

MATLAB-BASED SOFTWARE FOR ESTIMATION OF QUASIPOLYNOMIALS IN NOISE

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Abstract. The goal of this work is to describe the underlying theoretical and algorithmic basis of a MATLAB-based software developed by the authors. The software is intended for investigation of time series (signals) which can be modeled as the sum of real-valued quasipolynomials plus white noise. With the help of the software described, one can compute the expressions of the Cramér–Rao lower bound on the covariance matrix of the estimation error of unbiased estimates of damping factors and frequencies of quasipolynomials and to obtain estimates of these parameters using three versions of Prony method. Using this software, one can generate various models of quasipolynomials, obtain plots of their poles with respect to the unit circle, compute and plot 2σ -bounds (where σ is given by the CRB formula) around each pole, and also pole estimates obtained in each realization. Results of numerical experiments are presented.

Key words: MATLAB, superimposed signals, quasipolynomials, parametric estimation, Cramér–Rao bound.

1. Introduction. Models of noisy superimposed signals are very popular in the signal processing literature (see, for example, Kumaresan *et al.*, 1986; Bresler and Macovski, 1986; Parthasarathy and Tufts, 1985; Stoica *et al.*, 1989; Stoica and Nehorai, 1989; Yau and Bresler, 1992; Steedly and Moses, 1993). One of the simplest model is the sum of sinusoids in noise. Usually the noise is considered to be white Gaussian with zero mean and a finite variance. The main parameters to be estimated are the frequencies. An example of the model of two sinusoids in noise is presented in Fig. 1. A more complicated model is obtained when the damping is introduced. Then the main parameters to be estimated are the frequencies and damping factors. An example of the model of two damped sinusoids in noise is presented in Fig. 2. With each model of undamped/damped sinusoids, one can associate a linear stationary dynamic system whose impulse response can be described by that model. The poles of the system are called the

poles of the signal model (or simply of the signal). It is clear that the dynamic systems associated with the models of this kind have poles which are simple. In practice, however, one can encounter signals with multiple poles. Such signals are considered in the antenna data processing literature (Van Blaricum and Mitra, 1982). As it is pointed out in (Van Blaricum and Mitra, 1982), Tesche (1973) “has shown that a dipole can be resistively loaded in such a way as to make it critically damped; that is, to have a double pole on the negative real axis. Multiple poles also result in the transient response of a system if the system is driven by a signal which itself has a multiple pole. The most common example of this would be a system excited by a ramp waveform. The ramp waveform has a double pole located at the origin”. The impulse responses of linear stationary systems are, in general, signals that can have multiple poles. Such signals appear also in some biological systems.

So-called quasipolynomials are generalization of damped sinusoids for the case of multiple poles. The simplest example of a quasipolynomial is a signal of the form

$$A \exp(\lambda t) t^{K-1} \sin(\omega t + \varphi), \quad t = 0, 1, 2, \dots \quad (1.1)$$

It has a pole $z = \exp(\lambda + j\omega)$ the multiplicity of which is equal to K . An example of a quasipolynomial with $K = 2$ is shown in Fig. 3. We can take several signals of the form (1.1) with the same damping factors and the same frequencies. Their sum will make a new signal of the following form:

$$e^{\lambda t} (A_1 \sin(\omega t + \varphi_1) + A_2 t \sin(\omega t + \varphi_2) + \dots + A_K t^{K-1} \sin(\omega t + \varphi_K)). \quad (1.2)$$

This is the general form of a quasipolynomial of degree $K - 1$. Thus we can define a quasipolynomial as a polynomial with time-varying coefficients multiplied by an exponential function with a real exponent. So, in general case, a quasipolynomial is determined by the following parameters: the degree $K - 1$, the frequency ω , the damping factor λ , the amplitudes A_1, \dots, A_K , and the phases $\varphi_1, \dots, \varphi_K$. The main parameters of a quasipolynomial model are frequencies and damping factors. They enter the model in a nonlinear way. The remaining parameters (amplitudes and phases) enter the model in a linear way. The polynomial degree $K - 1$ is usually assumed to be known.

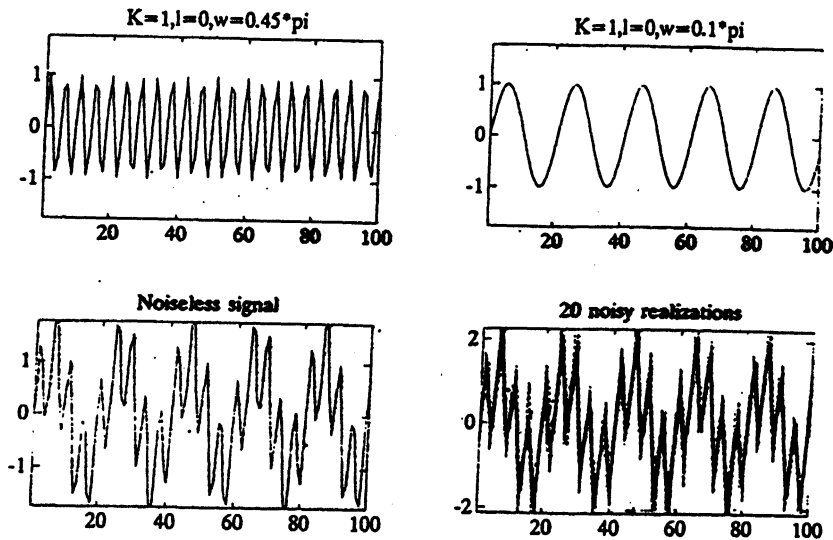


Fig. 1. Two undamped sinusoids and their sum (noiseless and noisy versions). $K = [1, 1]$, $A = [1, 1]$, $\varphi = [0, 0]$, $\lambda = [0, 0]$, $\omega = [0.45\pi, 0.1\pi]$, SNR = 10 dB, $N = 100$.

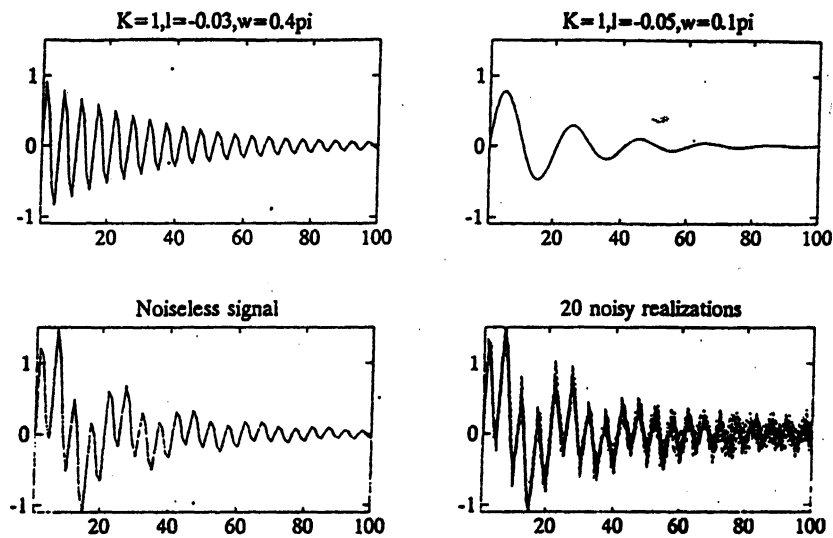


Fig. 2. Two damped sinusoids and their sum (noiseless and noisy versions). $K = [1, 1]$, $A = [1, 1]$, $\varphi = [0, 0]$, $\lambda = [-0.03, -0.05]$, $\omega = [0.4\pi, 0.1\pi]$, SNR = 10 dB, $N = 100$.

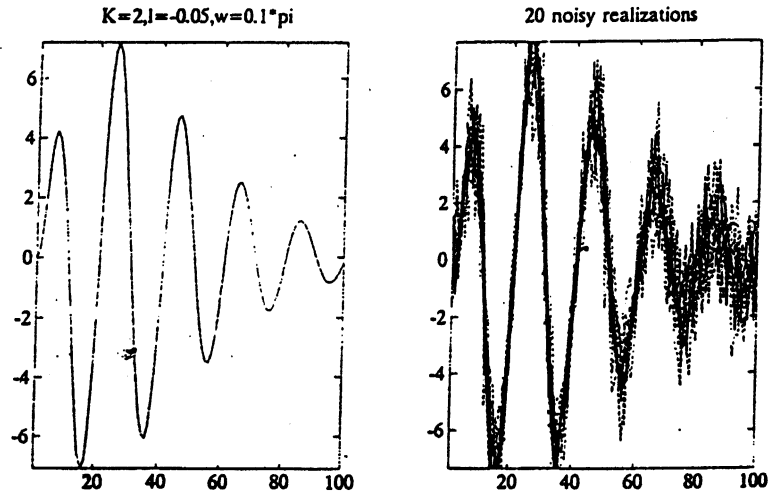


Fig. 3. A quasipolynomial of degree one (noiseless and noisy versions).

$K = 2$, $A = 1$, $\varphi = 0$, $\lambda = -0.05$, $\omega = 0.1\pi$, SNR = 10 dB, $N = 100$.

The main problem is to estimate the quasipolynomial parameters from noisy data, to obtain lower estimation accuracy bounds and to investigate statistical performance and asymptotic properties of parameter estimates.

With the help of the MATLAB software described, one can compute the Cramér–Rao lower bound on the covariance matrix of the estimation error of unbiased estimates of damping factors and frequencies and to obtain estimates of these parameters using three versions of Prony method: 1) Prony method, 2) Prony method with prefiltering, 3) Prony method with prefiltering and optimization. In particular case when the degree of a quasipolynomial is equal to 0, the model of quasipolynomials in noise becomes the model of damped sinusoids in noise. For such a model, an asymptotic Cramér–Rao bound formula has been derived by Wigren and Nehorai (1991). This formula can also be computed with the help of the software considered. We present some demo files. With the help of this software, one can generate various models of quasipolynomials, obtain plots of their poles with respect to the unit circle, compute and plot 2σ -bounds (where σ is given by the CRB formula) around each pole, show on the same plot not only poles and their estimation bounds but also pole estimates obtained for each noise realization. For simpler models when there are only one

or two quasipolynomials, one can compute the root-mean-square errors (RMSEs) of estimation of damping factors and frequencies and compare them with the corresponding values of the Cramér–Rao bound for various signal-to-noise ratios (SNRs). Also, the software allows one to compute the RMSEs of damping factors and frequencies for various SNRs and various frequency separations $\Delta\omega$. The RMSEs so obtained can be plotted in a three-dimensional space.

