INFORMATICA, 1995, Vol. 6, No. 3, 323–359

POLE-PLACEMENT CONTROLLERS FOR LINEAR SYSTEMS WITH POINT DELAYS

Manuel de la SEN and Josu JUGO

Departamento de Electricidad y Electrónica, Facultad de Ciencias,
Universidad del Pais Vasco, Apdo. 644. 48080 Bilbao (Leioa), SPAIN

Abstract. This paper presents a robust control algorithm for plants involving both internal (i.e., in the state) and external (i.e., in the output or input) known point delays. Several stabilizing controller structures are given and analyzed for the case of perfectly modelled plants with known parameters. The plant is assumed to be of known order and relative order. The parametrized parts of two of the controller structures involve delays while those of the two remaining controllers are delay-free. However, auxiliary compensating signals which weight the plant input and output integrals are incorporated in all the controller structures for stabilization and model matching purposes.

Key words: delayed systems, internal and external delays, robust control.

1. Introduction. In the last years, a number of papers have dealt with the problem of presence of delays in the controlled plant and the related properties of controllability, observability and stabilizability have been investigated in [1–13]. The stabilizability for known plants by using matrix Lyapunov equations and some delay-independent stabilization results have been addressed in [1–2]. Also, the relationships between the stabilization of systems with point delays have been studied in [5–6] by establishing an equivalent model, subject to point delays only, for the class of systems originally possessing exponentially distributed delays. The spectrum assignability has been investigated in [8–12] for systems with commensurate and noncommensurate delays. A system is finite spectrum assignable if it is reachable, and then, spectrally controllable. Also, if it is reachable with a closed loop finite spectrum can be achieved with a control function based on polynomials on the delay operator, [12]. However, the more serious stabilizability problems arise from the presence of internal delays, [7], since unsuitable infinite closed-loop spectra can be generated even if they are not established as a control objective.

In this paper, four robust time-invariant controller structures are given and those controller structures for the nominal known plants are studied. They consist of a parameterized part plus memory-type signal, which involve weighted integrals on the plant input and output which extend the scheme proposed in [14] for the single external point delay case. The contributions of this paper related to previous work, [7–12], [17–22], are the following: *(a) Some of the proposed Controller Structures (namely, the so-called controller structures I and II) are able to achieved prefixed (internal delay dependent) infinite or (delay independent) finite closed-loop spectra indistinctly, depending on the particular controller parametrization, in the nominal situation of known parameters and perfectly modelled dynamics. (b) The nominal closed-loop system can also match reference models whose zeros include those unstable ones of the plant. (c) Both single-internal and single-external point delay are considered in the plant state-space description.*

The control problem can be considered as non restrictive in the sense that n delays, being integer multiple of the internal delay, (n being the plant order) are automatically generated in the plant transfer function and, on the other hand, it has been proved in the literature that some distributed-delay systems with exponential distribution can be described through equivalent point delay systems [5–6]. Also, it has been proven that the stabilization of open-loop stabilizable systems subject to internal delays is ensured by the use of distributed-delay controllers even if the plant possesses point delays only (see [8–12] and [15–21]).

The paper is organized as follows. Section 2 deals with the state and input/output nominal descriptions of the plant and the various proposed controller structures as well as the statement and conditions of achievement of the control objectives. Those objectives are closed-loop model matching with (internal delay-dependent) infinite spectrum and (delay-independent) finite spectrum. Section 3 contains the proofs of the results of Section 2. In Section 4 several simulation examples are proposed to emphasize the efficiency of the proposed control schemes. Finally, in Section 5, conclusions end the paper.

Notation

- The Laplace transform of $f(t)$ is denoted by $f(s)$ or $L\{f(t)\}$ and the Laplace transform of $f(-t)$ for $t > 0$ is denoted by $\overline{f}(s)$.
- $\deg_\mu[p(\mu, s)]$ and $\deg_s[p(\mu, s)]$ stand for the degrees of the quasipoly-

nomial (or two-variable polynomial) $p(\mu, s)$ with respect to μ and s, respectively. If both degrees are identical or the polynomial is of one variable, then subindices are not used.

- Det$(\cdot)$ and Adj$(\cdot)$ stand for the determinant and Adjoint of the $(\cdot)$ matrix.
- C denotes the set of complex numbers. C^+ and C^- are, respectively, the open left-half plane and its complement in C. $C_\nu^+ = \{z \in C : \text{Re}(z) \leqslant -\nu\}$, and C_ν^- is the complement of C_ν^+ in C for any real constant ν. R and R^+ denote, respectively, the set of real and positive real numbers and $R_0^+ = R^+ \cup \{0\}$.
- Transfer functions involving internal and external delays h and h' are denoted by $G(s) = G(\mu, \mu', s)$, where $\mu = \exp(-hs)$ and $\mu' = \exp(-h's)$. The equivalent input-output differential-difference description is $y(t) = G(D, q^{-1}, q'^{-1})u(t)$ with $D = \frac{d}{dt}$, q^{-1}, and q'^{-1} being, respectively, the differential and the internal and external delay operator; i.e., $\dot{z} = Dz(t)$, $z(t-h) = q^{-1}z(t)$ and $z(t-h') = q'^{-1}z(t)$ for any signal $z(t)$.

2. Models for plants and controllers. This section describes the structures of a single-input single-output plant involving single point internal (i.e., in the state) and external (i.e., in the input or output), delays as well as four controller structures which lead to the achievement of model matching objectives with finite or infinite closed-loop spectra. Although the plant possess a single internal point delay, a general single-input single-output system with n internal delays is first analyzed in Subsection 2.1. The reason is that the controller structures of Subsection 2.2 below are required to have n internal delays in order to achieve (delay-dependent infinite-spectrum) model matching, if desired, as discussed later. Such a requirement arises naturally, in order to get sufficient conditions for the achievement of (infinite-spectrum) closed-loop pole-placement. Delays in the parametrized part of the controller will be not required when the control objective is the achievement of delay-independent finite closed-loop spectra.

2.1. Dynamic systems with internal and external point delay. Consider the next transfer function involving internal and external point delays

$$G(s) = G(\mu, \mu', s) = \frac{B(\mu, \mu', s)}{A(\mu, s)} =$$

$$=K_p\left[\frac{B_1(\mu,s)+\sum\limits_{j=1}^{n_1}\mu'^l B_2^{(l)}(\mu,s)}{A(\mu,s)}+\theta''\right] \quad (1.a)$$

$$=K_p\left[\frac{\sum\limits_{i=0}^{m} b_i(\mu,\mu')s^i}{s^n+\sum\limits_{i=0}^{n-1} a_i(\mu)s^i}+\theta''\right], \quad (1.b)$$

with $m < n$, K_p being the static gain and θ'' being scalar (if $\theta'' = 0$ then $G(s)$ is strictly proper), $\mu = \exp(-hs)$ and $\mu' = \exp(-h's)$ are complex variables dependent on s and the internal and external known delays h and h', and

$$b_i(\mu,\mu') = b_i^{(1)}(\mu) + \sum_{j=1}^{n_1} \mu'^j b_i^{(2j)}(\mu), \quad (2.a)$$

$$b_i^{(1)}(\mu) = \sum_{k=0}^{n_{i_2}} b_{m-i,k}^{(1)}\mu^k = \sum_{k=0}^{n_2} b_{m-i,k}^{(1)}\mu^k, \quad (2.b)$$

$$(b_{00}^{(1)} \neq 0 \quad \text{if} \quad h \neq 0),$$

$$b_i^{(2l)}(\mu) = \sum_{k=0}^{n_{i_2}} b_{m-i,k}^{(2l)}\mu^k = \sum_{k=0}^{n_2} b_{m-i,k}^{(2l)}\mu^k, \quad (2.c)$$

$$(b_{00}^{(2)} \neq 0 \quad \text{if} \quad h' \neq 0),$$

$$a_j(\mu) = \sum_{k=0}^{n_{j_3}} a_{n-j,k}\mu^k = \sum_{k=0}^{n_3} a_{n-j,k}\mu^k, \quad (2.d)$$

$$(a_n(\mu) = a_{00} = 1),$$

for $l = 1, 2, \ldots, n_1$; $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$ where $b_{(\cdot)}^{(1)}(\mu)$, $b_{(\cdot)}^{(2l)}(\mu)$ and $a_{(\cdot)}(\mu)$ are one-variable polynomials in μ and n_{i_2} and n_{i_3} are the numbers of internal delays acting from each state variable x_{i+1} through the state feedback and output measurement with the polynomials $b_i^{(1)}$ and a_{n-i-1} ($i = 0, 1, \ldots, n$), respectively, with $b_i^{(1)} = 0$ for $i > m$. Note that $n_2 = \max\limits_{0\leqslant i\leqslant m}(n_{i_2})$ and $n_3 = \max\limits_{0\leqslant i\leqslant n-1}(n_{i_3})$ so that $b_{i,k}^{(1)} = b_{i,k}^{(2l)} = 0$ for $k = n_{i_2}+1,\ n_{i_2}+2,\ \ldots, n_2$ and $a_{i,k} = 0$ for $k = n_{i_3}+1,\ n_{i_3}+2,\ \ldots, n_3$; all $l = 1, 2, \ldots, n_1$. If the values of n_{i_2} in (2.b) – (2.c) are different, they can be unified provided that the appropriate $b_{(\cdot)}^{(\cdot)}$ are zeroed. The two-variable polynomials $b_1(\mu, s)$, $b_2^{(l)}(\mu, s)$

and $A(\mu, s)$ in (1.a) are then given by

$$B_1(\mu, s) = \sum_{i=0}^{m} b_i^{(1)}(\mu)s^i = \sum_{i=0}^{n_2} B_i^{(1)}(s)\mu^i, \tag{3.a}$$

$$B_2^{(l)}(\mu, s) = \sum_{i=0}^{m} b_i^{(2l)}(\mu)s^i = \sum_{i=0}^{n_2} B_i^{(2l)}(s)\mu^i, \tag{3.b}$$

$$A(\mu, s) = s^n + \sum_{i=0}^{n-1} a_i(\mu)s^i = \sum_{i=0}^{n_3} A_i(s)\mu^i, \tag{3.c}$$

$l = 1, 2, \ldots, n_1$, the last right-hand-sides of (3) being obtained from the identities (2). Thus, $A(\mu, s)$ and $B_{(\cdot)}^{(\cdot)}(\mu, s)$ can be described indistinctly by two-variable polynomials in s with coefficients being one variable polynomials in μ or viceversa and

$$\begin{aligned} B_i^{(1)}(s) &= \sum_{k=0}^{m} b_{m-k,i}^{(1)} s^k, \\ B_i^{(2l)}(s) &= \sum_{k=0}^{m} b_{m-k,i}^{(2l)} s^k, \\ A_j(s) &= \sum_{k=0}^{n} a_{m-k,j} s^k, \end{aligned} \tag{4}$$

for $l = 0, 1, \ldots, n_1$; $i = 0, 1, \ldots, n_2$; $j = 0, 1, \ldots, n_3$. The two variables μ and s are not independent from the definition of μ. However, a two-variable description of $A(\mu, s)$ and $B_{(\cdot)}^{(\cdot)}(\mu, s)$ is used to facilitate the mathematical developments associated with the solution of two-variable diophantine equations for pole-placement where the second right-hand sides of (3) with (4) are used (see Section 2.3 below). From (3) and (4), Eq. 1 can be rewritten as

$$G(s) = G_1(s) + G_2(s), \tag{5.a}$$

$$G_1(s) = k_p \left[\frac{B_1(\mu, s)}{A(\mu, s)} + \theta'' \right], \quad G_2(s) = \frac{k_p \sum_{j=0}^{n_1} B_2^{(j)}(\mu, s)\mu'^j}{A(\mu, s)} \tag{5.b}$$

subject to (3) – (4). A direct decomposition of (3) – (4) in a controllability canonical form leads directly to the following state-space description (see Fig. 1):

$$\dot{x}(t) = \underset{\sim}{A}\left(q^{-1}\right) x(t) + du(t), \tag{6.a}$$

$$y(t) = \underset{\sim}{c}^T \left(q^{-1}, q'^{-1}\right) x(t) + \theta' u(t) \tag{6.b}$$

with $\theta' = k_p\theta''$, where $x(t)$ is the n-state vector and $\underset{\sim}{c}$ is a polynomial vector of dimension n in the time-delay operators q'^{-1} and q^{-1}, and

$$\begin{aligned}\underset{\sim}{A}(q^{-1}) &= \begin{bmatrix} 0 & I_{n-1} \\ -a_{n-1}(q^{-1}) & -a_{n-2}(q^{-1}), \ldots, -a_0(q^{-1}) \end{bmatrix} \\ &= A + \sum_{i=1}^{n_3} A_i q^{-i},\end{aligned} \tag{7.a}$$

$$\begin{aligned}A &= \begin{bmatrix} 0 & I_{n-1} \\ -a_{n-1,0} & -a_{n-2,0}, \ldots, -a_{00} \end{bmatrix}, \\ A_i &= \begin{bmatrix} 0_{(n-1)x(n-2)} \\ -a_{n-1,i}, -a_{n-2,i}, \ldots, -a_{0i} \end{bmatrix},\end{aligned} \tag{7.b}$$

$$d = \left[0_{n-1}^T, 1\right]^T, \tag{7.c}$$

$$\begin{aligned}\underset{\sim}{c}(q^{-1}, q'^{-1}) &= \sum_{j=0}^{n_1} \sum_{i=0}^{n_2} c_{ij} q^{-i} q'^{-j}, \\ c_{ij} &= \left[b_{mij}\ b_{m-1,ij}\ \ldots\ b_{0ij}\right]^T.\end{aligned} \tag{7.d}$$

