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STATISTICAL ESTIMATES OF AN UNKNOWN AUTOREGRESSION FUNCTION

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Abstract. In the paper a general approach to identification of non-linear autoregression processes in the class of parametric and non-parametric mathematical models is formulated. With the help of mathematical simulation the estimates of the processes of this class are studied: a nuclear estimate, an estimate of least squares projective estimates. Some statistical properties of these estimates are indicated.

Key words: non-linear autoregression processes, parametric and non-parametric mathematical models, mathematical simulation, statistical estimation.

Introduction. Practical and theoretical investigations show that many real physical processes (mechanical, chemical, thermophysical, plasmochemical, economic, biological, etc.) are non-linear stochastic processes (NSP) – (Neimark and Landa, 1987; Tong, 1983). Up to now in the studies of stochastic processes (if time is the argument, then time series) on the whole there prevailed linear mathematical autoregression models or moving average autoregression. However, a more deep study and knowledge of the essence of physical processes and phenomena indicate an essential shortage and boundedness in the application of linear models. Namely non-linear models may reveal interesting types of behaviour of physical processes and phenomena, such as: limit cycles, strange attractors, bifurcation, turbulence, etc. (Lichtenberg and Liberman, 1984; Neimark and Landa, 1987), which can be never revealed by means of linear models. Physical processes of the mentioned class may be sufficiently well described by a non-linear autoregression process with the help of the following difference equation:

$$X_t = f(X_{t-1}, \ldots, X_{t-n}) + \varepsilon_t, \tag{1}$$

where $t = 1, 2, \ldots$ is a discrete physical argument, $f(x)$ is an unknown function, $\{\varepsilon_t,\ t = 1, 2, \ldots\}$ is a sequent of independent random variables.

Let us note the case when $\varepsilon_t \equiv 0,\ \forall t$, then expression (1) relates to irregular oscillations (consequently called random) in the determinated dynamic systems of different physical nature which would promote a rapid development of this sphere of science.

It is possible to solve the problem estimating the unknown function $f(x)$ of process (1) according to the data of the observation $\{X_t,\ t = \overline{1, N}\}$ both by parametric and nonparametric methods and the class of mathematical models of process (1) is rather diverse.

The complexity of the estimation problem $f(\cdot)$ of process (1) consists in the first place in the approximation of multivariate non-linear functions with their peculiarities: continuous, discontinuous; in the second place the process $\{X_t\}$ is a dynamic process, i.e. the argument values of the function $f(\cdot)$ are random variables and the distribution of this argument $\{X_{t-1}, \ldots, X_{t-n}\}$ depends on the estimated function f; in the third place for $n > 2$ the obviousness of presentation of the function f is lost. Therefore in order to identify (analyze) the processes of type (1), we develop a complex approach: a non-parametric (nuclear), a parametric (projective) or their combination, i.e. a combined method. Its essence is in the following. In the statistical analysis of experimental data the estimate of the function $f(\cdot)$ is constructed by a non-parametric (nuclear) method. After studying the peculiarities of the obtained estimate $f(\cdot)$, an adequate parametric model is selected and the estimate of the function $f(\cdot)$ is calculated, e.g. of projective type. An important problem is the comparison of efficiency of the methods and algorithms applied in the sense of exactness of estimation of $f(\cdot)$ and (high speed of) algorithms on a computer.

Main suppositions

Let

$$V_t = (X_{t-1}, \ldots, X_{t-n}) + \varepsilon_t. \tag{2}$$

While studying statistical properties of estimates $\widehat{f}(V_t)$ of the function $f(V_t)$ of the non-linear stochastic process of type (1), we shall consider that following conditions are fulfilled.

1. The sequence of the white noise $\{\varepsilon_t,\ t = 1, 2, \ldots\}$ has a zero value and finite dispersion, i.e.

$$E(\varepsilon_t) = 0,\ D(\varepsilon_t) = \sigma_\varepsilon^2,\ 0 < \sigma_\varepsilon^2 < \infty, \tag{3}$$

where E is the symbol of mathematical expectation, D is the dispersion symbol.