The paper is organized as follows. In Section 2, the signal model is presented. In Section 3, we describe estimation methods used in the software. Section 3.1 deals with Prony method. Section 3.2 is based on Prony method with prefiltering (Kumaresan and Feng, 1991). The main ideas of Prony method with prefiltering are briefly described. The estimation error of the model is discussed in Section 3.3. The iterative pole optimization procedure which is used in the third estimation method is explained in Section 3.4. The CRB formulas associated with the estimation problem considered are analyzed in Section 4. A short description of the main software units is given in Section 5. In Section 6, some numerical examples are given.

We use the following notation:

A^T – the transpose of matrix A ;

A^\dagger – the pseudoinverse of an $n \times m$ matrix A

$$A^\dagger = \begin{cases} (A^T A)^{-1} A^T & \text{if } \text{rank } A = m, \\ A^T (A A^T)^{-1} & \text{if } \text{rank } A = n; \end{cases}$$

$\mathbf{I}_{n \times n}$ – the $n \times n$ identity matrix;

$\text{blockdiag}(A_1, \dots, A_n)$ – the blockdiagonal matrix with A_1, \dots, A_n on the diagonal;

$$j = \sqrt{-1};$$

$a_n * b_n$ – the convolution operation of two sequences $\{a_n\}$, $\{b_n\}$ ($n = \dots, -1, 0, 1, \dots$): $a_n * b_n = \sum_{k=-\infty}^{\infty} a_{n-k} b_k$. Note that $a_n * b_n = b_n * a_n$;

δ_n – the Kronecker delta ($= 1$ if $n = 0$, $= 0$ otherwise).

2. Signal model. Consider the following noise corrupted signal

$$v_n(q) = u_n(q) + e_n(q), \quad (n = 0, 1, \dots, N-1; q = 0, 1, \dots, Q), \quad (2.1)$$

where

$$u_n(q) = \sum_{l=1}^L h_{l,n}(q) \quad (2.2)$$

is the sum of real-valued quasipolynomials, i.e., signals of the form:

$$h_{l,n}(q) = e^{\lambda_l n} (a_{l1}(q) \sin(\omega_l n + \varphi_{l1}(q)) + a_{l2}(q) n \sin(\omega_l n + \varphi_{l2}(q)) \\ + \dots + a_{lK_l}(q) n^{K_l-1} \sin(\omega_l n + \varphi_{lK_l}(q))) \quad (2.3)$$

with the damping factors $\lambda_l < 0$, frequencies $\omega_l > 0$, amplitudes $a_{lk}(q) \in \mathbf{R}$ ($a_{lK_l}(q) \neq 0$), and phases $\varphi_{lk}(q) \in [-\pi, \pi)$, $k = 1, \dots, K_l$ (we assume the amplitudes and phases to be nonrandom); $e_n(q)$ stands for additive real-valued white Gaussian noise with zero mean and variance σ^2 satisfying the following condition:

$$E(e_n(q)e_m(s)) = \sigma^2 \delta_{nm} \delta_{qs}. \quad (2.4)$$

We assume also that $N \geq K_1 + \dots + K_L$, and that all the parameter pairs (λ_l, ω_l) are different. In a particular case when all $K_l = 1$, our model becomes the sum of real damped sinusoids in noise. The variable q in (2.1)–(2.3) stands for the variable of an experiment. In general case, the model (2.1)–(2.3) means that we have Q experiments and that in each of them we observe N noisy equally-spaced samples of the sum of L quasipolynomials. If $Q = 1$, we obtain the case of a single experiment, thus the variable q may be omitted.

Equations (2.1)–(2.3) can be written in a vector form:

$$\mathbf{v}(q) = \mathbf{u}(q) + \mathbf{e}(q), \quad (2.5) \\ \mathbf{u}(q) = \mathbf{h}_1(q) + \dots + \mathbf{h}_L(q),$$

where

$$\mathbf{v}(q) = [v_0(q), v_1(q), \dots, v_{N-1}(q)]^T, \quad (2.6) \\ \mathbf{h}_i(q) = [h_{i,0}(q), h_{i,1}(q), \dots, h_{i,N-1}(q)]^T, \quad i = 1, \dots, L.$$

The vectors $\mathbf{u}(q)$ and $\mathbf{e}(q)$ are defined analogously.

Define the vector \mathbf{d}_{ii}^c associated with (λ_l, ω_l) and with the power n^{i-1} in the following way:

$$\mathbf{d}_{ii}^c = [\delta_{i1}, e^{\lambda_l} \cos(\omega_l), 2^{i-1} e^{2\lambda_l} \cos(2\omega_l), \dots, \\ (N-1)^{i-1} e^{(N-1)\lambda_l} \cos((N-1)\omega_l)]^T, \quad (2.7)$$

for $i = 1, 2, \dots, K_l$. Introduce also the vector \mathbf{d}_{ii}^s which is defined analogously as \mathbf{d}_{ii}^c with the only difference that “cos” is replaced by “sin” and δ_{i1} by 0. Denote by \mathbf{D} a matrix whose columns are the vectors $\{\mathbf{d}_{ii}^c, \mathbf{d}_{ii}^s\}$, i.e.,

$$\mathbf{D} = [\mathbf{d}_{11}^c, \mathbf{d}_{11}^s, \dots, \mathbf{d}_{1K_1}^c, \mathbf{d}_{1K_1}^s, \dots, \mathbf{d}_{L1}^c, \mathbf{d}_{L1}^s, \dots, \mathbf{d}_{LK_L}^c, \mathbf{d}_{LK_L}^s]. \quad (2.8)$$

We call \mathbf{D} the matrix of basis signals.

Using the notations introduced, the noiseless observation $\mathbf{u}(q)$ may then be written as

$$\mathbf{u}(q) = \mathbf{D}\mathbf{a}(q), \quad (2.9)$$

where

$$\begin{aligned} \mathbf{a}(q) = [& a_{11}(q) \sin(\varphi_{11}), a_{11}(q) \cos(\varphi_{11}), \dots, \\ & a_{1K_1}(q) \sin(\varphi_{1K_1}), a_{1K_1}(q) \cos(\varphi_{1K_1}), \dots, \\ & a_{L1}(q) \sin(\varphi_{L1}), a_{L1}(q) \cos(\varphi_{L1}), \dots, \\ & a_{LK_L}(q) \sin(\varphi_{LK_L}), a_{LK_L}(q) \cos(\varphi_{LK_L})]^T. \end{aligned} \quad (2.10)$$

It follows from (2.9) that $\mathbf{u}(q)$ belongs to the linear subspace of \mathbf{R}^N spanned by the columns of \mathbf{D} . This subspace is called the signal subspace and is denoted \mathcal{E}_D . The complimentary subspace is called the noise subspace and denoted \mathcal{E}_E . Denote by $\mathbf{\Pi}_D$ and $\mathbf{\Pi}_E$ the projectors on \mathcal{E}_D and \mathcal{E}_E . We have $\mathbf{\Pi}_D + \mathbf{\Pi}_E = \mathbf{I}_{N \times N}$, the identity matrix. The projector $\mathbf{\Pi}_D$ is a known function of $\{(\lambda_l, \omega_l)\}$ since

$$\mathbf{\Pi}_D = \mathbf{I}_{N \times N} - \mathbf{\Pi}_E = \mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T. \quad (2.11)$$

3. Estimation of damping factors and frequencies of quasipolynomials.

Consider the quasipolynomial model (2.1)–(2.3). Assume that $Q = 1$, i.e., that we have a single experiment case. We are interested in estimating the damping factors and frequencies of this model. For this purpose, we use three variants of Prony method: 1) least-squares (LS) Prony method, 2) LS Prony method with prefiltering, and 3) LS Prony method with prefiltering and optimization. For a quick reference, we present a short description of these methods.

3.1. Prony method. Prony method (Prony, 1795; Hildebrand, 1956; Kay and Marple, 1981) is a method which models data (equally spaced samples) by a linear combination of exponentials. This procedure fits an exponential curve

of p exponentials to N data points. Since each exponential term $a_k e^{s_k n}$ has two parameters: an amplitude a_k and an exponent s_k , Prony method requires $N \geq 2p$ data measurements. The original Prony method exactly fitted the curve consisting of p exponential terms to a data set of $N = 2p$ elements. When $N > 2p$, a least squares technique is used.

Suppose that we have measurement results v_0, v_1, \dots, v_{N-1} of some function $v(t)$ at equally spaced points

$$v_n = v(n\Delta t), \quad n = 0, 1, \dots, N-1. \quad (3.1)$$

We want to approximate the measured data $\{v_n\}$ by the following exponential model

$$x_n = \sum_{k=1}^p a_k e^{s_k n}, \quad (3.2)$$

where $a_k, s_k \in C$, and

$$\begin{aligned} a_k &= A_k \exp(j\varphi_k), \\ s_k &= (\lambda_k + j\omega_k)\Delta t. \end{aligned} \quad (3.3)$$

A natural way of finding estimates of $\{A_k, \varphi_k, \lambda_k, \omega_k\}$ and p is to minimize the error

$$\sum_{n=0}^{N-1} |v_n - x_n|^2. \quad (3.4)$$

However, this is a difficult nonlinear least-squares problem. In the literature, one can find some iterative procedures to solve (3.4). In these procedures, an initial estimate of the unknown parameters is successively improved. Prony method is often used to estimate the parameters of (3.2). Although this method gives a suboptimal solution which does not minimize (3.4), its results, however, are quite satisfactory in many cases.

Denote

$$z_k = e^{s_k} \quad (k = 1, \dots, p). \quad (3.5)$$

Then equation (3.2) can be written as

$$x_n = \sum_{k=1}^p a_k z_k^n. \quad (3.6)$$

In the best case when the model (3.6) exactly fits the measured data v_0, v_1, \dots, v_{N-1} , we would have

$$v_n = \sum_{k=1}^p a_k z_k^n \quad (n = 0, 1, \dots, N-1). \quad (3.7)$$

Rewrite this as a system of equations

$$\begin{aligned} a_1 + a_2 + \dots + a_p &= v_0, \\ a_1 z_1 + a_2 z_2 + \dots + a_p z_p &= v_1, \\ a_1 z_1^2 + a_2 z_2^2 + \dots + a_p z_p^2 &= v_2, \\ \vdots & \quad \quad \quad \ddots \quad \quad \quad \vdots \\ a_1 z_1^{N-1} + a_2 z_2^{N-1} + \dots + a_p z_p^{N-1} &= v_{N-1}. \end{aligned} \quad (3.8)$$

If z_1, z_2, \dots, z_p were known, then this set of equations would make a system of N linear equations in the p unknowns a_1, a_2, \dots, a_p . If $N = p$, then this system could be solved exactly, and if $N > p$, then it could be solved approximately in a least squares sense.