By using (7), Eqs. (6) can be rewritten as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n_3} A_i x(t - ih) + du(t), \tag{8.a}$$

$$y(t) = \sum_{j=0}^{n_1} \sum_{i=0}^{n_2} c_{ij}^T x(t - ih - jh') + \theta' u(t) \tag{8.b}$$

leading to the transfer function

$$G(s) = \sum_{j=0}^{n_1} \sum_{i=0}^{n_2} \mu'^j \mu^i c_{ij}^T \left(sI - A - \sum_{k=1}^{n_3} \mu^k A_k\right)^{-1} d + \theta'. \tag{8.c}$$

REMARK 1. Note that system (8) has multiple delays. The particular values $n_1 = 1$, $n_{i_2} = m - i$ $(i = 0, 1, \ldots, m)$, $n_{i_3} = n - i$ $(i = 0, 1, \ldots, n-1) \Rightarrow n_2 = m$; $n_3 = n$ lead to a single-internal single-external delayed system with $a_{00} = 1$, $b_{00}^{(1)} \neq 0$ and, if $h' \neq 0$ then $b_{00}^{(2)} \neq 0$. Then, the transfer function

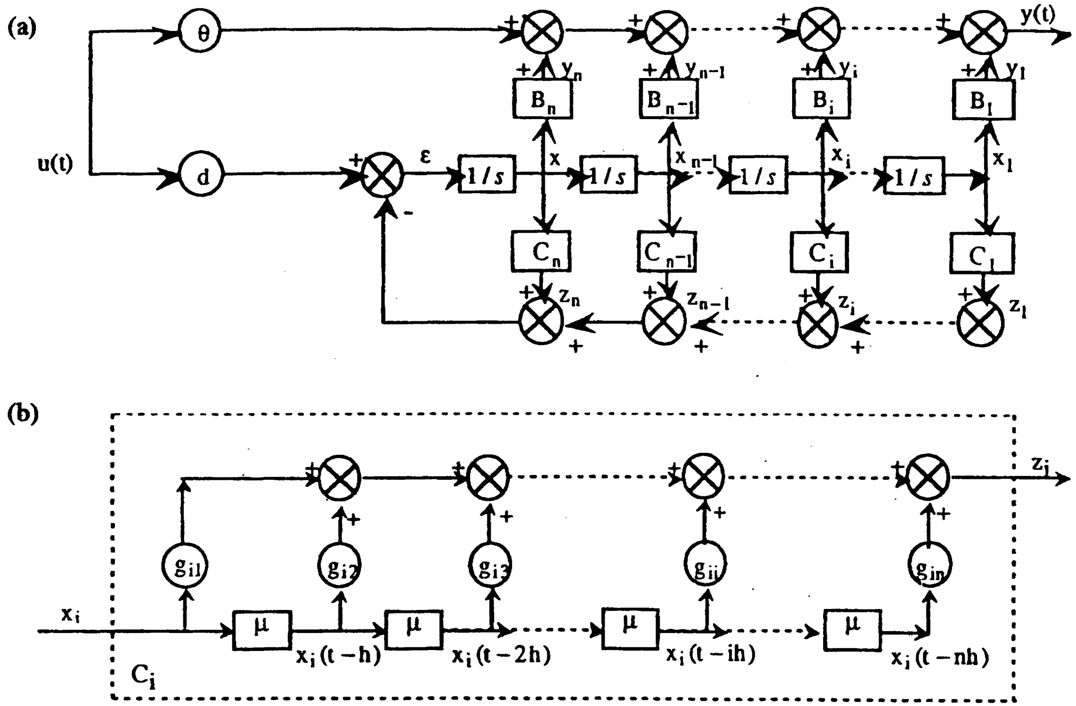

Fig. 1. Realization of system Eqs. 6 (to be continued).

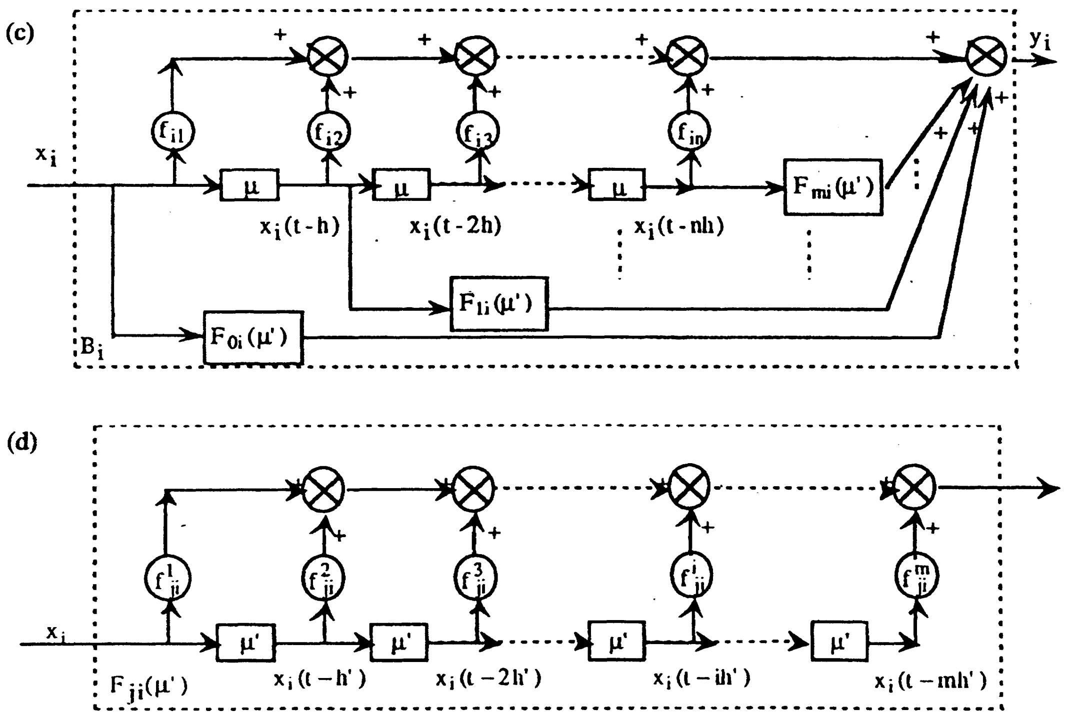

Fig. 1. Realization of system Eqs. 6 (continuation).

(8.c) becomes

$$\begin{aligned} G(s) &= \sum_{j=0}^{1}\sum_{i=0}^{m} \mu'^{j}\mu^{i} c_{ij}^{T}\left(sI - A - \sum_{k=1}^{n}\mu^{k}A_{k}\right)^{-1} d + \theta' \\ &= k_p \frac{B_1(\mu, s) + \mu' B_2(\mu, s)}{A(\mu, s)}, \end{aligned} \tag{9}$$

with $B_i(\mu, s)$ $(i = 1, 2)$ and $A(\mu, s)$ defined in (3) with the current values of n_1, n_2 and n_3, and the second superscript in $B_{(\cdot)}^{(\cdot)}$ being deleted since now there is just one external delay, and

$$b_i^{(l)}(\mu) = \sum_{k=0}^{m-i} b_{m-i,k}^{(l)}\mu^k, \quad a_j(\mu) = \sum_{k=0}^{n-j} a_{n-j,k}\mu^k, \tag{10.a}$$

$$B_i^{(l)}(s) = \sum_{k=0}^{m-i} b_{m-k,i}^{(l)} s^k, \quad A_j(s) = \sum_{k=0}^{n-j} a_{n-k,j} s^k, \tag{10.b}$$

for $i = 0, 1, \ldots, m$; $j = 0, 1, \ldots, n$; $l = 1, 2$ with $A_n(s) = a_{0n}$ and $a_n(\mu) = a_{00} = 1$, where $B_l(\mu, s)$ $(l = 1, 2)$ and $A(\mu, s)$ are given by (3) with $l = 1$ and $n_2 = m$, $n_3 = n$, $B_i^{(l)}(s)$ $(l = 1, 2)$ and $A_i(s)$ are given by (4). Also, $b_i^{(l)}(\mu)$ $(l = 1, 2)$ and $a_j(\mu)$ are given by (2.b) – (2.d) with $n_{i_2} = m - i$ and $n_{j_3} = n - j$ $(i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n - 1)$. Eqs. 10.b have been proposed in [8] for a transfer function (9). Note that they are particularizations of (4) for b_{ki} and a_{kj} being zero for $k < m - i$ and $k < n - j$, respectively. A state-space realization of (9) – (10) being algebraically equivalent to (8) is

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} A_i x(t - ih) + du(t), \tag{11.a}$$

$$y(t) = \sum_{j=0}^{m} c_{j0}^{T} x(t - jh) + \sum_{j=0}^{m} c_{j1}^{T} x(t - h' - jh) + \theta' u(t), \tag{11.b}$$

so that the output becomes undelayed while a single-internal delay is generated in the state equation. Furthermore, the two-variable $(\cdot)$-polynomials appearing as entries in $\sum c_{(\cdot)}^{T}\mathrm{Adj}(sI - A - \mu A_1)^{-1} d$ and $\mathrm{Det}(sI - A - \mu A_1)^{-1}$ have degrees verifying $\deg_s(\cdot) + \deg_\mu(\cdot) \leqslant m$ and $\deg_s(\cdot) + \deg_\mu(\cdot) \leqslant n$, respectively, for all real $n \times n$-matrices A and A_1, the above inequalities arising from the fact that all the entries in the matrix $(sI - A - \mu A_1)$ are of degree one in s and at most of degree one in μ.

REMARK 2. If $A_1 = 0$ in (11) then there is no internal delay ($h = 0$) and $G(s) = G_2(s)$. If there is no external delay ($h' = 0$) then $c_{ij} = 0,\ i \neq 0$ and $j \neq 0$, and $b^{(2)}_{(\cdot)} = 0$. If $A = 0$ and $c_{ij} = 0,\ i \neq 0$ and $j \neq 0$, then the pure internal plant delay h generates its n first integer multiples as additional delays, [8], leading to the transfer function $G(s) = c^T(sI - \mu A_1)^{-1}d$ explicited by

$$G(s) = \frac{\sum\limits_{i=0}^{m} b_i(\mu)s^i}{s^n + \sum\limits_{i=0}^{n-1} a_i(\mu)s^i} = \frac{\sum\limits_{i=0}^{m}\sum\limits_{k=i}^{m} b_{m-i,k-i}s^i\mu^{k-i}}{s^n + \sum\limits_{i=0}^{n-1}\sum\limits_{k=i}^{n} a_{n-i,k-i}s^i\mu^{k-i}}, \tag{12}$$

where some of the $a_{(\cdot)}$ and $b_{(\cdot)}$ could be zero.