2. Function f satisfies the inequality

$$\forall x,\ y \in R^n |f(x+y) - f(y)| \leqslant \alpha_1|x_1| + \cdots + \alpha_n|x_n|, \tag{4}$$

where $f(0) = 0, \alpha_1 + \cdots + \alpha_n = \alpha < 1$. Let us note that in this case processes (1) are stable (see, Baltrūnas and Rudzkienė, 1984) and the strong intermixing function $\rho(\tau)$ of process

(1) decreases with an exponential speed (see, Baltrūnas and Rudzkienė, 1986):

$$\rho(\tau) = \sup_{A \in F_{-\infty}^{0}, B \in F_{\tau}^{\infty}} |P(AB) - P(A)P(B)| \leqslant c\epsilon^{-\delta\tau}, \qquad (5)$$

where c and δ are positive constants, while by F_a^b δ-algebras are denoted which are generated by the random variables $\{X_t, a \leqslant t \leqslant b\}$.

Formulation of the problem. By observations $\{X_t, t = \overline{1, N}\}$ of process (1) in the domain $x \in W$ it is necessary to estimate the function $f(\cdot)$.

Let us consider the most widely spread methods of getting the estimates of the function $f(\cdot)$ and some their statistical properties and also formulate a more general approach to process identification of type (1).

1. Nuclear estimate (NE)

$$\widehat{f}(x) = \sum_{t=1}^{N} X_t K(\|V_t - x\|) \Big/ \sum_{t=1}^{N} K(\|V_t - x\|), \qquad (6)$$

where $\|z\|^2 = z_1^2 + \cdots + z_n^2$, $K(\cdot) = K_N(\cdot)$.

Usually kernel $K(\cdot)$ is defined by the equality $K(x) = \Psi(x/h)$, where $\Psi(\cdot)$ is an even, positive independent on N function, called the kernel form, $h = h(N)$.

Let estimating shortages be defined by the equality

$$E \int_W [f(x) - \widehat{f}(x)]^2 dx \triangleq \Delta^2(f, \widehat{f}). \qquad (7)$$

From the results of Collomb (1981) it follows that if there exists the second derivative f'' and $h \asymp N^{-1/(4+n)}$, then $\Delta^2(f, \widehat{f}) = O(N^{-4/(4+n)})$ for $N \to \infty$.

2. Projective estimates

2.1. Projective estimate 1(PE1). Let $\{e_k(x)\}_{k=1}^{\infty}$ be some orthonormalized base in the space $L_2(W)$. Then the expansion

$$f(x) = \sum c_k e_k(x), \quad x \in W, \tag{8}$$

holds, where

$$c_k = \int\limits_W e_k(x) f(x) dx.$$

For $n = 1$, $W = [-1, 1]$ the estimate $\widehat{f}(x)$ is defined by the equality

$$\widehat{f}(x) = \sum_{k=1}^{m} \widehat{c}_k e_k(x), \tag{9}$$

where $m = m(N)$,

$$\begin{aligned} c_k &= \sum_{t=1}^{N} X_t e_k(V_t) \cdot (b_t - a_t) \cdot \mathbf{1}_{\{V_t \in W\}}, \\ a_t &= \inf[x : -1 \leqslant x, (2x - V_t, V_t) \cap \{V_1, \dots, V_N\} = 0], \\ b_t &= \sup[x : x \leqslant 1, (V_t, 2x - V_t) \cap \{V_1, \dots, V_N\} = 0]. \end{aligned} \tag{10}$$

Baltrūnas and Rudzkienė (1987) have studied asymptotic properties of the estimates of shape (9). It is shown that

$$\Delta^2(f, \widehat{f}) = O(\frac{m}{N}) + \sum_{|k|>m} c_k^2. \tag{11}$$

Consequently, for a sufficient smoothness of the function f and the corresponding choice of $m(N)$ for the estimates of shape (9), it is possible to obtain a better rate of convergence in comparison with nuclear estimates (6). E.g. if $c_k = O(e^{-k})$ for $k \to \infty$, then for $m = \ln N$ we get

$$\Delta^2(f, \widehat{f}) = O(\ln N/N) \qquad \text{for estimates (9)}$$

and

$$\Delta^2(f, \widehat{f}) = O(N^{-4/5}) \qquad \text{for the nuclear estimate.}$$

Estimates of shape (9) are generalized for the case $n > 1$, however the algorithms for calculating the estimate $f(\cdot)$ become more complicated, therefore it is desirable to consider another projective method, according to which the estimates of the function $f(\cdot)$ are calculated considerably simply and for $n > 1$.

