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SHANURSA – A New Approach to Multi-Attribute Ranking and Selection in the Setting of Shadowed Set Theory

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Abstract. *Shadowed set* (SHS) theory was introduced by Witold Pedrycz in 1998 to play a vital role in handling vagueness and providing a way to approximate a given *fuzzy set* (FS) by a construct, which is easy to deal with in practice. Nonetheless, the application of SHSs and shadowed numbers (SHNs) is surprisingly underrepresented in fuzzy multi-attribute decision-making (FMADM) literature. To address this gap, the current research aims to introduce and demonstrate a new FMADM method called the ***SHA****dowed* ***NU****mber based* ***R****anking and* ***S****election* ***A****pproach* (SHANURSA). The methodology involves, *inter alia*, transforming FNs into their corresponding SHNs using Grzegorzewski's (2013) approach, then implementing an efficient weighted Minkowski Distance Metric (MDM) for SHNs to better gauge the proximity/remoteness between them. Additionally, a new and unusual closeness coefficient is introduced and defined to enhance the ranking accuracy of alternatives. Finally, the applicability and effectiveness of this closeness coefficient and proximity-based FMADM method is demonstrated by means of three distinct case studies from literature. The findings of this research show that the suggested method is trustworthy.

Key words: multi-attribute decision-making, fuzzy numbers,, fuzzy sets, fuzzy set approximation, ranking and selection, shadowed numbers, shadowed sets.

1. Introduction

Decision-making in fuzzy environments is a mature field of research. Over the last decades, a plethora of methods, based on commonly exploited trapezoidal fuzzy numbers (TrFNs) and triangular fuzzy numbers (TFNs) has been conceived to solve fuzzy multi-attribute decision-making (FMADM) problems (Zavadskas *et al.*, 2020; Mohammadian *et al.*, 2021; Keshavarz-Ghorabaee *et al.*, 2022; Damjanović *et al.*, 2024; Erdmann *et al.*,

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2024; Zhang *et al.*, 2025). As is known, many researchers and practitioners use the so-called *defuzzification to a point approximation method* in order to compare and rank fuzzy numbers (FNs) in the early or later stage of the decision-making process. By defuzzification to a point it is meant the approximation of a given FN by a single deterministic real number (Chen and Hwang, 1992). However, this approximation exhibits serious shortcomings. Most notably, it often leads to the inevitable loss of information existing in the original data, which may seriously affect the *reliability* and *validity* of the results obtained by applying FMADM methods (see, e.g. Dymova *et al.*, 2015; Wang, 2015; Sotoudeh-Anvari, 2020; Yatsalo and Martinez, 2018; Çelikbilek, 2019; Sharaf, 2020; Wang *et al.*, 2020). Therefore, it would be of great value to make use of some approximation method, which could lead to more accurate results.

Shadowed sets (SHSs), first introduced by W. Pedrycz in 1998 (see also Pedrycz, 2005; 2009), were proposed as a knowledge-granulation mechanism intended both to capture vagueness in representation and to transform fuzzy sets (FSs) into simpler, more intuitive and more convenient practical use. Several works have studied the SHSs (Cai *et al.*, 2017; Kong and Chen, 2017; Yao *et al.*, 2017; Ibrahim *et al.*, 2020; Gao *et al.*, 2020; Patel and Shah, 2021; Wu *et al.*, 2021; Yang and Yao, 2021; Boffa *et al.*, 2022).

Recently, El-Hawy *et al.* (2015), Wang *et al.* (2018) and Landowski (2020) established formal definitions of shadowed numbers (SHNs), shadowed sets of the real line $\mathbb{R}$ and their algebraic operations. Any FN can be transformed into a corresponding three-region SHN. Later, Ibrahim *et al.* (2020, 2021) introduced models with as many as five-region SHSs, or more broadly, n-region SHSs.

While n-region SHSs provide for $n > 3$, a more precise representation of uncertainty than three-region SHSs, they increase computational complexity and challenges in optimizing multiple thresholds. These factors tend to lead to practical difficulties in interpreting and processing of FSs for at least three reasons. Firstly, processing large amounts of distinct grades poses a significant challenge (Pedrycz, 1998). Secondly, an increased number of thresholds is associated with a more complex approximation of FNs. Finally, additional thresholds increase the risk of misclassification due to the difficulty in distinguishing between slightly different membership grades (William-West and Ibrahim, 2023a; Zhang *et al.*, 2024). For higher efficiency and practicality, this work is confined to three-region SHSs induced from the existing FSs.

In the literature on SHSs, trisecting a FS $\tilde{A}$ can be achieved by applying the principle of uncertainty relocation or by determining optimal partition thresholds α and $\beta \in [0, 1]$ such that $\alpha < \beta$ (possibly with $\beta = 1-\alpha$). Using the membership function $\mu_{\tilde{A}}$ of the fuzzy set $\tilde{A}$, the membership grades $\mu_{\tilde{A}}(x)$ higher than or equal to β are elevated to 1 and those lower than or equal to α are reduced to 0, while those in between are set to the entire open unit interval $(0, 1)$. For a review and a comparison of different optimized threshold-based methods for constructing SSs from FSs, the reader is referred to William-West and Ibrahim (2023b). While many trisecting methods rely on complex optimization approaches, Grzegorzewski (2013) proposed an alternative framework based on successive approximation intervals. The current research advocates the use of the Grzegorzewski's approach because, unlike others, it provides analytic solutions to directly calculate the four key points of a three-region SHS.

Compared to Pedrycz's method, this approach offers at least three distinct advantages. Firstly, it provides direct formulas for SHS parameters applicable to any type of FN. Secondly, it offers a superior representation of uncertainty by preserving the expected interval and width of FN. Finally, through the inverse transform, it allows for the reconstruction of FNs from SHSs – a feature not supported by Pedrycz's method (Grzegorzewski, 2013).

While FMADM has been extensively studied, its application within the framework of SHS theory has received considerably less interest. Since the inception of the SHS theory in 1998, and over the course of nearly a quarter century, few researchers have proposed and developed FMADM methods based on this mathematical framework. The existing FMADM methods involving SHNs are summarized as follows.

El-Hawy *et al.* (2015) introduced an enhanced version of shadowed fuzzy numbers (SHFNs), extending the concept to account for multiple types of uncertainty. They also proposed a novel method for ranking SHFNs. Wang *et al.* (2018) defined Pythagorean shadowed set (PSHS) and presented a data-driven approach for constructing PSHSs models for linguistic terms. Moreover, they have developed a score function for Pythagorean shadowed numbers (PSHNs) and introduced a new FMADM method based on PSHNs. Subsequently, Wang *et al.* (2020) proposed a q-rung orthopair shadowed set (q-ROSHS) and extended the VIKOR method to solve q-ROSHS FMADM problems involving q-ROSHS. On the other hand, He *et al.* (2021) modelled linguistics using SHSs and developed a SHS-based clustering model. They also introduced a centroid-based generalized Minkowski distance measure for SHSs and extended the TODIM method to accommodate the SS framework.

To fill the research gap on SHN FMADM, this paper proposes a new FMADM method within the SHN framework called the ***SHA****dowed* ***NU****mber based* ***R****anking and* ***S****election* ***A****pproach* (*SHANURSA*). This approach is designed to ensure sound and defensible selection decisions in a fuzzy environment. Notably, the SHANURSA method is grounded on transforming the most commonly used FNs into three-region SHNs via Grzegorzewski's (2013) approach, which then serve as the foundational elements of the method. Subsequently, the well-known Minkowski distance metric (MDM) is extended to the weighted MDM tailored for three-region SHNs. A new closeness coefficient is then introduced and defined to rank the available alternatives. lastly, the decision is made by selecting the alternative that maximizes the closeness coefficient (see details in Section 3).

The *novelty* of this research lies in the simultaneous use of Grzegorzewski's approach, the weighted MDM for SHNs, in addition to a new and an unusual closeness coefficient. The results obtained in the three illustrative examples presented in Section 4 clearly demonstrate the stability and robustness of the SHANURSA method. Additionally, the proposed method stands out as an effective tool for solving FMCDM problems in the setting of SHS theory without the typical transformation of the FMCDM decision matrix into a *beneficial decision matrix* beforehand, a step commonly seen in the FMCDM literature.

The remainder of the paper is as follows. We first give clear definitions and notations to key terms used herein. Then, we present a comprehensive, detailed description of the SHANURSA method. Next, we demonstrate the effectiveness of the SHANURSA method using three illustrative examples from literature. Lastly, we summarize key findings and provide directions for future research.

2. Preliminaries

This section briefly reviews several fundamental theoretical concepts, including the definitions of FSs, FNs, nonnegative fuzzy numbers (NNFNs), trapezoidal fuzzy numbers (TrFNs) and triangular fuzzy numbers (TFNs), as well as three-region SHSs and three-region SHNs. Additionally, we introduce a weighted MDM designed for three-region SHNs. Other related terms essential to understanding the proposed method are also defined.

2.1. *Fuzzy Sets and Fuzzy Numbers*

Definition 1 (Zadeh, 1965). Let U be a classical set of objects, called the universe of discourse. An FS $\tilde{A}$ in U is characterized by a membership function, $\mu_{\tilde{A}} : U \to [0, 1]$, which assigns to each object $x \in U$ a real number $\mu_{\tilde{A}}(x) \in [0, 1]$, so as $\mu_{\tilde{A}}(x)$ represents the degree of membership of x into $\tilde{A}$.

Moreover, there are some other useful definitions, which are presented below.

Definition 2 (Nasseri, 2008). A real FN $\tilde{a} = (a_1, a_2, a_3, a_4)$ is said to be an NNFN if its membership function $\mu_{\tilde{a}}$ satisfies $\mu_{\tilde{a}}(x) = 0$ for all $x < 0$.

Below, we present the most-used forms of FNs, namely: TrFNs and TFNs, which are of particular relevance to this work and defined as follows.

Definition 3. A TrFN $\tilde{a}$ is an FN with a membership function $\mu_{\tilde{a}}(x)$ expressed as

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & \text{if } x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leqslant x \leqslant a_2, \\ 1, & \text{if } a_2 \leqslant x \leqslant a_3, \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 \leqslant x \leqslant a_4, \\ 0, & \text{if } x > a_4, \end{cases} \tag{1}$$

with $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. In what follows, the TrFN $\tilde{a}$ will be denoted in brief as $\tilde{a} = (a_1, a_2, a_3, a_4)$.

Definition 4. A TFN $\tilde{a}$ is an FN with a membership function $\mu_{\tilde{a}}(x)$ defined by

$$\mu_{\tilde{a}}(x) = \begin{cases} 0, & \text{if } x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leqslant x \leqslant a_2, \\ \frac{a_3-x}{a_3-a_2}, & \text{if } a_2 \leqslant x \leqslant a_3, \\ 0, & \text{if } x > a_3, \end{cases} \tag{2}$$

with $a_1 \leqslant a_2 \leqslant a_3$. In what follows, the TFN $\tilde{a}$ will be denoted as $\tilde{a} = (a_1, a_2, a_3)$.

2.2. *Algebraic Operations on Nonnegative TrFNs/TFNs*

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ be two nonnegative TrFNs, and a real number ω. Then, the algebraic operations on nonnegative TrFNs are defined as follows (Keshavarz Ghorabaee *et al.*, 2016):

(1) Equality
$\tilde{a} = \tilde{b} \Leftrightarrow a_k = b_k$ for $k = 1, 2, 3,$ and 4.
(2) Addition
$\tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$.
(3) Translation
$\tilde{a} \oplus \omega = (a_1 + \omega, a_2 + \omega, a_3 + \omega, a_4 + \omega)$.
(4) Multiplication
$\tilde{a} \otimes \tilde{b} = (a_1 \cdot b_1, a_2 \cdot b_2, a_3 \cdot b_3, a_4 \cdot b_4)$.
(5) Scalar multiplication
$\omega \otimes \tilde{a} = (\omega \cdot a_1, \omega \cdot a_2, \omega \cdot a_3, \omega \cdot a_4), \omega > 0$.

Note that the algebraic operations on nonnegative TFNs can be defined in the same manner.

2.3. *Shadowed Sets and Shadowed Numbers*

A three-region SHS can be formally defined as follows.

Definition 5 (Pedrycz, 1998). Let X be the universe of discourse. A three-region SHS S on the universe X is defined as a mapping $S : X \to \{0, (0, 1), 1\}$. Hence, a three-region SHS is partitioned into three pairwise disjoint regions, which consist of the *core* (region of full belongingness), *shadow* (region of uncertainty), and region of *full exclusion*.

(1) The core is denoted as $core(S) = \{x \in X | S(x) = 1\}$, which is the subset of objects that certainly belong to the SHS S.
(2) The shadow is denoted as $sh(S) = \{x \in X | S(x) = (0, 1)\}$, which is the subset of objects with completely uncertain belongingness to the SHS S.
(3) The exclusion region is the subset of objects that certainly do not belong to the SHS S.

Definition 6 (*Wang* et al.*, 2018*). A SHN SN, denoted as $SN = [S_*, C_*, C^*, S^*]$, is a SHS of the real line $\mathbb{R}$.

Hence, a SHN SN is a mapping $SN : \mathbb{R} \to \{0, (0, 1), 1\}$.

As depicted in Fig. 1 above (Landowski, 2020), the three regions of the SHN SN are the following:

(1) the core of $SN : core(SN) = [C_*, C^*]$;
(2) the shadow of $SN : sh(SN) =]S_*, C_*[\cup]C^*, S^*[$;
(3) the support of $SN : supp(SN) = [S_*, S^*]$.

