glasson [49]). The periodic gains of the regulator are

$$C_k = \begin{pmatrix} C_{fil} & C_{sfil} \\ & C_{sil} \end{pmatrix}, \quad k = il; \quad (3.7)$$

$$C_k = [C_{fk} \quad C_{sfk}], \quad k \neq il.$$

The closed-loop dynamics of the multirate system is determined by the choice of the performance index

$$J = (1/2) \sum_{k=0}^{\infty} (x_k^T, u_{sk}^T) Q_k (x_k^T, u_{sk}^T)^T + 2 (x_k^T, u_{sk}^T) M_k \tilde{u}_k + \tilde{u}_k^T R_k \tilde{u}_k, \quad (3.8)$$

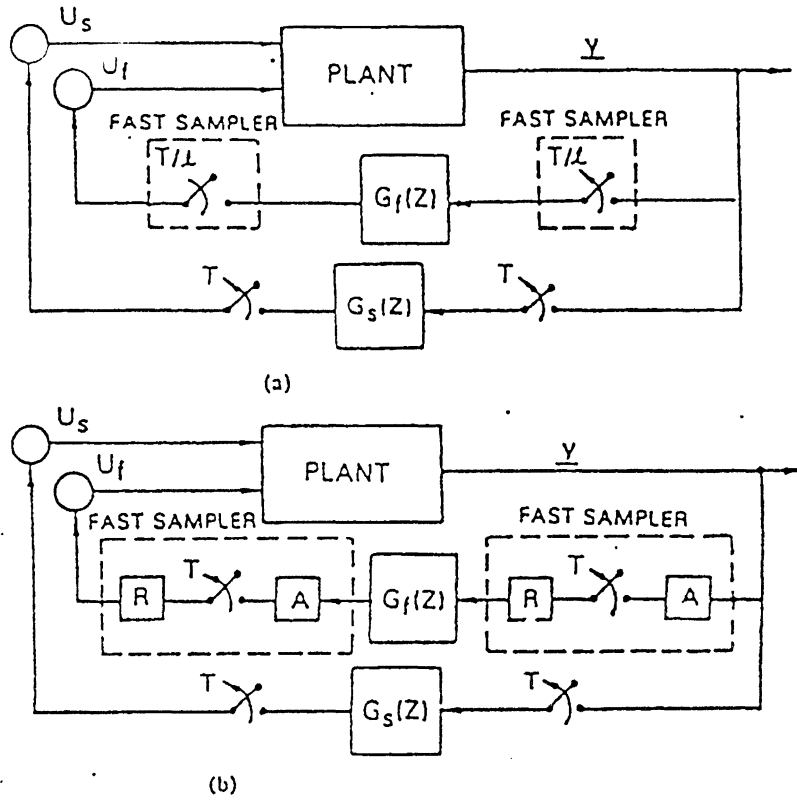


Fig. 3.4. Vector-switch representation of a multirate controller: (a) multi-rate structure; (b) equivalent single-rate structure.

with  $\tilde{u}_k = [u_{f,k}^T | v_k^T]^T$  being the equivalent input (see Section 5.3 below). The discrete-time periodic weighting matrices used to calculate the multirate gains are built from the single rate matrices [49]. The multirate gains are derived from the periodic steady-state solution of the discrete-time Riccati equation. In the stochastic case, with the state-disturbance being  $w_i \sim N(0, \sigma_i)$ , the Error Rejection Functional

$$J = (1/l) \sum_{k=1}^l W_k P_k W_k^T = E(a^2); \quad a_k = W_k x_k \quad (3.9)$$

is introduced in the control design functional in [44], [49], subject to the com-



putational constraint

$$(f_c T_s / n t_m) - (1/l) \sum_{i=1}^m N_i = 0 \quad (m \text{ control channels}), \quad (3.10)$$

where  $f_c$  is the fraction of the computation rate capability allocated to control,  $n$  is the number of multiplications per control channel,  $t_m$ , the execution time for multiplication,  $T_s$  is the base sample and  $N_i$ , the number of times control  $i$  is computed per control period. The goal of the optimization is to determine the base sampling period and the sample policies  $N_i$  that minimize the error rejection cost functional under the computational constraint (3.10) while maintaining the Riccati-type solution for the optimal control. The elements of the weighting array  $W(\cdot)$  are chosen accordingly to the judgement of the designer, or a physical basis, *for instance,  $a(\cdot)$  may be a component of vehicle acceleration* and the elements of  $W(\cdot)$  can be derived from the coefficients of the differential equation describing that acceleration. The first term of the constraint equation is the computational budget. The second one is the total number of control channel computations over the 1-cycle control schedule normalized to the single base sampling period. A generalization of the above idea has been proposed by using combined schemes that involve the use of the Kalman filter for state estimation in a noisy environment and a multirate controller which uses such an information have been also proposed in some papers (Glasson and Dowd [10], Glasson [49]).

**3.4. Pulse frequency modulation (PFM).** Pulse frequency modulated (PFM) actuators lie in the class of signal-dependent sample-data systems PFM-controllers which are developed in the discrete domain, [64], but they can be also analyzed from an initial statement in the continuous domain, [62]. PFM actuators are being used in technical and biological control systems. This leads to non-linear sampled-data systems in which the actuating signal is a train of constant pulses whose polarity and spacing are modulated. Biological and neuronal applications are commented in [68] where a view of applications with references is also given. Some applications are discussed in [85]. The operation of a PFM controller is summarized as follows. Firstly, the output signal  $u(t)$ , which is the input to the plant, is generated as an equally shaped pulse train. The absolute value of the frequency is a function of the absolute value of the (continuous) input signal  $e(t)$ . Then, the polarity of the pulses is generated as a function of the sign of  $e(t)$ .