However, if $\{z_k\}$ are unknown, we need to have at least $2p$ equations. Also, $\{z_k\}$ enters (3.6) in a nonlinear way what creates additional difficulties. Minimization of these difficulties is the essence of Prony method.

Let z_1, z_2, \dots, z_p be the roots of the following algebraic equation

$$\begin{aligned} z^p - \beta_1 z^{p-1} - \beta_2 z^{p-2} - \dots - \beta_{p-1} z - \beta_p \\ = (z - z_1) \dots (z - z_p) = 0. \end{aligned} \quad (3.9)$$

We need to determine the coefficients β_1, \dots, β_p . For this purpose, multiply the first equation of (3.8) by β_p , the second by β_{p-1} , the p th by β_1 , and the $(p+1)$ th by (-1) , and add the results. Since z_1, z_2, \dots, z_p are the roots of equation (3.9), we get the following relationship

$$v_p - \beta_1 v_{p-1} - \dots - \beta_p v_0 = 0. \quad (3.10)$$

Repeating the same procedure starting from the second, third, \dots , $(N-p)$ th equation, we will obtain $N-p-1$ additional equations. Thus, using (3.8) and

(3.9), we get the following system of $N - p$ linear equations

$$\begin{aligned} v_{p-1}\beta_1 + v_{p-2}\beta_2 + \dots + v_0\beta_p &= v_p, \\ v_p\beta_1 + v_{p-1}\beta_2 + \dots + v_1\beta_p &= v_{p+1}, \\ \vdots & \\ v_{N-2}\beta_1 + v_{N-3}\beta_2 + \dots + v_{N-p-1}\beta_p &= v_{N-1}, \end{aligned} \quad (3.11)$$

or, in a shortened form,

$$\sum_{k=1}^p v_{n-k}\beta_k = v_n, \quad n = p, p+1, \dots, N-1. \quad (3.12)$$

We can rewrite equation (3.12) in a more compact form

$$\sum_{k=0}^p v_{n-k}\beta_k = 0, \quad n = p, p+1, \dots, N-1, \quad (3.13)$$

where $\beta_0 = -1$.

Since v_0, v_1, \dots, v_{N-1} are known, system of linear equations (3.11) can be solved exactly if $N = 2p$, or solved approximately, by a least-squares method, if $N > 2p$.

After the coefficients β_1, \dots, β_p have been found, the unknowns z_1, \dots, z_p can be computed as the roots of equation (3.8). They are real or complex. Then (3.8) becomes a system of N linear equations in p unknowns a_1, \dots, a_p . They can be determined by using the first p equations of system (3.8), or, preferably, by applying a least-squares method to the entire system (3.8).

The method described is called Prony method. One of its main features is that nonlinearity of equations (3.8) is transferred into determining of the roots of algebraic equation (3.9). Below we present the *summary of Prony method*:

- Step 1.* Solve equation (3.11) to find the unknown coefficients of the polynomial in (3.9).
- Step 2.* Root the polynomial in (3.9) to determine its roots z_1, \dots, z_p .
- Step 3.* Solve system (3.8) to find the complex amplitudes a_1, \dots, a_p .

3.2. Prony method with prefiltering. It is well-known that Prony method is very sensitive to errors in the measured data – the resulting estimates are highly

biased. In order to improve performance of this method, Kumaresan and Feng (1991) proposed to use prefiltering of data before applying it. The rationale of the prefiltering is to suppress some of the noise. They have suggested two methods for determining the impulse response $1, w_1, \dots, w_q$ of the FIR prefilter $W(z) = 1 + w_1 z^{-1} + \dots + w_q z^{-q}$ where $0 \leq q \leq N - 2p$. In the first method, the prefilter $W(z)$ is determined off-line before estimating the poles. This is a disadvantage of the method. In the second method, the prefilter is iteratively determined from the data itself. We use the second method. The impulse response of the prefilter $1, w_1, \dots, w_q$ is determined in the following way. First, it is assumed that $w_0 = 1$ and $w_n = 0$ for $n = 1, \dots, q$. Then note that the polynomial $X(z)$, the z-transform of x_n (3.2),

$$X(z) = \sum_{n=0}^{N-1} \sum_{k=1}^p a_k e^{s_k n} z^{-n} \quad (3.14)$$

can be represented as a ratio of two polynomials

$$X(z) = \frac{C(z)}{B(z)}, \quad (3.15)$$

where

$$B(z) = \prod_{k=1}^p (1 - e^{s_k} z^{-1}) = 1 + \sum_{k=1}^p b_k z^{-k} \quad (3.16)$$

is a p th degree polynomial with roots at e^{s_1}, \dots, e^{s_p} and $C(z)$ is a $N + p - 1$ st degree polynomial

$$C(z) = \sum_{k=0}^{N+p-1} c_k z^{-k}, \quad (3.17)$$

such that

$$c_p = c_{p+1} = \dots = c_{N-1} = 0. \quad (3.18)$$

Relationships (3.18) give the following $N - p$ Prony difference equations:

$$\sum_{k=0}^p x_{n-k} b_k = 0, \quad n = p, p+1, \dots, N-1 \quad (b_0 = 1). \quad (3.19)$$

They are exact only for noiseless complex exponential signals. Equation (3.19) can be rewritten as follows

$$B(z)x_n = 0, \quad n = p, p+1, \dots, N-1. \quad (3.20)$$

We see that the polynomial $B(z)$ tries to annul the signal. Therefore the zeros of $B(z)$ are close to the regions where the signal energy is concentrated, and $1/B(z)$ when calculated around the unit circle shows a tendency to have peaks at these locations. Thus, in the next iteration, $W(z)$ is chosen so that it approximates $1/B(z)$:

$$B(z)W(z) = 1 + E(z), \quad (3.21)$$

where $E(z)$ is the z-transform of an error sequence e_n . In time domain, (3.21) corresponds to

$$b_n * w_n = \delta_n + e_n. \quad (3.22)$$

The impulse response of the prefilter w_0, w_1, \dots, w_q is obtained by minimizing

$$\sum_{n=0}^{p+q} |e_n|^2 = \sum_{n=0}^{p+q} |\delta_n - b_n * w_n|^2. \quad (3.23)$$

Note that $w_0 = 1$. This LS problem can be solved by the Levinson–Durbin recursive algorithm (Durbin, 1959) which ensures the obtained prefilter has its zeros inside the unit circle.

Once the prefilter is computed, the data is filtered and the Prony equations for prefiltered data $\{x'_n\}$

$$\sum_{k=0}^p x'_{n-k} b_k = 0, \quad n = p+q, p+q+1, \dots, N-1 \quad (3.24)$$

are solved to obtain new coefficients b_1, \dots, b_p ($b_0 = 1$). The procedure is repeated until the coefficients $\{b_i\}$ do not change considerably from iteration to iteration. Then the roots of the polynomial $B(z)$ are calculated. Once e^{s_1}, \dots, e^{s_p} are found, the determination of a_1, \dots, a_p reduces to a simple least-squares fit.

Note that in order to get the multiplicities of the estimated poles we need a special procedure to select closely-spaced poles and to take them as multiple. As we have already seen, Prony method with prefiltering gives estimates of the poles as the roots of the polynomial $B(z)$. Usually, when rooting a polynomial with the help of a computer one seldom gets multiple roots. However, some of the roots can be rather close. One might expect to get better results when taking closely-spaced roots as a single multiple root. We use a procedure of

determination of multiple poles in our implementation of Prony method with prefiltering.

3.3. Signal model estimation error. It follows from equations (2.5) and (2.9) that the noisy observation $\mathbf{v} = [v_0, v_1, \dots, v_{N-1}]^T$ can be written in the following form:

$$\mathbf{v} = \mathbf{D}\mathbf{a} + \mathbf{e}, \quad (3.25)$$

where $\mathbf{e} = [e_0, e_1, \dots, e_{N-1}]^T$ is the noise vector (we omit the variable q in (3.25) since we consider a single-experiment case). Once we have estimated the poles of the model, we get also the estimate $\hat{\mathbf{D}}$ of the basis signal matrix \mathbf{D} . Then equation (3.25) (with \mathbf{D} replaced by $\hat{\mathbf{D}}$) can be used to determine the LS estimate $\hat{\mathbf{a}}$ of the $2(K_1 + \dots + K_L) \times 1$ vector \mathbf{a} . The solution is given by

$$\hat{\mathbf{a}} = (\hat{\mathbf{D}}^T \hat{\mathbf{D}})^{-1} \hat{\mathbf{D}}^T \mathbf{v}. \quad (3.26)$$

The $N \times 1$ estimation error vector \mathbf{err} is then obtained as

$$\mathbf{err} = \mathbf{v} - \hat{\mathbf{D}}\hat{\mathbf{a}}, \quad (3.27)$$

and the $N \times 1$ estimated signal model vector as

$$\hat{\mathbf{u}} = \mathbf{v} - \mathbf{err} = \hat{\mathbf{D}}\hat{\mathbf{a}} = \hat{\mathbf{D}}(\hat{\mathbf{D}}^T \hat{\mathbf{D}})^{-1} \hat{\mathbf{D}}^T \mathbf{v} = \hat{\mathbf{D}}\hat{\mathbf{D}}^\dagger \mathbf{v}. \quad (3.28)$$

The root-mean-square error (RMSE) of estimation of parameters of the model (3.25) is computed as

$$\text{RMSE} = \sqrt{(\mathbf{err}^T \cdot \mathbf{err})/N}. \quad (3.29)$$

3.4. Prony method with prefiltering and optimization. Consider again the model (2.1)–(2.3). The method under the title above is comprised of two stages. In the first stage, the initial estimates of the poles are obtained by Prony method with prefiltering. In the second stage, they are improved via an iterative optimization procedure.