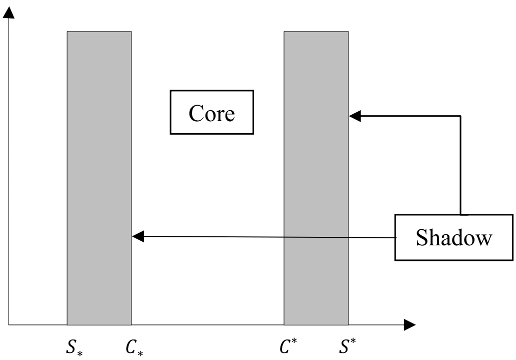

Fig. 1. Graphical representation of SHN $SN = [S_*, C_*, C^*, S^*]$.

Then, the membership function $S(x)$ of the SHN $SN = [S_*, C_*, C^*, S^*]$ can be expressed as

$$S(x) = \begin{cases} 0, & \text{if } x \leqslant S_*, \\ (0, 1), & \text{if } S_* < x < C_*, \\ 1, & \text{if } C_* \leqslant x \leqslant C^*, \\ (0, 1), & \text{if } C^* < x < S^*, \\ 0, & \text{if } x \geqslant S^*. \end{cases} \tag{3}$$

Remark 1. An SHN $SN = [S_*, C_*, C^*, S^*]$ is a degenerate SHN if $S_* = C_* = C^* = S^* = x$. Then, $SN = [x, x, x, x]$ is identified with the real number x.

Remark 2. An SHN $SN = [S_*, C_*, C^*, S^*]$ is said to be *nonnegative* if $S_* \geqslant 0$.

2.4. *Algebraic Operations on Shadowed Numbers*

Let $SN = [S_*, C_*, C^*, S^*]$ and $SN' = [S'_*, C'_*, C'^*, S'^*]$ be two SHNs. Then, the equality, addition, translation, and multiplication by a scalar of SHNs are respectively defined by the following:

(1) Equality
$SN = SN' \Leftrightarrow S_* = S'_*, C_* = C'_*, C^* = C'^*$, and $S^* = S'^*$.
(2) Addition
$SN + SN' = [S_* + S'_*, C_* + C'_*, C^* + C'^*, S^* + S'^*]$.
(3) Translation
$SN + \omega = [S_* + \omega, C_* + \omega, C^* + \omega, S^* + \omega]$.
(4) Scalar multiplication
$\omega \cdot SN = [\omega \cdot S_*, \omega \cdot C_*, \omega \cdot C^*, \omega \cdot S^*], \omega > 0$.

2.5. *Grzegorzewski's Approach*

This section introduces the approach proposed by Grzegorzewski (2013) for approximating FNs using SHSs. To facilitate the description of this approximation method, we first present the necessary notations and concepts.

Let $\tilde{a} = (a_1, a_2, a_3, a_4)$ be an arbitrary FN. Given a number $\alpha \in [0, 1]$, the α-cut of FN $\tilde{a}$ is defined as the closed inteval $\tilde{a}(\alpha) = [\tilde{a}_L(\alpha), \tilde{a}_U(\alpha)]$ whose lower endpoint is $\tilde{a}_L(\alpha) = inf\{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geqslant \alpha\}$ and upper endpoint $\tilde{a}_U(\alpha) = sup\{x \in \mathbb{R} : \mu_{\tilde{a}}(x) \geqslant \alpha\}$.

Let us note that $\tilde{a}(1) = core(\tilde{a}) = \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) = 1\}$ and $\tilde{a}(0) = supp(\tilde{a}) = \{x \in \mathbb{R} : \mu_{\tilde{a}}(x) > 0\}$.

In order to determine the SN that best fits the original FN, Grzegorzewski (2013) proposed an approximation method, which seeks to reach two successive approximation intervals $I_*(\tilde{a}) = [C_*(\tilde{a}), C^*(\tilde{a})]$ and $I^*(\tilde{a}) = [S_*(\tilde{a}), S^*(\tilde{a})]$. This was achieved by minimizing

$$d_\alpha^2\big(\tilde{a}, I_*(\tilde{a})\big) = \int_0^1 \alpha\big(A_L(\alpha) - C_*(\tilde{a})\big)^2 \mathrm{d}\alpha + \int_0^1 \alpha\big(A_U(\alpha) - C^*(\tilde{a})\big)^2 \mathrm{d}\alpha \tag{4}$$

with respect to $C_*(\tilde{a})$ and $C^*(\tilde{a})$, and by minimizing

$$d_{(1-\alpha)}^2\big(\tilde{a}, I^*(\tilde{a})\big) = \int_0^1 (1-\alpha)\big(A_L(\alpha) - S_*(\tilde{a})\big)^2 \mathrm{d}\alpha + \int_0^1 (1-\alpha)\big(A_U(\alpha) - S^*(\tilde{a})\big)^2 \mathrm{d}\alpha \tag{5}$$

with respect to $S_*(\tilde{a})$ and $S^*(\tilde{a})$.

In the setting of Grzegorzewski's optimization-based framework, the shadowed number $SN\ (\tilde{a})$ induced by fuzzy number $\tilde{a}$, written here as $SN(\tilde{a}) = [S_*(\tilde{a}), C_*(\tilde{a}), C^*(\tilde{a}), S^*(\tilde{a})]$, is characterized by means of the four points given by the following analytical formulas:

$$(1)\ S_*(\tilde{a}) = 2\int_0^1 \tilde{a}_L(\alpha)d\alpha - 2\int_0^1 \alpha\tilde{a}_L(\alpha)d\alpha, \tag{6}$$

$$(2)\ C_*(\tilde{a}) = 2\int_0^1 \alpha\tilde{a}_L(\alpha)d\alpha, \tag{7}$$

$$(3)\ C^*(\tilde{a}) = 2\int_0^1 \alpha\tilde{a}_U(\alpha)d\alpha, \tag{8}$$

$$(4)\ S^*(\tilde{a}) = 2\int_0^1 \tilde{a}_U(\alpha)d\alpha - 2\int_0^1 \alpha\tilde{a}_U(\alpha)d\alpha. \tag{9}$$

Since we confine ourselves to approximating TrFNs/TFNs, we establish in Statement 1 below the explicit expressions of the four points which characterize their corresponding SNs.

Statement 1. For any TrFN $\tilde{a} = (a_1, a_2, a_3, a_4)$, the corresponding induced SHN $SN(\tilde{a})$ is characterized by means of the four points given by the following algebraic expressions:

$$\begin{aligned} S_*(\tilde{a}) &= \frac{2a_1 + a_2}{3}, \qquad C_*(\tilde{a}) = \frac{a_1 + 2a_2}{3}, \\ C^*(\tilde{a}) &= \frac{2a_3 + a_4}{3}, \quad \text{and} \quad S^*(\tilde{a}) = \frac{a_3 + 2a_4}{3}. \end{aligned} \tag{10a}$$

Similarly, for any TFN $\tilde{a} = (a_1, a_2, a_3)$, the corresponding points are given by the following algebraic expressions:

$$\begin{aligned} S_*(\tilde{a}) &= \frac{2a_1 + a_2}{3}, \quad C_*(\tilde{a}) = \frac{a_1 + 2a_2}{3}, \\ C^*(\tilde{a}) &= \frac{2a_2 + a_3}{3}, \quad \text{and} \quad S^*(\tilde{a}) = \frac{a_2 + 2a_3}{3}. \end{aligned} \tag{10b}$$

Proof. See Appendix A for proof. □

Some immediate mathematical properties of the approximation operator SN are stated in Statement 2.

Statement 2. Consider three TrFNs/TFNs $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$, and a real number ω. Then, we have that

(1) One-to-one correspondence
$SN(\tilde{a}) = SN(\tilde{b}) \Leftrightarrow \tilde{a} = \tilde{b}$.

(2) Linearity
$SN(\tilde{a} + \tilde{b}) = SN(\tilde{a}) + SN(\tilde{b})$ (*additivity*).
$SN(\omega \cdot \tilde{c}) = \omega \cdot SN(\tilde{c})$, $\omega > 0$ (*scale invariance*).

(3) Translation invariance
$SN(\tilde{c} + \omega) = SN(\tilde{c}) + \omega$.

Proof. See Appendix A for proof. □

2.6. *A Distance Between Shadowed Numbers*

The mathematical concept of a distance metric is essential to the development of the SHANURSA method. Distance metrics are also an important tool for many FMADM methods proposed in the literature. Here we adopt Muscat's (2014) definition of a distance metric (2014).

Definition 7 (Muscat, 2014). Let X be an arbitrary nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a *distance metric* on X if, for all $x, y, z \in X$, it holds:

- $d(x, y) = d(y, x)$ (*symmetry*);

- $d(x, y) \leqslant d(x, z) + d(z, y)$ (*triangle inequality*);
- $d(x, y) = 0$ if and only if $x = y$ (*identity of indiscernibles*).

Consequently, we have that $d(x, y) \geqslant 0$ (non-negativity) for any $x, y \in X$. This follows from axioms 1 and 2.

To implement the SHANURSA method, it is essential to employ a suitable distance metric to evaluate the proximity/remoteness between SHNs. Let $SN(\tilde{a}) = [S_*(\tilde{a}), C_*(\tilde{a}), C^*(\tilde{a}), S^*(\tilde{a})]$ and $SN(\tilde{b}) = [S_*(\tilde{b}), C_*(\tilde{b}), C^*(\tilde{b}), S^*(\tilde{b})]$ be two SHNs induced by two arbitrary nonnegative TrFNs/TFNs $\tilde{a}$ and $\tilde{b}$, respectively. We propose a four-parameter generalized MDM between $SN(\tilde{a})$ and $SN(\tilde{b})$, denoted by $d_{SHN}(SN(\tilde{a}), SN(\tilde{b}))$ and defined as follows:

$$d_{SHN}\big(SN(\tilde{a}), SN(\tilde{b})\big) = \sqrt[p]{(1-\theta)\big\{(1-\kappa)|S_*(\tilde{a}) - S_*(\tilde{b})|^p + \kappa|S^*(\tilde{a}) - S^*(\tilde{b})|^p\big\} + \theta\big\{(1-\kappa')|C_*(\tilde{a}) - C_*(\tilde{b})|^p + \kappa'|C^*(\tilde{a}) - C^*(\tilde{b})|^p\big\}}, \tag{11}$$

where the symbol $|\,|$ denotes the absolute value function, and where θ, κ and κ' are weighting constants satisfying $0.5 \leqslant \kappa, \kappa' < 1$, $0.5 < \theta < 1$ and $1 \leqslant p$.

Remark 3. Let us remark that by varying the parameters, θ, κ, κ' and p, different distances between SHNs can be obtained. Note that here the upper bounds of the core and support are given more importance than their lower bounds, and the bounds of the core are given more importance than those of the support.

Having introduced the d_{SHN} distance, we can state Statement 3.

Statement 3. Let $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ be three nonnegative TrFNs/TFNs, and let ω be a positive real number. The function d_{SHN} defined as in the equation (11) satisfies all of the following metric axioms: symmetry, triangle inequality, and identity of indiscernibles, along with the following mathematical properties:

- Homogeneity

$$d_{SHN}\big(SN(\omega.\tilde{a}), SN(\omega.\tilde{b})\big) = \omega.d_{SHN}\big(SN(\tilde{a}), SN(\tilde{b})\big).$$

- Translation invariance
  - $d_{SHN}(SN(\tilde{a} + \omega), SN(\tilde{b} + \omega)) = d_{SHN}(SN(\tilde{a}), SN(\tilde{b}))$,
  - $d_{SHN}(SN(\tilde{a} + \tilde{c}), SN(\tilde{b} + \tilde{c})) = d_{SHN}(SN(\tilde{a}), SN(\tilde{b}))$.

Proof. Proof of Statement 3 follows directly and is therefore omitted. □

For operational purposes, the upper and lower bounds are considered to be of equal importance ($\kappa = \kappa' = 0.5$). Hence, the distance between pairs of induced SHNs is taken as:

$$
\begin{aligned}
&d_{SHN}\big(SN(\tilde{a}), SN(\tilde{b})\big) \\
&\quad = \sqrt{\frac{1}{3}\left\{\frac{(S_*(\tilde{a}) - S_*(\tilde{b}))^2 + (S^*(\tilde{a}) - S^*(\tilde{b}))^2}{2}\right\} + \frac{2}{3}\left\{\frac{(C_*(\tilde{a}) - C_*(\tilde{b}))^2 + (C^*(\tilde{a}) - C^*(\tilde{b}))^2}{2}\right\}} \\
&\quad = \sqrt{\frac{1}{6}\left\{\big(S_*(\tilde{a}) - S_*(\tilde{b})\big)^2 + \big(S^*(\tilde{a}) - S^*(\tilde{b})\big)^2\right\} + \frac{1}{3}\left\{\big(C_*(\tilde{a}) - C_*(\tilde{b})\big)^2 + \big(C^*(\tilde{a}) - C^*(\tilde{b})\big)^2\right\}}.
\end{aligned}
\tag{12}
$$

3. Shadowed Number Based Ranking and Selection Approach

A wide spectrum of methods to solve FMADM problems using the most commonly exploited TrFNs/TFNs has been developed over the years. What is at issue in current research is solving the FMADM problem of choosing a 'best' alternative from among a finite, given and fixed *choice set* (i.e. the predetermined set of feasible alternatives). As is known, to solve the above multi-attribute choice problem, most of the methods proposed in the FMADM literature adopt the 'rank then select' view. In fact, the ranking-based view states that at first a ranking of the alternatives from the most preferred to the least preferred (possibly with ties) is obtained; then, an alternative ranked the highest is chosen. Therefore, in the remainder of this work, we concentrate on ranking m ($\geqslant 2$) discrete multi-attribute alternatives ($A_1, A_2, \dots, A_{m-1},$ and A_m) characterized by fuzzy attribute values (performance-values)—measured in the same unit or transformed to a common scale—with respect to n ($\geqslant 2$) attributes $C = (C_1, C_2, \dots, C_{n-1}, \text{and} C_n)$. Thus, the set of attributes is divided into a subset C_C of cost attributes and a subset C_B of benefit attributes. Furthermore, we assume the representability of the fuzzy attribute values $\tilde{a}_{ij}$'s of every alternative A_i with respect to the attribute C_j by nonnegative TrFNs/TFNs. Moreover, we need to mention that the types of weights, which are acceptable to reflect the degrees of importance that are associated with the selection attributes, could be either crisp or nonnegative TrFNs/TFNs.