REMARK 3. In the more general situation than (8) when the delays h_i and h'_i are not multiple of h and h', the realization of Fig. 1 can still be obtained by substituting the blocks μ and μ' by operators $\exp\left[-(h_i/h_{i-1})s\right]$ and $\exp\left[-(h'_i/h'_{i-1})s\right]$ corresponding to pairs of consecutive delays and the associate time-delay operators q^{-1} and q'^{-1}.

2.2. Plant model. The plant is assumed to be strictly proper (i.e., $\theta' = \theta'' = 0$) and described by the single-internal single-external point delay system (9)–(11) of Remark 1. The associate degree constraints $\deg(a_i(\mu)) = n - \deg(A_i(s)) = i$, $\deg(b^{(l)}_j(\mu)) = m - \deg(B^{(\cdot)}_j(s)) = j$; $i = 0, 1, \ldots, n-1$; $j = 0, 1, \ldots, m$ in (10) is then used to solve two-variable diophantine equations arising in solving the closed-loop pole-placement.

REMARK 4. In the general case that coefficients of the suited closed-loop characteristic equation be dependent on the internal delay, the numerator of the transfer function of the proposed controller is required to have p ($p \leqslant 2n - 1$) one-variable polynomials in s, of degree $n - 1$, n being the plant order, similar to (4), p being the higher power in μ in the characteristic equation. *Thus, the controller to be synthesized will be typically a general multi-point delay system (1)–(4) even if the plant has only single internal and external delays and the suited closed-loop equation has an infinite spectrum*, [8–13].

Since time-delay systems are infinite-dimensional, [3–4], [7]–[13], it is not evident, without a proof, that multi-variable polynomials may be factorized as products of multi-variable polynomials. The following result extends a well-known one for delay-free systems (see, for instance, [20], [21]) and establishes the fact that any plant transfer function numerator can be factorized into unique (except for a constant) complex-variable functions having their zeros within preassigned (disjoint) stability and instability subsets of C.

Lemma 1. *Define* $C_p^+ = \{z \in C : \operatorname{Re}(z) \leqslant -p\}$ *and* C_p^- *as being the complement of* C_p^+ *in* C *for any given real* $p > 0$*. Then, the numerators of the transfer function (8) (or (11)) has the following properties:*

(i) *If* $h \neq 0$*, then a unique (except for a nonzero constant) factorization* $B(s) = B(\mu, \mu', s) = B_p^+(s)B_p^-(s)$ *exists where the zeros of the complex variable functions* $B_p^+(s)$ *and* $B_p^-(s)$ *are in* C_p^+ *and* C_p^-*, respectively.*

(ii) *If* $h = 0$ *and* $h' \neq 0$*, then* $B(s) = \exp(-h's)B_p^+(s)B_p^-(s)$ *with the functions* $B_p^+(s)$ *and* $B_p^-(s)$ *being characterized as in* (i).

REMARK 5. The above result is a direct application of the auxiliary Lemma 2 of Section 3 below by particulariting polynomial degrees. The notation in the arguments (μ, μ', s) is not kept for the factors $B_p^+(s)$ and $B_p^-(s)$ since their explicit dependence on $\mu = \exp(-hs)$ and $\mu' = \exp(-h's)$ as two-variable polynomials is not proven in Lemma 2. However, note from Remark 1 that $B_p^+(s) = k_+[s^{m^+} + \widetilde{B}_p^+(s)]$ and $B_p^-(s) = \frac{k_p}{k_+}[s^{m-m^+} + \widetilde{B}_p^-(s)]$ for some complex functions $\widetilde{B}_p^+(s)$ and $\widetilde{B}_p^-(s)$ of respective degrees in s being less than m^+ and $(m - m^+)$, respectively. Such a property will then allow us to address realizability issues when $B(\mu, \mu', s)$ is involved for calculations.

2.3. Controller structures. Four stabilizing controller structures valid for the achievement of the pole-placement and model-matching with infinite or finite spectrum are now described.

General framework. Although the plant can be (state-space) realized with a unique internal delay since $\deg(a_i(\mu)) = n - \deg(A_i(s)) = i$ and $\deg(b_j^{(l)}(\mu)) = n - \deg(B_j^{(\cdot)}(s)) = j$ $(i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, m)$, according to the modelling issues of Section 2.2, the stabilizing controller is required to posses p internal delays provided that the degree in μ of the prefixed closed-loop characteristic equation is $p \leqslant 2n - 1$ (see Remark 4). Four parametrized controller structures are presented in this section being generically described as $\sum_c(\theta, \Lambda_\omega(\theta, t))$, within the general framework proposed in [23] for delay-free systems, where θ is a parameter vector of dimension n_c and $\Lambda_\omega(\theta, t) = \{r(t), \omega(h, h', \theta, t), \overline{\Lambda}(h, \theta, t)\}$ is an extended regressor parametrized also in $\theta \in R^{n_c}$ with $r(t)$ being the uniformly bounded external reference which will be taken as the input to an explicit reference model, $\omega(h,h',\theta,t)$ being the regressor of the parametrized controller given by a delayed (or undelayed in the case of finite spectra objectives) differential system and $\overline{\Lambda}(h,h',\theta,t) = \{c(h,h',\theta,t),\Lambda(h,\theta,t)\}$ with $\Lambda(h,\theta,t) = \{\lambda(h,\theta,t),\ \lambda_i(h,\theta,t),\ \lambda'_j(h,\theta,t);\ i = p_1+1,\ p_1+2,\ldots,2n-1;\ \ j = p_2+1,\ p_2+2,\ldots,2n-1;\ \ p_1 \geqslant 0;\ p_2 \geqslant 0$ is a set of auxiliary weighting functions which are used to compensate for the transmission of unsuited delays through the loop by weighting the time integrals of the reference signal and the plant input under the appropriate time-intervals related to plant delays. Such a strategy is adopted since, apart from the original internal delay, this one together with their integer multiples and their combinations with the external delay are transmitted through the feedback loop. In particular, the $c(\cdot)$-function weights the reference signal related to the external delay and the $\lambda_{(\cdot)}(\cdot)$-functions weight the plant input related to the transmission of the internal delay and its multiples while the $\lambda(\cdot)$ and $\lambda'_{(\cdot)}(\cdot)$-functions are used to compensate for the combined effects of the internal and external delays. In the auxiliary Λ-set, the non-negative integers p_1 and p_2 are chosen so that all the powers of the internal delay greater than p_1 and all their combinations with the external delay being greater than p_2 are cancelled by the sets $\lambda_{(\cdot)}(\cdot)$ and $\lambda'_{(\cdot)}(\cdot)$, respectively, while such a combined effect for powers of $\exp(-hs)$ less than p_2 are cancelled by $\lambda(h,\theta,t)$. The reason for a separate choice of the $c(\cdot)$ and the set $\lambda'_{(\cdot)}(\cdot)$ is that if the $\lambda'(t)$ were omitted, then the unsuitable powers of $\exp(-hs)$ in the infinite spectra objective could not be zeroed what would introduce "a priori" constrains in the choice of the reference model. Thus, p_1 and p_2 are chosen by the designer with $p = \max(p_1,\ p_2) \leqslant 2n-1$ and are related to the suited maximum power of the internal delay in the closed-loop characteristic two-variable polynomials. Details about particular structures and initializations of the functions of Λ_ω are given later for each particular controller structure. Note that ω and the elements of $\overline{\Lambda}$ are parametrized by θ. Thus, the control law to be adopted has the generic form which includes an auxiliary signal $\nu(\cdot)$, apart from the standard parametrized part, as follows

$$u(t) = \theta^T\omega(h,h',\theta,t) + \nu\left(\Lambda_\omega(h,h',\theta,t)\right). \tag{13}$$

The two first structures allow the achievement of both infinite and finite closed-loop spectra. The maximum power of the internal delay in the objective is chosen by the designer by the choice of p_1 and p_2 in the Λ-set and the regressor is the state of a differential system involving the internal delays of the plant and $(n-1)$ of its integer multiples. The two last controller structures can only be used for (delay-independent) finite spectra control objectives.

Controller structure I. The particular control law of (13) is given by

$$u(t) = \theta^T \omega(t) + \left[\int_{-h'}^{0} \lambda(\tau)u(t+\tau)d\tau + \sum_{i=p_1+1}^{2n-1} \int_{-ih}^{0} \lambda_i(\tau)u(t+\tau)d\tau \right.$$
$$+ \sum_{i=p_2+1}^{n+m-1} \int_{-(ih+h')}^{0} \lambda'_i(\tau)u(t+\tau)d\tau$$
$$\left. + c_0\left(r(t) + \int_{-h'}^{0} c(\tau)r(t+\tau)d\tau \right) \right], \tag{14}$$

where $\nu(\cdot)$ in (13.a) is the signal in brackets, c_0 is a scalar parameter, being unity when the static gain of the plant and the reference model are identical, $c(\tau)$ is a reference weighting function, and

$$\theta = \left[\overline{\theta}^T, \theta'_1, \theta'_2\right]^T, \quad \overline{\theta}^T = \left[\overline{\theta}^{(0)T}, \overline{\theta}^{(1)T}, \ldots, \overline{\theta}^{(n-1)T}\right], \tag{15.a}$$

$$\omega(t) = \left[\overline{\omega}^T(t), \overline{\omega}^T(t-h), \ldots, \overline{\omega}^T(t-(n-1)h), u(t), y(t)\right]^T, \tag{15.b}$$

$$\overline{\omega}(t) = \left[\overline{\omega}^{(1)T}(t), \overline{\omega}^{(2)T}(t)\right], \quad \overline{\theta}^{(i)T} = \left[\overline{\theta}^{(i_1)T}, \overline{\theta}^{(i_2)T}\right], \quad (i = 0, 1, \ldots, n-1), \tag{15.c}$$

$$\dot{\overline{\omega}} = \overline{F}\overline{\omega}(t) + \sum_{i=1}^{n-1} \overline{F}_i \overline{\omega}(t-ih) + \overline{q}\overline{u}(t), \quad \overline{w}(t) = 0, \quad t \in [(n-1)h, 0], \tag{15.d}$$

$$\overline{F} = \text{Diag}(F; F), \quad \overline{F}_i = \text{Diag}(F_i; F_i) \quad (i = 1, 2, \ldots, n-1), \tag{15.e}$$

$$\overline{u}(t) = [u(t), y(t)]^T, \quad \overline{q} = [\widetilde{q}|\widetilde{q}]^T, \quad \widetilde{q} = \text{diag}(q_1, q_2), \tag{15.f}$$

$$F = \begin{bmatrix} 0 & I_{n-2} \\ -f_{n-2,0}, -f_{n-3,0}, \ldots, -f_{00} & \end{bmatrix}, \quad F_i = \begin{bmatrix} 0_{(n-2)x(n-1)} \\ -f_{n-2,i}, -f_{n-3,i}, \ldots, -f_{0i} \end{bmatrix}, \quad (i = 1, 2, \ldots, n-1), \tag{15.g}$$

where $\overline{\omega}^{(l)}(t)$ and $\overline{\theta}^{(il)}$ $(i = 0, 1, \ldots, n-1;\ l = 1, 2)$ are $(n-1)$-vectors and F and $F_{(\cdot)}$ are $(n-1) \times (n-1)$-matrices. Thus, the number of design parameters in θ is $n_{c_1} \leqslant n_c \leqslant n_{c_2}$ with $n_{c_1} = 2n(n-1)$ (i.e., $\theta'_1 = \theta'_2 = 0$) and $n_{c_2} = 2n(n-1) + 2$ (i.e., $\theta'_1 \neq 0$ and $\theta'_2 \neq 0$) and the numbers of $\lambda_{(\cdot)}(\cdot)$ and $\lambda'_{(\cdot)}(\cdot)$-functions are, respectively, $(2n - p_1 - 1)$ and $(n + m - p_2 - 1)$. Note that the first right-hand side of (14) is the output of a general system involving internal delays only (i.e., a particularization of (8) to the external delay-free case) driven by $\overline{u}(t) = [u(t), y(t)]^T$ and whose dynamics is given by (15) with $n_1 = 1$ and $n_2 = n_3 = n - 1$. The elements of the auxiliary set Λ are proven to exist and defined in Section 3.2, Eq. 31 in order to keep a clear exposition of the main text. On the other hand, the choice of the $c(\cdot)$-function as well as the overall role of the $\overline{\Lambda}$-set in the "a priori" closed-loop spectra, in the sense that the degree of $\exp(-hs)$ is constrained in the closed-loop transfer function denominator prior to the choice of θ, is later summarized in Proposition 1 for this particular controller structure as well as for the remaining ones.