The optimization procedure is as follows. Select a pole estimate obtained in the first stage and compute the estimation error in the three cases when the absolute value of the pole estimate is 1) increased by p percents, 2) decreased by p percents, and 3) not changed. The value of p can be taken arbitrarily

– for example, 10, 15, or 20. Then the three estimation errors obtained are computed, and the absolute value corresponding to the case with the smallest estimation error is selected. For the pole with the modified absolute value, the procedure is continued in the following way. The angle of the pole is 1) increased by p percents, 2) decreased by p percents, and 3) not changed. Again, the three errors obtained are compared, and the angle corresponding to the case with the smallest estimation error is selected. After the procedure has been repeated for each pole, the final estimation error is compared to the initial error (obtained when the poles were not changed). If the final error is smaller than the initial one, then the final error is taken to be the initial one, and the procedure is repeated again one more iteration. In case the final error has not become smaller than the initial one after the iteration, p is divided by 2, and the next iteration is started.

4. CRB of the poles. In order to evaluate the performance of our method, we compare it with the CRB. The CRB expression for estimation of frequencies and damping factors of the model of real quasipolynomials (2.1) – (2.3) is derived in (Slivinskas and Šimonytė, 1995) and is given by

$$\text{CRB}(\theta) = \sigma^2 (\mathbf{P}^T \mathbf{T}^T \mathbf{\Pi}_E \mathbf{T} \mathbf{P})^{-1}, \quad (4.1)$$

where

$$\theta = (\lambda_1, \omega_1, \dots, \lambda_L, \omega_L), \quad (4.2)$$

$$\mathbf{T} = [d_{1K_1}^c, -d_{1K_1}^s, d_{2K_2}^c, -d_{2K_2}^s, \dots, d_{LK_L}^c, -d_{LK_L}^s], \quad (4.3)$$

$$d_{mK_m}^c = \frac{d}{d\lambda_m} \mathbf{d}_{mK_m}^c, \quad d_{mK_m}^s = \frac{d}{d\lambda_m} \mathbf{d}_{mK_m}^s, \quad (4.4)$$

($\mathbf{d}_{mK_m}^c$ and $\mathbf{d}_{mK_m}^s$ are defined by (2.7)),

$$\mathbf{\Pi}_E = \mathbf{I}_{N \times N} - \mathbf{D}(\mathbf{D}^T \mathbf{D})^{-1} \mathbf{D}^T, \quad (4.5)$$

$$\mathbf{P} = \text{blockdiag}(\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_L), \quad (4.6)$$

$$\mathbf{P}_i = a_{iK_i} \begin{pmatrix} \sin \varphi_{iK_i} & \cos \varphi_{iK_i} \\ -\cos \varphi_{iK_i} & \sin \varphi_{iK_i} \end{pmatrix}. \quad (4.7)$$

Since we are more interested in estimating not the parameter θ but the poles $e^{\lambda_1 + j\omega_1}, \dots, e^{\lambda_L + j\omega_L}$, we will derive the CRB expression with respect to the parameter

$$\begin{aligned} \theta^* &= (\text{Re}(e^{\lambda_1 + j\omega_1}), \text{Im}(e^{\lambda_1 + j\omega_1}), \dots, \text{Re}(e^{\lambda_L + j\omega_L}), \text{Im}(e^{\lambda_L + j\omega_L})) \\ &= (e^{\lambda_1} \cos \omega_1, e^{\lambda_1} \sin \omega_1, \dots, e^{\lambda_L} \cos \omega_L, e^{\lambda_L} \sin \omega_L). \end{aligned} \quad (4.8)$$

We have the transformation

$$\boldsymbol{\theta}^* = \mathbf{g}(\boldsymbol{\theta}) = (g_1(\boldsymbol{\theta}), g_2(\boldsymbol{\theta}), \dots, g_{2L}(\boldsymbol{\theta})), \quad (4.9)$$

where

$$\mathbf{g} : \mathbf{R}^{2L} \rightarrow \mathbf{R}^{2L} \quad (4.10)$$

is defined as follows

$$\begin{cases} g_{2l-1}(\boldsymbol{\theta}) = e^{\lambda_l} \cos \omega_l, \\ g_{2l}(\boldsymbol{\theta}) = e^{\lambda_l} \sin \omega_l, \end{cases} \quad (l = 1, \dots, L). \quad (4.11)$$

The CRB with respect to the parameter $\boldsymbol{\theta}^*$ is then expressed in the following way

$$\text{CRB}(\boldsymbol{\theta}^*) = \mathbf{g}'^T \text{CRB}(\boldsymbol{\theta}) \mathbf{g}', \quad (4.12)$$

where

$$\mathbf{g}' = \begin{pmatrix} \frac{\partial g_1}{\partial \lambda_1} & \frac{\partial g_1}{\partial \omega_1} & \frac{\partial g_1}{\partial \lambda_2} & \frac{\partial g_1}{\partial \omega_2} & \cdots & \frac{\partial g_1}{\partial \lambda_L} & \frac{\partial g_1}{\partial \omega_L} \\ \frac{\partial g_2}{\partial \lambda_1} & \frac{\partial g_2}{\partial \omega_1} & \frac{\partial g_2}{\partial \lambda_2} & \frac{\partial g_2}{\partial \omega_2} & \cdots & \frac{\partial g_2}{\partial \lambda_L} & \frac{\partial g_2}{\partial \omega_L} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_{2L}}{\partial \lambda_1} & \frac{\partial g_{2L}}{\partial \omega_1} & \frac{\partial g_{2L}}{\partial \lambda_{2L}} & \frac{\partial g_{2L}}{\partial \omega_{2L}} & \cdots & \frac{\partial g_{2L}}{\partial \lambda_L} & \frac{\partial g_{2L}}{\partial \omega_L} \end{pmatrix} \\ = \text{blockdiag}(\mathbf{G}_1, \dots, \mathbf{G}_L). \quad (4.13)$$

Matrices $\{\mathbf{G}_l, l = 1, \dots, L\}$ are defined as follows

$$\mathbf{G}_l = \begin{pmatrix} e^{\lambda_l} \cos \omega_l & -e^{\lambda_l} \sin \omega_l \\ e^{\lambda_l} \sin \omega_l & e^{\lambda_l} \cos \omega_l \end{pmatrix}. \quad (4.14)$$

For each noise realization, we get the pole estimates $\{e^{\hat{\lambda}_l + j\hat{\omega}_l}, l = 1, \dots, L\}$ in the complex plane. These estimates make a “cloud” around each pole. It is of interest to know how widely scattered the “cloud” of each pole may be. For this purpose, we select a pole $z = e^{\lambda + j\omega}$ and decompose it into its real and imaginary parts $(\text{Re}z, \text{Im}z) = (e^\lambda \cos \omega, e^\lambda \sin \omega)$. We compute the CRB for $\text{Re}z, \text{Im}z$ according to (4.12)–(4.14) and plot the $2\sqrt{\text{CRB}(\text{Re}z)}$ and $2\sqrt{\text{CRB}(\text{Im}z)}$ in the complex plane. This procedure is repeated for each pole.

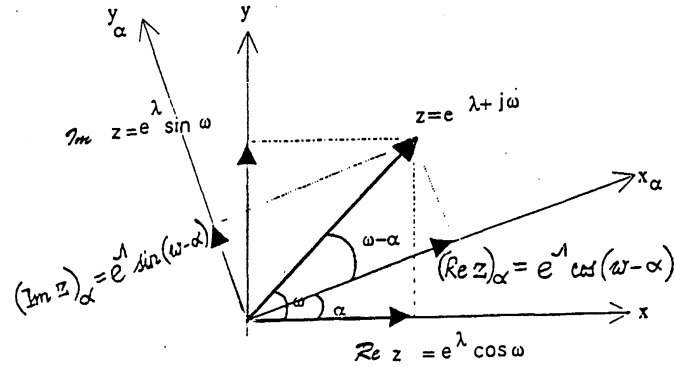


Fig. 4. Decomposition of a pole with respect to the old and new coordinate systems.

Then we turn the coordinate axes x, y around the origin at an angle α to obtain the new axes x_α, y_α and decompose again each pole into its real and imaginary parts with respect to the new coordinate system (see Fig. 4). Denote the new real and imaginary parts by $(\text{Re}z)_\alpha, (\text{Im}z)_\alpha$. One can easily see from Fig. 4 that

$$\begin{aligned} (\text{Re}z)_\alpha &= e^\lambda \cos(\omega - \alpha), \\ (\text{Im}z)_\alpha &= e^\lambda \sin(\omega - \alpha). \end{aligned} \quad (4.15)$$

It is not difficult to see that the $\text{CRB}(\theta_\alpha^*)$ where

$$\begin{aligned} \theta_\alpha^* &= (e^{\lambda_1} \cos(\omega_1 - \alpha), e^{\lambda_1} \sin(\omega_1 - \alpha), \dots, \\ &e^{\lambda_L} \cos(\omega_L - \alpha), e^{\lambda_L} \sin(\omega_L - \alpha)) \end{aligned} \quad (4.16)$$

is given by (4.12)–(4.14) with θ^* replaced by θ_α^* and ω_l ($l = 1, \dots, L$) by $\omega_l - \alpha$.

Having computed the $\text{CRB}(\theta_\alpha^*)$, we can now take its diagonal entries and plot $2\sqrt{\text{CRB}((\text{Re}z)_\alpha)}$ and $2\sqrt{\text{CRB}((\text{Im}z)_\alpha)}$. When α runs from 0 to $\pi/2$, the values of $2\sqrt{\text{CRB}((\text{Re}z)_\alpha)}$ and $2\sqrt{\text{CRB}((\text{Im}z)_\alpha)}$ form a closed curve which surrounds the pole z . One can consider that curve as the 2σ -bound of the pole in the complex plane. As we will see in the numerical simulations section, the curve is not necessarily a circle; sometimes it becomes an ellipsis, a modified ellipsis, and the like.