Definition 8. The nonnegative TrFNs $\tilde{W}_j = (W_{j1}, W_{j2}, W_{j3}, W_{j4})$, $j = 1, 2, \dots, n$, are declared acceptable to weight the selection attributes if and only if

1) $0 < W_{j1} \leqslant W_{j2} \leqslant W_{j3} \leqslant W_{j4} \leqslant 1$ ($j = 1, 2, \dots, n$);
2) $1 \leqslant \sum_{j=1}^{j=n} W_{j1}$;
3) $\sum_{j=1}^{j=n} w_j^4 \leqslant n$.

Definition 9. The triangular fuzzy numbers $\tilde{W}_j = (W_{j1}, W_{j2}, W_{j3})$, $j = 1, 2, \dots, n$, are declared acceptable to weight the selection attributes if and only if

1) $0 < W_{j1} \leqslant W_{j2} \leqslant W_{j3} \leqslant 1$ ($j = 1, 2, \dots, n$);
2) $1 \leqslant \sum_{j=1}^{j=n} W_{j1}$;
3) $\sum_{j=1}^{j=n} w_j^3 \leqslant n$.

REMARK 4. The above two definitions cover all possible fuzzy attribute weights in which one can be interested.

Now, let A_h and A_l be two alternatives and $\tilde{SN}_h = (SN_{h1}, SN_{h2}, \dots, SN_{hn})$ and $\tilde{SN}_l = (SN_{l1}, SN_{l2}, \dots, SN_{ln})$ their representing n-dimensional SHNs vectors (SHNVs) then, a *global distance* $d_G(A_h, A_l)$ between A_h and A_l can be taken as:

$$d_G(A_h, A_l) = d_{SHNV}(\tilde{SN}_h, \tilde{SN}_l) = \sum_{j=1}^{n} d_{SHN}(SN_{hj}, SN_{lj}). \tag{13}$$

Next, let $SN_{ij} = [S_{ij*}, C_{ij*}, C^*_{ij}, S^*_{ij}]$, $i = 1, 2, \dots, m$, be a sequence of m SHNs, then the j-th *maximal* and *minimal* SHNs, denoted as $SN^j_{\max}$ and $SN^j_{\min}$, are respectively defined by:

- $SN^j_{\max} = \Big[\max\limits_{1\leqslant i\leqslant m} S_{ij*}, \max\limits_{1\leqslant i\leqslant m} C_{ij*}, \max\limits_{1\leqslant i\leqslant m} C^*_{ij}, \max\limits_{1\leqslant i\leqslant m} S^*_{ij}\Big], \quad j = 1, 2, \dots, n,$ (14)
- $SN^j_{\min} = \Big[\min\limits_{1\leqslant i\leqslant m} S_{ij*}, \min\limits_{1\leqslant i\leqslant m} C_{ij*}, \min\limits_{1\leqslant i\leqslant m} C^*_{ij}, \min\limits_{1\leqslant i\leqslant m} S^*_{ij}\Big], \quad j = 1, 2, \dots, n,$ (15)

where max and min stand for the maximum and minimum operators.

As the SHANURSA method rests on ideal and anti-ideal concepts, the j-th positive ideal SHN, which is written SN^+_j, will be defined by:

$$SN^+_j = \begin{cases} SN^j_{\max}, & \text{if } C_j \in C_B, \\ SN^j_{\min}, & \text{if } C_j \in C_C. \end{cases} \tag{16}$$

Likewise, the j-th negative ideal SHN, which is written SN^-_j, will be given by:

$$SN^-_j = \begin{cases} SN^j_{\min}, & \text{if } C_j \in C_B, \\ SN^j_{\max}, & \text{if } C_j \in C_C. \end{cases} \tag{17}$$

Therefore, to define the positive ideal alternative (PIA) A^+ and the negative ideal alternative (NIA) A^-, we have to represent them by SHNVs. Therefore, A^+ (n-dimensional reference vector of desirable SHNs) and A^- (n-dimensional reference vector of undesirable SHNs) will be described respectively by the two n-dimensional SHNVs:

- $\tilde{SN}^+ = \big(SN^+_1, SN^+_2, \dots, SN^+_n\big),$ (18)
- $\tilde{SN}^- = \big(SN^-_1, SN^-_2, \dots, SN^-_n\big).$ (19)

Following, we define the *proximity value* $P(A_i)$ of the alternative A_i to A^+ as follows:

$$P(A_i) = d_{SHNV}\big(\tilde{SN}_i, \tilde{SN}^+\big) = \sum_{j=1}^{n} d_{SHN}\big(SN_{ij}, SN_j^+\big), \quad i = 1, 2, \dots, m. \tag{20}$$

Similarly, the *remoteness value* $R(A_i)$ of A_i from A^- can be defined as follows:

$$R(A_i) = d_{SHNV}\big(\tilde{SN}_i, \tilde{SN}^-\big) = \sum_{j=1}^{n} d_{SHN}\big(SN_{ij}, SN_j^-\big), \quad i = 1, 2, \dots, m. \tag{21}$$

To rank the alternatives, a new closeness coefficient relative to the ideal alternative is defined. This ranking index incorporates the relative importance of the separation measures from both PIA and NIA. In doing so, it addresses a key limitation of the conventional TOPSIS index. Specifically, the traditional TOPSIS index implicitly assumes that both distances are equally important (Kuo, 2017). In some situations, decision-makers may have distinct preferences regarding proximity to PIA versus avoidance of NIA. This necessity has prompted research efforts. For example, Doukas *et al.* (2010) developed a formula for separating weights to rank alternatives. A ranking formula was proposed by Kuo (2017) based on the method of Doukas *et al.* (2010), which normalizes the distance measures of each alternative relative to the positive and the negative ideal alternatives. Nevertheless, this index exhibits a notable drawback as the number of alternatives increases, the value of the ranking index decreases, thereby reducing its interpretability (Sadabadi *et al.*, 2020). To address these drawbacks, Sadabadi *et al.* (2020) proposed a novel ranking index by modifying the normalization process of the distance measures.

Obviously, the previous research focuses on the simple additive weighting for constructing the closeness coefficient. However, the SAW is a fully compensatory method. To mitigate the compensation effect, this study proposes a different closeness coefficient based on the Weighted Product Model (WPM), which is less compensatory and enhances the robustness of the ranking results. The formula of the proposed closeness coefficient is as follows:

$$C(A_i) = \left(\frac{R(A_i)}{\max_k R(A_i)}\right)^{\lambda} \times \left(\frac{\min_k P(A_i)}{P(A_i)}\right)^{1-\lambda}, \quad i = 1, 2, \dots, m, \ 0 < \lambda < 1. \tag{22}$$

The parameter λ denotes the weight representing the importance the decision maker assigns to the distance from NIA. Additionally, $1 - \lambda$ is the weight representing the importance assigned to the distance to PIA. In cases where the decision-makers do not show a preference for either the positive or negative ideal alternative ($\lambda = 0.5$), the closeness coefficient becomes:

$$C(A_i) = \sqrt{\frac{R(A_i)}{\max_k R(A_i)} \times \frac{\min_k P(A_i)}{P(A_i)}}, \quad i = 1, 2, \dots, m. \tag{23}$$

This corresponds to an equal weighting of the distances to both PIA and NIA.

3.1. *The SHANURSA Method Steps*

The eight steps of the SHANURSA method are shown in the Fig. 2 below. At first, the SHANURSA method starts with the identification of feasible alternatives and relevant attributes, in addition to the selection of (fuzzy) attribute weights. Then follows the construction of a fuzzy decision matrix. And, the weighted FNs are converted into SHNs using Grzegorzewski's (2013) approach. Once the SHN matrix is determined, the maximal and minimal SNs, $SN^j_{\max}$ and $SN^j_{\min}$, are determined. Next, the ideal and anti-ideal alterna-

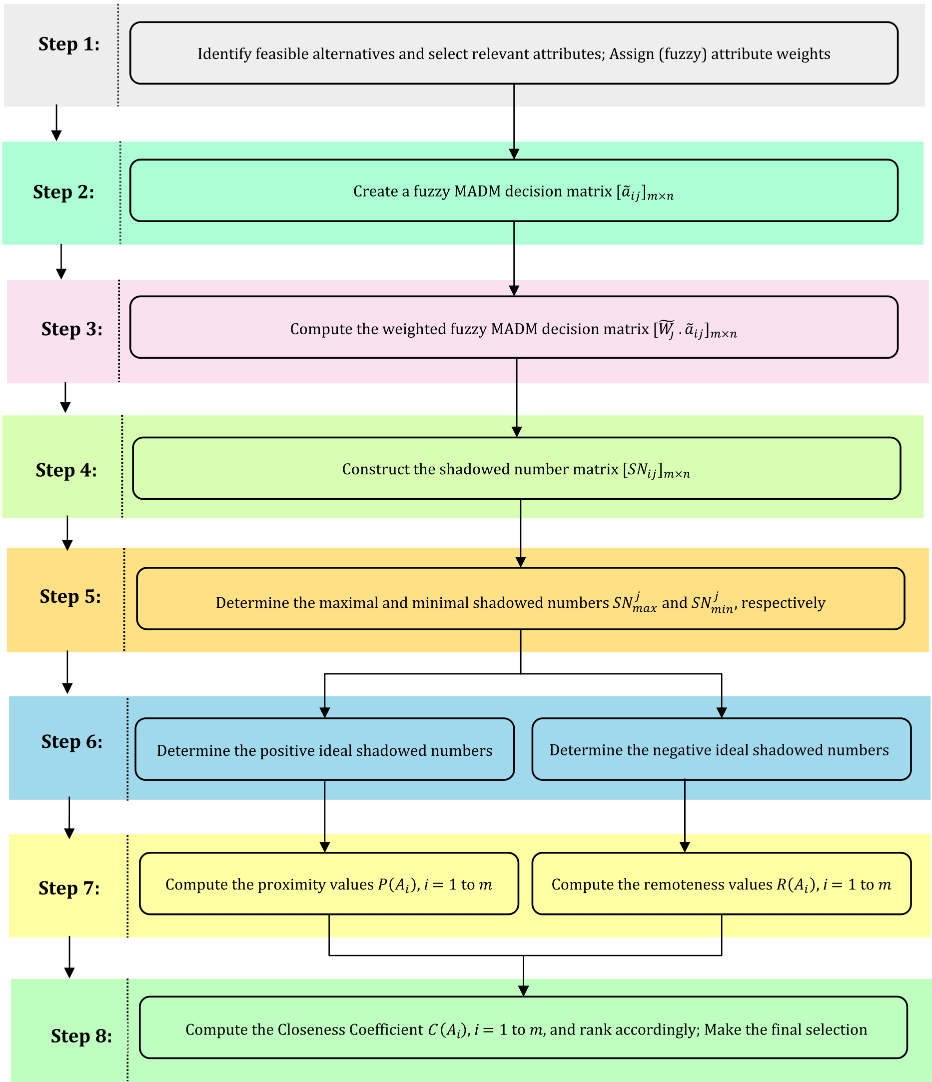

Fig. 2. The SHANURSA eight-step process.

tives PIA and NIA are identified. The proximity value $P(A_i)$ and the remoteness value $R(A_i)$ of the alternative A_i are then computed. Lastly, the closeness coefficient value for each alternative is computed using, and the alternatives are ranked accordingly.

The computational complexity of the SHANURSA method is polynomial with respect to the number of alternatives m and criteria n. The Steps (1)–(7) require O $(m\,n)$ computational cost. The final ranking stage requires sorting the alternatives, which adds O $(m$ Log $m)$. Hence, the overall complexity of the proposed method is O $(m\ n\ {+}m$ Log $m)$. This shows that the proposed approach is efficient and appropriate for large-scale FMADM problems.

4. Numerical Illustrations, Sensitivity Analysis and Comparative Analysis

In this section, we demonstrate the effectiveness of the SHANURSA method using the following three different illustrative examples:

(1) *Example 1*: The selection of a heating, ventilating and air conditioning (HVAC) and air handling unit (AHU) system supplier (Polat and Bayhan, 2020);
(2) *Example 2*: The selection of a transportation company (Ulutaş *et al.*, 2021);
(3) *Example 3*: The selection of an optimal reclamation strategy for the Tamnava-West field (Tamnava Zapadno polje) open-pit mine (Dimitrijević *et al.*, 2024).

The results are examined from the following three distinct perspectives.