Controller structure II. The control law is now

$$u(t) = \theta^T \omega(t) + \Bigg[\int_{h'}^{t} \lambda(\tau) u(t-\tau) d\tau + \sum_{i=p_1+1}^{2n-1} \int_{ih}^{t} \lambda_i(\tau) u(t-\tau) d\tau + \sum_{i=p_2+1}^{n+m-1} \int_{(ih+h')}^{t} \lambda'_i(\tau) u(t-\tau) d\tau + c_0 \left(r(t) + \int_{0}^{t} c(\tau) r(t-\tau) d\tau \right) \Bigg], \qquad (16)$$

subject to (15). Note that Controller II, compared to Controller I, contains convolution integral-type terms constructed with the elements of the Λ-set which are defined in Section 3.2, Eq. 34.

The two next structures are particularizations of Controllers I and II, respectively, and they involve delay-free controller dynamics. It will be seen that closed-loop, internal delay-dependent dynamics is unachievable by the use of those controllers.

Controller structure III. The control law is now

$$u(t) = \theta^T \omega(t) + \left[\sum_{i=1}^{n} \int_{-ih}^{0} \lambda_i(\tau) u(t+\tau) d\tau + \sum_{i=0}^{m} \int_{-(ih+h')}^{0} \lambda'_i(\tau) u(t+\tau) d\tau + c_0 \left(r(t) + \int_{-h'}^{0} c(\tau) r(t+\tau) d\tau \right) \right], \tag{17}$$

where

$$\theta = \left[\overline{\theta}^{(1)T}, \overline{\theta}^{(2)T}, \theta'_1, \theta'_2 \right]^T, \qquad \omega(t) = \left[\overline{\omega}^{(1)T}(t), \overline{\omega}^{(2)T}(t), u(t), y(t) \right]^T, \tag{18.a}$$

$$\dot{\overline{\omega}}(t) = \overline{F}\overline{\omega}(t) + \overline{q}\overline{u}(t), \quad \overline{\omega}(t) = \left[\overline{\omega}^{(1)T}(t), \overline{\omega}^{(2)T}(t) \right]^T, \tag{18.b}$$

with $\overline{F}$, $\overline{q}$ and $\overline{u}$ being defined as for Controllers I–II and θ'_l $(l = 1, 2)$ being zero or nonzero scalars (see (15.e) to (15.g)). In this case, the dimension of n_c is constrained to $2n - 2 \leqslant n_c \leqslant 2n$, n_c being the dimension of θ and $\omega(t)$ and there are n $\lambda_{(\cdot)}$-function and m $\lambda'_{(\cdot)}$-functions defined in Section 3.2 (Eq. 36).

Controller structure IV. The control law is now

$$u(t) = \theta^T \omega(t) + \left[\sum_{i=1}^{n} \int_{ih}^{t} \lambda_i(\tau) u(t-\tau) d\tau + \sum_{i=0}^{m} \int_{(ih+h')}^{t} \lambda'_i(\tau) u(t-\tau) d\tau + c_0 \left(r(t) + \int_{0}^{t} c(\tau) r(t-\tau) d\tau \right) \right], \tag{19}$$

with θ and ω being defined as in (18) and F and q being defined as in (15.g) – (15.e). The Λ-set is defined by (38).

REMARK 6. Note that if $p_1 = 2n - 1$ and $p_2 = m + 1$ then all the functions $\lambda_{(\cdot)}(\cdot)$ and $\lambda'_{(\cdot)}(\cdot)$ in Eq. 14 and Eq. 16 are deleted then the closed-loop spectrum would be dependent of the higher powers of $\exp(-hs)$. Also, $\max(p_1, p_2) < n$ in order that the two-variable diophantine equations associated with the pole-placement problem have a solution for some parametrization θ of the controller.

The next assumptions about both plant and controller are made to address the model matching problem in the case of known plants.

Assumption 1. (a) Both the internal and external delay, are bounded and the plant is strictly proper (i.e., $n > m$ and (open-loop) stabilizable; i.e., $\text{rank}\big[sI - A - \exp(-hs)A_1, d\big] = n$ for all $s \in C$ with Re $s \geqslant 0$ if Controllers I or III are used.

(b) The delays are bounded and the plant is strictly proper and strictly stable if Controllers II or IV are used; i.e., $\det\big(sI - A - \exp(-hs)A_1\big) = 0$ has all roots in Re $s < 0$.

Assumption 2. The parameterized part of the controller is spectrally controllable and strictly stable; i.e., F, F_i $(i = 1, 2, \ldots, n-1)$ and q are chosen verifying

$$\text{rank}\left[sI - F - \sum_{i=1}^{n-1} \exp(-ihs)F_i, q\right] = n - 1, \quad \text{all } s \in C, \tag{20.a}$$

and

$$\begin{aligned} D(s) &= \text{Det}\left(sI - F - \sum_{i=1}^{n-1} \exp(-ihs)F_i\right) \\ &= \sum_{i=0}^{n-1} D_i(s)m^i = \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} d_{ki}s^k m^i = 0 \end{aligned} \tag{20.b}$$

has all roots in Re $s < 0$, with $F_i = 0$ $(i = 1, 2, \ldots, n-1)$ for Controllers III and IV and, furthermore, all $D_{(\cdot)}(\cdot)$-polynomials are monic; i.e., $D_i(s) = s^{n-1} + \overline{D}_i(s)$ with deg $\overline{D}_i(s) = n - 2$.

The stabilizability Assumption 1(a) is necessary for stabilization purposes. If could be substituted by the stronger one of plant spectral controllability, namely, the given rank condition holds for all $s \in C$. Assumption 1(b) is stronger than Assumption 1(a) and it is motivated by the fact that Controller II and IV involve the use of convolution integrals over increasing time intervals.

The next result, which is proved in Section 3.2, is related to the role of the auxiliary set $\overline{\Lambda}(\theta,t)$ for synthesis of the closed-loop transfer function.

Proposition 1. (Achievable Transfer function from $\overline{\Lambda}(\theta,t)$). Assume that Assumptions 1–2 hold and consider the factorizations $B(\mu,\mu',s) = B_p^+(s)$ $B_p^-(s)$ of Lemma 1 for a given $p>0$ and $D(\mu,s) = D_1(s)D_2(s)$ of Lemma A.1(i) with all the zeros of $D_1(s)$ in $C_1 \subset C_{p'}^+$, and those of $D_2(s)$ in $C_2 \subset C_{p'}^+$ and $C_{p'}^+ = \{z \in C;\ \mathrm{Re}\, z \leqslant -p'\}$ some $p'>0$. Thus, for any Controller structure I to IV parameterized at $\theta \in R^{n_c}$, these exits an auxiliary set $\overline{\Lambda}(\theta,t)$ such that the closed-loop transfer function is given by

$$\frac{y(s)}{r(s)} = \frac{c_0 k_p D(\mu,s)B(\mu,\mu',s)}{M(s)\,[Q_0(s)+Q_1(\mu,s)]} = \frac{c_0 k_p B_p^-(s)D_2(s)}{Q_0(s)+Q_1(\mu,s)}, \tag{21}$$

where

$$M(s) = D_1(s)B_p^+(s) = \begin{cases} \dfrac{1}{[1+(1-\mu')\overline{c}(s)]} & \text{(Controllers I and III)}, \\ \dfrac{1}{[1+c(s)]} & \text{(Controllers II and IV)}, \end{cases} \tag{22.a}$$

so that

$$\overline{c}(s) = L\{c(-t)\} = \frac{1}{1-\mu'}\left[\frac{1-D_1(s)B_p^+(s)}{D_1(s)B_p^+(s)}\right], \tag{22.b}$$

$$c(s) = L\{c(t)\} = \frac{1-D_1(s)B_p^+(s)}{D_1(s)B_p^+(s)}, \tag{22.c}$$

and $Q_0(s)+Q_1(\mu,s)$ is calculated from (32), (33), (35) and (37) (Section 3.2) for the Controller structures I to IV, respectively with $Q_1(\mu,s)=0$ being delay-independent and for Controller structures II and IV. Note that (22.b) is a synthesizable signal since $L\{(1-q'^{-1})c(-t)\} = (1-D_1(s)B_p^+(s))/D_1(s)B_p^+(s)$ is a filter of zero relative degree having impulse response $v(t)$ so that $c(-t) = c(-t-h') + v(t)$, or, $c(t) = c(t-h') + v(-t)$ for all $t \geqslant 0$, the negative time argument meaning that t is changed to $(-t)$ when the filter impulse response is calculated (see Notation Section).

2.4. Control objectives. A reference model defining the suitable behaviour for the plant is defined by the transfer function

$$G_m(s) = G_m(\mu,\mu',s) = k_m \frac{B_m(\mu,s)}{A_m(\mu,s)}, \tag{23}$$

with $A_m(\mu, s) = A_{m0}(s) + A_m^1(\mu, s) = \sum_{i=0}^{n_m} A_{mi}(s)\mu^i$. Although the numerator of $G_m(s)$ depends, in general, on the internal delay, its possible two-variable polynomial structure is not obvious, as an achievable objective, from the factorization of the numerator of the closed-loop transfer function (21) and the considerations of Remark 5. This point will be discussed later since for model matching purposes, the numerator of (23) has to be fixed to that of (21). The next assumption is introduced.

Assumption 3. The reference model is strictly stable and realizable so that all the roots of $A_m(\mu, s) = 0$ are in Re $s < 0$ and $n_m = \deg_s(A_m(\mu, s)) \geqslant m_m = \deg(B_m(\mu, s))$. Furthermore, $n_m - m_m \geqslant n - m$ (thus, the controller is strictly proper from Assumption 1(a)).

Note that the reference model is realizable and model matching is an "a priori" achievable objective since $n_m - m_m \geqslant n - m$. The control objectives are:

Objective 1 (*Model matching with infinite spectrum independent of the external delay*). In this case, $\theta \in R^{n_c}$ exists such that (21) equalizes (23) with $A_m^1(\mu, s) \neq 0$. It can be divided into two subobjectives, namely:

(a) Gain and zero matching: The following design constraints hold $c_0 = k_m k_p^{-1}$ and $B_m(s) = B_p^-(s) D_2(s)$. Since $D(\mu, s)$ is freely designed by choosing the matrices F and $F_{(\cdot)}$, the reference model includes the unstable plant zeros related to the stability domain C_p^+ plus a number of free stable zeros given by $D_2(s)$. Note that $\deg_s(D_2) = m_m - \deg_s(B_p^-) \leqslant m + n_m - n - \deg(B_p^-)$.

(b) Pole-assignment: It is achieved with a particular controller within the Controller structures I and II provided that there exists a parameter vector $\theta \in R^{n_c}$ such that $Q_0(s) + Q_1(\mu, s) = A_m(\mu, s)$. This objective is unachievable with $Q_1(\mu, s) \neq 0$ from the Controller structures III and IV since those ones do not possess internal delays.

Objective 2 (*Model matching with finite spectrum*). It is similar to Objective 1 but $A_m^1(\mu, s) = 0$ so that the closed-loop characteristic equation can match to that of a delay-independent reference model.

Objective 2 will be proven to be achievable with any of the given controller structures under a set of (rather weak) Assumptions including Assumptions 1–3. The existence of particular parametrizations θ of Controllers I–IV leading to the achievement of Objectives 1–2 is addressed in the next main result of this section which is proven in Section 3.2.