2.2. Projective estimate II (PE2). Let p_{XV}, p_V be distribution densities of the random vectors (X_t, V_t), V_t, respectively. Since $f(x) = E(X_t | V_t = x)$, having denoted $g(x) = \int\limits_{-\infty}^{\infty} z p_{XV}(z, x)\, dz$ we have

$$f(x) = g(x)/p_V(x). \tag{12}$$

Suppose that the expansions take place

$$g(x) = \sum_k a_k e_k(x), \quad p_V(x) = \sum_k b_k e_k(x), \tag{13}$$

where

$$a_k = \int_W g(x) e_k(x)\, dx = E X_t e_k(V_t) \cdot \mathbf{1}_{\{V_t \in W\}},$$

$$b_k = \int_W p_V(x) e_k(x)\, dx = E e_k(V_t) \cdot \mathbf{1}_{\{V_t \in W\}}.$$

Let us define the estimate of function $f(x)$ by the equality

$$\widehat{f}(x) = \widehat{g}(x) \Big/ \widehat{p}_V(x) = \sum_{i=1}^{m_1} \widehat{a}_i e_i(x) \Big/ \sum_{j=1}^{m_2} \widehat{b}_j e_j(x), \qquad (14)$$

where

$$\widehat{a}_i = \frac{1}{N} \sum_{i=1}^{N} X_t e_i(V_t) \cdot \mathbf{1}_{\{V_t \in W\}},$$

$$b_j = \frac{1}{N} \sum_{t=1}^{N} e_j(V_t) \cdot \mathbf{1}_{\{V_t \in W\}}.$$

According to (14) $\widehat{g}(x)$ and $\widehat{p}_V(x)$ are simple projective estimates of the functions g and p_V. Chencov (1962) has proposed a projective method for the estimation of distribution density. It is known that for smooth densities this method of estimation gives good results (Doukhan, 1986; Chencov, 1962). Some results concerning the investigation of the properties of estimates $\widehat{g}$, $\widehat{p}_V$ in the case, when as base functions $e_k(x)$ Hermit's polynomials are used, given by Doukhan (1986).

Note that estimate (14) may strongly differ from $f(x)$ in the case, when values $\widehat{p}(x)$ are close to zero. This has been also proved by the experimental investigation of estimate (14), carried out by the authors of the paper. In this connection we have performed the simulation by a computer not of estimate (14) but of its modified variant

$$\widehat{f}(x) = \widehat{g}(x)/[\widehat{p}_V(x) + \bar{\alpha} + \beta\sqrt{N^*}], \qquad (15)$$

where $\bar{\alpha} = \max(-\alpha; 0)$, $\alpha = \min \widehat{p}_V(u)$, $u \in W$, $\beta = \max_{u \in W} \widehat{p}_V(u)$, N^* is a number of points V_t, occurring in the interval W.

2.3. Estimate of least squares (ELS). For relatively small values of the variable m it is possible to modify estimates

(9) or (14), replacing the estimates of coefficients c_k, defined by formula (10), by the estimates, obtained with the help of the least square method. ELS is defined by the formula

$$f(x) = \sum_{k=1}^{m} c_k^* e_k(x), \tag{16}$$

while the coefficients $c_k^*(k = \overline{1,m})$ are calculated according to the expression

$$(c_1^*, \dots, c_m^*) = \arg \left\{ \min_{c_1,\dots,c_m} \sum_{t=1}^{M} \left[X_t - \sum_{k=1}^{m} c_k e_k(V_t)\right]^2 \right\}. \tag{17}$$

In the conclusion of this section some words can be said concerning spline estimates. These estimates become rather popular; especially in regression problems. Though their properties in connection with the estimation of the function $f(\cdot)$ of process (1) in this paper are not studied, we shall present the main idea for the construction of such estimates.