(1) A sensitivity analysis was performed to determine how changes in the parameters k, k', p and θ impacted the final rankings. The parameters were explored over predefined domains: k and k' were selected from the set $\{0.5, 0.6, 0.7, 0.8, \text{ and } 0.9\}$, p is varied within $\{1, 2\}$ and θ takes values of $\frac{2}{3}, 0.7, 0.8, \text{ and } 0.9$. The reliability of the SHANURSA method was assessed through the systematic combination of these values which yielded a total of 200 experimental scenarios denoted by $S_1, S_2, \ldots, S_{200}$.
(2) The so-called *dynamic decision matrix effects* were used to evaluate the *ranking stability* of the SHANURSA method. This analysis involved the alteration of the components of the initial decision matrix using a series of scenarios. The number of alternatives varies across scenarios. In each instance, the lowest alternative is excluded from subsequent consideration to observe potential shifts in the ranking results. In particular, the indication of the most preferred alternative was analysed in each scenario (Ecer, 2021).
(3) The result obtained using the SHANURSA method were benchmarked against ten (10) known FMADM methods:
    - Fuzzy Weighted Aggregated Sum-Product ASsessment (F-WASPAS) (Turskis *et al.*, 2015);
    - Fuzzy Evaluation based on Distance from Average Solution (F-EDAS) (Keshavarz-Ghorabaee *et al.*, 2016);
    - Fuzzy Multi-Objective Optimization on the basis of Ratio Analysis (F-MOORA) (Siddiqui and Tyagi, 2016);

- Fuzzy Additive Ratio ASsessment (F-ARAS) (Rostamzadeh *et al.*, 2017);
- Fuzzy COmbinative Distance based ASsessment (F-CODAS) (Keshavarz-Ghorabaee *et al.*, 2017);
- Fuzzy COmplex PRoportional ASsessment (F-COPRAS) (Tolga and Durak, 2019);
- Fuzzy Measurement Alternatives and Ranking according to the COmpromise Solution (F-MARCOS) (Stanković *et al.*, 2020);
- Fuzzy COmbined COmpromise SOlution (F-CoCoSo) (Ulutaş *et al.*, 2021);
- Fuzzy Technique for Order of Preference by Similarity to Ideal Solution (F-TOPSIS) (Cakar and Çavuş, 2021);
- Fuzzy Compromise Ranking of Alternatives from Distance to Ideal Solution (F-CRADIS) (Puška *et al.*, 2022).

The Spearman rank correlation coefficient (r_S) was used to assess the similarity between the rankings yielded by the SHANURSA method and those returned by the above ten methods.

$$r_S = 1 - \frac{6\sum d_i^2}{m(m^2-1)}, \tag{24}$$

where d_i is the difference in rankings for each alternative A_i, $i = 1, 2, \dots, m$.

4.1. *Analysis of Example 1*

Example 1 consists in choosing the most suitable heating, ventilating and air conditioning (HVAC) and air handling unit (AHU) system along with its corresponding supplier.

Step 1. In this example, six alternatives $\{A_i\}_{i=1}^{i=6}$ are to be evaluated on the basis of the following eight (8) attributes:

- Purchase price (C_1);
- Warranty period (C_2);
- Delivery lead time (C_3);
- Conformity with the specifications (C_4);
- Quality of supplier's communication (C_5);
- Quality and availability of after-sales support and spare parts (C_6);
- Conformity with energy performance directives for green production, design and supply (C_7);
- Motors efficiency level (C_8).

Purchase price and *Delivery lead time* are cost attributes, meaning lower values are more desirable. Conversely, the remaining six attributes are benefit attributes, where higher values indicate better performance. Both the performance ratings and the attribute weights are modelled by TrFNs. The TrFN weights are recorded in Table 1.

Step 2. The fuzzy decision matrix is displayed in Table 2.

Table 1
Trapezoidal fuzzy weights.

Attribute	Fuzzy weight
C_1	(0.7, 0.8, 0.86, 0.9)
C_2	(0.66, 0.76, 0.78, 0.88)
C_3	(0.54, 0.64, 0.72, 0.8)
C_4	(0.8, 0.9, 1, 1)
C_5	(0.48, 0.58, 0.66, 0.76)
C_6	(0.68, 0.78, 0.82, 0.9)
C_7	(0.72, 0.82, 0.84, 0.92)
C_8	(0.7, 0.8, 0.86, 0.92)

Table 2
Fuzzy decision matrix—Example 1.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(3.2, 4.2, 4.6, 5.6)	(1.2, 2.2, 2.4, 3.4)	(2.8, 3.8, 4.4, 5.4)	(2, 2.6, 3.2, 4.2)	(3.4, 4.4, 4.4, 5.4)	(2.8, 3.8, 4.4, 5.4)
C_2	(6.2, 7.2, 7.6, 8.6)	(7, 8, 8.6, 9.2)	(6.6, 7.6, 7.8, 8.8)	(2.4, 3.4, 4.2, 5.2)	(4.8, 5.8, 6, 7)	(8, 9, 10, 10)
C_3	(1.8, 2.8, 3.6, 4.6)	(0.8, 1.4, 2, 3)	(1, 2, 2, 3)	(0.4, 0.8, 1.4, 2.4)	(4.8, 5.8, 6.6, 7.4)	(4.4, 5.4, 6.4, 7.2)
C_4	(7.4, 8.4, 8.8, 9.4)	(5.8, 6.8, 7.4, 8.4)	(7.2, 8.2, 8.4, 9.2)	(3. 4, 4.4, 5, 6)	(5.2, 6.2, 6.2, 7.2)	((4.4, 5.4, 5.8, 6.8)
C_5	(8, 9,10, 10)	(5.6, 6.6, 7, 8)	(7.2, 8.2, 8.4, 9.2)	(5, 6, 7, 8)	(5.8, 6.8, 6.8, 7.8)	(6.2, 7.2, 7.6, 8.2)
C_6	(5.8, 6.8, 6.8, 7.8)	(6.4, 7.4, 8, 8.8)	(5.6, 6.6, 7, 8)	(2.4, 3.4, 3.6, 4.6)	(4.4, 5.4, 6.4, 7.2)	(7, 8, 8, 9)
C_7	(6.8, 7.8, 8.2, 9)	(5, 6, 6.4, 7.4)	(6.4, 7.4, 8, 8.8)	(4.2, 5.2, 5.4, 6.4)	(6.4, 7.4, 8, 8.8)	(4.6, 5.6, 6.2, 7.2)
C_8	(7.2, 8.2, 8.4, 9.2)	(7.4, 8.4, 9.4, 9.6)	(7, 8, 8, 9)	(4.6, 5.6, 6.2, 7.2)	(5.2, 6.2, 6.8, 7.8)	(5, 6, 6.4, 7.4)

Table 3
Weighted fuzzy decision matrix.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	(2.24, 3.36, 3.96, 5.38)	(0.84, 1.76, 2.1, 3.26)	(1.96, 3.04, 3.78, 5.18)	(1.4, 2.08, 2.752, 4.03)	(2.38, 3.5, 3.78, 5.18)	(1.96, 3.04, 3.78, 5.184)
C_2	(4.1, 5.5, 5.93, 7.57)	(4.62, 6.1, 6.71, 8.1)	(4.36, 5.78, 6.1, 4.74)	(1.58, 2.58, 3.28, 4.58)	(3.17, 4.4, 4.68, 6.16)	(5.28, 6.84, 7.8, 8.8)
C_3	(9.72, 1.79, 2.59, 3.62)	(0.43, 0.896, 1.44, 2.4)	(0.54, 1.28, 1.44, 2.4)	(0.216, 0.51, 1, 1.92)	(2.6, 3.71, 4.75, 5.92)	(2.38, 3.46, 4.61, 5.76)
C_4	(5.92, 7.56, 8.8, 9.4)	(4.64, 6.12, 7.8, 8.4)	(5.76, 7.38, 8.4, 9.2)	(2.72, 3.96, 5, 6)	(4.16, 5.58, 6.2, 7.2)	(3.52, 4.86, 5.8, 6.8)
C_5	(3.84, 5.2, 6.6, 7.6)	(2.4, 3.83, 4.62, 6.1)	(3.46, 4.76, 5.54, 6.99)	(2.4, 3.48, 4.62, 6.1)	(2.78, 3.94, 4.49, 5.9)	(2.98, 4.18, 5.02, 6.23)
C_6	(3.94, 5.3, 5.58, 7.02)	(4.35, 5.77, 6.56, 7.92)	(3.81, 5.15, 5.74, 7.2)	(1.63, 2.65, 2.95, 4.14)	(2.99, 4.2, 5.25, 6.48)	(4.76, 6.24, 6.56, 8)
C_7	(4.9, 6.4, 6.9, 8.3)	(3.61, 4.92, 5.38, 6.81)	(4.61, 6.1, 6.72, 8.1)	(3.02, 4.26, 4.54, 5.9)	(4.61, 6.1, 6.72, 8.1)	(3.3, 4.6, 5.21, 6.62)
C_8	(5.04, 6.56, 7.2, 8.46)	(5.18, 6.72, 8.1, 8.83)	(4.9, 6.4, 6.9, 8.3)	((3.22, 4.48, 5.3, 6.62)	(3.64, 4.96, 5.85, 7.2)	(3.5, 4.8, 5.5, 6.81)

4.1.1. *Solving the Problem Given in Example 1*

Following, we proceed with the remaining steps of the SHANURSA method.

Step 3. In this step, the weighted fuzzy decision matrix is constructed by applying the multiplication operation. The resulting matrix is displayed in Table 3.

Step 4. The SHN matrix is generated by transforming TrFNs using the direct formulas for the four characteristic points (see Statement 1). The resulting SHN matrix is displayed in Table 4.

Step 5. The resulting maximal and minimal SHNs are as presented in Table 5.

Step 6. The positive and negative ideal SHNs are identified and presented in Table 6.

Steps 7 and 8. The proximity and remoteness values, the *closeness coefficient values*, and the final rankings of the alternatives are recorded in Table 7.

Table 4
Shadowed number matrix.

	A_1	A_2	A_3	A_4	A_5	A_6
C_1	[2.613, 2.987, 4.317, 4.679]	[1.15, 1.45, 2.4, 2.73]	[2.32, 2.68, 4.143, 4.5]	[1.63, 1.85, 3.1, 3.44]	[2.76, 3.14, 4.14, 4.5]	[2.32, 2.68, 4.14, 4.5]
C_2	[4.55, 5.01, 6.475, 7.02]	[5.11, 5.59, 5.17, 7.63]	[4.83, 5.3, 6.64, 7.19]	[1.92, 2.25, 3.71, 4.14]	[3.58, 3.99, 5.17, 5.67]	[5.8, 6.32, 8.13, 8.47]
C_3	[2.32, 2.68, 4.143, 4.5]	[0.11, 0.13, 0.33, 0.41]	[0.11, 0.13, 0.25, 0.32]	[0.15, 0.18, 0.61, 0.81]	[0.04, 0.042, 0.067, 0.075]	[0.04, 0.044, 0.07, 0.08]
C_4	[6.47, 7.01, 9, 9.2]	[5.13, 5.63, 8, 8.2]	[6.3, 6.84, 8.67, 8.93]	[3.13, 3.45, 5.33, 5.67]	[4.63, 5.11, 6.53, 6.87]	[3.97, 4.14, 6.13, 6.47]
C_5	[4.3, 4.76, 6.93, 7.27]	[2.88, 3.35, 5.11, 5.59]	[3.89, 4.32, 6.03, 6.51]	[2.76, 3.12, 5.11, 5.59]	[3.17, 3.56, 4.97, 5.45]	[3.38, 3.78, 5.42, 5.83]
C_6	[4.4, 4.85, 6.06, 6.54]	[4.82, 5.3, 7.01, 7.47]	[4.25, 4.7, 6.23, 6.7]	[1.97, 2.31, 3.45, 3.74]	[3.4, 3.8, 5.66, 6.07]	[5.25, 5.75, 7.07, 7.59]
C_7	[5.4, 5.9, 7.35, 7.82]	[4.04, 4.48, 5.85, 6.33]	[5.1, 5.58, 7.18, 7.64]	[3.44, 3.85, 4.99, 5.44]	[5.09, 5.58, 7.18, 7.64]	[3.74, 4.16, 5.68, 6.15]
C_8	[5.55, 6.05, 7.64, 8.05]	[5.69, 6.21, 8.33, 8.58]	[5.4, 5.9, 7.35, 7.81]	[3.64, 4.06, 5.76, 6.2]	[4.08, 4.52, 6.29, 6.73]	[3.93, 4.37, 5.94, 6.37]

Table 5
Maximal and minimal shadowed numbers.

Attribute	$SN^j_{\max}$	$SN^j_{\min}$
$\boldsymbol{C_1}$	[2.76, 3.14, 4.32, 4.68]	[1.15, 1.45, 2.4, 2.73]
$\boldsymbol{C_2}$	[5.8, 6.32, 8.13, 8.47]	[1.92, 2.25, 3.71, 4.14]
$\boldsymbol{C_3}$	[2.97, 3.34, 5.14, 5.53]	[0.31, 0.41, 1.31, 1.62]
$\boldsymbol{C_4}$	[6.47, 7.01, 9, 9.2]	[3.13, 3.55, 5.33, 5.67]
$\boldsymbol{C_5}$	[4.3, 4.76, 6.93, 7.27]	[2.76, 3.12, 4.97, 5.45]
$\boldsymbol{C_6}$	[5.25, 5.75, 7.07, 7.59]	[1.97, 2.31, 3.35, 3.74]
$\boldsymbol{C_7}$	[5.4, 5.9, 7.35, 7.82]	[3.44, 3.85, 4.99, 5.44]
$\boldsymbol{C_8}$	[5.69, 6.21, 8.33, 8.58]	[3.64, 4.06, 5.76, 6.19]

Table 6
Positive and negative ideal shadowed numbers.

Attribute	SN^+_j	SN^-_j
C_1	[1.15, 1.45, 2.4, 2.73]	[2.76, 3.14, 4.32, 4.68]
C_2	[5.8, 6.32, 8.13, 8.47]	[1.92, 2.25, 3.71, 4.14]
C_3	[0.31, 0.41, 1.31, 1.62]	[2.97, 3.34, 5.14, 5.53]
C_4	[6.47, 7.01, 9, 9.2]	[3.13, 3.55, 5.33, 5.67]
C_5	[4.3, 4.76, 6.93, 7.27]	[2.76, 3.12, 4.97, 5.45]
C_6	[5.25, 5.75, 7.07, 7.59]	[1.97, 2.31, 3.35, 3.74]
C_7	[5.4, 5.9, 7.35, 7.82]	[3.44, 3.85, 4.99, 5.44]
C_8	[5.69, 6.21, 8.33, 8.58]	[3.64, 4.06, 5.76, 6.19]

Table 7
Results of the SHANURSA method.