In contrast, in classical Communication Theory, PFM is defined as a pulse train (carrier signal  $P(\omega t + \phi)$ ) whose frequency or phase angle is modulated due to a given input. Thus, in those terms, a PFM controller, which gives positive pulses for positive inputs, negative pulses for negative inputs and no pulses for zero input, can be interpreted as a phase angle modulator with a carrier frequency equal to zero (Dillmann and Frank [62]). For technical implementability reasons, a so-called filling time  $t_p$  must pass between two successive pulses of equal sign. The modulation is a nonlinear process which involves a great mathematical effort for a precise understanding. Other similar approaches are the so-called PAM and PWM. The PAM is used in the standard sampled-data systems, while in a PWM system, the width (duration) of each pulse is proportional to the absolute value of the sampled signal at the sampling instants. The combination of PFM and PWM is called CPFWM. A characteristic property of PFM's is that, due to their non-linear nature, they may present undamped oscillations as well as aperiodic limit cycles [62]. This fact can be linked with the nonlinear models for adaptive sampling systems to be then described in Section 5.3. Related stability studies have been reported in [61] and [65]. On the other hand, the usual optimization and design methods involving PFM controllers use balance harmonic linearization or Lyapunov functions to compute the stability regions, [63], [66–67]. The mathematical model for a satellite motion has been described through a double integrator. The controller strategy is summarized as follows: (1) if an initial disturbance is fed into the control loop, then a pulse is fired; (2) at firing time, the values of the state components are fixed to constant values and the controller parameters are calculated; (3) two gains of the feedback loop are computed according to a parametrical index of the controller which is modified in the controller computation.

There have been many attempts to develop mathematical models of PFM controllers (see, for instance, [57], [67–68]). In [67], the so-called sigma PFM ( $\Sigma$ PFM) is introduced. In [68], the following classification was made by reviewing several preceding papers:

(a) *Integral PFM* (IPFM). The input signal is integrated and a pulse is generated whenever the value of that integral reaches a threshold magnitude. Note that such a philosophy is quite similar to that used in adaptive sampling criteria with constant amplitude difference given in Section 2.2. The integrator resets after each pulse so that successive integrations start from zero. When the

pulse train has only one polarity, one has the single signed IPFM (ISSIPFM), and when the pulse train is bipolar, one has the so-called double signed IPFM (DSIPFM).

(b) *Sigma PFM* ( $\Sigma$ PFM). It is a generalization of the IPFM which incorporates a nonlinear negative feedback signal. Firing may be deleted by using a threshold level with the additional feedback playing the role of an input threshold, [85].

(c) *Complete Reset PFM* (CRPFM). It is another generalization of the PFM which incorporates a memoryless actuator.

(d) *Other PFM*. The functional PFM (FPFM) is a generalization of  $\Sigma$ PFM (and so, it also involves IPFM), and the delta PFM ( $\delta$ PFM) in which the emission of a pulse is decided by observing the value of the input integral at certain times.

(e) *Dillmann and Frank's PFM-controller* ([62]). It has the same basic philosophy as that used in Communication Theory. The frequency of pulse train is modulated by a given input. The output of the PFM-controller, which is the input to the continuous process, is given by the convolution integral  $u(t) = q(t) * dp(t)/dt$ . In this formula,  $p(t)$  is the desired pulse from  $u(t)$  and  $q(t)$  is a zero or nonzero constant plus a nonlinear function of the time-integral of the output of a transducer which converts the closed-loop error into frequency.

The most typical result of PFM is that it is characterized by a *high degree of noise immunity*. Hence, PFM controllers are useful in applications where there exist strong noise. Several applications will be summarized in Section 6.

#### 4. Fundamental properties of dynamic systems

**4.1. General formulations with non-periodic sampling.** Several papers have been interested in the following main problem: *Assume that a property holds for a continuous dynamic system. Thus, which is the way to select a finite set of sampling points in such a way that such a property still holds?* The second question closely related to the above problem arises naturally, namely, *How those sampling points have to be selected in such a way that either the information about the particular problem or the transmission of the errors from the data be as accurate as possible?* The choice of those sampling points is performed only if the property holds in the continuous case. For instance, if the system is observable or controllable what is taken as a necessary condition

prior to convert the problem into one taking as data a set of samples. Note that

(1) The problem can be solved algebraically. This would lead to a simpler tool than the usual numerical methods. In addition, the associate technique can be applied either for discrete systems or for continuous ones (through the use of a discrete set of measurements but without discretizing the plant inputs). In this way, discrete techniques and their associate computer technology can be easily extended to problems of a continuous nature (see, for instance, [71–73], [76], [81]).

(2) The sampling instants can be designed according to optimization tools either by using a numerical procedure being applied for a given time interval or sample per sample which leads to an easy scheme's implementation (see the same above references).

(3) The algebraic problem of solving a set of linear equations, associated with each sampling point, is reduced to one of ensuring the fulfillment of both the parametrical condition of the continuous problem (see [72], [77], [79], [80–81]) plus extra conditions on the particular set sampling points (see, [72–73], [76], [81]). In particular, the identifiability and model matching problems were studied from an algebraic point of view by selecting a set of coherent sampling points in [72–73] while those of controllability and observability were first studied in [81].

For instance, consider the identifiability of the linear and time-invariant dynamic system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \quad x(t_0) = d, \quad \forall t \in [t_0, t_f], \\ y(t) &= Cx(t), \end{aligned} \quad (4.1)$$

where  $x(t)$  is the state  $n$ -vector,  $u(t)$  is the scalar input,  $y(t)$  is the output  $s$ -vector and  $A$ ,  $b$ ,  $C$  and  $d$  are real, constant and parameterized by the real  $r$ -vector  $p$  of nominal value  $p_0$ . It is assumed that within the neighborhood of interest of  $p_0$ ;  $A$ ,  $b$ ,  $C$  and  $d$  are continuously differentiable with respect to every parameter component and all their partial derivatives are uniformly bounded and piecewise continuous, [79]. Consider the linear dynamic sensitivity system

$$\begin{aligned} \dot{X}(t) &= \bar{A}X(t) + \bar{b}u(t), \quad X(t_0) = \bar{d}, \quad \forall t \in [t_0, t_f], \\ Y(t) &= \bar{C}X(t) \end{aligned} \quad (4.2)$$