The idea to construct spline estimates of the function $f(\cdot)$ of process (1) consists in the partitioning of the domain W into non-intersecting subdomains $w_l(l = \overline{1,L})$, i.e.

$$W = \cup_{l=1}^{L} w_l(X_{t-1}, \dots, X_{t-n}), \ w_i \cap w_j = \varnothing \text{ for } i \neq j. \tag{18}$$

Then according to the partitioning on the domain W (18) for the realization $\{X_t, \ t = \overline{1,N}\}$ of process (1), we shall write the following expression:

$$(y)_l = f(V)_l + (\varepsilon)_l, \quad l = \overline{1,L}, \tag{19}$$

where $(y)_l = (X_{t_1}, \dots, X_{t_s(l)})$, $(\varepsilon)_l = (\varepsilon_{t_1}, \dots, \varepsilon_{t_s(l)})$, $f(V)_l =$ $= (f(V_{t_1}), \dots, f(V_{t_s(l)}))$, here by $t_1, \dots, t_{s(l)}$ denoted the indices of elements of the set $\{V_{n+1}, \dots, V_N\}$, belonging to the domain w_l.

According to (19) the problem of estimating the function $f(V_t)$ by the realization $\{X_t,\ t=\overline{1,N}\}$ of process (1) is transformed to the problem of estimating $f(V)_l$ in the subdomains $\{w_l,\ l=\overline{1,L}\}\in W$.

Problem (19) referres to the class of regressive problems, while estimation of $f(\cdot)$ of process (1) in the shape of (19) is not a formal transformation. This is connected with the peculiarities of solution (19) and with the statistical properties of the estimates $\widehat{f}(V)_l$ $(l=\overline{1,L})$.

According to expression (19) it is possible to define the estimate $\widehat{f}(V)_l$ in different ways: in the subdomains $w_l(\cdot)$ $(l=\overline{1,L})$ to approximate $f(V)_l$ by a multivariate polynomial, splines or the method of finite elements.

Experimental methods. Mathematical simulation realized in the given paper is applied to the studies of statistical properties of the above estimates of the function f for nonlinear autoregression process (1) for $n\in\{1,2\}$, $W=[-1,1]^n$, and also to the calculation of the efficiency of the considered estimation algorithms of $f(x)$. In the capacity of $f(x)$ the following functions were used:

$$
\begin{aligned}
&f_1(x)=|\cos(\pi x)|,\quad f_2(x)=x/2-5xe^{-x^2},\\
&f_3(x)=\operatorname{sign}(x)\cdot|x|^{1/2},\quad f_4(x_1,x_2)=\cos\,\pi(x_1+x_2),\\
&f_5(x_1,x_2)=|x_1+x_2|\cdot(2-|x_1+x_2|)\cdot\mathbf{1}_{\{|x_1+x_2|\leqslant 2\}},\\
&f_6(x_1,x_2)=0.6\ \operatorname{sign}(x_1)+0.3\ \operatorname{sign}(x_2).
\end{aligned}
\tag{20}
$$

In one–dimensional case $(n=1)$ for each of the functions $f_1(x)$, $i=\overline{1,3}$ all the four mentioned estimates were calculated for the sample size $N\in\{100,1000,5000,10000\}$. In two-dimensional case $(n=2)$ for each of the functions $f_i(x_1,x_2)$, $i=\overline{4,6}$ three types of estimates were considered: NE, PE2, LSE for the sample size $n\in\{500,1000,5000,10000\}$.

Note that for the estimated functions f_i and their estimates $\widehat{f}_i$ it is rather difficult to calculate a theoretical value $\Delta^2(f,\widehat{f})$. In order to estimate the statistic $\Delta^2(f,\widehat{f})$ for fixed parameters of the experiment, it has been repeated 10 times by independent realizations $\{X_t^{(\nu)},\ t=\overline{1,N},\ \nu=\overline{1,10}\}$ and the following values were calculated

$$\delta^2 = \begin{cases} \frac{1}{100} \sum\limits_{k=-50}^{50} \left[f(x_k)-\widehat{f}(x_k)\right]^2, & \text{if n=1,} \\ \frac{1}{900} \sum\limits_{i,j=-15}^{15} \left[f(x_{1i},x_{2j})-\widehat{f}(x_{1i},x_{2j})\right]^2, & \text{if n=2.} \end{cases} \tag{21}$$

The value $\bar{\delta}^2 = \frac{1}{10}\sum\limits_{\nu=1}^{10} \delta_\nu^2$ was considered as an estimate $\Delta^2(\cdot)$. Empirical dispersion of the estimates δ_ν^2, i.e. $\bar{\sigma}^2 = \frac{1}{10}\sum\limits_{\nu=1}^{10}\left[\delta_\nu^2-\delta^2\right]^2$ was calculated, too.