5. Short description of the main software units. The software considered contains the functions for computation of:

- the Cramér–Rao bound for the covariance matrix of unbiased estimates of the damping factors and frequencies of quasipolynomials in noise (function *cramrao*);
- the denominator of the transfer function estimate from a data using LS Prony or LS Prony method with prefiltering. A FIR-prefilter is iteratively computed from the data samples (function *denomf*);
- the nominator of the transfer function estimate by a least-squares method (function *optim*);
- the Cramér–Rao bound for the covariance of unbiased estimates of the real and imaginary parts of the poles of a quasipolynomial model. The real and imaginary parts of the poles are computed with respect to the coordinate system turned at an angle α around the origin (function *pagcrb*);
- the basis signal matrix from the signal poles and their multiplicities (function *sysmatri*);
- the roots from the upper half-plane and their multiplicities given all the roots of a quasipolynomial (function *verify*);
- the poles of the model from the upper half-plane after optimization, their multiplicities, the estimated signal model and its root-mean-square error. A simple iterative pole optimization method is used (function *opti*);
- the impulse response vector of length N of the system whose transfer function is of the form $1/a(z)$ (function *vectorh*).

Besides the functions described, the work contains MATLAB *.m files. These files include two demo files for estimation of the damping factors and frequencies of the sum of two damped sinusoids embedded in white Gaussian noise by LS Prony method with prefiltering and optimization. There is also a demo file illustrating how to compute the CRB for the error covariance matrix of unbiased estimates of damping factors and frequencies of the sum of quasipolynomials in noise. The *.m files compute the RMSEs of the dampings and frequencies for noisy damped sinusoid models as a function of the frequency separation and signal-noise ratio and compares them with the CRB.

6. Numerical examples

6.1. Example 1. In this example, consider the case of one damped sinusoid with the following parameters

$$K = 1, A = 1, \varphi = 0, \lambda = -0.03, \omega = 0.45\pi. \quad (6.1)$$

The model considered is the following

$$v_n = \exp(-0.03n) \sin(0.45\pi n) + e_n \quad (n = 0, 1, \dots, N - 1), \quad (6.2)$$

where $\{e_n\}$ is Gaussian white noise sequence with zero mean and variance σ^2 . The standard deviation σ is computed from the SNR expression as follows: the SNR is defined by the formula

$$\text{SNR} = 10 \log_{10} \frac{S}{\sigma^2}, \quad (6.3)$$

where

$$S = \frac{1}{N} \sum_{n=0}^{N-1} |v_n|^2, \quad (6.4)$$

then σ is computed using the expression

$$\sigma = \sqrt{S} \cdot 10^{-\text{SNR}/20}. \quad (6.5)$$

The data length $N = 100$. We have estimated the damping factor and frequency of the model for various SNR values. Five hundred trials were used in each case. To estimate the parameters mentioned above, we have used three methods: LS Prony, LS Prony with prefiltering, and LS Prony with prefiltering and optimization. For the prefiltering and optimization procedures, the following parameters were used: the prefilter order $q = 21$, the maximal number of optimization iterations $iternum = 10$. Since the optimization procedure is rather costly, we have made an experiment to ascertain the minimal number of the optimization iterations for which the CRB is attained. We obtained that $iternum = 10$ is optimal for the case of SNR = 2 dB. It is clear that this value of $iternum$ will give good results for higher SNR, too. The estimated poles along with the true poles and the 2σ -bounds (where σ is given by the

CRB formula (4.1)) are shown in Fig. 5. The plots of the RMSE as a function of the SNR are shown in Fig. 6.

6.2. Example 2. In this example, the case of two damped sinusoids with the following parameters

$$K = [1, 1], A = [1, 1], \varphi = [0, 0], \lambda = [-0.1, -0.05], \omega_1 = 0.4\pi \quad (6.6)$$

was considered (ω_2 is specified below). Thus the following model of the damped sinusoids was generated:

$$v_n = \exp(-0.1n) \sin(0.4\pi n) + \exp(-0.05n) \sin(\omega_2 n) + e_n, \quad (6.7)$$

where $\{e_n\}$ is white noise sequence with zero mean and standard deviation σ computed according to the expression (6.5). The number of observations $N = 100$. We have investigated how the performance of estimation of the damping factors and frequencies depends upon the frequency separation $\Delta\omega = |\omega_1 - \omega_2|$. We have selected SNR=30 dB and investigated the behaviour of the RMSE for various $\Delta\omega$. The number of the trials used was 500, the range of $\Delta\omega = \{0.02\pi, 0.04\pi, \dots, 0.2\pi\}$. The plots of the poles and their estimates for the case $\Delta\omega = 0.2\pi$ are shown in Fig. 7. Again, the three versions of Prony method – LS Prony method, LS Prony method with prefiltering and LS Prony method with prefiltering and optimization have been used. The RMSEs for damping factors and frequencies are shown in Fig. 8. In these figures, we omit the RMSEs of the estimates obtained by LS Prony method. These RMSEs are rather far from the CRB curve and their behaviour is not regular. The RMSEs obtained by LS Prony method with prefiltering are much closer to the CRB. If compared with the RMSEs of LS Prony method, their behaviour is more regular. Its monotonic character, however, is still missing (especially for the first damped sinusoid). As one can see from the figures, optimization improves the estimation accuracy significantly. Having applied a hundred optimization iterations, the CRB is attained already for $\Delta\omega = \Delta\omega_{\text{periodogram}}$ (and, of course, for the wider separations).

6.3. Example 3. Wigren and Nehorai (1991) have derived the CRB on the parameters of superimposed real-valued damped sine waves in noise. When the multiplicities of quasipolynomials in our model (2.1)–(2.3) are equal to unity,

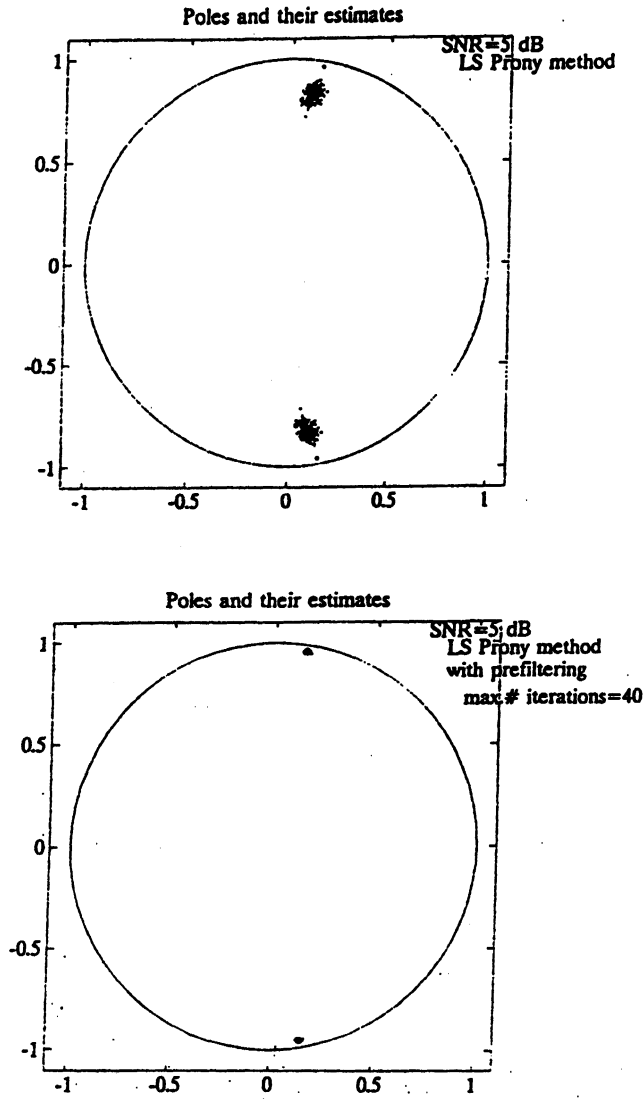


Fig. 5. Poles and their estimates obtained by three versions of Prony method for a single damped sinusoid with parameters: $K = 1$, $A = 1$, $\varphi = 0$, $\lambda = -0.03$, $\omega = 0.45\pi$, $N = 100$ (to be continued).

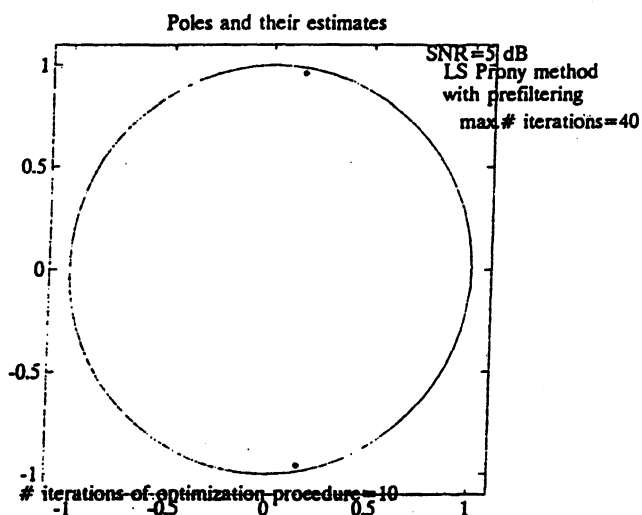


Fig. 5. Continuation.

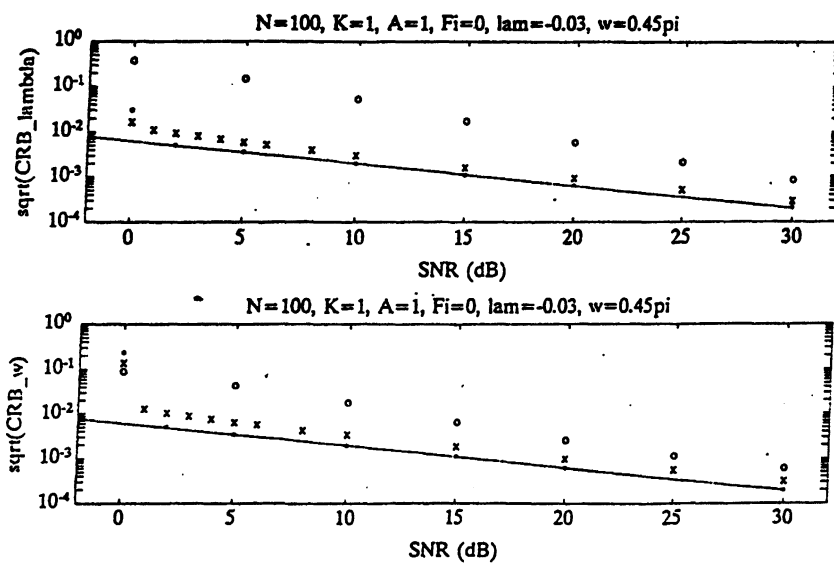


Fig. 6. The RMSE for λ and ω versus the SNR (solid line = the $\sqrt{\text{CRB}}$, 'o' = the RMSE of Prony method, 'x' = the RMSE of Prony method with prefiltering, '*' = the RMSE of Prony method with prefiltering and optimization; number of optimization iterations =10).