with  $X(t) = [x^T(t), x_{(1)}^T(t), \dots, x_{(r)}^T(t)]^T$ ,  $Y(t) = [y^T, y_{(1)}^T, \dots, y_{(r)}^T]^T$ ,  $\bar{b} = [b^T, b_{(1)}^T, \dots, b_{(r)}^T]^T$ ,  $\bar{d} = [d^T, d_{(1)}^T, \dots, d_{(r)}^T]^T$  and,

$$\bar{A} = \begin{pmatrix} A & 0 & \dots & \dots & 0 \\ A_{(1)} & A & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ A_{(r)} & 0 & \dots & \dots & A \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} C & 0 & \dots & \dots & 0 \\ C_{(1)} & C & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ C_{(r)} & 0 & \dots & \dots & C \end{pmatrix},$$

where the subscripts  $(i)$  mean partial derivatives with respect to the  $i$ th component of  $p_0$  and  $T$  transposition. In this case, it is well-known that the first problem referred to above about the “parametrical property” for the identifiability around  $p_0$  to hold is guaranteed for the unforced system under the sufficient condition ([72], [79]):

$$\text{rank} \begin{pmatrix} (Cd)_{(1)} & \dots & (Cd)_{(r)} \\ (CA d)_{(1)} & \dots & (CA d)_{(r)} \\ \vdots & \ddots & \vdots \\ (CA^{2n-1}d)_{(1)} & \dots & (CA^{2n-1}d)_{(r)} \end{pmatrix} = r = \dim(p). \quad (4.3)$$

If the partial derivatives are not dependent on  $p_0$ , then the identifiability becomes global ([79-80]). The power  $2n - 1$  can be (perhaps) reduced substituting  $2n$  by the degree of the minimal polynomial of  $A$ , [79]. The same philosophy is applicable for the forced system through the change of  $d$  by  $b$  in (4.3) (see [72]). Now, if a set of samples of the output is taken then the property is kept, provided that (4.3) holds iff the set of sampling points satisfies the condition

$$\text{Det} [a_i(t_j)] \neq 0 \quad (i = 0, 1, \dots, N - 1; j = 1, 2, \dots, N). \quad (4.4)$$

In the above expression,  $N$  is any finite integer greater than or equal to the degree of  $\bar{P}(A)$ , i.e., the minimal polynomial of  $\bar{A}$ , the  $a(\cdot)$  being linearly independent scalar functions arising from the expansion of  $e^{\bar{A}t} = \sum_{k=0}^N a_k(t)\bar{A}^k$ ,  $\forall$  integer  $N \geq \text{degree}(P(\bar{A}))$ . They are dependent on the chosen number samples  $N$ . For the forced system, the extended parametrical rank condition from (4.3) becomes more involved since it contains a richness condition on the input. In particular, for zero initial conditions, a non-zero input satisfying  $\text{Det} [\int_0^{t_j} a_i(t_j - \tau)u(\tau)d\tau] \neq 0$  has to be selected at the chosen sampling

points. The way to compute such sampling points can be found in [72] for the identifiability and model matching problems and in [76] and [81] for the observability and controllability problems. For the last two problems, the minimum number of required sampling points is the degree of the minimal polynomial of the  $A$ -matrix of (4.1), which is typically smaller than that required for the identifiability and model matching problems. This feature is motivated by the global nature of the problem which implies that a “sensitivity model” similar to (4.2) is not required. In such cases, the necessary parametrical condition is the usual rank condition guaranteeing observability or controllability. Note that such conditions are similar to (4.3) except that partial derivatives, with respect parameters, are not involved and the maximum power of the  $A$ -matrix is  $(n - 1)$ . As pointed out in [79], the eigenvalues of  $\bar{A}$  are those of  $A$  so that the  $a(\cdot)$ -functions are computed for the local identifiability problem in the same way as for the observability which can be considered as one of global identifiability of a parametrization consisting of the initial conditions of (4.1) only (see [79], [81]). In the case of model matching, two dynamic systems are involved (namely, the current system and its reference model) so that the parametrical condition of the algebraic problem is more involved and the minimum number of samples required to keep the property under discretization is increased accordingly, [72–73]. The algebraic problem in this case reduces to compute the forward and feedback compensator matrices for a set of non-periodically distributed set of samples (see [72]) as shown in Fig. 4.1. Applications to the case of the determination of such matrices for regulation of a group of amplidyne acting on a synchronous motor has been studied in [73]. It was pointed out in [81] and then in [72–73] that the  $a(\cdot)$ -functions are a Chebyshev system on each interval of finite length. As a result, it suffices to take for the unforced system all the sampling points being distinct within the real interval  $(\gamma, \gamma + \pi/\omega^*)$  with  $\gamma > 0$  being any nonnegative real number and  $\omega^*$  a known upper-bound of a maximum system eigenfrequency. For the forced system, a more involved generalization using similar considerations is applicable ([72–73], [76], [81]).

**4.2. Accuracy aspects.** In a schematic reasoning, the various papers cited in Section 4.1 concerned with the algebraic solution of the controllability, observability, identifiability model matching problems deal with the abstract mathematical question of ensuring that two applications, namely,  $q$  (being dependent

Time	1	2	3	4	5
Periodic design					
0.5	37.75	30.72	26.64	33.62	101.09
1.0	115.43	131.17	110.58	164.20	108.84
1.5	195.43	291.92	235.43	379.78	833.74
2.0	275.43	485.41	393.12	653.84	1327.85
Non-periodic design					
0.5	37.75	29.60	19.21	11.79	43.32
1.0	115.43	124.50	85.41	53.75	241.21
1.5	195.43	274.97	194.58	122.30	594.81
2.0	275.43	453.94	336.16	207.21	1065.72

**Fig. 4.1.** Loss functions: (1) Model. (2) Open-loop physical process. (3) Closed-loop compensated system without measuring errors. (4) Closed-loop compensated system (measuring errors 3%). (5) Closed-loop compensated system (measuring errors 4.2%).

The loss functions are defined as the accumulated areas of the outputs with respect to time.