Nuclear estimates were constructed with the kernel of type

$$\Psi(x) = \begin{cases} 1-|x|, & |x|<1, \\ 0, & |x|>1 \end{cases} \tag{22}$$

for different values of h. Projective estimates and the estimate of least squares were calculated for the number of coefficients of expansion (10) $m=2,\ 3,\dots,16$ (in the case $n=1$) and $m=2,\ 3,\dots,55$ (in the case $n=2$), where for PE2 $m_1=m_2=m$ was accepted. In the capacity of the base functions $e_k(x)$ the orthonormalized Lagrange polynomials were used, while for the recurrent functions f_1 and f_4 the estimates were constructed with the base from trigonometrical functions. For the estimate $\widehat{f}_4$ a smaller error was obtained on the base of $e_k(x)$ from Lagrange polynomials, while for $\widehat{f}_1$ on the base from trigonometrical functions.

Estimates PE1, PE2 and LSE are defined if the base $e_k(x)$ and number m are given. These estimates are rather sensitive to the value m. In the paper the following method of the optimal selection of m by the sample $\{X_t, t = \overline{1, N}\}$ is investigated. By the first half of the sample $\{X_t,\ t = \overline{1, N/2}\}$ for $m = 2, 3, \dots$ the corresponding estimates $\widehat{f}(V_t, m)$ are calculated, further, in the second half of the sample $\{X_t,\ t = \overline{N/2, N}\}$ for the obtained $\widehat{f}(V_t, m)$ $\rho^2(m) = \frac{2}{N} \sum\limits_{t=N/2}^{N} \left[X_t - \widehat{f}(V_t, m)\right]^2$ are calculated, while m^* is defined by the equality

$$m^* = \arg\{\min_m \rho^2(m)\}. \qquad (23)$$

The variable m^* was compared with the variable $\underline{m}$, where by number $\underline{m}$ such a number $m \in \{2, 3, \dots\}$ was denoted, for which the variable $\bar{\delta}^{-2}$ is minimal.

Results of investigation. Due to the limited volume of the paper one part of the results is presented here.

One–dimensional case ($n = 1$). From the simulation results (Fig. 1, 2) it follows that the exactness of estimation of the functions of type $f_i(x)$, $i = \overline{1, 3}$, little depends on the type of the estimates and, naturally, it strongly depends on the length of the realization N. In short realizations (hundreds of discretes of N–order) for projective estimates and LSE the approximating polynomical optimal degree $\underline{m}$ oscillates from 2 to 6, while for $10000 > N > 1000$ the variable $\underline{m}$, corresponding to the minimum of the variable $\bar{\delta}^{-2}$, varies within the limits $\overline{6, 16}$.

The way to define m^* by formula (23) is reliable. According to the simulation results either $m^* = \underline{m}$ or the value $\bar{\delta}^2$ for $m = m^*$ little differs from the value $\bar{\delta}^2$ for $m = \underline{m}$.

Two–dimensional case ($n = 2$). From the simulation results (Fig. 3, 4) it follows that for all the cases under consideration $N \in \{500, 1000, 5000, 10000\}$ nuclear estimates have the best accuracy, characterized by the variable $\bar{\sigma}^2$ and the

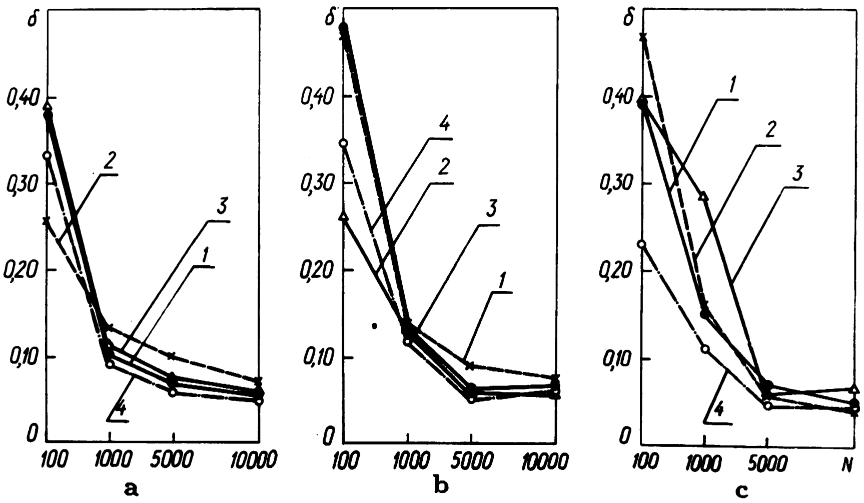

Fig. 1. Dependence of the statistics $\bar{\delta}$ for $m = \underline{m}$ on the length of the realization N for the functions: a) $-f_1(x)$, b) $-f_2(x)$, c) $-f_3(x)$. Functions approximated: 1 - LSE, 2 - PE1, 3 - PE2, 4 - NE

