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INVERSION OF LINEAR PERIODICALLY TIME–VARYING DIGITAL FILTERS

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Abstract. This paper discusses the inversion of linear periodically time–varying (LPTV) digital filters using the idea of converting the LPTV filter to the block time–invariant filter. Explicit expressions are given to determine the inversion of LPTV filters. Controllability, observability and stability of the inversion of LPTV filters are discussed.

Key words: digital filters, block model, inversion, controllability, observability.

Introduction. Most of the analysis methods for a discrete–time signal processing assume that the filter is time invariant or close enough to neglect the effects. In fact, there is a large number of important applications, where the filter is time–varying. LPTV filters form an important class of time–variant filters. Many mechanical and chemical processes exhibit a periodical behavior. This has motivated the development of the methods for the analysis of LPTV filters (Acha, 1989; Al-Rachmani and Franklin, 1989; Barnes and Shinnaka, 1980; Bolzern, Colaneri and Scattolini, 1986; Critchley and Rayner, 1988; Gnanasekaran, 1988; Nikias, 1985; Vaidyanathan and Mitra, 1988).

LPTV filters are difficult to analyze like the general time-varying case. Most of the analysis of LPTV filters has been based on the idea of converting a LPTV filter to a block time-invariant filter (Friedland, 1961; Jury and Mullin, 1958; Meyer and Burrus,1976). One type of LPTV filters is commonly referred to as multirate digital filters. Efficient realizations of multirate digital filters have been studied by Crochiere and Rabiner (1983). Miyawaki and Barnes (1983) developed block-state time–invariant structures. Meyer and Burrus (1975) used the idea of block processing to analyze multirate and LPTV filters. Liu and Franaszek (1969) considered time–varying filters as mapping filters.

In this paper the idea of converting a LPTV filter to a block time–invariant filter is used for the determination of the inversion of LPTV filters. Controllability, observability and inversion of LPTV filters are analyzed.

Block model of LPTV filters. A linear N–th order time–varying filter can be described by a set of state equations

$$x(k+1) = A(k)x(k) + B(k)v(k), \tag{1a}$$

$$y(k) = C^T(k)x(k) + d(k)v(k), \; k = 0, 1, 2, \ldots, \tag{1b}$$

where $x(k)$ is of dimension $N \times 1$, $A(k)$ is of dimension $N \times N$, $B(k)$ and $C(k)$ are of dimension $N \times 1$, $d(k)$, $v(k)$ and $y(k)$ are scalars.

The filter has single–input sequence $\{v(0), v(1), \ldots, v(k), \ldots\}$, single–output sequence $\{y(0), y(1), \ldots, y(k), \ldots\}$ and zero initial conditions.

Let $k = mL + n$, where L denotes the size of the block, m is a variable of the block and n is a variable inside the block. From (1) we have

$$\begin{aligned} x(mL+n+1) = A(mL+n)x(mL+n)+ \\ + B(mL+n)v(mL+n), \; m = 0, 1, 2, \ldots, \end{aligned} \tag{2a}$$

$$y(mL+n) = C^T(mL+n)x(mL+n) + \\ + d(mL+n)v(mL+n),\ n = 0,1,\dots,L-1. \tag{2b}$$

LPTV filter in the state space is described by (1) and is a partial case of the time-varying filter. For LPTV filter $A(mL+n) = A(n)$, $B(mL+n) = B(n)$, $C^T(mL+n) = = C^T(n)$, $d(mL+n) = d(n)$, where L denotes the period of the coefficients variation. Therefore

$$x(mL+n+1) = A(n)x(mL+n) + \\ + B(n)v(mL+n),\ m = 0,1,2,\dots, \tag{3a}$$

$$y(mL+n) = C^T(n)x(mL+n) + \\ + d(n)v(mL+n),\ n = 0,1,\dots,L-1. \tag{3b}$$

In this paper we shall refer to (3) as the canonical form of LPTV filters.

From (3), it can be easily obtained

$$x(mL+L) = A(L-1)\cdots A(0)x(mL) + A(L)\cdots \\ \cdots A(1)B(0)v(mL) + \ldots + B(L-1)v(mL+L-1). \tag{4}$$

Define $x(mL+L) = \nu(m+1), x(mL) = \nu(m)$,

$$V(m) = \Big[v(mL),\dots,v(mL+n),\dots,v(mL+L-1)\Big]^T,$$

$$Y(m) = \Big[y(mL),\dots,y(mL+n),\dots,y(mL+L-1)\Big]^T, \\ m = 0,1,2,\dots,$$

where $V(m)$ is the input block sequence and $Y(m)$ is the output block sequence.