$K$  = Feedforward Compensator,  $\bar{K}$  = Feedback Compensator

periodic sampling ( $K = 0.057, \bar{K} = 0.76$ )

non-periodic sampling ( $K = 0.075, \bar{K} = 0.782$ ).

on the system structure and associated parameters); and  $s$  (being dependent on a set of sampling points and on the system structure/parameters as well) are both bijective in such a way that “some vector to be determined” (v.t.b.d.) can be computed from the next chain of causality:

$$\begin{array}{ccc}
 \text{v.t.b.d.} & \xrightarrow{q} & \text{Markov parameters} \\
 & & \text{associated with the} \\
 & & \text{samples} \\
 & & \xrightarrow{s} \text{Set of output(s)} \\
 & & \text{(or impulse response)} \\
 & & \text{(continuous) property}
 \end{array} \quad (4.5)$$

with

$$\text{v.t.b.d.} = \begin{cases} \text{(local) Identifiability: } \begin{cases} \text{Current parameter vector } p \\ \text{Nominal parameter vector } p \end{cases} \\ \text{Observability: } x_0 \text{ (initial Conditions)} \\ \text{Controllability: } U = (u_0, u_1, \dots, u_{n-1})^T \\ \text{Model Matching: } K \text{ and } \tilde{K} \text{ Feedforward and Feedback} \\ \text{compensator matrices} \end{cases}$$

In this process, several problems occur, namely:

(1) The chosen set of sampling points can greatly influence the coefficient matrix of the algebraic problem at hand. It is well known from Numerical Analysis, [72–73], [76], [81], that when solving the algebraic system  $y = Cx$ , one has for the obtained estimations  $\tilde{x}$ ,  $\tilde{y}$ ,  $\tilde{C}$  of  $x$ ,  $y$  and  $C$ , that

$$\frac{\|x - \tilde{x}\|}{\|x\|} \leq k(C) \left[ \frac{\|C - \tilde{C}\|}{\|C\|} + \frac{\|y - \tilde{y}\|}{\|y\|} \right]; \quad k(C) = \|C\| \|C^{-1}\| \quad (4.6)$$

in a first-order error approximation where  $k(\cdot)$ , as defined in (4.6) is the condition number of the given norm of the  $(\cdot)$ -vector of matrix. Eq. 4.6 indicates that  $k(C)$  bounds the transmission of measuring/rounding errors from the data towards the results through the computation procedure. In [81], the use of performance indices to be minimized was also proposed so that the absolute (rather than the relative) errors were minimized when solving the observability and controllability problems.

(2) Since the cases when the input is a continuous signal are also considered, it is known that, through the discretization, the overall information about the solution of a given property of a continuous problem cannot be obtained, in general, from discrete data.

(3) In the local identification problem, a first-order approximation of the parameter variation is computed for the real parametrical variation because of the involved “sensitivity” method.

One concludes from those considerations that the fulfillment of a property does not ensure a good estimation method. Therefore, the possible strategies should increase sufficiently the number of samples  $n$  in order to achieve a good accuracy in the results and to distribute the set of samples in some optimal way such that the condition number of the coefficient matrix be diminished as much



as possible. Also, the maintenance of this number under a finite upper-bound automatically ensures both the parametrical and sampling conditions to hold. For this purpose, a suboptimization procedure is proposed in [72–73] where an upper-bound of the weighted sensitivity matrix is minimized sample-per-sample leading to the computation of a (sub)optimal sampling period at each new sample.

**4.3. Further applications.** An interesting application of arbitrary but fixed sampling is that it can be used for decoupling a large class of system when they cannot be decoupled in continuous time. In [26], the case of a 2-input 2-output completely controllable and observable system is considered and a sufficient condition for decoupling is established. Once, the decoupling at sampling instants has been achieved, it is found that the coupling between sampling instants is comparatively small. The same authors present results in [27] for multirate control. It is seen in that paper that the Nyquist criterion has in the multirate case a parallel form as in the continuous-time case which is useful for both qualitative and quantitative analysis of the stability of feedback systems. The pole-placement problem and the design of the so-called “serial-sampling” type observers, which renew their internal state when new data are obtained while the plant outputs are detected serially, have been recently considered in [25] and [43], respectively. In self-tuning control of fast responding systems, a trade-off is proposed between small periods leading to uncertain quasi-unstable models and (possibly instabilizing) large periods [19]. The use of random sampling with its related properties are studied in [71].

**Modelling results.** The use of non-periodic sampling in dynamic systems cannot be addressed with the use of parametrical descriptions based on z-transforms which inherently assume that the sampling rate is constant (see [31–32]). A time-varying difference equation describes parametrically a non-periodic sampling system in those papers which is formally similar to the standard difference equation for constant sampling systems. These models have been used for the improvement of the transients in adaptive control through the real time design of the sampling period since a parametrical model is required for parameter estimation in the adaptive system [5–6]. However, impulse response methods are still applicable as a direct generalization of discrete systems involving constant sampling and have been used in some of the designs

of compensators using time-domain techniques (see [60, [95 – 96], [98 – 99]). Note that those methods require infinite memory as time increases and, therefore, they can only be properly used in systems with finite transients, such as minimum or dead-beat response systems, or in situations when the response becomes close to the steady-state in a reasonably small time. On the other hand, the use of some adaptive sampling criteria based on a constant amplitude differences make the sampler and zero-order-hold together with the adaptive sampling law to be nonlinear and, consequently, nonlinear phenomena, such as limit cycles, can appear [42]. Also, a sampling period variation with respect to a nominal value can be designed in real time by optimizing a standard quadratic loss function if such a (sample per sample) sampling increment is considered as an extra input. Thus, the system appears modelled as a bilinear one (see [33]). On the other hand, the use of multirate sampling in different problems including adaptive control, filtering, decoupling, pole-placement and optimization (see, for instance, [25], [27 – 28], [36 – 37], [39], [40 – 41], [89]) can require a description based on the use of extended models as seen in Section 2 (see [49]).