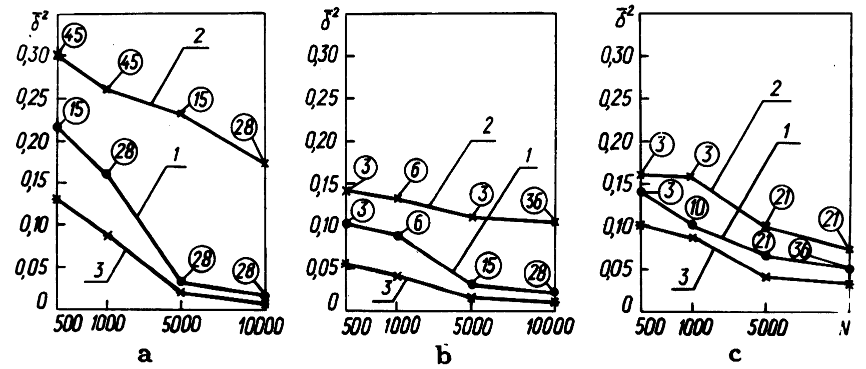

Fig. 2. Dependence of approximation accuracy (values of the statistics $\bar{\delta}^2$ for $m = \underline{m}$) of process (1) on the length of realization N for the functions: a) $-f_4(x_1,\ x_2)$, b)$-f_5(x_1,\ x_2)$, c)$-f_6(x_1,\ x_2)$. Functions approximated: 1 - LSE, 2 - PE2, 3 - NE

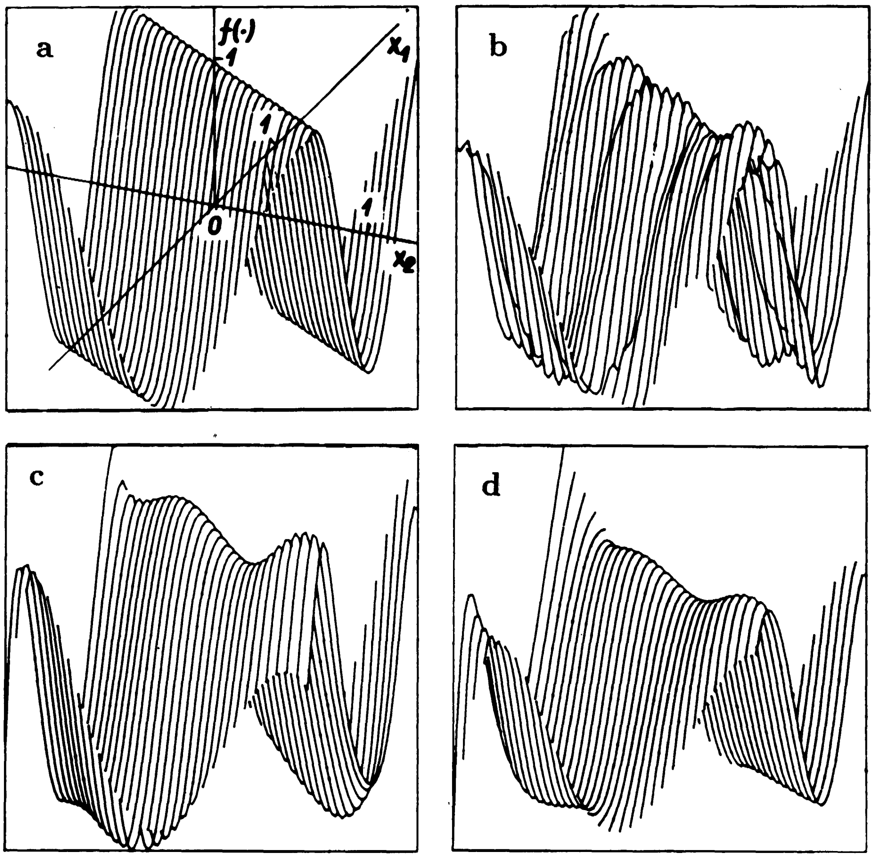

Fig. 3. Shape of the function $f_4(x_1,\ x_2) = \cos\pi(x_1+$ $+x_2)$ in the rectangular axonometric view (dimetric) a) and its approximations for $N = 5000$: b) – NE, c) PE2, $m = 28$, d) – LSE, $m = 28$