Hence from (4) we get the first block equation of LPTV filter in the state space

$$\nu(m+1) = F\nu(m) + GV(m),\ m = 0,1,2,\dots, \tag{5a}$$

where matrix F is of dimension $N \times N$: $F = A(L-1) A(1) \cdots A(0)$, matrix G is of dimension $N \times L$: $G = [G_1, \ldots, G_j, \ldots, G_L]$, where $G_j = A(L-1)A(L-2)\cdots A(j)B(j-1)$, $j = 1, 2, \ldots, L-1$, $G_L = B(L-1)$.

We find the second block equation of LPTV filter in the state space from (3). A substitution of the values $x(mL+1), \ldots, x(mL+L-1)$ from (3a) to (3b) leads to the expression

$$Y(m) = H\nu(m) + RV(m), \; m = 0, 1, 2, \ldots, \qquad (5b)$$

where matrix H is of dimension $L \times N$: $H = [H_1, \ldots, H_j, \ldots, H_L]^T$, $H_j = C^T(j-1)A(j-2)\cdots A(0)$, $j = 1, 2, \ldots, L$. Matrix R is of dimension $L \times L$: $R = \{r_{ij}\}$, where $r_{ij} = 0$, if $i < j; r_{ij} = d(i-1)$, if $i = j; r_{ij} = C^T(i-1)B(j-1)$, if $i = j+1$; $r_{ij} = C^T(i-1)A(i-2)\cdots A(j)B(j-1)$, if $i > j+1$.

The matrices in (5) are with the constant elements. Hence for the analysis of LPTV filter we can use ordinary methods and find the inversion of LPTV filter and the conditions of controllability and observability.

Corollary 1: If $L = 1$, for (5) we have $F = A$, $G = B$, $H = C^T$, $R = d$. The class of filters with constant parameters is a subclass of LPTV filters.

Corollary 2: If $A(k) = A$, $B(k) = B$, $C^T(k) = C^T$, $d(k) = d$ and $L > 1$, we have block-state model (5) of filters with constant parameters, where $F = A^L$, $G = [A^{L-1}B, \ldots, AB, B]$, $H = [C^T, C^TA, \ldots, C^TA^{L-1}]^T$, $R = \{r_{ij}\}$, where $r_{ij} = 0$, if $i < j$; $r_{ij} = d$, if $i = j$; $r_{ij} = C^TB$, if $i = j+1$; $r_{ij} = C^TA^{i-j-1}B$, if $i > j+1$.

Definition and design of inversion of LPTV filters. Consider another filter which is described by the equations (Jaksoo, 1980)

$$\nu(m+1) = \bar{F}\nu(m) + \bar{G}V(m), \qquad (6a)$$

$$Y(m) = \bar{H}\nu(m) + \bar{R}V(m),\ m = 0, 1, 2, \dots, \qquad (6b)$$

where $\bar{F}$, $\bar{G}$, $\bar{H}$, $\bar{R}$ are matrices of dimension $N \times N$, $N \times L$, $L \times N$ and $L \times L$, respectively.

The transfer function of filter (6) is described by equation (7)

$$\bar{K}(z) = \bar{H}(z^L I - \bar{F})^{-1}\bar{G} + \bar{R}. \qquad (7)$$

The transfer function of LPTV filter (5) is described by equation (8)

$$K(z) = H(z^L I - F)^{-1}G + R. \qquad (8)$$

Filter (6) is the inversion of LPTV filter (5), if $\bar{K}(z) = [K(z)]^{-1}$.

From (5) we obtain

$$\nu(m+1) - GV(m) = F\nu(m),$$

$$RV(m) = H\nu(m) - Y(m).$$

Hence it follows

$$\begin{pmatrix} I & -G \\ 0 & R \end{pmatrix} \begin{pmatrix} \nu(m+1) \\ V(m) \end{pmatrix} = \begin{pmatrix} F\nu(m) & \\ H\nu(m) & -Y(m) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \nu(m+1) \\ V(m) \end{pmatrix} = \begin{pmatrix} I & -G \\ 0 & R \end{pmatrix}^{-1} \begin{pmatrix} F\nu(m) & \\ H\nu(m) & -Y(m) \end{pmatrix}. \qquad (9)$$

The solution of (9) exists and is unique respectively of the pair vectors $\nu(m+1), V(m)$ iff the matrix $\begin{pmatrix} I & -G \\ 0 & R \end{pmatrix}$ is nonsingular. As shown in the book (Graybill, 1969)

$$\begin{pmatrix} I & -G \\ 0 & R \end{pmatrix}^{-1} = \begin{pmatrix} I & GR^{-1} \\ 0 & R^{-1} \end{pmatrix}.$$

Hence the inversion of LPTV filter exists iff the matrix R is nonsingular. From (9) we obtain

$$\begin{aligned} \bar{F} &= F + GR^{-1}H, \quad \bar{G} = -GR^{-1}, \\ \bar{H} &= R^{-1}H, \quad \bar{R} = -R^{-1}. \end{aligned} \tag{10}$$

Controllability and observability of inversion of LPTV filters. The pair $(\bar{F}, \bar{G})$ is said to be controllable iff $\operatorname{rank}(\,\lambda_i I - \bar{F} \quad \bar{G}\,) = N$ for all eigenvalues λ_i of the matrix $\bar{F}$, where I is the $N \times N$ identity matrix (Zadeh and Desoer, 1970).