**5.1. Linear models.** In many problems, like those of time-sensitivity compensation (non-parametrical) impulse response models, which are direct extensions from the uniform sampling case, were used in [8] and [47]. Afterwards, it was claimed to make more powerful the analysis of adaptive sampling systems. An analytic method leading to a recursive time-varying input/output equation was presented in [8], [95 – 96] and [97 – 98]. The method is based on the use of the Laplace transform and the discrete convolution theorem in the time domain. A time-varying difference equation for the input/output sequence is obtained via Cramer's rule from solving a determinant which involves differences between sampling points. This computation is used to calculate the time-varying parameters. In [31 – 32], an equivalent time-varying equation is calculated which is applicable to systems of any order eventually being multivariable. The parameters which appear in the input/output difference equation are dependent on the transition matrices and the model was derived by following the next steps:

(1) *Discretization of the continuous plant equations:* The state and control transition matrices are obtained from the continuous plant and control matrices by assuming that a zero-order-hold acts on the continuous plant input.

(2) *Definition of the mean state transition matrix and sampling period:* Using the properties of the fundamental (or state-transition) matrices of differential

systems, the next equalities were stated:

$$\phi_{i,k} = \phi \left( \sum_{j=k}^{i-1} T_j \right) = \phi^{i-k} \left( \frac{1}{i-k} \sum_{j=k}^{i-1} T_j \right) = \phi^{i-k} (\bar{T}_{i,k}) = \bar{\phi}_{i,k}^{i-k} \quad (5.1)$$

for the state transition matrix from  $t_k$  to  $t_i$ . The superscripts  $(\bar{\cdot})$  indicate mean values. By using (5.1), one establishes the state transition equations from  $x_k$  to  $x_{n+k-i}$ ,  $\forall i = 0, 1, \dots, n-1$ ;  $\forall k \geq 0$  in a standard way.

(3) *Definition of time shifts relative to the mean transition matrices.* For all sampling instants  $t_{n+k-1} \in [t_n, t_{n+k}]$ , i.e., the current modelling interval at the  $(n+k)$ -th sampling point, one computes time shifts

$$\Delta \bar{t}_{n+k-i}(k) = \frac{n-i}{n} t_{n+k} + \frac{i}{n} t_k - t_{n+k-i}, \quad (5.2)$$

such that  $\Delta \bar{\phi}_{n+k,k}^{n-i} = \bar{\phi}_{n+k-i,k}^{n-i} \phi(\Delta \bar{t}_{n+k-i}(k))$ .

(4) *Application of the Cayley-Hamilton theorem to the mean transition matrix.* The characteristic equation of  $\bar{\phi}_{n+k,k}$  is

$$p_\lambda(\bar{\phi}_{n+k,k}) = \lambda^n + \sum_{i=1}^{n-1} a_i(n+k, k) \lambda^{n-i} = 0, \quad (5.3)$$

where the  $a_i(\cdot, \cdot)$ -parameters are dependent on the eigenvalues of the dynamics matrix of the continuous system and their multiplicities. The Cayley-Hamilton theorem applied to (5.3) leads to a similar equation for the  $n$ th power of the mean state transition matrix and their lower powers which substituted into the  $n$  state equations describing the state transition from  $x_k$  to  $x_{k+n}$  and the use of the output equation leads to the time-varying input-output equation of Section 2.2, Eq. 2.4. Thus, the last step is:

(5) *Derivation of the difference equation matrix.* from steps 1-4, the following time-varying difference equation is obtained

$$A[q'(k), k] y_{n+k} = B(q, k) u_{n+k}, \quad \forall k \geq 0, \quad (5.4)$$

where

$$A[q'(k), k] = 1 + \sum_{j=1}^n a_j(n+k, k) q'^j(k), \quad (5.5a)$$

$$B(q, k) = \sum_{j=1}^n b(n+k, k)q^j, \quad (5.5b)$$

where the time delay operators  $q$  and  $q'(\cdot)$  are defined according to  $qy_{n+k-i} = y_{n+k-i-1}$  and  $q'y_{n+k-i} = y(t'_{n+k-i}(k))$ , and the  $b_j(\cdot)$ -parameters in (5.5b) are dependent on the transition matrices and on the  $a_j(\cdot)$ -parameters. The  $t'(\cdot)$  are the so-called induced sampling points which are dependent on the real sampling points of each current modelling interval  $[k, k+n]$  and are computed by using the time shifts (5.2) as follows

$$t'_{n+k-i}(k) = t_{n+k-i} + \Delta t'_{n+k-i}(k) = t_k + (n-i)\bar{T}_{n+k,k} \quad (5.6)$$

As pointed out in Section 2.2, the output must be measured at the induced sampling points. Note that the last induced sampling point of a modelling interval is coincident with the current real one. Eq. 5.4 is directly applicable to discretization under zero-order-holds of strictly proper transfer matrices. Note finally that under adaptive sampling leads to moderate computational effort since the (time-varying) plant parameters are not obtained from the above formulas but from the parameter-adaptive scheme. Thus, the only necessary relation in those applications, from the above model, is (5.6) to calculate the induced sampling points governing the output and the sampling points governing the input. Extensions to the non-strictly proper case are also given in [31]. In [33], a bilinear model was given for a discrete system in which the sampling period acts as an extra input which can be optimized step-by-step via regulation criteria with sliding planning horizons.