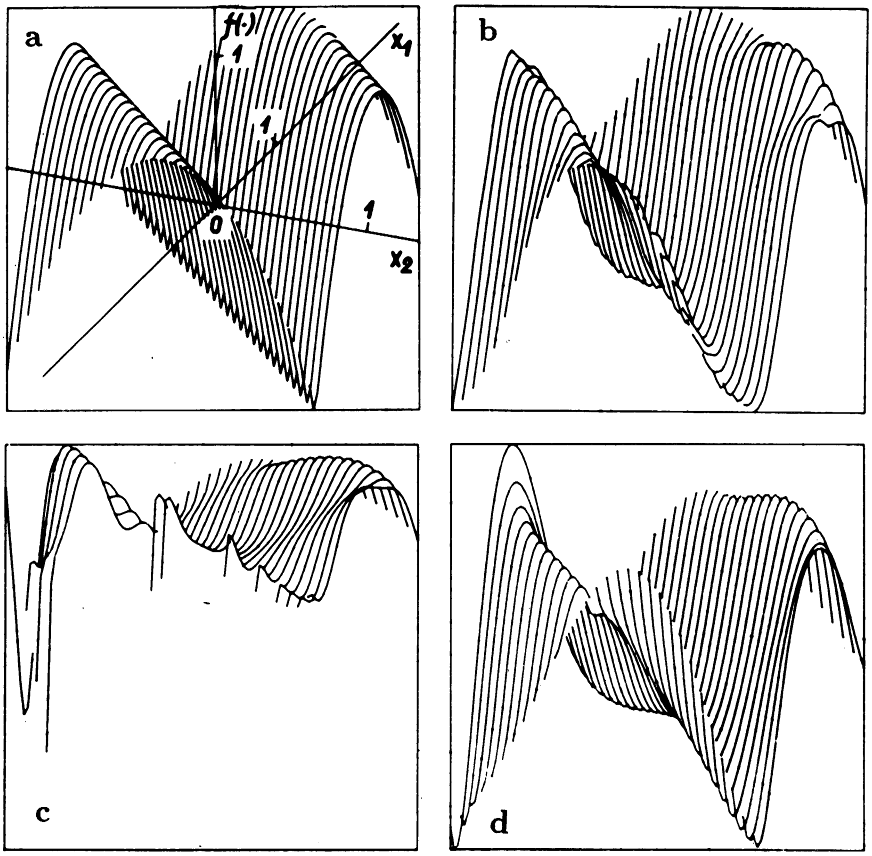

Fig. 4. Shape of the function $f_5(x_1, x_2) = |x_1 + x_2| \cdot (2 - |x_1 + x_2|) \cdot \mathbf{1}_{\{|x_1+x_2| \leqslant 2\}}$ a) and its approximations: b) – NE c) – PE2, $m = 36$ d) – LSE, $m = 28$, $N = 5000$