Alt.	$P(A_i)$	$R(A_i)$	$C(A_i)$	$Rank(A_i)$
A_1	5.990	17.33	0.975	**2**
A_2	5.780	17.58	1	**1**
A_3	6.060	17.37	0.970	**3**
A_4	18.10	5.28	0.310	**6**
A_5	15.25	8.32	0.420	**5**
A_6	12.44	10.98	0.730	**4**

The final ranking of the alternatives is as follows: $A_2 \succ A_1 \succ A_3 \succ A_6 \succ A_5 \succ A_4$, where the symbol $\succ$ stands for "is preferred to". Based on these results, A_2 is clearly identified as the best-ranked alternative, while A_4 is the lowest-raked one.

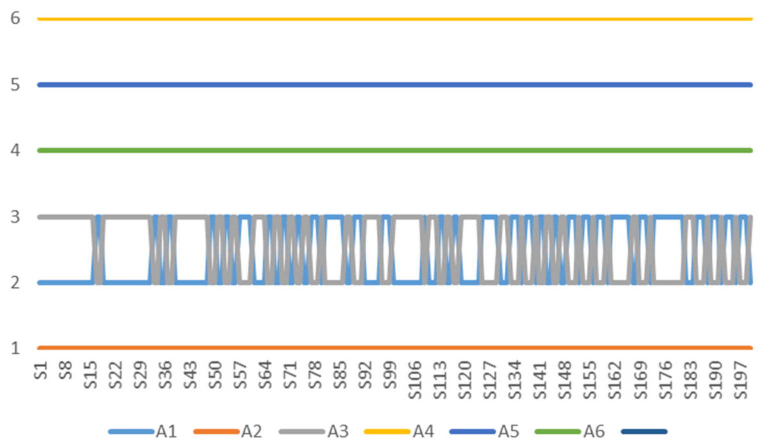

Fig. 3. Parametric sensitivity results—Example 1.

4.1.2. *Parameter Sensitivity*

Figure 3 below shows how the final rankings of the alternatives change when the parameter of the weighted MDM are varied.

All in all, the impact of changing the MDM distance parameters is not substantial. In fact, across all scenarios, alternative A_2 remains first ranked confirming its superiority over the others. The alternatives A_4, A_5, and A_6 maintain the same low ranks uniformly across all scenarios. Moreover, one can notice that the ranking orders produced by the SHANURSA method present pairwise inversions between the alternatives A_1 and A_3. However, these variations do not significantly affect the final ranking results of the SHANURSA method. Therefore, one can conclude that the proposed method remains broadly stable despite altering the distance parameters.

4.1.3. *Dynamic Decision Matrix Effects*

As known, rank reversal is an important tool for judging the trustworthiness of MADM methods. To this end, we calculate the effects of dynamic decision matrices on the final ranking of the alternatives. We form, in this example, a total of five different scenarios by removing each time the worst-ranked alternative from subsequent consideration and ranking the remaining ones using the SHANURSA method. In the original scenario (initially obtained ranking), A_4 comes out to be the worst-ranked alternative. Thus, in scenario 1, alternative A_4 is dropped from further consideration in order to create a new decision matrix with five alternatives. This time, the obtained ranking is $A_2 \succ A_1 \succ A_3 \succ A_6 \succ A_5$. Obviously, alternative A_5 is the new worst-ranked alternative. Thus, in scenario 2, A_5 is removed in order to create a new decision matrix. The process continues until the alternative A_2 itself remains. Table 8 depicts all the rankings generated under the five scenarios. In fact, Table 8 shows that A_2 is the best-ranked alternative uniformly under all the scenarios. This signifies that the SHANURSA method maintains the indication of the 'best' alternative. Evidently, SHANURSA did not present any rank reversal in this example. This confirms the stability and robustness of the proposed method in the dynamic environment.

Table 8
Ranking results under different scenarios.

Scenario	Ranking
Original	$A_2 > A_1 > A_3 > A_6 > A_5 > A_4$
Scenario-1	$A_2 > A_1 > A_3 > A_6 > A_5$
Scenario-2	$A_2 > A_1 > A_3 > A_6 > A_5$
Scenario-3	$A_2 > A_1 > A_3$
Scenario-4	$A_2 > A_1$
Scenario-5	A_2

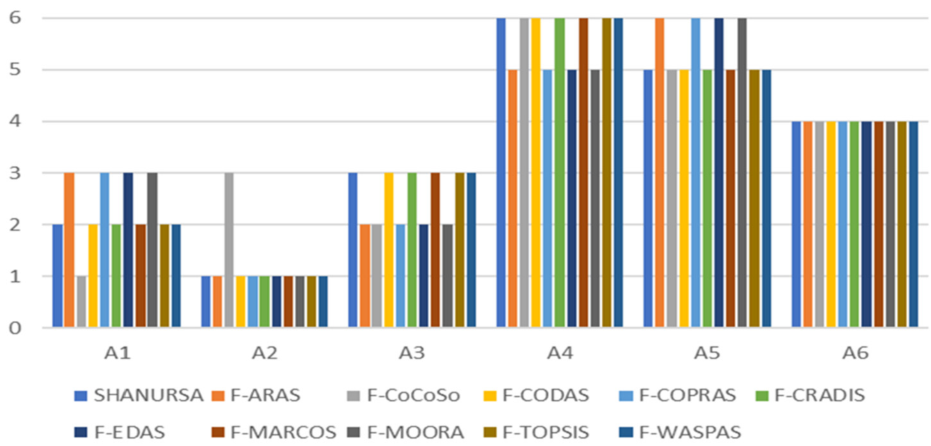

Fig. 4. Rankings results from different FMADM methods—Example 1.

4.1.4. *Comparison with Other FMADM Methods*

As said earlier, the prescriptions of the SHANURSA method will be compared to those of other well-known methods in the literature. Figure 4 depicts all the rankings prescribed by the various FMADM methods being used. It is noteworthy that, in Example 1, SHANURSA, F-CODAS, F-CRADIS, F-MARCOS, F-TOPSIS, and F-WASPAS produce the same ranking of the alternatives. Moreover, as shown in Fig. 4, alternative A_2 is ranked first by 10 out of the 11 (total number) FMADM methods being used, including the SHANURSA method. In contrast, alternative A_1 is ranked first only by F-CoCoSo. Whilst, alternative A_2 is ranked third. Lastly, alternative A_4 comes out to be ranked last by seven out of the 11 FMADM methods being used, including the SHANURSA method.

4.1.5. *Correlation Analysis*

The Spearman rank correlation results between the rankings delivered by the SHANURSA method and the rankings produced by the other FMADM methods used herein are recorded in Table 9.

The Spearman rank correlation results reveal a perfect match between the rankings yielded by the SHANURSA method and those produced by F-CODAS, F-CRADIS, F-MARCOS, F-TOPSIS, and F-WASPAS. Additionally, the SHANURSA method exhibits a

Table 9
Spearman rank correlation results.

FMADM method	r_s
F-ARAS	0.886
F-CoCoSo	0.83
F-CODAS	1
F-COPRAS	0.886
F-CRADIS	1
F-EDAS	0.886
F-MARCOS	1
F-MOORA	0.886
F-TOPSIS	1
F-WASPAS	1

very strong agreement with all the remaining methods with a degree of correlation exceeding 0.83. These results ascertain that the SHANURSA method provides broadly reliable results.

4.2. *Analysis of Example 2*

In this example, the multi-attribute choice problem consists in selecting the most suitable transportation company.

Step 1. In this case, five alternatives $\{A_i\}_{i=1}^{i=5}$ are evaluated against the following seven selection attributes:

- Cost of service (C_1);
- Flexibility (C_2);
- Complementary service (C_3);
- Work experience (C_4;)
- Delivery time (C_5);
- Reputation (C_6);
- Transportation capacity (C_7).

The *Cost of service* and *Delivery time* are cost attributes while the remaining attributes are benefit attributes. Both the attribute values and the attribute weights are modelled as TFNs. The fuzzy attribute weights are those presented in Table 10.

Step 2. The fuzzy MADM decision matrix is displayed in Table 11.

4.2.1. *Solving the Problem Given in Example 2*

Step 3. The weighted fuzzy decision matrix is displayed in Table 12.

Step 4. By applying the transformation of induced SNs based on the four characteristic points, the SN matrix is obtained. It is displayed in Table 13.

Step 5. The maximal and minimal SHNs are determined and recorded in Table 14.

Step 6. The positive and negative ideal shadowed numbers are determined and recorded in Table 15.

Table 10
Triangular fuzzy weights.

Attribute	Fuzzy weight
C_1	(0.056, 0.259, 0.852)
C_2	(0.026, 0.073, 0.252)
C_3	(0.038, 0.114, 0.393)
C_4	(0.021, 0.07, 0.283)
C_5	(0.047, 0.18, 0.634)
C_6	(0.039, 0.138, 0.537)
C_7	(0.051, 0.167, 0.638)

Table 11
Fuzzy decision matrix —Example 2.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.193, 0.408, 0.612)	(0.3, 0.5, 0.7)	(0.408, 0.612, 0.814)	(0.125, 0.332, 0.535)	(0.125, 0.332, 0.535)
C_2	(0.572, 0.774, 0.939)	(0.572, 0.774, 0.939)	(0.572, 0.774, 0.939)	(0.241, 0.451, 0.654)	(0.241, 0.451, 0.654)
C_3	(0.5, 0.7, 0.9)	(0.408, 0.612, 0.814)	(0.612, 0.814, 0.959)	(0.612, 0.814, 0.959)	(0.572, 0.774, 0.939)
C_4	(0.5, 0.7, 0.9)	(0.1, 0.3, 0.5)	(0.856, 0.979, 1)	(0.1, 0.3, 0.5)	(0.535, 0.736, 0.919)
C_5	(0.193, 0.408, 0.612)	(0.368, 0.572, 0.774)	(0.612, 0.814, 0.959)	(0.368, 0.572, 0.774)	(0.408, 0.612, 0.814)
C_6	(0.332, 0.535, 0.736)	(0.5, 0.7, 0.9)	(0.856, 0.979, 1)	(0.572, 0.774, 0.939)	(0.814, 0.959, 1)
C_7	(0.368, 0.572, 0.774)	(0.612, 0.814, 0.959)	(0.814, 0.959, 1)	(0.612, 0.814, 0.959)	(0.612, 0.814, 0.959)

Table 12
Weighted fuzzy decision matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	(0.011, 0.106, 0.521)	(0.017, 0.13, 0.596)	(0.023, 0.159, 0.694)	(0.007, 0.086, 0.456)	(0.007, 0.086, 0.456)
C_2	(0.015, 0.057, 0.237)	(0.015, 0.057, 0.237)	(0.015, 0.057, 0.237)	(0.006, 0.033, 0.156)	(0.006, 0.033, 0.156)
C_3	(0.019, 0.08, 0.354)	(0.016, 0.07, 0.32)	(0.023, 0.093, 0.377)	(0.023, 0.093, 0.377)	(0.022, 0.088, 0.369)
C_4	(0.011, 0.049, 0.255)	(0.002, 0.021, 0.142)	(0.018, 0.069, 0.283)	(0.002, 0.021, 0.142)	(0.011, 0.052, 0.26)
C_5	(0.009, 0.073, 0.388)	(0.019, 0.103, 0.491)	(0.029, 0.147, 0.608)	(0.017, 0.103, 0.491)	(0.019, 0.103, 0.491)
C_6	(0.013, 0.074, 0.395)	(0.02, 0.097, 0.481)	(0.033, 0.135, 0.537)	(0.022, 0.107, 0.504)	(0.032, 0.132, 0.537)
C_7	(0.019, 0.096, 0.494)	(0.031, 0.136, 0.612)	(0.042, 0.16, 0.638)	(0.031, 0.136, 0.612)	(0.031, 0.136, 0.612)

Table 13
Shadowed number matrix.

	A_1	A_2	A_3	A_4	A_5
C_1	[0.0442, 0.074, 0244, 0.383]	[0.054, 0.092, 0.285, 0.441]	[0.068, 0.113, 0.337, 0.515]	[0.033, 0.06, 0.209, 0.33]	[0.033, 0.06, 0.21, 0.333]
C_2	[0.029, 0.043, 0.117, 0.177]	[0.029, 0.043, 0.117, 0.177]	[0.029, 0.043, 0.117, 0.177]	[0.015, 0.024, 0.077, 0.124]	[0.015, 0.024, 0.077, 0.121]
C_3	[0.039, 0.06, 0.171, 0.262]	[0.034, 0.052, 0.153, 0.273]	[0.046, 0.07, 0.187, 0.282]	[0.046, 0.07, 0.187, 0.282]	[0.044, 0.066, 0.182, 0.275]
C_4	[0.023, 0.036, 0.118, 0.186]	[0.008, 0.015, 0.0610.101]	[0.035, 0.052, 0.14, 0.212]	[0.008, 0.015, 0.061, 0.101]	[0.025, 0.038, 0.121, 0.19]
C_5	[0.031, 0.052, 0.178, 0.283]	[0.047, 0.075, 0.232, 0.361]	[0.068, 0.107, 0.3, 0.454]	[0.046, 0.074, 0.232, 0.361]	[0.05, 0.08, 0.245, 0.38]
C_6	[0.033, 0.054, 0.181, 0.288]	[0.045, 0.071, 0.226, 0.354]	[0.067, 0.101, 0.269, 0.403]	[0.05, 0.079, 0.239, 0.372]	[0.065, 0.099, 0.267, 0.402]
C_7	[0.0440.07, 0.228, 0.361]	[0.066, 0.1, 0.295, 0.453]	[0.081, 0.121, 0.32, 0.48]	[0.066, 0.101, 0.295, 0.453]	[0.066, 0.101, 0.295, 0.453]

Table 14
Maximal and minimal shadowed numbers.