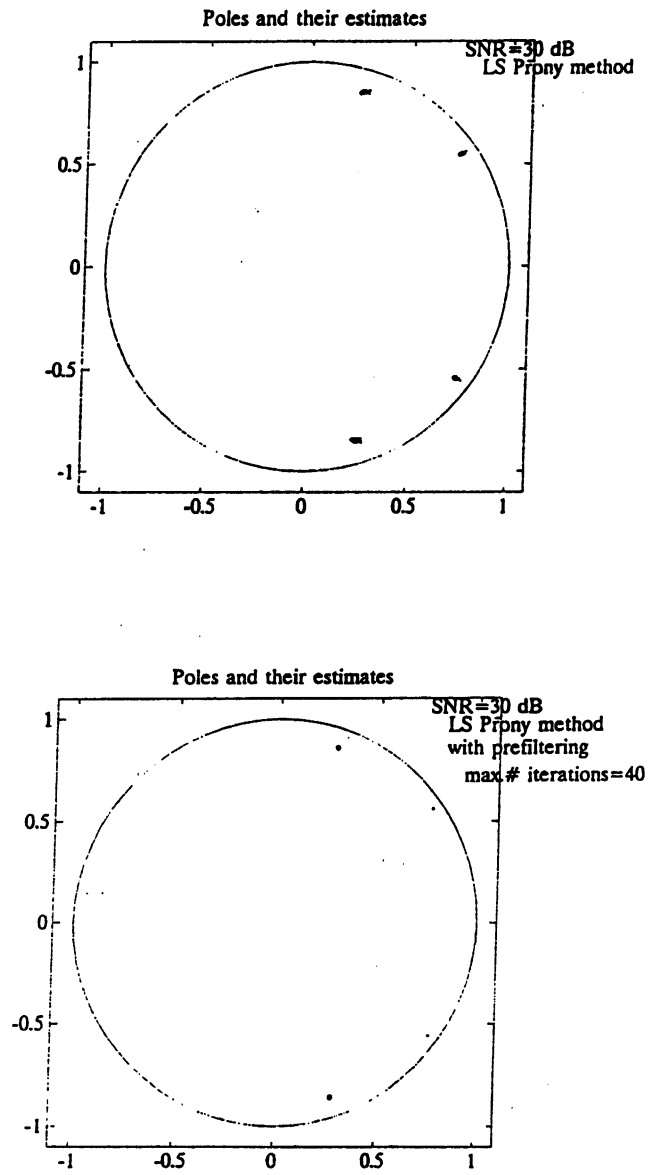


Fig. 7. To be continued.

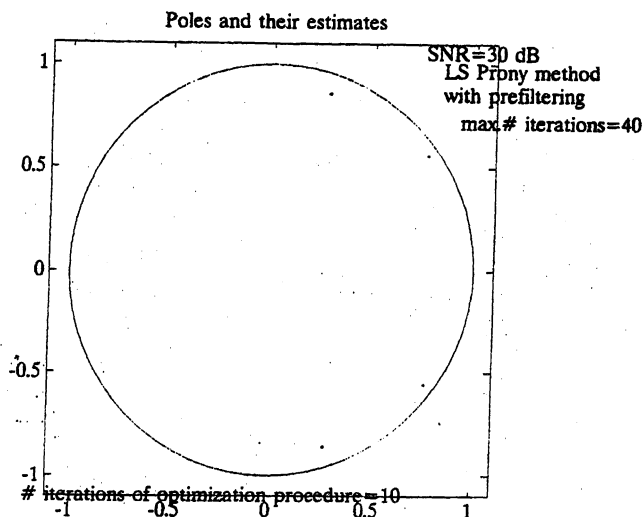


Fig. 7. Poles and their estimates obtained by three versions of Prony method for the case $\Delta\omega = 0.2\pi$ (continuation).

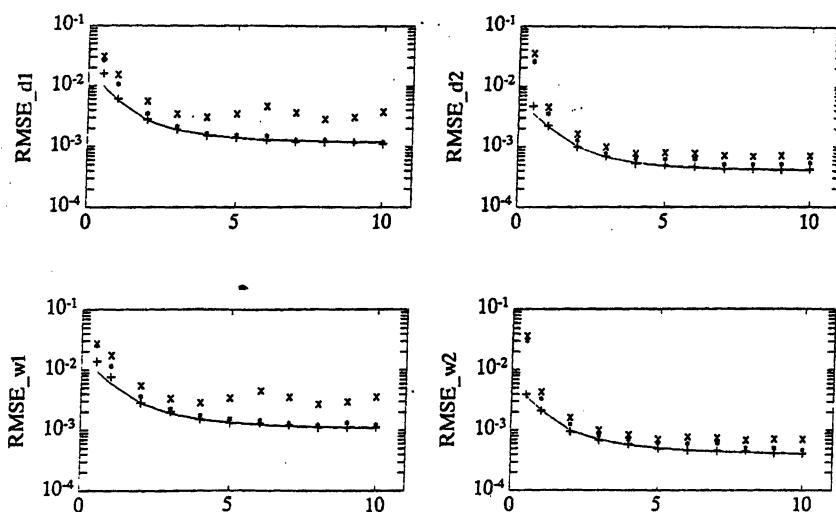


Fig. 8. The RMSE for $\{\lambda_k, \omega_k\}$ as a function of normalized frequency separation $\Delta\omega$ expressed in Fourier bins. (For example, the point 5 on the horizontal axis means $\Delta\omega = 5 \frac{2\pi}{N}$). SNR = 30 dB.

both models coincide and we can compare the result given by (4.1) to that of Wigren–Nehorai.

Consider the model (2.1)–(2.3) with the following parameters: $N = 1000$, $L = 2$, $K_1 = K_2 = 1$, $a_{11} = a_{21} = 1$, $\lambda_1 = \lambda_2 = -0.03$, $\omega_1 = 0.25\pi$, $\omega_2 = 0.75\pi$, $\varphi_{11} = \varphi_{21} = 0$, $\sigma = 0.5$. SNR is defined as $10 \log_{10}(1/\sigma^2)$. In this case, SNR = 6 dB.

The asymptotic CRB expression derived by Wigren and Nehorai was obtained under a number of assumptions (see Assumptions A1–A5 in (Wigren and Nehorai, 1991)). Let's check their validity for our model. Assumptions A1 and A3 are satisfied. Now check assumptions A2, A4 and A5. The expression of assumption A2 is

$$N^2 e^{\lambda_k N} = 1000^2 e^{-0.03 \cdot 1000} = 9.35 \cdot 10^{-8} \ll 1,$$

for $N = 1000$, $\lambda_k = -0.03$ ($k = 1, 2$).

Thus assumption A2 is satisfied. Now check assumption A4:

$$\left| \frac{\lambda_k}{\omega_k} \right| = \begin{cases} \frac{0.03}{0.25\pi} = 0.038 \ll 1, \\ \frac{0.03}{0.75\pi} = 0.012 \ll 1. \end{cases} \quad (6.8)$$

As we see from (6.8), this assumption is also satisfied. Finally, check A5. First compute

$$\Delta\omega_{kl} = \min(|\omega_k - \omega_l|, |\omega_k + \omega_l|) \quad \text{for } k = 1, 2 \quad (k \neq l).$$

We have

$$\Delta\omega_{12} = \Delta\omega_{21} = \min(0.5\pi, \pi) = 0.5\pi.$$

Now

$$\left| \frac{\lambda_1 + \lambda_2}{\Delta\omega_{12}} \right| = \left| \frac{\lambda_2 + \lambda_1}{\Delta\omega_{21}} \right| = \frac{0.06}{0.5\pi} = 0.038 \ll 1.$$

Assumption A5 is also satisfied. Thus all the assumptions are satisfied and we can use the asymptotic CRB expression (Wigren and Nehorai, 1991). In our case, the asymptotic CRB for damping factors and frequencies are given by

$$\text{CRB}_{\lambda_k} = \text{CRB}_{\omega_k} = \sigma^2 \cdot (16|\lambda_k|^3) \quad (k = 1, 2),$$

where σ^2 is the noise variance. The plots of the asymptotic and exact CRB on the damping factors and frequencies are presented in Fig. 9. We can see that there is a good agreement between the exact and asymptotic CRB.

Now change the damping factor values. Take $\lambda_1 = \lambda_2 = -0.003$. The corresponding plots of the CRB are shown in Fig. 9. We see that the exact CRB starts approaching the asymptotic CRB after a longer time period than in the previous example. When the damping decreases 10 times, the values of N smaller than 1000 become too small for the asymptotic formula to be valid.

6.4. Example 4. In this example, we consider the case of five quasipolynomials. As we have mentioned in Section 4, the 2σ -bound (where σ is given by the CRB formula) of a pole in the complex plane is not necessarily a circle – sometimes it becomes an ellipsis, a modified ellipsis, and the like. To illustrate this statement, we investigate the following model

$$v_n = h_{1,n} + h_{2,n} + h_{3,n} + h_{4,n} + h_{5,n} + e_n \quad (n = 0, 1, \dots, N - 1), \quad (6.9)$$

where

$$\begin{aligned} h_{1,n} &= \exp(-0.1n) \sin(0.4\pi n + 0.5\pi), \\ h_{2,n} &= \exp(-0.07n) \sin(0.38\pi n + \pi/3), \\ h_{3,n} &= 0.1 \exp(-0.15n) \sin(0.2\pi n + \pi), \\ h_{4,n} &= 0.1 \exp(-0.18n) \sin(0.1\pi n + \pi/4), \\ h_{5,n} &= \exp(-0.05n) \sin(0.05\pi n + \pi/6), \end{aligned} \quad (6.10)$$

and $\{e_n\}$ is Gaussian white noise sequence with zero mean and variance σ^2 . Thus we have the following parameters of our model: the vector $K = [1, 1, 2, 2, 1]$, the vector of amplitudes $A = [1, 1, 0, 0.1, 0, 0.1, 1]$, the vector of phases $\varphi = [\pi/2, \pi/3, 0, \pi, 0, \pi/4, \pi/6]$, the vector of damping factors $\lambda = [-0.1, -0.07, -0.15, -0.18, -0.05]$, and the vector of frequencies $\omega = [0.4\pi, 0.38\pi, 0.2\pi, 0.1\pi, 0.05\pi]$. Note that since the degree of the third and fourth quasipolynomial is equal to one, i.e., $K_3 = K_4 = 2$, they have two amplitudes (0, 0.1) and two phases (0, π) and (0, $\pi/4$) each. The first, second, and fifth quasipolynomials are simply damped sinusoids as $K_1 = K_2 = K_5 = 1$. The SNR is equal to 10 dB. The standard deviation σ is computed using this SNR value according to expression (6.5). The data length $N = 100$. The plots of the poles surrounded by 2σ -bounds are shown in Fig. 10.