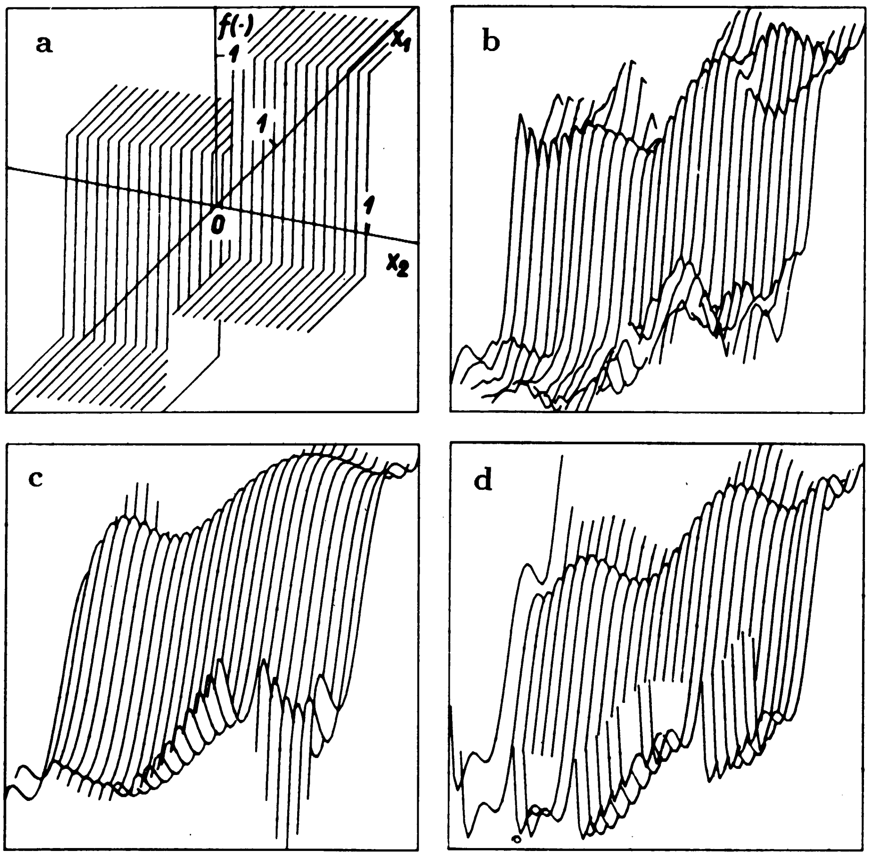

Fig. 5. Shape of the function $f_6(x_1, x_2) = 0.6\,\mathrm{sign}\,(x_1)+$ $+0.3\,\mathrm{sign}\,(x_2)$ a) and its approximations: b) – NE, c) – PE2, $m = 28$, d) – LSE, $m = 36$, $N = 5000$

smallest empirical dispersion $\bar{\delta}^2$ of the estimates $\widehat{f}_i(x_1, x_2)$ for the functions $f_i(x_1, x_2)$, $i = \overline{4,6}$. Further according to the accuracy follows the estimates of least squares, which for $N \asymp \asymp 10000$ according to the accuracy are close to nuclear estimates. The projective estimate PE2 is more rough. It is connected with the fact that the denominator's values in expression (14) are close to zero or even more scattered. Therefore the dispersion of the estimates PE2 is larger than that of the remaining. The optimal degree $\underline{m}$ at which the statistics $\bar{\delta}^2$ is minimal for the estimates PE2 and LSE strongly depends not only on N but also on the function shape. For f_4 the variable $\bar{\delta}^2$ achieves minimum for sufficiently large values $m \in \{15, \ldots, 45\}$. For f_5, f_6 when $N \in \{500, 1000\}$ the statistics $\bar{\delta}^2$ is minimal for a small $m \in \{3, \ldots, 10\}$ Fig. 3, 4. Almost in the cases NE have the least empirical dispersion ($\bar{\delta}^2$ from 10^{-3} to 10^{-5} for $N = 500$ and of order $10^{-5} - 10^{-6}$ for $N = 10000$). LSE dispersion slightly differs from NE. PE2 dispersion for all N is of order $10^{-3} - 10^{-4}$. The same as one–dimensional case the estimation of the variable m^* according to (23) is reliable.

Conclusions

1. In one–dimensional case ($n = 1$) all the considered estimates (NE, PE1, PE2, LSE) are approximately identical by accuracy and may be used for practical applications. In two–dimensional case ($n = 2$) the most exact is the nuclear estimate, for $N \asymp 10000$ the estimate of least squares approximates to it by accuracy.

2. The approximating polynomial optimal degree $\underline{m}$ for which the variable $\bar{\delta}^2$ is minimal, in the case $n = 1$ basically depends on the length of the realization N, while in the case $n = 2$ the optimal $\underline{m}$ essentially depends both on N and the estimated function.

3. The way of defining the approximating polynomial optimal order m^* according to expression (23) for the estimates

of LSE, PE1, PE2 is reliable and may be used for the analysis of physical processes.

All the conclusions are obtained for the sample size $100 \leqslant \leqslant N \leqslant 10000$.

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