Theorem 1. *Under Assumptions 1–3, consider the factorization for* $B(\mu,\mu',s)$ *and* $D(\mu,s)$ *of Proposition 1 for arbitrary real positive constants* p_1 *and* p_2 *with* $c_0 = k_m k_p^{-1}$ *and* $c(-t)$ *(Controllers I–III) or* $c(t)$ *(Controllers II–IV) being defined by (22.b) and (22.c), respectively. Thus, the next propositions hold.*

(i) *If* $n-1 \geqslant p_1 \geqslant p_2 \geqslant 0$ *and* $\theta'_2 \neq 0$, *then a (nonnecessary unique) parameter vector* θ *and a set auxiliary functions,* $\overline{\Lambda}(\theta,t)$ *parameterized at* θ, *exist for Controller I so that Objectives 1–2 are achieved for any strictly Hurwitz* $A_m(\mu,s) = A_{m0}(s) + A_m^1(\mu,s) = \sum_{i=0}^{m} A_{mi}(s)\mu^i$ $(A_m^1(\mu,s)$ *being zero for Objective 2) provided that the next two conditions hold:*

1. $\deg(A_{mi}(s)) = 2n-1$ *if* $\theta'_1 \neq 0$ *and* $\deg(A_{mi}(s)) = 2n-2$ *if* $\theta'_1 = 0$ *for* $i = 0,1,\ldots,p_1$.
2. *Each polynomial pair* $\left(A_0(s),\ B_0^{(1)}(s)+\delta_i B_0^{(2)}(s)\right)$ *is a coprime pair;* $i = 0,1,\ldots,p_1$ *with* $\delta_i = 1$ *for* $i = 0,1,\ldots,p_2$ *and* $\delta_i = 0$ *for* $i = p_2+1, p_2+2,\ldots,p_1$. *A sufficient condition is that* $A(0,s)$, $B(0,1,s)$ *and* $B(0,0,s)$ *be coprime in the variable* s.

(ii) *Proposition* (i) *holds for some parametrization of Controller II provided that Condition 1 holds and* $\left(A_0(s),\ B_0^{(1)}(s)\right)$ *or, equivalently,* $A(0,s)$ *and* $B(0,0,s)$ *are coprime pairs.*

(iii) *Objective 2 is achieved for some parametrization* θ *of Controller III provided that* $A_m^1(\mu,s) = 0$, *Condition 1 holds for* $i = 0$ *and the maximum common factor of* $\left(\sum_{i=1}^{n} A_i(s)\right)$ *and* $\left(\sum_{i=0}^{m}\left(B_i^{(1)}(s)+B_i^{(2)}(s)\right)\right)$ *or, equivalently, that of* $A(1,s)$ *and* $B(1,1,s)$ *divides* $A_{m0}(s)$.

(iv) *Proposition* (iii) *holds for some parametrization of Controller IV provided that the last condition is changed by the maximum common factor of* $A_0(s)$ *and* $B_0^{(1)}(s)$ *or, equivalently, that of* $A(0,s)$ *and* $B(0,0,s)$ *dividing* $A_{mo}(s)$.

The alternative conditions on quasipolynomials which guarantee the fulfilment of Theorem 1 [(i) – (iv)] follow directly by comparison with those established for one-variable polynomials. In the same way, the replacement of the coprimeness conditions of (i) – (ii) by more general ones is obvious by first cancelling all the common factors in the diophantine equations so that the resulting conditions become converted into coprimeness conditions of the coefficient polynomials. Each (one-variable) solution can only be explicited after expliciting the associated ones which those corresponding to preceding powers of μ.

3. Mathematical proofs

3.1 Preliminary technical result to Lemma 1. The proof of the next result leads directly to Lemma 1.

Lemma 2. *Consider disjoint sets C_ν^+ and C_ν^- for any real constant ν and any two-variable polynomial*

$$Q(s) = Q(\mu, \mu', s) = Q_1(\mu, s) + \mu' Q_2(\mu, s)$$
$$= \sum_{i=0}^{m_q} \left[q_i^{(1)}(\mu) + \mu' q_i^{(2)}(\mu)\right] s^{m_q - i}, \qquad (24)$$

with

$$q_i^{(1)}(\mu) = \sum_{k=0}^{n_{q_{i1}}} q_{m_q-i,k}^{(1)} \mu^k$$
$$= \sum_{k=0}^{n_q} q_{m_q-i,k}^{(1)} \mu^k \quad (q_{00}^{(1)} \neq 0 \quad \text{if} \quad h \neq 0), \qquad (25.a),$$
$$q_i^{(2)}(\mu) = \sum_{k=0}^{n_{q_{i2}}} q_{m_q-i,k}^{(2)} \mu^k$$
$$= \sum_{k=0}^{n_q} q_{m_q-i,k}^{(2)} \mu^k \quad (q_{00}^{(2)} \neq 0 \quad \text{if} \quad h' \neq 0), \qquad (25.b)$$

and $n_q = \max_{0 \leqslant i \leqslant m_q} (n_{q_{i1}}, n_{q_{i2}})$ so that some of the $b_{(\cdot)}$-coefficients in (25.a) – (25.b) *can be structurally zero. Then, the following propositions hold:*

(i) *If $h \neq 0$, then a unique (except for a nonzero constant) factorization $Q(s) = Q_\nu^+(s) Q_\nu^-(s)$ exist with the zeros (which can, possible, be an infinite number) of the complex variable functions $Q_\nu^+(s)$ and $Q_\nu^-(s)$ being in $C_\nu^+(s)$ and $C_\nu^-(s)$, respectively.*

(ii) *If $h = 0$ and , then there exist $Q_\nu^+(s)$ and $Q_\nu^-(s)$ with the same properties as in (ii) such that $Q(s) = \exp(-h's) Q_\nu^+(s) Q_\nu^-(s)$.*

(iii) *$Q(s)$ can be uniquely factorized, except for a nonzero constant, as $Q_1(s)Q_2(s)$ with $Q_1(s)$ and $Q_2(s)$ being complex-variable functions which have zeros on (nonnecessarily connected) arbitrary disjoint subsets of C.*

Proof. (i) For all complex $s = \sigma + j\omega$, $|\mu| = |\exp(-h\sigma)|$, $|\mu'| = |\exp(-h'\sigma)|$. From (24)–(25), $Q(s) = s^{m_q}\left[\sum_{i=0}^{m_q}\left(q_i^{(1)}(\mu) + \mu' q_i^{(2)}(\mu)\right)s^{-i}\right]$ since $h \neq 0$, $q_{00}^{(1)} \neq 0$. Thus, note the following facts: (a) $|Q(s)| \to \infty$ at the rate $|\sigma|^{m_q}|q_{00}^{(1)}|$ as $\sigma = \mathrm{Re}(s) \to +\infty$, since $\mu \to 0$, $\mu' \to 0$ as $\sigma \to +\infty$, namely, there exist two positive real constant k_1 and k_2 ($k_2 \geqslant k_1$) such that for some positive real constant σ_1 and all $\sigma = \mathrm{Re}(s) \geqslant \sigma_1$, $k_1|\sigma|^{m_q}|q_{00}^{(1)} \leqslant |Q(s)| \leqslant k_2|\sigma|^{m_q}|q_{00}^{(1)}|$.

(b) $|Q(s)| \to \infty$ as $\sigma = \mathrm{Re}(s) \to -\infty$ at the rate of

$$\max\Big[\max_{\substack{0\leqslant j\leqslant n_q\\0\leqslant i\leqslant m_q}}\left(\exp(jh|\sigma|)|q_{0j}^{(1)}||\sigma|^{m_q-i}\right),$$

$$\max_{\substack{0\leqslant j\leqslant n_q\\0\leqslant i\leqslant m_q}}\left(\exp\left((jh+h')|\sigma|\right)\right)\left(q_{0j}^{(2)}||\sigma|^{m_q-i}\right)\Big].$$

In particular, that divergence occurs at the rates of $\exp(hn_q|\sigma|)|q_{0n_q}^{(1)}|$ or $\exp[(n_q h + h')|\sigma|]|q_{0n_q}^{(2)}|$ if $q_{0n_q}^{(1)} \neq 0$ and $q_{0n_q}^{(2)} = 0$, or $q_{0n_q}^{(1)} = 0$ and $q_{0n_q}^{(2)} \neq 0$, respectively.

By combining (a) and (b), $|Q^{-1}(s)| \to 0 \Rightarrow Q^{-1}(s) \to 0$ for $\sigma \to \pm\infty$ as $|s| \to \infty$. Now, proceed by contradiction by assuming that $Q(s)$ has no zeros so that $Q^{-1}(s)$ is analytic in C and converges to zero as $|s| \to \infty$ and, thus, is bounded in C. Then, by Liouville's theorem [17], the analytic and bounded function $Q^{-1}(s)$ is constant in C which is a contradiction to its convergence to zero as $|s| \to \infty$. Thus, $Q(s)$ has a zero $s_0 \in C$ so that $Q(s) = (s - s_1)Q_1(s)$ for some complex-variable function $Q_1(s)$. Now, proceed recursively from $j = 0$ to $j = \alpha$ to yield $Q(s) = \left[\prod_{i=0}^{j-1}(s - s_i)\right]Q_j(s) = \left[\prod_{i=0}^{j}(s - s_i)\right]Q_{j+1}(s)$, where $s_j \in C$ is a zero of $Q_{j+1}(s)$, where a is some finite or (denumerable) infinite nonnegative integer such that $Q_{\alpha+1}(s) = k_Q$, some real nonzero constant k_Q. Thus, $Q(s) = k_Q\left[\prod_{i=0}^{\alpha}(s - s_i)\right] = k_Q\left[\prod_{s_i C_\nu^+}(s - s_i)\right]\left[\prod_{s_i \in C_\nu^-}(s - s_i)\right]$ and the proof of (i) is complete with $Q_\nu^+(s) = \prod_{s_i \in C_\nu^+}(s - s_i)$ and $Q_\nu^-(s) = \prod_{s_i \in C_\nu^-}(s - s_i)$.

(ii) For $h = 0$ and $h' \neq 0$, $Q(s) = \mu'\widetilde{Q}(s)$ and (i) can be applied to $\widetilde{Q}(s)$.

(iii) If follows directly from $Q(s) = k_Q\left[\prod_{i=0}^{\alpha}(s - s_i) = k_Q Q_1(s)Q_2(s)\right.$ with

$Q_j(s) = \prod_{s_{ij} \in D_j} (s - s_{ij})$ $(j = 1, 2)$ for arbitrary disjoint subsets D_1 and D_2 of C.

3.2. Achievable closed-loop transfer functions from the Controller structures I to IV

3.2.1. Preliminary technical lemmas. The next two result are then used in the next subsections:

Lemma 3. *Consider the Laplace transformable functions* ρ: $(-\infty, 0] \to R$, $\bar{\rho}$: $[0, \infty) \to R$ *and* ν: $[0, \infty) \to R$ *with* $\bar{\rho}(t) = \rho(-t)$ *for* $t \geqslant 0$. *Then,* $L\left\{ \int_{-\delta}^{0} \rho(\tau)\nu(t+\tau)\, d\tau \right\} = (1 - \exp(-\delta s))\bar{\rho}(s)\nu(s)$ *for any real constant* $\delta \geqslant 0$, *where* $\bar{\rho}(s) = L\{\bar{\rho}(t)\} = L\{\rho(-t)\}$. *Also, for any Laplace transformable function* ρ: $[0, \infty) \to R$, $L\left\{ \int_{\delta}^{\infty} \rho(\tau)\nu(t + \tau)\, d\tau \right\} = \exp(-\delta s)\rho(s)\nu(s)$.

Outline of proof. To prove the first identity, take Laplace transforms in the identity $\int_{-\delta}^{0} \rho(\tau)\nu(t + \tau)\, d\tau = \int_{0}^{\infty} \bar{\rho}(-\tau)\nu(t - \tau)\, d\tau - \int_{0}^{\infty} \bar{\rho}(-\tau)[\nu(t - \tau)U_\delta(t - \tau)]\, d\tau$ where $U_\delta(\tau)$ is the unity step function at $t = \delta$. The second identity can be found in any basic introductory text to Laplace transforms.