Attribute	$SN^j_{\max}$	$SN^j_{\min}$
C_1	[0.068, 0.113, 0.337, 0.515]	[0.033, 0.06, 0.209, 0.333]
C_2	[0.029, 0.043, .117, 0.177]	[0.015, 0.024, 0.077, 0.121]
C_3	[0.046, 0.07, 0.187, 0.282]	[0.034, 0.052, 0.153, 0.237]
C_4	[0.035, 0.052, 0.14, 0.212]	[0.008, 0.015, 0.061, 0.101]
C_5	[0.068, 0.107, 0.3, 0.454]	[0.031, 0.052, 0.187, 0.283]
C_6	[0.067, 0.101, 0.269, 0.403]	[0.033, 0.054, 0.181, 0.288]
C_7	[0.081, 0.12, 0.32, 0.48]	[0.044, 0.07, 0.228, 0.361]

Table 15
Positive and negative ideal shadowed numbers.

Attribute	SN_j^+	SN_j^-
C_1	[0.033, 0.06, 0.209, 0.333]	[0.068, 0.113, 0.337, 0.515]
C_2	[0.029, 0.043, .117, 0.177]	[0.015, 0.024, 0.077, 0.121]
C_3	[0.046, 0.07, 0.187, 0.282]	[0.034, 0.052, 0.153, 0.237]
C_4	[0.035, 0.052, 0.14, 0.212]	[0.008, 0.015, 0.061, 0.101]
C_5	[0.031, 0.052, 0.187, 0.283]	[0.068, 0.107, 0.3, 0.454]
C_6	[0.067, 0.101, 0.269, 0.403]	[0.033, 0.054, 0.181, 0.288]
C_7	[0.081, 0.12, 0.32, 0.48]	[0.044, 0.07, 0.228, 0.361]

Table 16
The SHANURSA method results.

Alt.	$P(A_i)$	$R(A_i)$	$C(A_i)$	$Rank(A_i)$
A_1	0.218	0.285	0.882	**4**
A_2	0.27	0.234	0.799	**5**
A_3	0.216	0.287	0.885	**3**
A_4	0.198	0.306	0.915	**2**
A_5	0.138	0.366	1	**1**

Steps 7–8. The proximity and remoteness values, the closeness coefficient values, along with the resulting ranks of the alternatives, are presented in Table 16.

4.2.2. *Parameter Sensitivity*

Figure 5 below depicts the rankings resulting from the weighted MDM parameter changes within the context of the transportation company selection problem. Figure 5 tells that the alternative, A_5, remains top-ranked regardless of the changes in the weighted MDM parameters. Furthermore, the rankings of the other alternatives do not exhibit major or significant inversions. This ascertains the overall order of the alternatives remains relatively stable and robust.

4.2.3. *Dynamic Decision Matrix Effects*

In this example, we form a total of four different scenarios by removing each time the worst-ranked alternative and examining the changes in the ranking results. The initial obtained ranking by applying the SHANURSA method (original scenario) is $A_5 \succ A_4 \succ A_3 \succ A_1 \succ A_2$. Thus, in scenario 1, alternative A_2 is dropped in order to create a new decision matrix with four alternatives. The resulting ranking is $A_5 \succ A_4 \succ A_3 \succ A_1$. Alternative A_1 is now the worst-ranked one. For the scenario 2, A_1 is removed. This process continues until the alternative A_5 itself remains. Table 17 depicts the different generated rankings. As can be seen in Table 17, the alternative A_5 remains the best-ranked across all the scenarios. Obviously, the SHANURSA method did not produce any rank reversal in this example. This confirms once again the stability and robustness of SHANURSA in the dynamic environment.

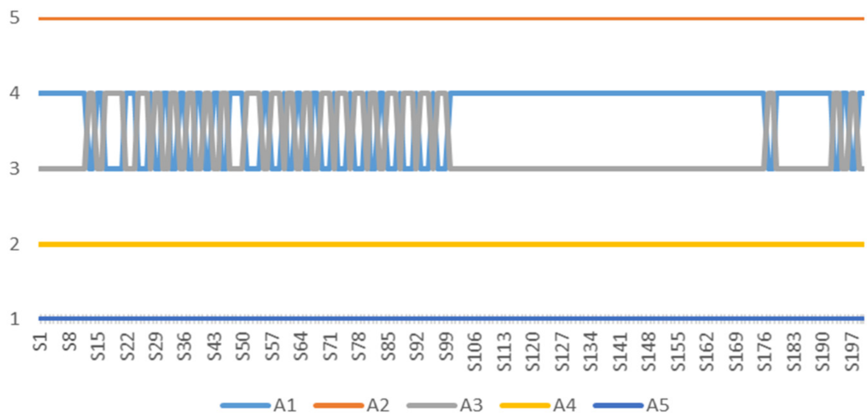

Fig. 5. Parametric sensitivity results—Example 2.

Table 17
Ranking results under different scenarios.

Scenario	Ranking
Original	$A_5 > A_4 > A_3 > A_1 > A_2$
Scenario-1	$A_5 > A_4 > A_3 > A_1$
Scenario-2	$A_5 > A_4 > A_3$
Scenario-3	$A_5 > A_4$
Scenario-4	A_5

4.2.4. *Comparison with Other FMADM Methods*

Figure 6 below depicts the rankings prescribed by all the FMADM methods being used. Noteworthy, in this example, SHANURSA, F-CoCoSo produce the same ranking of the alternatives. However, there exists slightly modified ranking in the ranked alternatives A_1 and A_3 when comparing SHANURSA with F-COPRAS, F-EDAS, and F-MOORA, F-WASPAS. Besides, as can be seen in Fig. 6, the alternative A_5 is ranked first by nine out of the 11 considered methods, including SHANURSA. At the same time, A_5 is ranked second by F-CRADIS and F-MARCOS. Additionally, F-CRADIS and F-MARCOS produce the same ranking and prescribe alternative A_1 as the preferred choice. Lastly, alternative A_2 comes out to be ranked last by 8 out of the 11 methods being used, including SHANURSA.

4.2.5. *Correlation Analysis*

The rankings obtained from the different FMADM methods being used were examined using the Spearman rank correlation coefficient. Table 18 indicates that the rankings obtained by the SHANURSA method exhibit a high degree of similarity with most of the ranking produced by the other methods. However, the rankings returned by the SHANURSA method show the smallest correlation scores with those delivered by F-MARCOS and F-CRADIS. Noteworthy, the Spearman correlation score between the rankings produced by the SHANURSA method and those of F-CoCoSo is equal to 1, indicating a perfect agreement in rankings. Moreover, the correlations between the rankings produced by

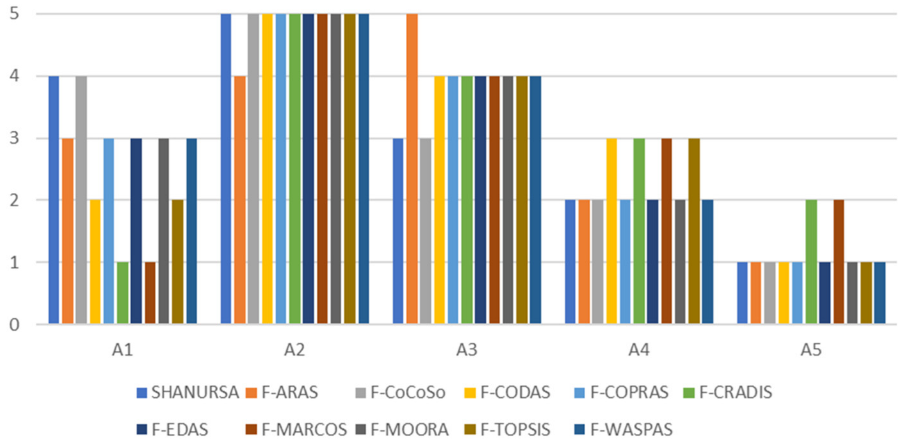

Fig. 6. Ranking results from different FMADM methods—Example 2.

Table 18
Spearman rank correlation results.

FMADM method	r_s
F-ARAS	0.7
F-CoCoSo	1
F-CODAS	0.7
F-COPRAS	0.9
F-CRADIS	0.4
F-EDAS	0.9
F-MARCOS	0.4
F-MOORA	0.9
F-TOPSIS	0.7
F-WASPAS	0.9

the SHANURSA method, F-WASPAS, F-COPRAS, F-EDAS, and F-MOORA were notably high, reflecting a strong similarity in rankings. And, only a small inconsistency was observed in the ranking of two specific alternatives. Despite the fact that relatively lower correlation scores were obtained with F-CODAS, F-TOPSIS, and F-ARAS, the overall consistency remains acceptable,

4.3. *Analysis of Example 3*

This example deals with the selection of an optimal investment sector (IS) by integrating environmental, social and governance (ESG) factors.

Step 1. Ten (10) investment sectors (ISs) are to be evaluated and prioritized based on the following twelve (12) ESG criteria:

- Climate change and carbon emissions (C_1) (cost attribute);
- Resource management and efficiency (C_2);

Table 19
Triangular fuzzy weights.

Attribute	Fuzzy weight
C_1	(0.089, 0.101, 0.118)
C_2	(0.071, 0.083, 0.1)
C_3	(0.065, 0.076, 0.092)
C_4	(0.114, 0.127, 0.145)
C_5	(0.047, 0.058, 0.071)
C_6	(0.133, 0.143, 0.193)
C_7	(0.035, 0.045, 0.057)
C_8	(0.032, 0.041, 0.053)
C_9	(0.065, 0.076, 0.092)
C_{10}	(0.021, 0.029, 0.042)
C_{11}	(0.041, 0.152, 0.172)
C_{12}	(0.029, 0.037, 0.047)

- Biodiversity and land use (C_3) (cost attribute);
- Pollution and waste management (C_4) (cost attribute);
- Labour practices and working conditions (C_5);
- Diversity and inclusion (C_6);
- Community engagement and social impact (C_7) (cost attribute);
- Customer relations and product safety (C_8) (cost attribute);
- Board composition and independence (9) (cost attribute);
- Executive compensation and incentives (C_{10});
- Transparency and disclosure (C_{11});
- Anti-corruption and ethical practices (C_{12}).

The Climate change and carbon emissions, Biodiversity and land use, Pollution and waste management, Community engagement and social impact, Customer relations and product safety, Board composition and independence, and Anti-corruption and ethical practices are cost attributes.

The triangular fuzzy attribute weights are recorded in Table 19.

Step 2. Construction of the FMADM decision matrix.

The fuzzy decision matrix build is displayed in Table 20.

4.3.1. *Solving the Problem Given in Example 3*

Step 3. The weighted fuzzy decision matrix is displayed in Table 21.

Step 4. By applying the transformation of induced SNs based on the four characteristic points, the SN matrix is obtained. It is displayed in Table 22.

Step 5. The maximal and minimal SHNs are determined and recorded in Table 23.

Step 6. The positive and negative ideal shadowed numbers are determined and recorded in Table 24.

Steps 7–8. The proximity and remoteness values, the closeness coefficient values, along with the resulting ranks of the alternatives, are presented in Table 25.

Table 20
Fuzzy decision matrix—Example 3.

Alt.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
C_1	(7, 7, 9)	(3, 3, 5)	(3, 3, 5)	(7, 7, 9)	(3, 3, 5)	(1, 1, 1)	(1, 3, 3)	(3, 5, 5)	(1, 3, 3)	(7, 9, 9)	(7, 9, 9)
C_2	(7, 9, 9)	(1, 1, 1)	(5, 7, 7)	(1, 1, 1)	(5, 5, 7)	(1, 1, 3)	(7, 9, 9)	(1, 1, 1)	(7, 9, 9)	(7, 7, 9)	(1, 1, 1)
C_3	(7, 9, 9)	(1, 1, 3)	(5, 7, 7)	(7, 9, 9)	(7, 9, 9)	(7, 9, 9)	(5, 5, 7)	(1, 1, 3)	(7, 9, 9)	(5, 5, 7)	(5, 5, 7)
C_4	(7, 9, 9)	(3, 5, 5)	(5, 7, 7)	(7, 9, 9)	(1, 1, 3)	(1, 1, 3)	(1, 1, 3)	(7, 9, 9)	(1, 1, 3)	(5, 5, 7)	(7, 9, 9)
C_5	(7, 9, 9)	(7, 9, 9)	(7, 9, 9)	(5, 7, 7)	(5, 7, 7)	(5, 7, 7)	(7, 9, 9)	(5, 7, 7)	(7, 9, 9)	(7, 9, 9)	(3, 5, 5)
C_6	(5, 5, 7)	(7, 7, 9)	(5, 7, 7)	(3, 3, 5)	(7, 9, 9)	(7, 9, 9)	(7, 9, 9)	(3, 5, 5)	(7, 9, 9)	(1, 3, 3)	(5, 5, 7)
C_7	(5, 7, 7)	(3, 5, 5)	(5, 5, 7)	(7, 9, 9)	(3, 3, 5)	(5, 7, 7)	(5, 7, 7)	(7, 9, 9)	(3, 3, 5)	(5, 5, 7)	(7, 9, 9)
C_8	(5, 7, 7)	(3, 5, 5)	(5, 5, 7)	(7, 9, 9)	(3, 3, 5)	(7, 7, 9)	(3, 3, 5)	(7, 9, 9)	(5, 5, 7)	(7, 7, 9)	(7, 9, 9)
C_9	(1, 3, 3)	(1, 3, 3)	(1, 3, 3)	(1, 3, 3)	(1, 1, 3)	(7, 7, 9)	(1, 3, 3)	(7, 9, 9)	(1, 1, 3)	(5, 5, 7)	(7, 7, 9)
C_{10}	(7, 7, 9)	(7, 9, 9)	(7, 9, 9)	(1, 1, 3)	(1, 3, 3)	(3, 5, 5)	(3, 5, 5)	(1, 1, 1)	(1, 1, 3)	(1, 3, 3)	(3, 3, 5)
C_{11}	(7, 9, 9)	(7, 9, 9)	(7, 9, 9)	(3, 5, 5)	(5, 7, 7)	(7, 9, 9)	(7, 9, 9)	(3, 5, 5)	(7, 7, 9)	(3, 5, 5)	(1, 1, 3)
C_{12}	(5, 7, 7)	(3, 5, 5)	(7, 9, 9)	(1, 1, 3)	(1, 1, 3)	(5, 7, 7)	(1, 1, 3)	(7, 9, 9)	(1, 1, 3)	(3, 5, 5)	(5, 5, 7)

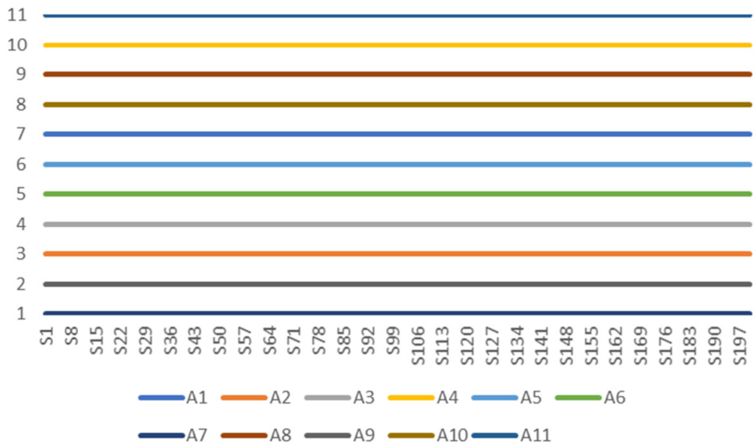

Fig. 7. Parametric sensitivity results—Example 3.