**5.2. Multirate control.** Multirate sampling systems can be easily modeled by using extended state vectors. In [27], several approaches have been supplied involving many variables with simple coefficients including stability studies. In [10–11], the discrete dynamics is described by

$$x_{k+1} = \phi x_k + \Sigma_f u_{fk} + \Sigma_s u_{sk}, \quad (5.7)$$

where  $\phi$  and  $\Sigma$  are the transition matrices, and the input  $u$  is partitioned into subcomponents  $u_f$  and  $u_s$  to signify those controls computed at the base sampling rate and those scheduled at smaller sampling frequencies, respectively. The development of the multirate regulator structure requires the augmentation

of the natural plant dynamics (see [10–11] and [27]), namely:

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} \phi & \Sigma_s \\ 0 & I \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + \begin{pmatrix} \Sigma_f & \Sigma_s \delta_{k,il} \\ 0 & I \delta_{k,il} \end{pmatrix} \begin{pmatrix} u_{fk} \\ v_k \end{pmatrix}, \quad (5.8)$$

where  $\delta_{k,il}$  is the Kronecker delta function. In this way,  $u_f$  is updated at a fast rate,  $(1/T_s)$  samples per second;  $u_s$  is computed at a slower rate  $(1/T_s)$  samples per second, and it is held constant between computations by a holding circuit. Recomputation of  $u_s$  is accomplished by adding and increment  $v_k$  to that circuit on cycles such that  $k = il$ , otherwise,  $v_k$  is set to zero. Such a model has been applied for analysis and synthesis purposes to different multirate control and filtering structures.

**5.3. Non-linear and random models.** In certain adaptive sampling systems (for instance, when the constant amplitude difference criterion is used), it is observed that self-sustained oscillations can appear if the sampling period violated the asymptotic stability domain. This phenomenon has been reported in [95–96] and [98–99] and treated analytically in [42] where a nonlinear model for the whole scheme was proposed. In that paper, the used model is based on the substitution of the zero order and hold plus the adaptive sampling law by a non-linear characteristics which is a generalization of a relay with hysteresis. Such a model allows the correct interpretation and detection of a possible limit cycles based on a first-order harmonic approximation. The approach consists of the calculation of a describing function for the nonlinearity is shown in Fig. 5.1

Also, random sampling is useful in nonlinear filtering and Communication Theory (see [71] and [100]). In estimation theory, it is assumed that the statistical properties of measurement noise is known deterministically. However, in real situations when the measuring device is subject to a random failure, the statistical property of the measurement noise will change at random from stage to stage. In this way, one has estimation in switching environments. In [71], a complete list of previous references is supplied. The paper by Reiser [100], because of its applied nature, is better described in the next section. In Akashi and Kumamoto, [71], the following considerations are used for design purposes: (1) the set of sequences which characterizes the minimum variance estimate is regarded as a population; (2) the estimate is calculated with a relatively small number of sequences sampled at random from the population; (3) the nonlinear signal model for output measuring is nonlinear and generates a noise driven by

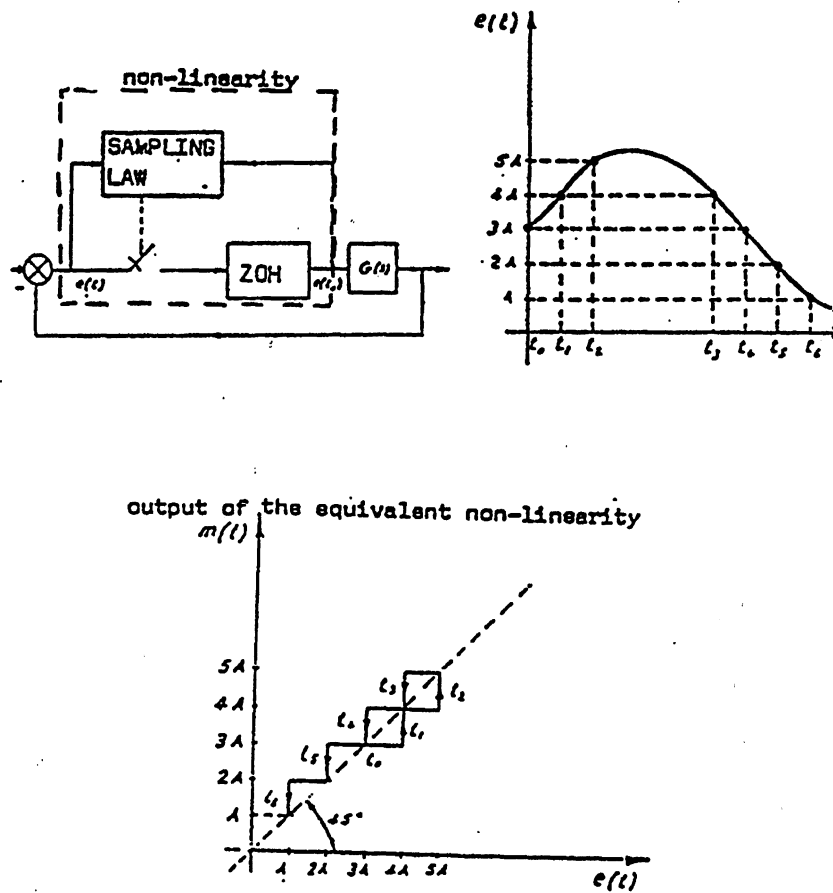


Fig. 5.1. Sampling criterion of constant difference of amplitudes.

a Markov sequence of random variables. The transition and initial probabilities are required as data and the state, measuring and driving noises and the random initial state are assumed mutually independent. The initial state, and the state and the measuring noises are assumed to be Gaussian processes. The proposed sampling strategy for state estimation consists of defining a sequence of random numbers for each sampling point candidate from a maximum set of  $N$  (possible) samples which is fixed by the designer. Each one of those random numbers is distributed with a rectangular distribution between 0 and 1. It is found that the proposed estimation algorithm does not usually lead to unbiased or optimal state estimation. Other classical applications of random sampling lie in fields like computing time-shared or elimination of hidden oscillations (see a list of references in [64]).

**6. Applications.** Techniques of non-periodic or adaptive sampling have not been widely used in Industry or Biomedical Applications. Several papers give methods to select the uniform sampling rates while others (especially in biological problems) use non-periodic sampling more due to experimental requirements than as a optimization or “behavior improvement” technique. It is our opinion that the absence of theoretical developments using nonuniform sampling techniques may be caused by two main reasons, namely: (1) the absence up till very few years ago of available input/output descriptions involving a finite number of parameters and input/output measurements only (In fact, the models referred to in Section 5 may require, in practice, the use of a computer program to evaluate the coefficients so that the computation problem is not trivial in many situations); (2) the possible difficulty for many applied researchers to precisely understand the deterministic and stochastic models necessary to analytically solve their problems.