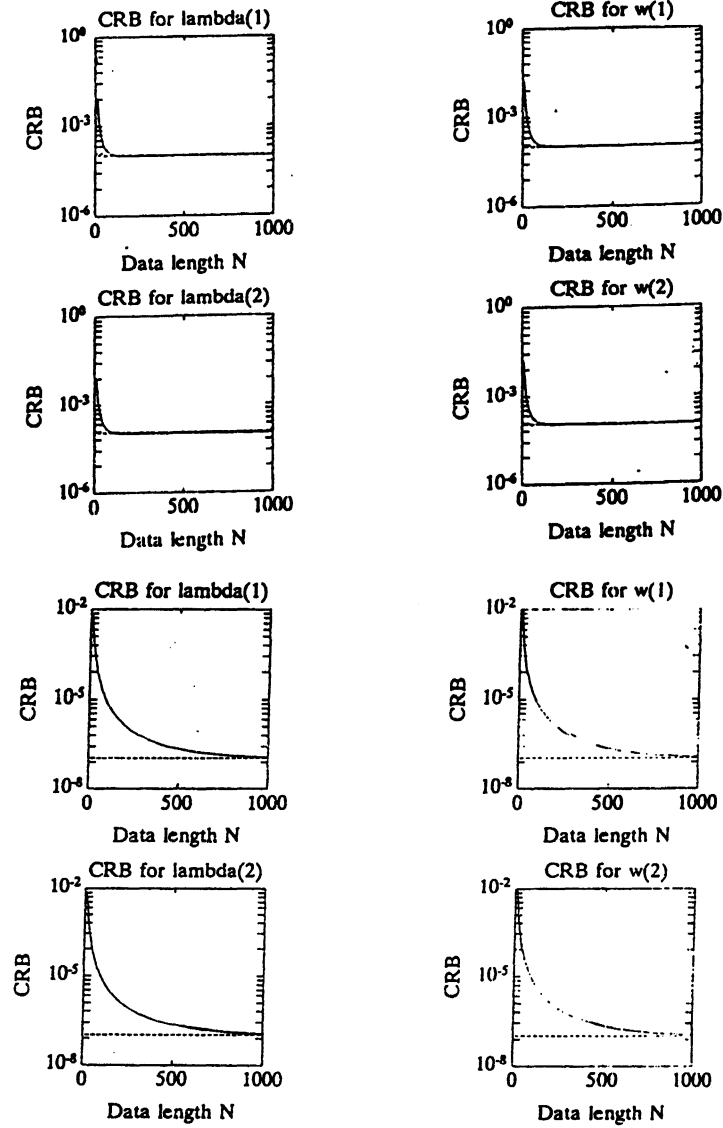


Fig. 9. Asymptotic and exact CRB on the damping factors and frequencies for the model of two damped sinusoids. Four upper plots: $\lambda_1 = \lambda_2 = -0.03$, four lower plots: $\lambda_1 = \lambda_2 = -0.003$.

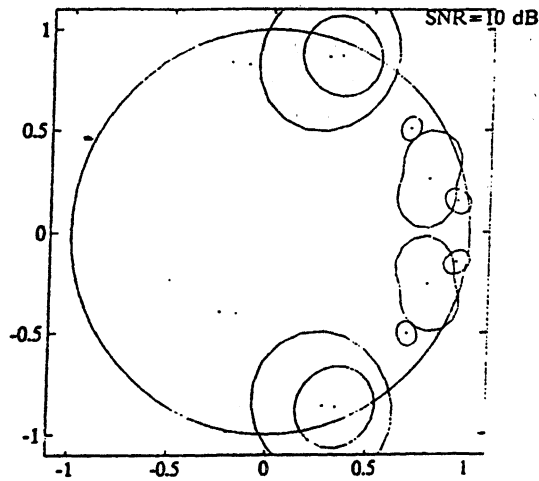


Fig. 10. Poles and their 2σ -bounds.

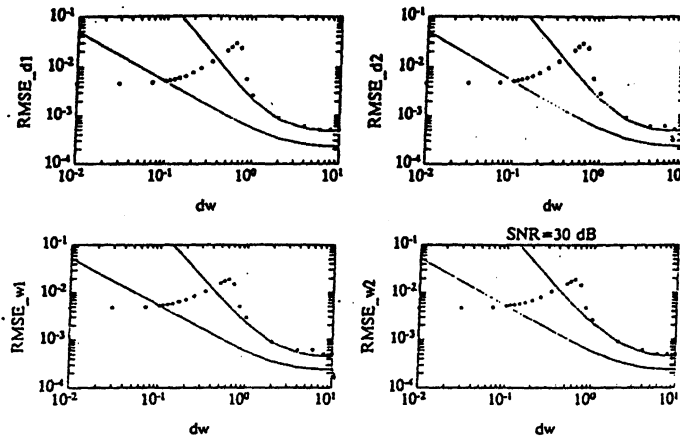


Fig. 11. The RMSEs for $\lambda_1, \lambda_2, \omega_1, \omega_2$ as a function of normalized frequency separation $\Delta\omega$. The model parameters: $K = [1 \ 1]$, $A = [1 \ 1]$, $\varphi = [0 \ 0]$, $\lambda = [-0.05 \ -0.05]$, $\omega_1 = 0.4\pi$, $\omega_2 = \omega_1 + \Delta\omega$. The higher solid line = the $\sqrt{\text{CRB}}$ of the original model, the lower solid line = the $\sqrt{\text{CRB}}$ of the approximating model with parameters: $K = 2$, $A = [2, \Delta\omega]$, $\varphi = [0, \pi/2]$, $\lambda = -0.05$, $\omega = 0.4\pi$, '*' = the RMSE of Prony method with prefiltering and optimization; number of optimization iterations = 10).

6.5. Example 5. In this experiment, we consider the case of the sum of two damped sinusoids with equal damping factors $\lambda_1 = \lambda_2$ and frequencies ω_1 and ω_2 . The number of the available noisy observations $N = 100$. The specific values of parameters of the signals are as follows

$$\begin{aligned} K &= [1, 1], \quad A = [1, 1], \quad \varphi = [0, 0], \\ \lambda &= [-0.05, -0.05], \quad \omega_1 = 0.4\pi, \quad \omega_2 = \omega_1 + \Delta\omega. \end{aligned} \quad (6.11)$$

The frequency separation $\Delta\omega$ is taken from a wide range:

$$\Delta\omega = [10^{-2}, 10]\Delta\omega_{\text{periodogram}} \quad (6.12)$$

where $\Delta\omega_{\text{periodogram}} = 2\pi/N = 0.02\pi$. So we consider the case of closely-spaced damped sinusoids as well as far-spaced damped sinusoids. Taking into account the specific values of the parameters, we get the following model

$$\begin{aligned} v_n &= \exp(-0.05n) \sin(0.4\pi n) \\ &+ \exp(-0.05n) \sin((0.4\pi + \Delta\omega)n) + e_n, \end{aligned} \quad (6.13)$$

where $\{e_n\}$ is Gaussian white noise sequence with zero mean and standard deviation σ computed according to the expression (6.5). The SNR = 30 dB. Note that this model has two pairs of complex-conjugate poles z_i, z_i^* ($i = 1, 2$), where $z_1 = \exp(-0.05 + j0.4\pi)$, $z_2 = \exp(-0.05 + j(0.4\pi + \Delta\omega))$. To estimate the poles of the model, we first applied LS Prony method with prefiltering. The result is either two pairs of complex-conjugate numbers \hat{z}_i, \hat{z}_i^* ($i = 1, 2$) or two real numbers r_1, r_2 and a pair of complex-conjugate numbers ζ, ζ^* . Before applying the optimization procedure, we must select poles from the upper half-plane. We use the RMSE obtained for a model with the selected poles as the selection criterion. We proceed in the following way: if the result of applying LS Prony method with prefiltering is two pairs of complex-conjugate numbers, we select an estimate \hat{z}_i ($i = 1, 2$) of a pole z_i from the upper half-plane, assign multiplicity 2 to it, and compute the difference between the data and the model corresponding to the selected pole. This difference is an $N \times 1$ estimation error vector err obtained from (3.27). For a pole estimate \hat{z} with multiplicity 2, the matrix $\hat{\mathbf{D}}$ from (3.27) has the following form (see (2.8), (2.9)):

$$\hat{\mathbf{D}} = \hat{\mathbf{D}}(\hat{z}) = [\hat{\mathbf{d}}_{11}^c, \hat{\mathbf{d}}_{11}^s, \hat{\mathbf{d}}_{12}^c, \hat{\mathbf{d}}_{12}^s] \quad (6.14)$$

with the following columns

$$\begin{aligned}
\hat{\mathbf{d}}_{11}^c &= [1, e^{\hat{\lambda}} \cos(\hat{\omega}), e^{2\hat{\lambda}} \cos(2\hat{\omega}), \dots, e^{(N-1)\hat{\lambda}} \cos((N-1)\hat{\omega})]^T, \\
\hat{\mathbf{d}}_{11}^s &= [0, e^{\hat{\lambda}} \sin(\hat{\omega}), e^{2\hat{\lambda}} \sin(2\hat{\omega}), \dots, e^{(N-1)\hat{\lambda}} \sin((N-1)\hat{\omega})]^T, \\
\hat{\mathbf{d}}_{12}^c &= [0, e^{\hat{\lambda}} \cos(\hat{\omega}), 2e^{2\hat{\lambda}} \cos(2\hat{\omega}), \dots, (N-1)e^{(N-1)\hat{\lambda}} \cos((N-1)\hat{\omega})]^T, \\
\hat{\mathbf{d}}_{12}^s &= [0, e^{\hat{\lambda}} \sin(\hat{\omega}), 2e^{2\hat{\lambda}} \sin(2\hat{\omega}), \dots, (N-1)e^{(N-1)\hat{\lambda}} \sin((N-1)\hat{\omega})]^T,
\end{aligned} \tag{6.15}$$