With respect to (10) we have the condition of controllability of the inversion of LPTV filter (6)

$$\operatorname{rank}(\,\lambda_i I - \bar{F} \quad \bar{G}\,) = \operatorname{rank}(\,\lambda_i I - F - GR^{-1}H \quad -GR^{-1}\,) =$$

$$= \operatorname{rank}(\,\lambda_i I - F \quad G\,)\begin{pmatrix} I & 0 \\ -R^{-1}H & -R^{-1} \end{pmatrix}.$$

If the matrix $\begin{pmatrix} I & 0 \\ -R^{-1}H & -R^{-1} \end{pmatrix}$ is nonsingular, then

$$\operatorname{rank}(\,\lambda_i I - \bar{F} \quad \bar{G}\,) = \operatorname{rank}(\,\lambda_i I - F \quad G\,)$$

and the pair $(\bar{F}, \bar{G})$ of the inversion of LPTV filter (6) is controllable iff the pair (F, G) of LPTV filter (5) is controllable, i.e., when $\operatorname{rank}(\,\lambda_i I - F \quad G\,) = N$ for all λ_i.

The pair $(\bar{F}, \bar{H})$ is said to be observable iff

$$\operatorname{rank}\begin{pmatrix} \lambda_i I - \bar{F} \\ \bar{H} \end{pmatrix} = N$$

for all eigenvalues λ_i of the matrix $\bar{F}$. With consideration to (10), we have

$$\begin{pmatrix} \lambda_i I - \bar{F} \\ \bar{H} \end{pmatrix} = \begin{pmatrix} I & -GR^{-1} \\ 0 & R^{-1} \end{pmatrix}\begin{pmatrix} \lambda_i I - F \\ H \end{pmatrix}.$$

If the matrix $\begin{pmatrix} I & -GR^{-1} \\ 0 & R^{-1} \end{pmatrix}$ is nonsingular, then

$$\operatorname{rank}\begin{pmatrix} \lambda_i I - \bar{F} \\ \bar{H} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \lambda_i I - F \\ H \end{pmatrix}.$$

The inversion of LPTV filter is observable iff the pair (F, H) of LPTV filter (5) is observable, i.e., when

$$\operatorname{rank}\begin{pmatrix} \lambda_i I - F \\ H \end{pmatrix} = N$$

for all λ_i.

Example. Consider the LPTV filter with the matrices

$$A(0) = \begin{pmatrix} 0 & 1 \\ 0.1 & 0.5 \end{pmatrix}, A(1) = \begin{pmatrix} 0.4 & 0 \\ 0.1 & 2 \end{pmatrix}, A(2) = \begin{pmatrix} 0.5 & 1 \\ 0.4 & 0 \end{pmatrix},$$

$$B^T(0) = (\,-1 \quad 0\,), B^T(1) = (\,0 \quad 2\,), B^T(2) = (\,2 \quad 1\,),$$

$$C^T(0) = (\,3 \quad 2\,), C^T(1) = (\,0.1 \quad 0\,), C^T(2) = (\,0 \quad 1\,),$$

$$d(0) = -2, \quad d(1) = 1, \quad d(2) = 2.$$

The matrices of block LPTV filter are

$$F = A(2)A(1)A(0) = \begin{pmatrix} 0.2 & 1.3 \\ 0 & 0.16 \end{pmatrix},$$

$$G = \begin{pmatrix} G_1 & G_2 & G_3 \end{pmatrix} = \begin{pmatrix} -0.3 & 2 & 2 \\ -0.16 & 0 & 1 \end{pmatrix},$$

where

$$G_1 = A(2)A(1)B(0) = \begin{pmatrix} -0.3 \\ -0.16 \end{pmatrix}, G_2 = A(2)B(1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

$$G_3 = B(2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$H = (\, H_1 \quad H_2 \quad H_3 \,)^T = \begin{pmatrix} 3 & 2 \\ 0 & 0.1 \\ 0.2 & 1.1 \end{pmatrix},$$

where

$$H_1 = C^T(0) = (\, 3 \quad 2 \,), H_2 = C^T(1)A(0) = (\, 0 \quad 0.1 \,),$$

$$H_3 = C^T(2)A(1)A(0) = (\, 0.2 \quad 1.1 \,),$$

$$R = \begin{pmatrix} d(0) & 0 & 0 \\ C^T(1)B(0) & d(1) & 0 \\ C^T(2)A(1)B(0) & C^T(2)B(1) & d(2) \end{pmatrix} =$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ -0.1 & 1 & 0 \\ -0.1 & 2 & 2 \end{pmatrix}.$$

An LPTV filter is stable iff all the eigenvalues of F have a modulus less than 1. The eigenvalues of the matrix F are equal to 0.2 and 0.16. Hence the LPTV filter is stable.