Lemma 4. *Consider two-variable polynomials* $Q(\mu, s) = \sum_{i=0}^{q} Q_i(s)\mu^i$ *and* $Q'(\mu, s) = \sum_{i=0}^{q'} Q'_i(s)\mu^i$. *Then,* $\overline{Q}(\mu, s) = Q(\mu, s)Q'(\mu, s) = \sum_{i=0}^{q+q'} \overline{Q}_i(s)\mu^i = \sum_{j=0}^{q} \sum_{i=j}^{j+q} Q_j(s)Q'_{i-j}(s)\mu^i = \sum_{j=0}^{q'} \sum_{i=j}^{j+q} Q'_j(s)Q_{i-j}(s)\mu^i$ *where* $\overline{Q}_i(s) = \sum_{j=\max(0,i-q')}^{\max(i,q)} Q_j(s)Q'_{i-j}(s) = \sum_{j=\max(0,i-q')}^{\max(i,q)} Q'_j(s)Q_{i-j}(s)$ $(i = 0, 1, \ldots, q + q')$.

The proof follows directly by expanding in powers of μ.

3.2.2. Calculations of $\overline{\Lambda}(q,t)$ for Controller Structures I–IV. Proof of Proposition 1

Controller structure I. By taking Laplace transforms in (15.b) – (15.d), with zero initial conditions so that $y(s) = G(s)u(s)$ with $G(s)$ defined in (9), one gets directly

$$\theta^T \omega(s) = [F_1(s) + F_2(s)G(s)]\, u(s) \tag{26}$$

with

$$F_l(s) = \frac{N_l(\mu, s)}{D(\mu, s)} = \frac{\overline{N}_l(\mu, s)}{D(\mu, s)} + \theta'_l$$
$$= \bar{\theta}^{(l)T}\left(sI - F - \sum_{i=0}^{n-1} \mu^i F_i\right)^{-1} q + \theta'_l \quad (l = 1, 2) \tag{27}$$

with $\overline{N}_l(\mu, s)/D(\mu, s)$ $(l = 1, 2)$ being strictly proper that implies the realizability of $F_{(\cdot)}(s)$ since $\deg_s(N_{(\cdot)}(\mu, s)) \leqslant \deg_s(D(\mu, s)) = n - 1$ with strict inequalities holding if and only if $\theta'_{(\cdot)}$ is zero. By taking Laplace transforms in (14) with zero initial conditions and the use of (26) – (27) together with Lemma 3, one gets directly

$$\begin{aligned} u(s) =& c_0[1 + (1 - \mu')\bar{c}(s)]r(s) \\ &+ \left\{ \sum_{i=p_1+1}^{2n-1} (1 - \mu^i)\bar{\lambda}_i(s) + (1 - \mu')\bar{\lambda}(s) + \sum_{i=p_2+1}^{n+m-1} (1 - \mu'\mu^i)\bar{\lambda}'_i(s) \right. \\ &+ \frac{1}{D(\mu, s)A(\mu, s)}[A(\mu, s)N_1(\mu, s) + B_1(\mu, s)N_2(\mu, s) \\ &\left. + \mu' B_2(\mu, s)N_2(\mu, s)] \right\} u(s), \end{aligned} \tag{28}$$

with $\bar{c}(s) = L\{c(-t)\}$, $\bar{\lambda}(s) = L\{\lambda(-t)\}$, $\bar{\lambda}_{(\cdot)}(s) = L\{\lambda_{(\cdot)}(-t)\}$ and $\bar{\lambda}'_{(\cdot)}(s) = L\{\lambda'_{(\cdot)}(-t)\}$. The particularization of (3) to the plant (9)-(11) and Controller I, (14) – (15), leads to

$$A(\mu, s) = \sum_{i=0}^{n} A_i(s)\mu^i; \quad B_l(\mu, s) = \sum_{i=0}^{m} B_i^{(l)}(s)\mu^i, \tag{29.a}$$

$$\begin{aligned} D(\mu, s) &= \sum_{i=0}^{n-1} D_i(s)\mu^i; \\ N_l(\mu, s) &= \sum_{i=0}^{n-1} N_i^{(l)}(s)\mu^i = \sum_{i=0}^{n-2} \overline{N}_i^{(l)}(s)\mu^i + \theta'_l D(\mu, s) \end{aligned} \tag{29.b}$$

for $l = 1, 2$. The substitution of (29) into (27) and the use of Lemma 4 yield

$$
\begin{aligned}
&\left\{\sum_{i=0}^{p_1}\left[\sum_{j=j_{i_1}}^{n-1}\left(D_j(s)-N_j^{(1)}(s)\right)A_{i-j}(s)-\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(1)}(s)\right]\mu^i\right.\\
&+\sum_{i=p_1+1}^{2n-1}\left[D(\mu,s)A(\mu,s)\bar{\lambda}_i(s)+\sum_{j=j_{i_1}}^{n-1}\left(D_j(s)-N_j^{(1)}(s)\right)A_{i-j}(s)\right.\\
&\left.-\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(1)}(s)\right]\mu^i-D(\mu,s)A(\mu,s)\left[\bar{\lambda}(s)\sum_{i=p_1+1}^{2n-1}\bar{\lambda}_i(s)\right.\\
&\left.+\sum_{i=p_2+1}^{n+m-1}\bar{\lambda}_i'(s)\right]+\sum_{i=p_2+1}^{n+m-1}\left[D(\mu,s)A(\mu,s)\lambda_i'(s)\right.\\
&\left.-\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(2)}(s)\right]\mu^i\mu'+\left[D(\mu,s)A(\mu,s)\bar{\lambda}(s)\right.\\
&\left.\left.-\sum_{i=0}^{p_2}\left(\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(2)}(s)\right)\mu^i\right]\mu'\right\}u(s)\\
=&c_0\left[1+(1-\mu')\bar{c}(s)\right]D(\mu,s)A(\mu,s)r(s),
\end{aligned}
\tag{30}
$$

with $j_{i_1}=\max(0,i-n)$, $j_{i_2}=\max(0,i-m)$. Now, the subset $\Lambda(\theta,t)$ of $\overline{\Lambda}(\theta,t)$ is determined for any controller parametrization θ by taking Laplace antitransforms of

$$
\bar{\lambda}(s)=\frac{1}{D(\mu,s)A(\mu,s)}\left[\sum_{i=0}^{p_2}\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(2)}(s)\right]\mu^i, \tag{31.a}
$$

$$
\begin{aligned}
\bar{\lambda}_i(s)=&\frac{1}{D(\mu,s)A(\mu,s)}\left[\sum_{j=j_{i_1}}^{n-1}\left(N_j^{(1)}(s)-D_j(s)\right)A_{i-j}(s)\right.\\
&\left.+\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(2)}(s)\right],\\
&i=p_1+1,p_1+2,\ldots,2n-1,
\end{aligned}
\tag{31.b}
$$

$$
\begin{aligned}
\bar{\lambda}_i'(s)=&\frac{1}{D(\mu,s)A(\mu,s)}\left[\sum_{j=j_{i_2}}^{n-1}N_j^{(2)}(s)B_{i-j}^{(1)}(s)\right],\\
&i=p_2+1,p_2+2,\ldots,n+m-1.
\end{aligned}
\tag{31.c}
$$

Now, substituting (22.b), (31) into (30) and the obtained u(s)/r(s) into the closed-loop transfer function $y(s)/r(s) = G(s)u(s)/r(s)$ leads directly to (21) with

$$\begin{aligned} Q_0(s) =& \left[D_0(s) - N_0^{(1)}(s)\right]A_0(s) - N_0^{(2)}(s)\left[B_0^{(1)}(s) + B_0^{(2)}(s)\right] \\ &+ \left\{ \sum_{i=p_1+1}^{2n-1} \left[\sum_{j=j_{i_1}}^{n-1} \left(D_j(s) - N_j^{(1)}(s)\right) A_{i-j}(s) \right.\right. \\ &- \sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(1)}(s) \\ &- \left.\left. \sum_{i=p_2+1}^{2n-1} \sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(2)}(s) \right] \right\}, \end{aligned} \tag{32.a}$$

$$\begin{aligned} Q_1(\mu, s) =& \sum_{i=1}^{p_1} \left[\sum_{j=j_{i_1}}^{n-1} \left(D_j(s) - N_j^{(1)}(s)\right) A_{i-j}(s) \right. \\ &- \left. \sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(1)}(s) \right] \mu^i \\ &- \sum_{i=1}^{p_2} \left[\sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(2)}(s) \right] \mu^i, \end{aligned} \tag{32.b}$$

and Proposition 1 is proved for Controller I.

Controller structures II-III and IV. The results for the remaining Controllers, obtained in a similar way as in the previous subsection, are the following:

(i) Controller II:

$$Q_0(s) = \left[D_0(s) - N_0^{(1)}(s)\right]A_0(s) - N_0^{(2)}(s)B_0^{(1)}(s), \tag{33.a}$$

$$\begin{aligned} Q_1(\mu, s) =& \sum_{i=1}^{\rho_1} \left[\sum_{j=j_{i_1}}^{n-1} \left(D_j(s) - N_{1j}(s)\right) A_{i-j}(s) \right. \\ &\left. - \sum_{j=j_{i_2}}^{n-1} N_{2j}(s) B_{i-j}^{(1)}(s) \right] \mu^i, \end{aligned} \tag{33.b}$$

with

$$\lambda(s) = - \frac{1}{D(\mu, s)A(\mu, s)} \left[\sum_{i=0}^{p_2} \left(\sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(2)}(s) \right) \mu^i \right], \tag{34.a}$$

$$\lambda_i(s) = \frac{1}{D(\mu,s)A(\mu,s)}\left[\sum_{j=j_{i_1}}^{n-1}\left(D_j(s) - N_j^{(1)}(s)\right)A_{i-j}(s) - \sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s)B_{i-j}^{(2)}(s)\right], \tag{34.b}$$

$$i = p_1+1, p_1+2, \ldots, 2n-1,$$

$$\lambda_i'(s) = -\frac{1}{D(\mu,s)A(\mu,s)}\left[\sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s)B_{i-j}^{(1)}(s)\right], \tag{34.c}$$

$$i = p_2+1, p_2+2, \ldots, n+m-1.$$

(ii) Controller III:

$$Q_0(s) = \left(\sum_{i=0}^{n} A_i(s)\right)[D(s)-N_1(s)] - \sum_{i=0}^{m}\left[B_i^{(1)}(s)+B_i^{(2)}(s)\right]N_2(s), \tag{35}$$

with

$$\bar{\lambda}_i(s) = \frac{A_i(s)[N_1(s)-D(s)] + B_i^{(1)}(s)N_2(s)}{D(s)A(\mu,s)}; \tag{36.a}$$

$$i = 1,2,\ldots,m,$$

$$\bar{\lambda}_i(s) = \frac{A_i(s)[N_1(s)-D(s)]}{D(s)A(\mu,s)}, \quad \bar{\lambda}_j'(s) = \frac{B_j^{(2)}(s)N_2(s)}{D(s)A(\mu,s)}, \tag{36.b}$$

$$i = m+1,\ldots,n \quad j = 0,1,\ldots,m.$$

(iii) Controller IV:

$$Q_0(s) = A_0(s)[D(s)-N_1(s)] - B_0^{(1)}(s)N_2(s), \tag{37}$$

with

$$\lambda_i(s) = \frac{[D(s)-N_1(s)]A_i(s) - N_2(s)B_i^{(1)}(s)}{D(s)A(\mu,s)}, \tag{38.a}$$

$$i = 1,2,\ldots,m$$

$$\lambda_i(s) = \frac{[D(s)-N_1(s)]A_i(s)}{D(s)A(\mu,s)}, \quad \lambda_j'(s) = \frac{N_2(s)B_j^{(2)}(s)}{D(s)A(\mu,s)}, \tag{38.b}$$