where $\hat{\lambda} = \text{Re}(\log(\hat{z}))$, $\hat{\omega} = \text{Im}(\log(\hat{z}))/j$. Thus, taking \hat{z}_1 with multiplicity 2 and \hat{z}_2 with multiplicity 2, we get two RMSE values – `rmse1` and `rmse2` – computed according to (3.29). We also compute `rmse3` (obtained from (3.29)) with `err` computed by (3.27) with \hat{D} of the following form

$$\hat{D} = \hat{D}(\hat{z}_1, \hat{z}_2) = [\hat{\mathbf{d}}_{11}^c, \hat{\mathbf{d}}_{11}^s, \hat{\mathbf{d}}_{21}^c, \hat{\mathbf{d}}_{21}^s] \tag{6.16}$$

where the four columns of \hat{D} are as follows (see Section 2)

$$\begin{aligned}
\hat{\mathbf{d}}_{11}^c &= [1, e^{\hat{\lambda}_1} \cos(\hat{\omega}_1), e^{2\hat{\lambda}_1} \cos(2\hat{\omega}_1), \dots, \\
&\quad e^{(N-1)\hat{\lambda}_1} \cos((N-1)\hat{\omega}_1)]^T, \\
\hat{\mathbf{d}}_{11}^s &= [0, e^{\hat{\lambda}_1} \sin(\hat{\omega}_1), e^{2\hat{\lambda}_1} \sin(2\hat{\omega}_1), \dots, \\
&\quad e^{(N-1)\hat{\lambda}_1} \sin((N-1)\hat{\omega}_1)]^T, \\
\hat{\mathbf{d}}_{21}^c &= [1, e^{\hat{\lambda}_2} \cos(\hat{\omega}_2), e^{2\hat{\lambda}_2} \cos(2\hat{\omega}_2), \dots, \\
&\quad e^{(N-1)\hat{\lambda}_2} \cos((N-1)\hat{\omega}_2)]^T, \\
\hat{\mathbf{d}}_{21}^s &= [0, e^{\hat{\lambda}_2} \sin(\hat{\omega}_2), e^{2\hat{\lambda}_2} \sin(2\hat{\omega}_2), \dots, \\
&\quad e^{(N-1)\hat{\lambda}_2} \sin((N-1)\hat{\omega}_2)]^T.
\end{aligned} \tag{6.17}$$

Such \hat{D} corresponds to the model with two pairs of complex-conjugate poles \hat{z}_1, \hat{z}_1^* and \hat{z}_2, \hat{z}_2^* of multiplicity one. We compare the `rmse1`, `rmse2`, `rmse3` and select that case which corresponds to the smallest `rmsei` ($i = 1, 2, 3$). If `rmse3` is equal to either `rmse1` or `rmse2`, then we choose the case corresponding to the multiple root. In case the result of applying LS Prony method with prefiltering is two real numbers r_1, r_2 and a pair of complex-conjugate ζ, ζ^* , we assign multiplicity 2 to r_1, r_2 and ζ , and compute the corresponding RMSEs which are denoted by `rmse1`, `rmse2` and `rmse4`. By `rmse3` we denote the RMSE

obtained when taking a model with the poles r_1 and r_2 of multiplicity 1. If the minimum of rmse1 and rmse2 is greater than rmse4 , we assign rmse4 to rmse1 . Now that rmse1 and rmse2 represent the RMSEs obtained for the models with poles of multiplicity 2, we compare them with rmse3 – the RMSE of the model with two simple poles. If the minimum of rmse1 and rmse2 is less than 2 times rmse3 , we select the pole corresponding to that minimum and start the optimization procedure (note that the pole selected has multiplicity 2). Otherwise, the optimization procedure is started with two simple poles. The number of the optimization iterations is equal to 10.

In Fig. 11, we present the RMSEs for four parameters $\lambda_1, \lambda_2, \omega_1, \omega_2$ obtained after the optimization procedure has been applied. The plots show the RMSE as a function of the frequency separation $\Delta\omega$ expressed in Fourier bins (one Fourier bin = $2\pi/N$). We also plot the $\sqrt{\text{CRB}}$. The higher solid line is the $\sqrt{\text{CRB}}$ when the CRB is computed for the model of two damped sinusoids with the parameters described in (6.11). The lower solid line is the $\sqrt{\text{CRB}}$ plotted for $\Delta\omega$ values from the interval $[10^{-2}, 10]$ Fourier bins where the CRB is obtained for the model of one quasipolynomial of multiplicity 2 with the damping factor $\lambda_1 = -0.05$, frequency $\omega_1 = 0.4\pi$, amplitudes $A = [2, \Delta\omega]$, phases $\varphi = [0, \pi/2]$. These values of parameters are obtained from the following reasoning. Consider model (6.13). When $\Delta\omega$ is small,

$$\begin{aligned} \sin(0.4\pi n + \Delta\omega n) &= \sin 0.4\pi n \cos \Delta\omega n + \cos 0.4\pi n \sin \Delta\omega n \\ &\approx \sin 0.4\pi n + \Delta\omega n \cos 0.4\pi n \\ &= \sin 0.4\pi n + \Delta\omega n \sin(0.4\pi n + \pi/2). \end{aligned} \quad (6.18)$$

So when $\Delta\omega$ is small, the model (6.13) can be approximated by the following model:

$$v_n = \exp(-0.05n)(2 \sin(0.4\pi n) + \Delta\omega n \sin(0.4\pi n + \pi/2)) + e_n. \quad (6.19)$$

But this is a noisy quasipolynomial of multiplicity 2 with the parameters specified above. Thus if we have two damped sinusoids with closely-spaced frequencies, then it is worthwhile to estimate them as a quasipolynomial of multiplicity 2 rather than two damped sinusoids with simple poles. In Fig. 12, we present the 2σ -bounds (where σ is given by the CRB formula (4.1)) for the models (6.13) and (6.19). The larger circles correspond to the CRB for the model with

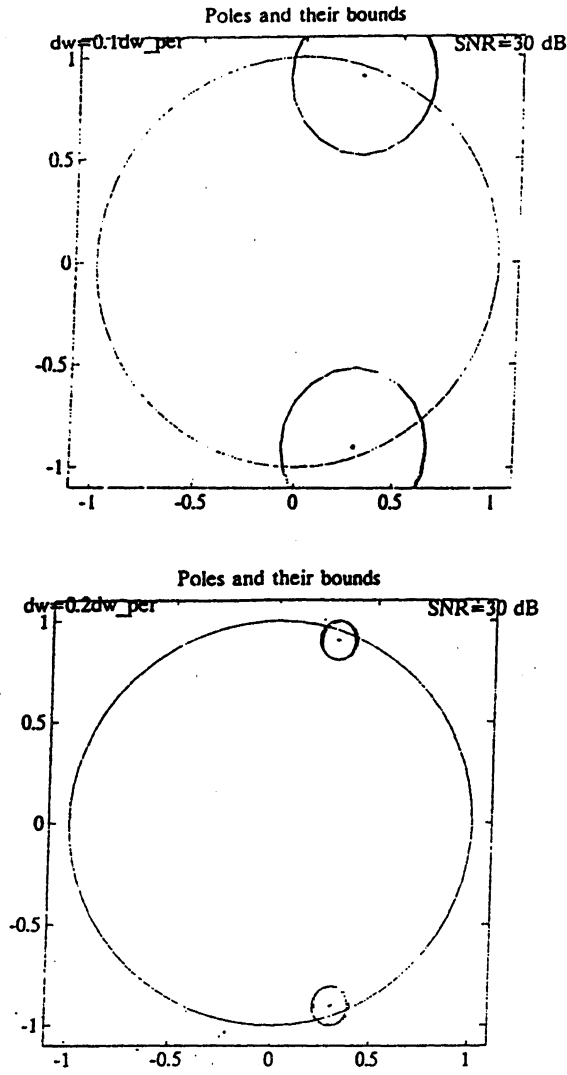


Fig. 12. The poles and their 2σ -bounds for the model with parameters: $N = 100$, $K = [1 \ 1]$, $A = [1 \ 1]$, $\varphi = [0 \ 0]$, $\lambda = [-0.05 \ -0.05]$, $\omega_1 = 0.4\pi$, $\omega_2 = \omega_1 + \Delta\omega$. The frequency separation $\Delta\omega = 0.1\omega_{per}$ (and $0.2\omega_{per}$) where $\omega_{per} = 2\pi/N$. The larger circles = the bounds for the original model, the smaller circles = the bounds for the approximating model.

simple poles (6.13) while the smaller circles to the CRB for the model of a quasipolynomial (6.19).

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**UŽTRIUKŠMINTŲ KVAZIPOLINOMŲ ĮVERTINIMO
PROGRAMINĖ SISTEMA, SUKURTA MATLAB PAGRINDU****Vytautas SLIVINSKAS ir Virginija ŠIMONYTĖ**

Šio darbo tikslas – aprašyti autorių sukurtos, naudojant MATLAB'ą, programinės sistemos teorinius ir algoritminius pagrindus. Programinė sistema yra skirta laiko eilučių (signalų), kurias galima modeliuoti kaip realiųjų kvazipolinomų suma plus baltas triukšmas, tyrimui. Aprašytosios programinės sistemos pagalba galima apskaičiuoti kvazipolinomų gesimo koeficientų ir dažnių nepaslinktųjų įverčių įvertinimo paklaidos kovariacijos matricos Kramero–Rao apatinio režio išraiškas ir gauti šių parametru įverčius, naudojant tris Prony metodo variantus. Naudojant šią programinę sistemą, galima generuoti įvairius kvazipolinomų modelius, gauti jų polių išsidėstymo vienetinio apskritimo atžvilgiu grafikus, apskaičiuoti ir nubraižyti aplink kiekvieną polių 2σ -režius (kur σ yra nusakoma KRR formule), o taip pat polių įverčius, gautus kiekvienoje realizacijoje. Pateikiami skaitinių eksperimentų rezultatai.