The condition of controllability of the pair (F, G) is

$$\operatorname{rank}(\, 0.2I - F \quad G \,) = \operatorname{rank} \begin{pmatrix} -1.3 & -0.3 & 2 & 2 \\ 0.04 & -0.16 & 0 & 1 \end{pmatrix} = 2,$$

$$\operatorname{rank}(\, 0.16I - F \quad G \,) =$$

$$= \operatorname{rank} \begin{pmatrix} -0.04 & -1.3 & -0.3 & 2 & 2 \\ 0 & 0 & -0.16 & 0 & 1 \end{pmatrix} = 2.$$

Hence the LPTV filter is controllable.

The condition of observability of the pair (F, G) is

$$\operatorname{rank}\begin{pmatrix} 0.2I - F \\ H \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & -1.3 \\ 0 & -0.04 \\ 3 & 2 \\ 0 & 0.1 \\ 0.2 & 1.1 \end{pmatrix} = 2,$$

$$\operatorname{rank}\begin{pmatrix} 0.16I - F \\ H \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -0.04 & -1.3 \\ 3 & 2 \\ 0 & 0.1 \\ 0.2 & 1.1 \end{pmatrix} = 2.$$

Hence the LPTV filter is observable.

The matrices of the inversion of the LPTV filter are

$$\bar{F} = F + GR^{-1}H = \begin{pmatrix} 0.7 & 2.6 \\ 0.415 & 0.82 \end{pmatrix},$$

$$\bar{G} = -GR^{-1} = \begin{pmatrix} -0.1 & 0 & -1 \\ -0.105 & 1 & -0.5 \end{pmatrix},$$

$$\bar{H} = R^{-1}H = \begin{pmatrix} -1.5 & -1 \\ -0.15 & 0 \\ 0.175 & 0.5 \end{pmatrix},$$

$$\bar{R} = -R^{-1} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.05 & -1 & 0 \\ -0.025 & 1 & -0.5 \end{pmatrix}.$$

The eigenvalues of the matrix $\bar{F}$ are $\lambda_1 = 0.776$ and $\lambda_2 = -0.624$. Since the eigenvalues of the matrix $\bar{F}$ are inside the unit circle, the inversion of the LPTV filter is stable.

Now consider the conditions of controllability and observability of the inversion of the LPTV filter.. From the condition of controllability of the pair $(\bar{F}, \bar{G})$

$$\operatorname{rank}\begin{pmatrix} 0.776I - \bar{F} & \bar{G} \end{pmatrix} =$$

$$= \operatorname{rank}\begin{pmatrix} 0.076 & -2.6 & -0.1 & 0 & -1 \\ -0.415 & -0.044 & -0.105 & 1 & -0.5 \end{pmatrix} = 2,$$

$$\operatorname{rank}\begin{pmatrix} -0.624I - \bar{F} & \bar{G} \end{pmatrix} =$$

$$= \operatorname{rank}\begin{pmatrix} -1.324 & -2.6 & -0.1 & 0 & -1 \\ -0.415 & -1.444 & -0.105 & 1 & -0.5 \end{pmatrix} = 2$$

we obtain that the inversion of the LPTV filter is controllable. It was expected, since the pair (F, G) is controllable.

From the condition of observability of the pair $(\bar{F}, \bar{H})$

$$\operatorname{rank}\begin{pmatrix} 0.776I - \bar{F} \\ \bar{H} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0.076 & -2.6 \\ -0.415 & -0.044 \\ -1.5 & -1 \\ -0.15 & 0 \\ 0.175 & 0.5 \end{pmatrix} = 2,$$

$$\operatorname{rank}\begin{pmatrix} -0.624I - \bar{F} \\ \bar{H} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -1.324 & -2.6 \\ -0.415 & -1.444 \\ -1.5 & -1 \\ -0.15 & 0 \\ 0.175 & 0.5 \end{pmatrix} = 2$$

we obtain that the inversion of the LPTV filter is observable. It was expected, since the pair (F, H) is observable.

Conclusions. An approach to the design of the inversion of LPTV digital filters has been introduced. It is based on the concept of converting a LPTV filter to a block time–invariant filter. The conditions of controllability, observability and stability of the inversion of LPTV filters are given. It is believed that this approach can be usefully applied to the analysis of LPTV filters. A further research is required to extend the results of this paper to more general filters such as multivariable ones.

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