$$i = m+1, m+2\ldots,n \quad j = 0,1,\ldots,m.$$

3.3. Proof of Theorem 1. (i) From (24), the numerator of (21) equalizes that of (23) and zero matching is achieved for any reference model including the unstable plant zeros and having free stable zeros at $D_2(\mu, s) = 0$. Note that the degree constraint on the factor D_2 of $D(\mu, s)$ ensures that the relative degree of the closed-loop transfer function is non less than that of the plant (Assumption 3). Thus, Objectives 1–2 are fulfilled for the auxiliary set $\overline{\Lambda}(\theta, t)$ generating the closed-loop transfer function (21) if and only if the denominator of (21) and (23) become equal for some θ (namely, if and only if pole-placement is achieved for some θ). Thus, Objective 1 is achieved if and only if for any arbitrary polynomial $\overline{A}_{m0}(s)$ of degree $2n-1$

$$Q_0(s) + Q_1(\mu, s) = \overline{A}_{m0}(s) + \left[A_{m0}(s) - \overline{A}_{m0}(s)\right] + A_m^1(\mu, s), \tag{39}$$

and Objective 2 is fulfilled if and only if (39) holds with $Q_1(\mu, s) = 0$. From (27), $N_j^{(l)} = \overline{N}_j^{(l}(s) + \theta_l' D_j(s);\quad j = 0, 1, \ldots, n-1,\ l = 1, 2$. Thus, the substitution of (32) into (39), by separating the powers of μ, yields the following set of equations

$$\begin{aligned}&\left[\overline{N}_0^{(1)}(s) + \theta_1' D_0(s)\right] A_0(s) + \left[B_0^{(1)}(s) + B_0^{(2)}\right]\left[N_0^{(2)}(s) + \theta_2' D_0(s)\right]\\ &\quad = D_0(s) A_0(s) - A_{m0}(s),\end{aligned} \tag{40.a}$$

$$\begin{aligned}&\left[\overline{N}_i^{(1)}(s) + \theta_1' D_i(s)\right] A_0(s) + \left[B_0^{(1)}(s) + \delta_i B_0^{(2)}(s)\right]\left[N_i^{(2)}(s) + \theta_2' D_i(s)\right]\\ &\quad = \sum_{j=0}^{i-1}\left[D_j(s) - N_j^{(1)}(s)\right] A_{i-j}(s) - N_j^{(2)}(s)\left[B_j^{(1)}(s) + \delta_i B_j^{(2)}(s)\right]\\ &\quad\quad + D_i(s) A_0(s) - A_{mi}(s);\quad i = 1, 2, \ldots, p_1 - 2,\end{aligned} \tag{40.b}$$

$$\begin{aligned}&\left[\overline{N}_{p_1-1}^{(1)}(s) + \theta_1' D_{p_1-1}(s)\right] A_0(s) + \left[B_0^{(1)}(s) + B_0^{(2)}(s)\right]\\ &\quad\times\left[N_{p_1-1}^{(2)}(s) + \theta_2' D_{p_1-1}(s)\right]\\ &\quad = \overline{A}_{m0}(s) + D_{p_1-1}(s) A_0(s) - A_{m0}(s) - \sum_{i=p_2+1}^{2n-1}\sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(2)}(s)\\ &\quad\quad - \sum_{i=p_1+1}^{2n-1}\left\{\sum_{\substack{j=j_{i_1}\\ j\neq p_1-1}}^{n-1}\left[N_j^{(1)}(s) - D_j(s)\right] A_{i-j}(s)\right.\\ &\quad\quad\quad \left. + \sum_{j=j_{i_2}}^{n-1} N_j^{(2)}(s) B_{i-j}^{(1)}(s)\right\},\end{aligned} \tag{40.c}$$

$$[\overline{N}_{p_1}^{(1)}(s)+\theta_1' D_0(s)]A_0(s)+[B_0^{(1)}(s)+\delta_{p_1}B_0^{(2)}(s)][N_0^{(2)}(s)+\theta_2' D_{p_1}(s)]$$
$$=\sum_{j=0}^{p_1-1}[D_j(s)-N_j^{(1)}(s)]A_{i-j}(s)-N_j^{(2)}(s)[B_j^{(1)}(s)+\delta_{p_1}B_j^{(2)}(s)]$$
$$+D_{p_1}(s)A_0(s)-A_{mp_1}(s)]\,. \qquad (40.d)$$

Remark in the proof. Although (40.a) and (40.c) are both obtained for the μ^0-power in (39), they cannot be jointly solved in the pairs $(N_{(\cdot)}^{(1)}, N_{(\cdot)}^{(2)}$ since the right-hand-side terms of (32) posses common triples $(N_{(\cdot)}^{(1)}, N_{(\cdot)}^{(2)}, D_{(\cdot)})$ of polynomials corresponding to powers μ^0 and μ^i $(i \neq 0)$. To achieves decoupling, the arbitrary polynomial $\overline{A}_{m0}(s)$ is used and the resulting equation for the μ^0-power is split into (40.a) and (40.c).

To pursue with the proof, note that $\deg(N_i^{(1)}(s)) \leqslant \deg(N_i^{(2)}(s)) = n-1 < \deg(A_0(s)) = n$ if $\theta_2' \neq 0$, with the first inequality becoming an equality if $\theta_1' \neq 0$ so that each Eq. (40) has a unique solution $(N_i^{(1)}(s), N_i^{(2)}(s))$ [or, equivalently, $(N_i^{(1)}(s), N_i^{(2)}(s), \theta_1', \theta_2')$ for $i = 0, 1, \ldots, p_1$ for any arbitrarily prefixed $(N_j^{(1)}(s), N_j^{(2)}(s))$; $j = p_1+1, p_1+2, \ldots, n-1$, $i = 0, 1, \ldots, n-1$, $l = 1, 2$ provided that the given coprimeness assumption holds for each prefixed set of polynomials $A_{mi}(s)$; $i = 0, 1, \ldots, p_1$ of degree $2n-1$ if $\theta_1' \neq 0$ and $\theta_2' \neq 0$ and $2n-2$ if $\theta_1' = 0$ and $\theta_2' \neq 0$, [23]. The solution is built recursively from $k = 0$ to $k = p_1$ by using for each equation in (40) corresponding to each k-th power of μ, the previously calculated to $(N_j^{(1)}(s), N_j^{(2)}(s))$; $j = 0, 1, \ldots, k-1$ calculate $(N_k^{(1)}(s), N_k^{(2)}(s)$ for each $k = 0, 1, \ldots, p_1$ for prefixed $D(\mu, s)$ and $(N_i^{(1)}(s), N_i^{(2)}(s))$; $i = p_1+1, p_1+2, \ldots, n-1$. Thus, proposition (i) is proved. The proofs of (ii) – (iv) follow similarly by using (33), (34) and (35), and Theorem 1 is proved.

4. Simulated examples. The main purpose of this section is to verify the theory presented in the previous sections by numerical simulations. In the following examples, we check the performance of the proposed controllers even for unstable plants, in the achievement of Objectives 1–2 by using the Controller structure I, in Examples 1 and 2, and the Controller structure II, in Example 3. Controllers III and IV could be considered as particular cases of Controllers I and II and have not been explicitly simulated.

Example 1. In this example, we analize the performance of Controller structure I for the transfer function $G(s) = 1/(s^2+2s-1-\mu)$.

The model reference transfer function is $G_m(s) = 1/(s^2 + 5s + 6)$.

We apply the proposed Controller structure I by choosing $F = -5,\ q = 1$. With this choice the parameter vector is $\theta = (-3, 42, -3, -17)$ and the $\lambda(t)$-function transform:

$$\bar{\lambda}(t) = \lambda(-t) \xrightarrow{L} \bar{\lambda}(s) = \frac{(s+5)}{(s^2 + 2s - 1 - \mu)(s+5)}.$$

Fig. 2 summarize the results of this numerical simulation for a step input.

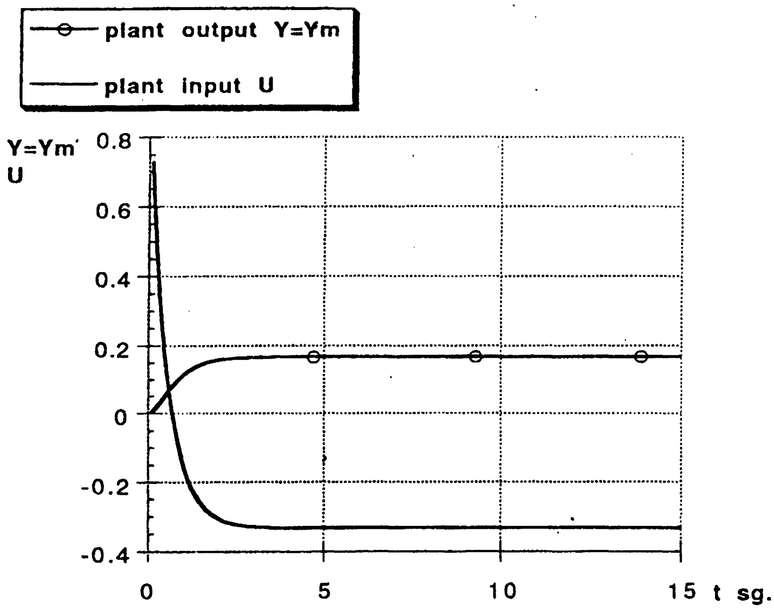

Fig. 2. Example 1. Plant input and output. Model output.

As a proof of robustness, we present the numerical simulation of the former plant with a perturbated plant $G(s) = (1 - \mu)/(s^2 + 2s - 1 - \mu)$ without modifying the nominal controller. In Fig. 3, we summarize the results of the robustness performance.

Example 2. The Match control for a linearized Wind Tunnel Model has been considered. In steady-state operating conditions, the dynamic response of the Match number perturbations δM to small perturbations in the guide vane

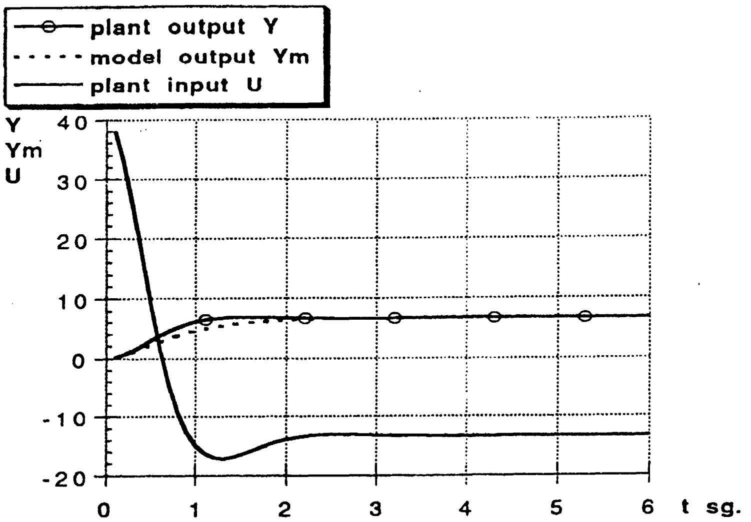

Fig. 3. Example 1. Robustness performance. Plant input and output. Model output.

angle actuator $\delta\theta_A$ is described by the next differential system:

$$\tau\delta\dot{M}(t) + \delta M(t) = k\delta\theta(t-h),$$
$$\delta\ddot{\theta}(t) + 2\zeta\omega\delta\dot{\theta}(t) + \omega^2\delta\theta(t) = \omega^2\delta\theta_A(t),$$

where $\delta\theta(t)$ is the guide vane angle, and $\tau, k, h, \zeta, \omega$ are parameters defining the operating point. These are considered constants with small perturbations. In the state variable form we have added a term depending on the Match number perturbation. The reason for this is the achievement of closed-loop transfer function for the plant with infinite or finite spectrum to the designer's choice. One of the more important features of the presented controller is its capacity of controlling that class of systems. The system in state variable form is written as

$$\dot{x}_1 = -ax_1 + kax_2(t-h),$$
$$\dot{x}_2 = x_3(\underline{+x_1}) \rightarrow \text{ added term},$$
$$\dot{x}_3 = -\omega^2 x_2 - 2\zeta\omega x_3 + \omega^2 u,$$

where $a = 1/\tau$, $x_1 = \delta M$, $x_2 = \delta\theta$, $x_3 = \delta\dot{\theta}$, $u = \delta\theta_A$. With $a = 1$, $k =$