### 6.1. Industry and communication applications.

**Communication theory and engineering.** In Reiser, [100], a list of 142 previous references is given with special emphasis on the use of sampling tools in Communication Theory. The major interest is devoted to the mathematical tools for applying sampling via the use of Markov chains to discrete-time systems and queueing theory. Two kinds of queues for messages are described, namely,  $M/D/1$ : Markovian inputs of exponentially distributed interarrival times and constant deterministic intervals; and  $M/M/1$ : Markovian inputs with expo-

nential arrival and service distributions. On the other hand, *the randomness of the sampling period explicitly appears in the so-called arrival theorem for messages and in the mean transit time for protocols to manage multidrop links as well as point-to-point connections.* It is given by

$$\bar{t} = E(T) + \frac{\lambda E(T^2)}{2(1 - \lambda E(T))}, \quad (6.1)$$

where  $E(T)$  and  $E(T^2)$  denote the first two moments of the virtual waiting time.

**PFM.** Several subsequent applications of PFM controller (see Section 3.4) are described with crossed references in [68]:

- Combined with a stepping motor provide a two-term proportional plus derivative controller which is useful for industrial control of multivariable systems.

- Using pulse tachometers as pulse frequency modulators, the multivariable PFM controller can be used for controlling the speeds of multiple motor systems in such a way that the discontinuity effect of the up-down counter method is avoided.

- They have given good performances in spacecraft attitude control using gas jets, [69]. Inherent oscillations to PFM control may occur, [57], and they could be deleted in some cases by using the variant of [62]. See Section 3.4.

- They are useful as an alternative to rate feedback or nonlinear compensation with feedforward compensator, [67], to overcome, for instance, the nonlinear characteristics of a hydraulic servovalve.

- They are useful to vary the power to the load in AC power control, for instance, by controlling the point at which the thyristor starts to conduct during each half cycle. The disadvantage is that radio interference increases.

- The IPFM and PFM (see Section 3.4) have been used to build several neural nets. For example, Pavlidi's model of a single neuron shows a chain of two neurons with backward inhibition, i.e., the input is used to excite the first neuron, [85], [101].

- PFM controllers have been also used in the former adaptive control strategies (see, for instance, Murphy and West [102]).

**Multirate Control.** It is widely used in spacecraft attitude and maneuver control as pointed out in Section 3 by reasons of accommodation of instrumen-



tation measurements and maneuvering rates as well as reasons of noise filtering ([10–12], [89]).

It is also of interest in adaptive flight control when using optimal quadratic regulator techniques in digital fly-by-wire flight control systems. For a fixed system, control gains are made stabilizing and while they are adjusted in response of parameter changes. The adjustment has been addressed through an interactive correction over the Riccati matrix. For a fourth order model with two control inputs such that a weighted least squares identification is performed on all four states, the following time-scheduling slices results in [88]:

State estimation .....	3.9 msec.
Control computation .....	0.41 msec.
Control gain adaptation .....	7.0 msec.
Parameter identification .....	1.5 msec.

**Bang-bang adaptation of the sampling period.** Some sampling period adaptations within constraints or prescribed intervals  $[T_{\min}, T_{\max}]$  have been described ([85–89]). In the application of [18], the process of liquid moving in a vessel is described in terms of inflow/outflow rates. If a load disturbance occurs immediately following a measurement, the liquid level will (open-loop) respond for the subsequent sampling period. The allowable sampling periods depend on the vessel size, the maximum anticipated disturbance at the equilibrium flow and the maximum and minimum controller gains. Note that, as pointed out in Section 2.2, this bang-bang sampling technique is a theoretical response to the requirements of bandwidth, stability and filtering required by the discrete systems.

**Self-tuning and pole-placement controllers.** In some papers (see, for instance, [19–20]), the influence of the sampling rate in self-tuning and pole-placement controllers has been investigated. In the above cited paper, a procedure for autoselection of the sampling period in pole-placement controllers is discussed for chemical process which consists of a tabular autothermal reactor with internal countercurrent heat exchange. Feed gases enter the bottom, travel up through an annular section, turn around at the top, and pass down through the catalyst bed. Non-periodic sampling is selected as being linked with the response associated with the dominant pole what leads to excellent results.

**6.2. Biomedical applications.** An intuitive and complete description of sampling rate techniques in Geology and Paleontology with abundant references has been supplied in [82]. The strategies followed in these disciplines fulfill faithfully Shannon's theorem for sampling with wide accuracy margins. The sampling rate is uniform, in general, what may be due to the fact that usually materials follow a cyclic sedimentological process and there is no "a priori" criteria for each novel experiment in updating sampling rates for data measurements. However, it seems plausible that, at a second stage, sampling rate could be chosen as time-varying by taking more data from geological (sub)-times where their accumulation was greater, [103]. On the other hand, non-periodic sampling has been used in several ways in biological and biomedical processes as listed below:

- (1) due to experimental clinical and data constraints ([74], [83], [86], [90 – 92], [101]);
- (2) through the use of PFM models ([84 – 85], [101]);
- (3) to optimally identify physiological systems from very limited data ([74], [90], [93]).

In some applications in biomedical engineering, sampling has not merited special attention, [87]. The following observations are the basis of a possible application of non-periodic sampling ([104] – [105]):

– *Biological signals are present in the brain, muscle and eye of humans and animals and are characterized by alternating periods of relative activity and inactivity.* Electroencephalograms, electrooculograms and electrocardiograms, reveal variable activity. For instance, an alert person displays a low amplitude (10 to 30 microvolts) of electroencephalogram signals of mixed frequency in the 13 to 18 Hz while a relaxed person produces large amounts of sinusoidal waves at a single frequency in the 8 to 13 Hz range (called alpha). As an individual sleeps, alpha-activity is replaced by several intermediate stages of repetitive cycles where the amplitudes and frequencies of the brain signals become modified.