4.3.2. *Parameter Sensitivity*

Figure 7 below shows the parameter sensitivity results when of the weighted MDM parameters are varied. The ranking orders were remarkably consistent at all tested settings. The systematic variations in parameter values did not alter the final rankings. This invariance proves that the SHANURSA method is stable and robust when the weighted MDM parameters vary.

4.3.3. *Dynamic Decision Matrix Effects*

The resilience of the SHANURSA mothed against the rank reversal problem was evaluated in this example through ten distinct scenarios. In each scenario, the worst-ranked alternative was excluded. The remaining alternatives were re-ranked by repeating the computational process. The resulting rankings are presented in Table 26. Based on the results in Table 26, the alternative A_7 remained the most preferred alternative across all scenar-

Table 21
Weighted fuzzy decision matrix.

Alt.	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
C_1	(0.567,0.672, 1.017)	(0.243, 0.288, 0.565)	(0.243, 0.288,0.565)	(0.567, 0.864,1.017)	(0.243, 0.288, 0.565)	(0.081, 0.096,0.113)	(0.081, 0.288, 0.339)	(0.243, 0.48, 0.565)	(0.081, 0.288, 0.339)	(0.567,0.864,1.017)	(0.567, 0.864, 1.017)
C_2	(0.43, 0.68, 0.82)	(0.06, 0.08, 0.09)	(0.31, 0.53, 0.64)	(0.06, 0.08, 0.09)	(0.31, 0.38, 0.64)	(0.43, 0.68, 0.82)	(0.43, 0.68, 0.82)	(0.06, 0.08, 0.09)	(0.43, 0.68, 0.82)	(0.43,0.53, 0.82)	(0.06, 0.08, 0.09)
C_3	(0.43, 0.68, 0.82)	(0.06, 0.08, 0.27)	(0.31, 0.38, 0.64)	(0.43, 0.53, 0.82)	(0.43, 0.68, 0.82)	(0.43, 0.68, 0.82)	(0.31, 0.38, 0.64)	(0.06, 0.08, 0.27)	(0.43, 0.53, 0.82)	(0.31, 0.38, 0.64)	(0.31, 0.38, 0.64)
C_4	(0.76, 1.12, 1.28)	(0.43, 0.37, 0.71)	(0.55, 0.87, 0.99)	(0.76, 1.12, 1.28)	(0.11, 0.12, 0.43)	(0.11, 0.12, 0.43)	(0.11, 0.12, 0.43)	(0.76, 1.12, 1.28)	(0.11, 0.12, 0.43)	(0.55, 0.62, 0.99)	(0.76, 1.12, 1.28)
C_5	(0.32, 0.5, 0.66)	(0.32, 0.5, 0.66)	(0.32, 0.5, 0.66)	(0.23, 0.39, 0.51)	(0.23, 0.39, 0.51)	(0.23, 0.39, 0.51)	(0.32, 0.5, 0.66)	(0.23, 0.39, 0.51)	(0.32, 0.5, 0.66)	(0.32, 0.5, 0.66)	(0.14, 0.28, 0.37)
C_6	(0.9, 0.97, 1.48)	(1.25, 1.35, 1.91)	(0.9, 1.35, 1.48)	(0.54,0.58, 1.1)	(1.25,1.74, 1.91)	(1.25,1.74, 1.91)	(1.25,1.74, 1.91)	(0.54,0.97, 1.1)	(1.25,1.74, 1.91)	(0.18, 0.58, 0.64)	(0.9, 0.97, 1.48)
C_7	(0.17, 0.31, 0.39)	(0.1, 0.22, 0.28)	(0.17, 0.31, 0.39)	(0.23, 0.4, 0.5)	(0.1, 0.13, 0.28)	(0.17, 0.31, 0.39)	(0.17, 0.31, 0.39)	(0.23, 0.4, 0.5)	(0.1, 0.13, 0.28)	(0.17, 0.22, 0.39)	(0.23, 0.4, 0.5)
C_8	(0.17, 0.31, 0.39)	(0.1, 0.22, 0.28)	(0.17, 0.31, 0.39)	(0.23, 0.4, 0.5)	(0.1, 0.13, 0.28)	(0.23, 0.31, 0.5)	(0.1, 0.13, 0.28)	(0.23, 0.4, 0.5)	(0.17, 0.22, 0.4)	(0.23, 0.31, 0.5)	(0.23, 0.4, 0.5)
C_9	(0.06, 0.23, 0.27)	(0.06, 0.23, 0.27)	(0.06, 0.23, 0.27)	(0.06, 0.23, 0.27)	(0.06, 0.08, 0.27)	(0.43, 0.53, 0.82)	(0.06, 0.23, 0.27)	(0.43, 0.68, 0.82)	(0.06, 0.08, 0.27)	(0.31, 0.38, 0.64)	(0.43, 0.68, 0.82)
C_{10}	(0.13, 0.23, 0.32)	(0.13, 0.23, 0.32)	(0.13, 0.23, 0.32)	(0.02, 0.03, 0.11)	(0.02, 0.08, 0.11)	(0.05, 0.13, 0.18)	(0.05, 0.13, 0.18)	(0.02, 0.03, 0.04)	(0.02, 0.03, 0.11)	(0.02, 0.08, 0.11)	(0.05, 0.13, 0.18)
C_{11}	(0.98, 1.4, 1.56)	(0.98, 1.4, 1.56)	(0.98, 1.4, 1.56)	(0.42, 0.78, 0.87)	(0.42, 0.78, 0.87)	(0.98, 1.4, 1.56)	(0.98, 1.4, 1.56)	(0.42, 0.78, 0.87)	(0.98, 1.09, 1.56)	(0.42, 0.78, 0.87)	(0.14, 0.16, 0.52)
C_{12}	(0.13, 0.23, 0.32)	(0.08, 0.18, 0.23)	(0.13, 0.18, 0.32)	(0.03, 0.04, 0.14)	(0.03, 0.04, 0.14)	(0.13, 0.25, 0.32)	(0.03, 0.11, 0.14)	(0.18, 0.32, 0.41)	(0.03, 0.04, 0.14)	(0.08, 0.18, 0.23)	(0.13, 0.18, 0.32)

Table 22
Shadowed number matrix.

		C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	$S_*(\breve{A}_1)$	0.6	0.51	0.51	0.88	0.38	0.92	0.21	0.21	0.12	0.16	1.12	0.17
	$C_*(\breve{A}_1)$	0.64	0.59	0.59	0.1	0.44	0.94	0.26	0.26	0.17	0.2	1.26	0.21
	$C^*(\breve{A}_1)$	0.079	0.72	0.72	1.17	0.55	1.14	0.34	0.34	0.24	0.26	1.45	0.27
	$S^*(\breve{A}_1)$	0.9	0.77	0.77	1.22	0.6	1.31	0.36	0.36	0.26	0.29	1.5	0.3
A_2	$S_*(\breve{A}_2)$	0.26	0.066	0.066	0.34	0.38	1.29	0.14	0.14	0.12	0.15	1.12	0.11
	$C_*(\breve{A}_2)$	0.27	0.07	0.07	0.36	0.44	1.32	0.18	0.18	0.17	0.16	1.26	0.14
	$C^*(\breve{A}_2)$	0.38	0.08	0.14	0.49	0.55	1.54	0.24	0.24	0.24	0.23	1.45	0.19
	$S^*(\breve{A}_2)$	0.047	0.09	0.21	0.6	0.6	1.44	0.26	0.26	0.26	0.28	1.5	0.21
A_3	$S_*(\breve{A}_3)$	0.26	0.38	0.33	0.65	0.38	1.05	0.18	0.18	0.12	0.16	1.12	0.14
	$C_*(\breve{A}_3)$	0.27	0.45	0.35	0.76	0.44	1.2	0.2	0.2	0.17	0.2	1.26	0.16
	$C^*(\breve{A}_3)$	0.38	0.56	0.46	0.91	0.55	1.4	0.28	0.28	0.24	0.26	1.45	0.22
	$S^*(\breve{A}_3)$	0.47	0.6	0.55	0.95	0.6	1.44	0.34	0.34	0.26	0.29	1.5	0.27
A_4	$S_*(\breve{A}_4)$	0.67	0.066	0.46	0.88	0.28	0.55	0.29	0.29	0.12	0.02	0.54	0.03
	$C_*(\breve{A}_4)$	0.77	0.07	0.49	0.1	0.33	0.57	0.34	0.34	0.17	0.023	0.66	0.032
	$C^*(\breve{A}_4)$	0.92	0.08	0.62	1.17	0.43	0.74	0.43	0.43	0.24	0.05	0.81	0.07
	$S^*(\breve{A}_4)$	0.97	0.09	0.72	1.22	0.47	0.9	0.47	0.47	0.26	0.081	0.84	0.1
A_5	$S_*(\breve{A}_5)$	0.26	0.33	0.51	0.11	0.28	1.41	0.11	0.11	0.066	0.04	0.54	0.03
	$C_*(\breve{A}_5)$	0.27	0.35	0.59	0.12	0.33	1.58	0.12	0.12	0.7	0.06	0.66	0.032
	$C^*(\breve{A}_5)$	0.38	0.46	0.72	0.23	0.43	1.79	0.18	0.18	0.14	0.09	0.81	0.07
	$S^*(\breve{A}_5)$	0.47	0.55	0.77	0.33	0.47	1.85	0.23	0.23	0.21	0.1	0.84	0.1
A_6	$S_*(\breve{A}_6)$	0.09	0.066	0.51	0.11	0.28	1.41	0.21	0.26	0.46	0.08	1.12	0.17
	$C_*(\breve{A}_6)$	0.091	0.07	0.59	0.12	0.33	1.58	0.26	0.28	0.49	0.11	1.26	0.21
	$C^*(\breve{A}_6)$	0.1	0.141	0.72	0.23	0.43	1.79	0.34	0.37	0.62	0.15	1.45	0.27
	$S^*(\breve{A}_6)$	0.11	0.21	0.77	0.33	0.47	1.85	0.36	0.44	0.72	0.16	1.5	0.3
A_7	$S_*(\breve{A}_7)$	0.15	0.51	0.33	0.11	0.38	1.41	0.21	0.11	0.12	0.08	1.12	0.05
	$C_*(\breve{A}_7)$	0.22	0.59	0.35	0.12	0.44	1.58	0.26	0.12	0.17	0.11	1.26	0.08
	$C^*(\breve{A}_7)$	0.31	0.72	0.46	0.23	0.55	1.79	0.34	0.18	0.24	0.15	1.45	0.12
	$S^*(\breve{A}_7)$	0.32	0.77	0.55	0.33	0.6	1.85	0.36	0.23	0.26	0.16	1.5	0.13
A_8	$S_*(\breve{A}_8)$	0.32	0.066	0.066	0.88	0.28	0.68	0.29	0.29	0.51	0.02	0.54	0.22
	$C_*(\breve{A}_8)$	0.4	0.07	0.07	0.1	0.33	0.82	0.34	0.34	0.59	0.023	0.66	0.27
	$C^*(\breve{A}_8)$	0.51	0.08	0.14	1.17	0.43	1	0.43	0.43	0.72	0.03	0.81	0.03
	$S^*(\breve{A}_/)$	0.54	0.09	0.21	1.22	0.47	1.03	0.47	0.47	0.77	0.033	0.84	0.032
A_9	$S_*(\breve{A}_9)$	0.15	0.51	0.51	0.11	0.38	1.41	0.11	0.18	0.066	0.02	1.02	0.03
	$C_*(\breve{A}_9)$	0.22	0.59	0.59	0.12	0.44	1.58	0.12	0.2	0.7	0.023	1.1	0.032
	$C^*(\breve{A}_9)$	0.31	0.72	0.72	0.23	0.55	1.79	0.18	0.28	0.14	0.05	1.24	0.07
	$S^*(\breve{A}_9)$	0.32	0.77	0.77	0.33	0.6	1.85	0.23	0.34	0.21	0.081	1.4	0.1
A_{10}	$S_*(\breve{A}_{10})$	0.67	0.46	0.33	0.57	0.38	0.31	0.18	0.26	0.33	0.04	0.54	0.11
	$C_*(\breve{A}_{10})$	0.77	0.49	0.35	0.6	0.44	0.45	0.2	0.28	0.35	0.06	0.66	0.14
	$C^*(\breve{A}_{10})$	0.92	0.62	0.46	0.75	0.55	0.6	0.28	0.37	0.46	0.09	0.81	0.19
	$S^*(\breve{A}_{10})$	0.97	0.72	0.55	0.87	0.6	0.62	0.34	0.44	0.55	0.1	0.84	0.21
A_{11}	$S_*(\breve{A}_{11})$	0.67	0.066	0.33	0.88	0.18	0.92	0.29	0.29	0.51	0.08	0.145	0.14
	$C_*(\breve{A}_{11})$	0.77	0.07	0.35	0.1	0.23	0.94	0.34	0.34	0.59	0.11	0.15	0.16
	$C^*(\breve{A}_{11})$	0.92	0.08	0.46	1.17	0.31	1.14	0.43	0.43	0.72	0.15	0.28	0.22
	$S^*(\breve{A}_{11})$	0.97	0.09	0.55	1.22	0.34	1.31	0.47	0.47	0.77	0.16	0.4	0.27

Table 23
Maximal and minimal shadowed numbers.