10, $\zeta = 3$, $\omega = 4$ rads/s and $h = 0.2$. The plant transfer function is:

$$G_m(s) = \frac{40}{s^3 + 13s^2 + 16s + 4 - 10s\mu - 120\mu},$$

and the model reference transfer function is:

$$G_m(s) = \frac{40}{s^3 + 2s^2 + 12s + 10}.$$

Choose $F = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$ and $q_1 = 0$, $q_2 = 1$. The parameter vector θ and the $\lambda(t)$-function Laplace transform, displayed on Fig. 4, are:

$$\theta = (-76, 10, -10, -5144, -3340, -10160, 360, -1200, -100, 842),$$

$$\lambda(-t) = \bar{\lambda}(t) \rightarrow \bar{\lambda}(s) = \frac{-60(s^2 + 5s + 6)}{(s^3 + 13s^2 + 16s + 4 - 10s\mu - 120\mu)(s^2 + 5s + 6)}.$$

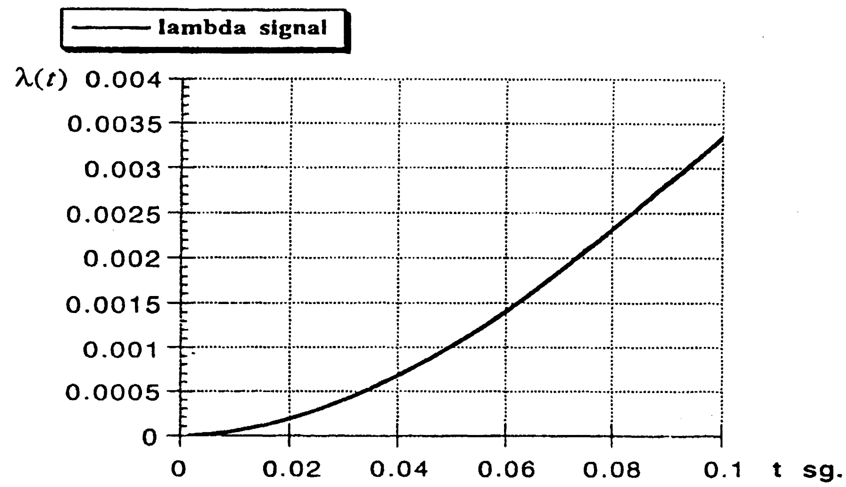

Fig. 4. Example 2. $\bar{\lambda}(t)$-function, being $\bar{\lambda}(t) = \lambda(-t)$.

In Fig. 5.1, the initial conditions are zero for the plant and for the reference model. Similar results will be obtained with the Controller structure II. In Fig. 5.2, initial conditions are zero for the reference model and $x_1(0) = 0.1$, $x_2(0) = x_3(0) = 0$ for the plant. In both cases, the reference input is $r(t) = 10\sin(0.1t)$.

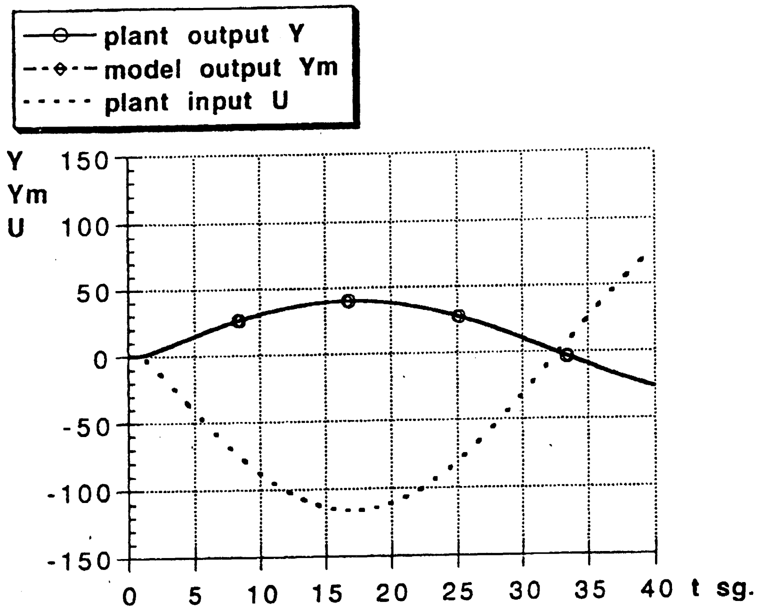

Fig. 5.1. Example 2. Zero I.C. Plant input and output. Model output.

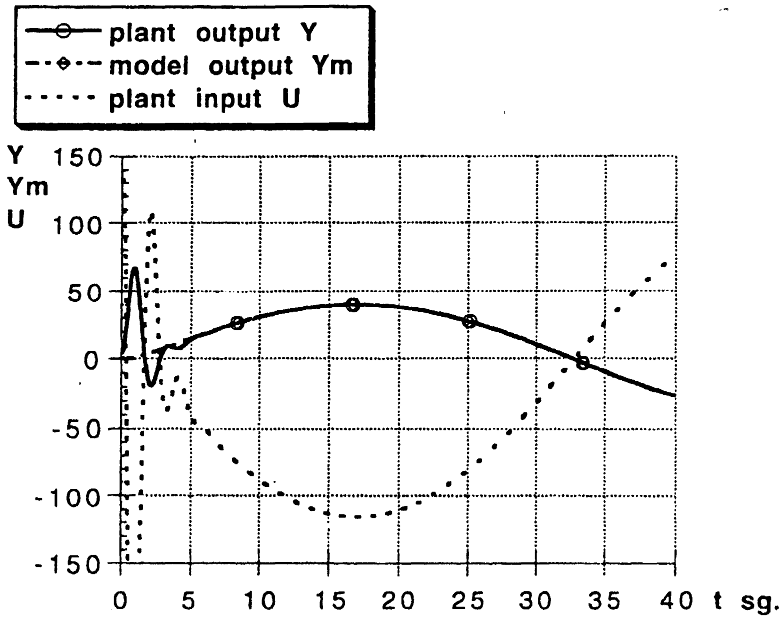

Fig. 5.2. Example 2. Non zero I.C. Plant Input and Output. Model Output.

Example 3. Now, the behaviour of the Controller structure II is proved by choosing the same data as in the previous example but using $x_1(0) = 0.2$, $x_2(0) = x_3(0) = 0$. Fig. 6 summarizes the results for this simulation. The parameter vector θ and the Laplace transform of $\lambda(t)$ are:

$$\theta = (-76, 10, -10, -5144, -3340, -10160, 360, -1200, -100, 902),$$
$$\lambda(s) = \frac{-60(s^2+5s+6)}{(s^3+13s^2+16s+4-10s\mu-120\mu)(s^2+5s+6)},$$

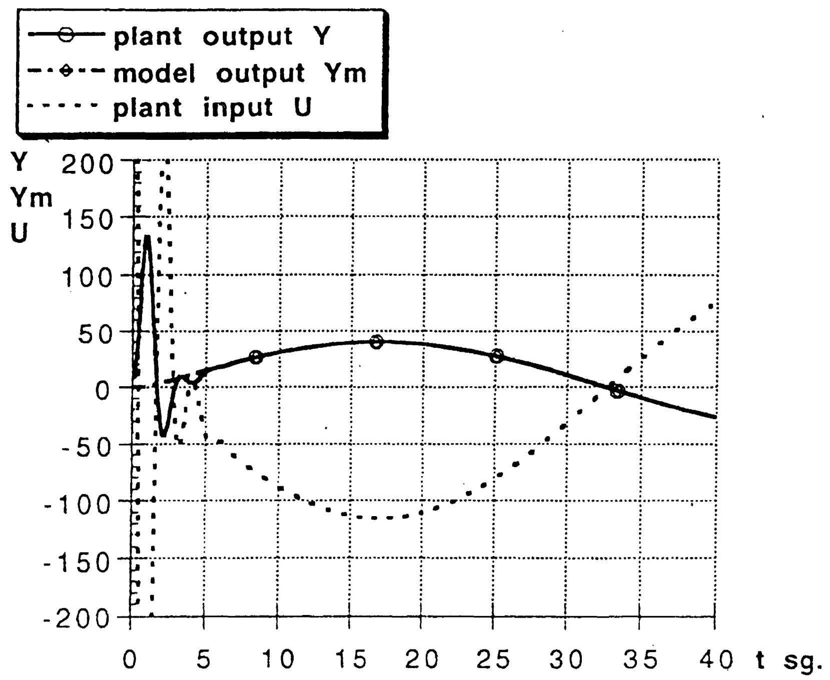

Fig. 6. Example 3. Plant input and output. Model output.

For the computation of the convolution integral, the next modification of the Controller structure is considered

$$\int_h^t \lambda(\tau)u(t-\tau)\,d\tau \xrightarrow{L} \mu\lambda(s)u(s) = \mu\frac{D(s)}{A(s)D(s)}\frac{A(s)}{A_m(s)}r(s)$$

$$= \frac{\mu r(s)}{A_m(s)} \xrightarrow{L^{-1}} y(t-h).$$

Since $\lambda(s) = 1/A(s)$ and $u(s) = A(s)/A_m(s)r(s)$ with A/A_m being realizable with $\deg_s(A) = \deg_s(A_m)$. The above change makes easier (and in this example equivalent to the theoretical one) the controller realization. Note that although it is not recommended the use of Controller II or IV for unstable plants (see Assumption 2.b), it is seen in this example that the associate performance is acceptable.

5. Conclusions. In this paper, four controller structures have been proposed for closed-loop plant delays when both delays are finite and known. The four controller structures involve a memory effect in the control action to compensate for the presence of delays. Such a memory acts, in general, in two ways, namely, the parameterized part of the controller can consist of a linear dynamic system involving (internal) delays and, furthermore, a set of weighting functions which ponderate the input time-integral is additively used to generate the plant input. The first memory effect is used to make possible the achievement of delay-dependent pole-placement control objectives and it can be omitted in the case when finite-spectrum assignability is suited. The second memory effect is used to cancel the unsuitable multiples of the internal delay and their combinations with the external one which are generated through the feedback loop and which are not prefixed in the control objective. Two important design features are that the plant is allowed to posses unstable zeros and the closed-loop characteristic polynomial can be of finite or infinite (namely, delay-dependent) spectrum in accordance to designer's choice.

Acknowledgements. The authors are very grateful to the Spanish Research Funding Agency DGICYT by its partial support of this work through Project PS90-0095 and through the program "Programa de Movilidad Temporal de Personal Funcionario Docente e Investigador" (ref. 93–008).The authors are also grateful to the Electric Company IBERDROLA by the financial support of the Ph. D. research of Mr. J. Jugo.

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Received May 1995

M. de la Sen was born in Arrigorriaga (Bizkaia) in the spanish Basque Country. He obtained the Ms.Sci. degree in Applied Physiscs in 1975, the Ph.D. degree in Applied Physics by the University of Basque Country in 1979 and the degree of Docteur d'Etat-es-Sciences Physiques by the University of Grenoble in 1987. He is currently full Professor for Systems and Control Engineering in the University of the Basque Country. His interest research field lies in the subjects of hereditary systems and adaptive control.

J. Jugo was born in Galdakao (Bizkaia) in the spanish Basque Country. He is Ms Sci. (Applied Physics) by the University of Basque Country where he is now completing his Ph.D. project in the subject of stabilization and adaptive control of time-delay systems. .

TIESINIŲ SISTEMŲ SU VĖLINIMU POLIŲ PADĖTIES VALDYMAS

Manuel de la SEN, Josu JUGO

Straipsnyje nagrinėjami tiesinių sistemų su vėlinimu polių padėties valdymo robastiniai algoritmai. Nagrinėjamos sistemos su vidiniu ir išoriniu vėlinimu, jų modeliai bei valdymo algoritmų struktūros, kai valdymo taisyklės apibūdinamos keturiais dėsniais. Pateikiami formuluojamų uždavinių matematiniai sprendimai, modeliavimo pavyzdžiai.