– *The limits of the thresholds of transient phenomena and sampling rates are difficult to set from provisions for viewing the data prior to digitizing.* Therefore, sampling is performed across channels of rapid sampling of the more rapidly changing process such as the electroencephalogram signals and lower sampling of more gradually changing phenomena such as respiration. Thus, sampling

rates are adapted to changes of each process.

– Neurons emit signals which are not uniformly time-spaced so that non-periodic sampling is a natural way to describe some related processes. For instance, neurons located in the sensory relay nuclei of the thalamus seem to change their firing pattern from one which is totally random to a bursting pattern, where two actions occur close together followed by a long interval before the next burst, [83]. The neuronal nets can be modelled by using PFM and IPFM-models ([84 – 85]) without ability for backward transmission.

– Compartmental models are widely used in Biology and Ecology (see Gowdy [86] where a complete list of previous references is given). It is pointed out that ecologists tend to know the turnovers for the species present in an ecosystem. Analytical relationships between these turnovers and the eigenvalues would lead to the computation of the eigenvalues bounds and, in turn, to determine bounds for sampling rates. The sampling period is taken inversely proportional to the ecosystem size and to the number of individuals and interactions between species. It is also suggested that non-periodic sampling can be used by interlacing multiple fixed sample periods in cases when the precalculated minimum and maximum sampling periods are widely separated. Also, in certain metabolical processes (for instance, the process of the “bromsulphalein” from the blood to the liver, [91 – 92]), sampled-data are not uniformly time-spaced and the parametrical identification is greatly dependent on the compartmental model size.

It may be deduced from the above observations that (a) *samples are separated by clinical constraints* and (b) *the sampling rate is governed by experimental constraints or system size*. If “adaptivity” of the sampling rate is allowed, then its “adaptivity degree” is usually low due to the difficulty in predicting the experiment evolution. Other considerations given in [74], [90] and [92 – 93] are:

(1) The data obtained are usually very few and quite noisy. Sampling is in many cases the only variable to test. The number of samples is limited in order to avoid significant systems alteration. Also, the types of test-input signals are also severely restricted by practical constraints (for instance, intravenous injection or infusion).

(2) The accuracy of each signal sampled at a given time is usually relatively poor. This typically due to severe methodological problems in the assessment

of molecular concentrations in biological fluid samples. Typically, data errors are of the order of 5 to 30 percent. In [93], the research is addressed to choose the test input and the output sample schedule which maximizes the expected accuracies of the parameter estimates obtained from data. Two steps are scheduled, namely, (a) formulation of the design procedure as a nonlinear optimization problem and (b) design of the nonuniform sampling interval by maximizing the determinant of the stochastic information matrix.

**6.3. Other recent results.** In the last years, some results concerned with statistical issues and multirate controllers have been published. In [106–107], the theory of locally asymptotically normal experiments is used to test hypotheses about the parameters of a controlled linear stochastic process. A constant bias property of the variation estimate of the parametrical local variance of the noise is discussed. Multirate control from the blood to the liver, [91–92]), sampled-data are not unifers have been revisited in [108–110] including the study of its relative stability in the frequency domain, [109]. In [111], a parametrization of all stabilizing controllers for a given discrete-time time-varying linear finite-dimensional multirate system is presented. The parametrization is an extension of the Youla parametrization of all stabilizing controllers. State-space formulas for the parametrization are given. Periodic discrete-time and the use of periodic controllers which operate at different sampling periods that the plant sampling interval have also received much attention in the last years. In particular, the block decoupling control of a linear  $\omega$ -periodic discrete state-feedback system is considered in [112] by invoking the notion of the maximal reachability subspace extended from the case of non-periodic systems. A necessary and sufficient solvability condition for block decoupling is represented in terms of such a subspace.

Other recent studies concerned with multirate issues are: (a) The study of basic properties of reachability and observability in hybrid systems, namely, those which involve both continuous and discrete (in general coupled) devices, [113]. A running closed-loop sampling period, which can be different from those used for the various information channels, can be used to define the suitable dynamics. Also, an effort has been given in obtaining external models for nonuniform sampling systems based on geometric approaches which are alternative to those discussed in Section 5 (see, for instance, [114–115], while the joint reachability and observability of nonuniformly sampled-data systems

has been revisited under new mathematical conditions in [116]. Finally, an extension of the sampling criteria of constant difference of amplitudes, briefly commented in Section 2.1 given in [9], [42], [96] and [98] was recently used in [117] successfully for self-tuning of PID's based on the limit cycles generated by the essentially non linear nature of the criterion.

**7. Conclusions.** This paper has brought into focus an overview of different theoretical and applied results about the modelling and use of non-periodic sampling in Control and System Theory. In particular, the problem of signal adaptation by adaptive sampling and their projection in the improvement of the transients in adaptive control have been dealt with. Another reviewed problem has been the compensation of local plant variations with appropriate sampling period variations while maintaining the nominal digital controllers. This allows a cheaper design compared to the whole redesign of the controller. Also, the problem of minimizing the transmission of errors in problem such as controllability, observability or identifiability and its practical applications has been overviewed.

**Acknowledgments.** The author is very grateful to Professor J.M. Tarela by his useful suggestions about PFM controllers. He is also grateful to the support by DGIC'YT (Project PB93-0005) and to the reviewers for their interesting observations to the first version of this manuscript.

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Received March 1996

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## NEPERIODINIO IR ADAPTYVAUS DISKRETIZAVIMO METODŲ APŽVALGA

Manuel de la SEN

Straipsnyje nagrinėjami neperiodinio ir adaptyvaus diskretizavimo metodų pritaikymai įvairiose sistemų teorijos ir valdymo uždaviniuose. Apžvelgiamos įvairios paskutiniųjų trisdešimties metų adaptyvaus signalų diskretizavimo schemas, kurios pagerina sistemų pereinamųjų procesų charakteristikas. Parodyta, kad neperiodinis signalų diskretizavimas gali būti naudojamas kompensavimo uždaviniuose, kaip alternatyva gerai žinomiems dažnuminiams metodams. Be to, neperiodinis diskretizavimas pagerina dinaminių sistemų apvalinimo paklaidas.