Attribute	$SN^j_{\max}$	$SN^j_{\min}$
C_1	[0.67, 0.77, 0.92, 0.97]	[0.086, 0.091, 0.1, 0.11]
C_2	[0.51, 0.59, 0.72, 0.77]	[0.066, 0.07, 0.08, 0.09]
C_3	[0.51, 0.59, 0.72, 0.77]	[0.066, 0.07, 0.14, 0.21]
C_4	[0.88, 1, 1.17, 1.22]	[0.11, 0.12, 0.23, 0.33]
C_5	[0.38, 0.44, 0.55, 0.6]	[0.18, 0.23, 0.31, 0.34]
C_6	[1.41, 1.58, 1.79, 1.85]	[0.312, 0.45, 0.6, 0.62]
C_7	[0.29, 0.34, 0.43, 0.47]	[0.11, 0.12, 0.18, 0.23]
C_4	[0.29, 0.34, 0.43, 0.47]	[0.11, 0.12, 0.18, 0.23]
C_5	[0.51, 0.59, 0.72, 0.77]	[0.066, 0.07, 0.14, 0.21]
C_6	[0.16, 0.2, 0.26, 0.29]	[0.02, 0.023, 0.03, 0.033]
C_7	[1.12, 1.26, 1.45, 1.5]	[0.145, 0.15, 0.28, 0.4]
C_{12}	[0.22, 0.27, 0.35, 0.38]	[0.03, 0.032, 0.07, 0.104]

Table 24
Positive and negative ideal shadowed numbers.

Attribute	SN_j^+	SN_j^-
C_1	[0.086, 0.091, 0.1, 0.11]	[0.67, 0.77, 0.92, 0.97]
C_2	[0.51, 0.59, 0.72, 0.77]	[0.066, 0.07, 0.08, 0.09]
C_3	[0.066, 0.07, 0.14, 0.21]	[0.51, 0.59, 0.72, 0.77]
C_4	[0.11, 0.12, 0.23, 0.33]	[0.88, 1, 1.17, 1.22]
C_5	[0.38, 0.44, 0.55, 0.6]	[0.18, 0.23, 0.31, 0.34]
C_6	[1.41, 1.58, 1.79, 1.85]	[0.312, 0.45, 0.6, 0.62]
C_7	[0.11, 0.12, 0.18, 0.23]	[0.29, 0.34, 0.43, 0.47]
C_8	[0.11, 0.12, 0.18, 0.23]	[0.29, 0.34, 0.43, 0.47]
C_9	[0.066, 0.07, 0.14, 0.21]	[0.51, 0.59, 0.72, 0.77]
C_{10}	[0.16, 0.2, 0.26, 0.29]	[0.02, 0.023, 0.03, 0.033]
C_{11}	[1.12, 1.26, 1.45, 1.5]	[0.145, 0.15, 0.28, 0.4]
C_{12}	[0.22, 0.27, 0.35, 0.38]	[0.03, 0.032, 0.07, 0.104]

ios. Notably, the SHANURSA method produced no rank reversal in this example, further confirming its stability and robustness in a dynamic decision matrix environment.

4.3.4. *Comparison with Other FMADM methods*

Figure 8 below depicts the results obtained by employing the SHANURSA method and the other FMADM methods applied in current research. Figure 8 shows a high degree of agreement across the different FMADM methods being used. The alternative A_7 is ranked first by all methods, including SHANURSA. The alternative A_9 is ranked second by nearly all methods, with the exception of F-CoCoSo, which identifies it as the fourth. The alternative A_2 is ranked third by SHANURSA, F-COPRAS, F-EDAS, and F-MOORA, while the remaining methods ranked it fifth and second. Finally, A_4 and A_{11} are consistently ranked at the bottom by all methods.

Table 25
The SHANURSA method results.

Alt.	$P(A_i)$	$R(A_i)$	$C(A_i)$	$Rank(A_i)$
A_1	3.104	3.62	0.456	**7**
A_2	1.67	5.06	0.735	**3**
A_3	2.11	4.61	0.624	**4**
A_4	5.38	1.352	0.212	**10**
A_5	2.179	4.45	0.609	**6**
A_6	2.142	4.558	0.618	**5**
A_7	1.016	5.703	1	**1**
A_8	4.54	2.168	0.292	**9**
A_9	1.416	5.31	0.817	**2**
A_{10}	4.294	2.425	0.317	**8**
A_{11}	5.669	1.047	0.181	**11**

Table 26
Ranking results under different scenarios.

Scenario	Ranking
Original	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5 > A_1 > A_{10} > A_8 > A_4 > A_{11}$
Scenario-1	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5 > A_1 > A_{10} > A_8 > A_4$
Scenario-2	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5 > A_1 > A_{10} > A_8$
Scenario-3	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5 > A_1 > A_{10}$
Scenario-4	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5 > A_1$
Scenario-5	$A_7 > A_9 > A_2 > A_3 > A_6 > A_5$
Scenario-6	$A_7 > A_9 > A_2 > A_3 > A_6$
Scenario-7	$A_7 > A_9 > A_2 > A_3$
Scenario-8	$A_7 > A_9 > A_2$
Scenario-9	$A_7 > A_9$
Scenario-10	A_7

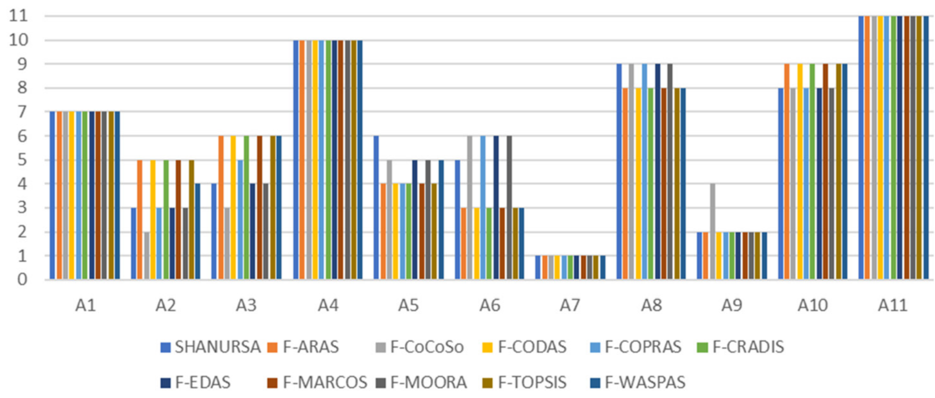

Fig. 8. Rankings results from different FMADM methods—Example 3.

Table 27
Spearman rank correlation results.

FMCDM method	r_s
F-ARAS	0.918
F-CoCoSo	0.964
F-CODAS	0.918
F-COPRAS	0.973
F-CRADIS	0.918
F-EDAS	0.991
F-MARCOS	0.918
F-MOORA	0.991
F-TOPSIS	0.918
F-WASPAS	0.946

4.3.5. *Correlation Analysis*

To further investigate the relationships between the SHANURSA method and the remaining ones, the Spearman rank correlation scores were calculated. The results are recorded in Table 27. Table 27 tells that all methods perform exceptionally well. More specifically, the correlations between the rankings yielded by SHANURSA method and those produced by the other methods reached the score 0.91 or higher. These strong correlations confirm that SHANURSA ranking results are broadly reliable.

5. Conclusion and Future Work

In current research, we have proposed a novel ranking and selection approach to solve fuzzy multi-attribute choice problems based on Grzegorzewski's (2013) shadowed set approximation of a fuzzy number. We have developed a FMADM method called SHANURSA able to produce reliable ranking results. In addition, we have demonstrated the effectiveness of this newly developed FMADM method using three illustrative examples taken from the literature. The prescriptions of the SHANURSA method were compared to those of ten existing FMADM methods, which include F-ARAS, F-CoCoSo, F-CODAS, F-COPRAS, F-CRADIS, F-EDAS, fuzzy F-MARCOS, F-MOORA, F-TOPSIS, and F-WASPAS. In light of the obtained results, it can be said that the application of the SHANURSA method guarantees sound and defensible multi-attribute selection decisions in a fuzzy environment. We believe that the future directions that need to be worked upon include:

- Extending the SHANURSA method for fuzzy multi-attribute decision-making (FMADM) problems with interval-valued trapezoidal/triangular fuzzy assessments.
- Extending the SHANURSA method for fuzzy multi-attribute group decision-making (FMAGDM) problems with interval-valued trapezoidal/triangular fuzzy assessments.

A. Appendix. Proofs of Statements

Proof of Statement 1. In the specialist literature, the endpoints of the α-cut $[\tilde{a}_L(\alpha), \tilde{a}_U(\alpha)]$ of a TrFN $\tilde{a} = (a_1, a_2, a_3, a_4)$ are given, for every $\alpha \in (0, 1]$, by Dutta *et al.* (2011):

(1) $\tilde{a}_L(\alpha) = a_1 + (a_2 - a_1)\alpha$;
(2) $\tilde{a}_U(\alpha) = a_4 - (a_4 - a_3)\alpha$.

Whereas, for a TFN $\tilde{a} = (a_1, a_2, a_3)$, we will have that

(1) $\tilde{a}_L(\alpha) = a_1 + (a_2 - a_1)\alpha$;
(2) $\tilde{a}_U(\alpha) = a_3 - (a_3 - a_2)\alpha$.

We already know that a SHN $SN(\tilde{a}) = [S_*(\tilde{a}), C_*(\tilde{a}), C^*(\tilde{a}), S^*(\tilde{a})]$ induced by a FN $\tilde{a}$ is characterized by means of the four points given formulated as (repeated):

(1) $S_*(\tilde{a}) = 2\int_0^1 \tilde{a}_L(\alpha)d\alpha - 2\int_0^1 \alpha\tilde{a}_L(\alpha)d\alpha$;
(2) $C_*(\tilde{a}) = 2\int_0^1 \alpha\tilde{a}_L(\alpha)d\alpha$;
(3) $C^*(\tilde{a}) = 2\int_0^1 \alpha\tilde{a}_U(\alpha)d\alpha$;
(4) $S^*(\tilde{a}) = 2\int_0^1 \tilde{a}_U(\alpha)d\alpha - 2\int_0^1 \alpha\tilde{a}_U(\alpha)d\alpha$.

So, we will have, for instance, that

$$S_*(\tilde{a}) = 2\int_0^1 \tilde{a}_L(\alpha)d\alpha - 2\int_0^1 \alpha\tilde{a}_L(\alpha)d\alpha =$$
$$S_*(\tilde{a}) = 2\int_0^1 \big(a_1 + (a_2 - a_1)\alpha\big)d\alpha - 2\int_0^1 \big(a_1 + (a_2 - a_1)\alpha\big)\alpha d\alpha.$$

Thus, by computing the above two integrals, we get

$$S_*(\tilde{a}) = 2\left(a_1 + \frac{a_2 - a_1}{2}\right) - 2\left(\frac{a_1}{2} + \frac{a_2 - a_1}{3}\right) = a_1 + \frac{a_2 - a_1}{3} = \frac{2a_1 + a_2}{3}. \quad \square$$

We can prove all other formulas of Statement 1 in the same way.

Proof of Statement 2. The one-to-one correspondence property can be easily verified. The other properties are immediate consequences of the below α-cut operations and the linearity of integration.

Let $\tilde{a}$ and $\tilde{b}$ be two arbitrary FNs, and ωa real number. For every $\alpha \in [0, 1]$, we have that:

(1) $(\tilde{a} + \tilde{b})_L(\alpha) = \tilde{a}_L(\alpha) + \tilde{b}_L(\alpha)$;
(2) $(\tilde{a} + \tilde{b})_U(\alpha) = \tilde{a}_U(\alpha) + \tilde{b}_U(\alpha)$;
(3) $(\tilde{a} + \omega)_L(\alpha) = \tilde{a}_L(\alpha) + \omega$;
(4) $(\tilde{a} + \omega)_U(\alpha) = \tilde{a}_U(\alpha) + \omega$;
(5) $(\omega \cdot \tilde{a})_L(\alpha) = \omega \cdot \tilde{a}_L(\alpha), \omega > 0$;
(6) $(\omega \cdot \tilde{a})_U(\alpha) = \omega \cdot \tilde{a}_U(\alpha), \omega > 0$. $\square$

Declaration of interests

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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