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Characterising Quasi-Closed Elements via Closure Systems on Complete Fuzzy Lattices

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Abstract. The notion of quasi-closed element plays a central role in several branches of mathematics and computer sciences, for instance, in the Duquenne-Guigues basis of attribute implications. This paper deals with the extension of quasi-closed elements to the fuzzy setting by extending the well-known characterisation of quasi-closed elements in the crisp case, which is given in terms of closure systems. Specifically, we provide two distinct definitions, one considering crisp closure systems and another for fuzzy ones. Finally, we obtain a characterisation for each one of these notions.

Key words: closure operator, complete lattice, fuzzy logic, quasi-closed.

1. Introduction

The notions of functional dependency, Horn clause or attribute implication resemble the idea of if-then rule in different fields of Mathematics and Computer Science, namely Relational Databases, Logic Programming and Formal Concept Analysis (FCA). In all these areas, the concept of basis is that of a subset that captures the knowledge of the whole data structure.

Duquenne and Guigues proved that a complete and non-redundant set of attribute implications can be derived from what they called *non-redundant nodes* (Guigues and Duquenne, 1986). In addition, the bases generated from these elements were minimal in the number of implications. These elements are now standard in FCA and are the so-called pseudointents, which were popularised by Ganter and Wille (1999) and are defined in a recursive manner. Moreover, it was proved that pseudointents can be obtained from quasi-closed elements, also called critical sets (Adaricheva and Nation, 2016), and these can be defined without recursion (Ganter, 2010; Kuznetsov and Obiedkov, 2008).

The main extension of these notions to the fuzzy framework is that of Vychodil and Bělohlávek (2005), who extended the notion of set of pseudointents via the recursive definition. The results in the cited paper include the completeness and non-redundancy of

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the basis. However, the minimality of the basis could be guaranteed only in very specific cases. As a matter of fact, in the cited paper the authors provide examples of formal contexts which have several sets of pseudointents with different cardinality. Nevertheless, the extension of quasi-closed elements to the fuzzy setting and whether they provide better results concerning bases of fuzzy attribute implications is still an open problem.

In this paper, we consider the extension of the notion of quasi-closed element to the fuzzy framework. A quasi-closed element is characterised by the fact that its addition to the set of closed elements is a closure system. Therefore, our goal is to extend this property. We consider two main possibilities, whether the resulting set is a classical closure system or a fuzzy one. Bělohlávek proposed two notions as extensions of closure systems in the fuzzy setting, namely L-closure systems (Bělohlávek, 2001) on the fuzzy powerset lattice. The extensions to general complete fuzzy lattices were introduced in Ojeda-Hernández *et al.* (2022b) and will be the ones used throughout the paper.

The paper is structured as follows. After the introduction, there is a brief section of preliminaries. In Section 3, we propose two possible definitions of quasi-closed element in the fuzzy framework. In the subsequent sections, we provide different characterisations of these definitions. First, we characterise the notion concerning classical closure systems, where we find similarities with both the crisp case and the approach by Bělohlávek and Vychodil. Second, we provide a characterisation for the fuzzy quasi-closed elements and we find again similarities with previous approaches in the literature and, as expected, we prove that being a fuzzy quasi-closed element is equivalent to being quasi-closed and an additional property. In the last section of the paper we derive our conclusions and showcase some possible lines for further work.

2. Preliminaries

In this section, we present the framework to which we are going to generalize the notion of quasi-closed element. It has been chosen with the idea of being as general as possible and thus having a wider range of possible applications. Specifically, we introduce complete residuated lattices (Bělohlávek, 2002; Hájek, 2013), the notions of fuzzy poset and fuzzy complete lattice, and some basic results that will be needed to follow the manuscript (Bělohlávek, 2004; Konecny and Krupka, 2017).

A complete residuated lattice is an algebra $\mathbb{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ such that

- $(L, \wedge, \vee, 0, 1)$ is a complete lattice with 0 and 1 being the least and the greatest elements of L, respectively,
- $(L, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and 1 is neutral with respect to $\otimes$), and
- $\otimes$ and $\to$ satisfy the so-called *adjointness property*: for all $a, b, c \in L$, we have that $a \otimes b \leqslant c$ iff $a \leqslant b \to c$.

This structure is utilized in mathematical fuzzy logics and their applications as structures of truth degrees with $\otimes$ and $\to$ used as truth functions of *fuzzy conjunction* and *fuzzy implication*, respectively (Hájek, 2013). The unit interval with the pairs of t-norms and implications introduced by Łukasiewicz, Gödel and Goguen are examples of complete residuated lattices.

In the study of residuated lattices, it is common to consider a *negation* in $\mathbb{L}$ as the antitone mapping $\neg\colon L \to L$ defined by $\neg a = a \to 0$.

Let U be a non-empty set, usually called universe. An $\mathbb{L}$-set, or fuzzy set, is a mapping $A\colon U \to L$. Let L^U be the set of $\mathbb{L}$-sets on U. A crisp set X is a fuzzy set such that $\mathrm{Im}(X) \subseteq \{0, 1\}$. Operations with $\mathbb{L}$-sets are defined element-wise. For instance, $A \cup B \in L^U$ is defined as $(A \cup B)(u) = A(u) \vee B(u)$ for all $u \in U$. In addition, given $\alpha \in L$ the α-cut of an $\mathbb{L}$-set A is defined as $A^\alpha = \{u \in U\colon A(u) \geqslant \alpha\}$.

Binary $\mathbb{L}$-relations (binary fuzzy relations) on a set U can be thought of as $\mathbb{L}$-sets on the universe $U \times U$. That is, a binary $\mathbb{L}$-relation on U is a mapping $\rho \in L^{U\times U}$ assigning to each $x, y \in U$ a truth degree $\rho(x, y) \in L$ (a degree to which x and y are related by ρ).

For ρ being a binary $\mathbb{L}$-relation in U, we say that

- ρ is *reflexive* if $\rho(x, x) = 1$ for all $x \in U$.
- ρ is *symmetric* if $\rho(x, y) = \rho(y, x)$ for all $x, y \in U$.
- ρ is *antisymmetric* if $\rho(x, y) \otimes \rho(y, x) = 1$ implies $x = y$ for all $x, y \in U$.
- ρ is *transitive* if $\rho(x, y) \otimes \rho(y, z) \leqslant \rho(x, z)$ for all $x, y, z \in U$.

Definition 1. Given a non-empty set A and a binary $\mathbb{L}$-relation ρ on A, the pair $\mathbb{A} = (A, \rho)$ is said to be a *fuzzy poset* if ρ is a *fuzzy order*, i.e. if ρ is reflexive, antisymmetric and transitive.

To present the notion of fuzzy lattice we need to generalize those of upper (lower) bound and supremum (infimum).

Definition 2. Given a fuzzy poset $\mathbb{A} = (A, \rho)$ and a fuzzy set $X \in L^A$, we define the *up-cone* X and the *down-cone* of X, respectively, as the fuzzy sets $X^\rho, X_\rho \in L^A$ where, for all $a \in A$,

$$X^\rho(a) = \bigwedge_{x\in A}\big(X(x) \to \rho(x, a)\big) \quad \text{and} \quad X_\rho(a) = \bigwedge_{x\in A}\big(X(x) \to \rho(a, x)\big).$$

Thus, $X^\rho(a)$ and $X_\rho(a)$ can be seen as the degree to which a is an upper bound and lower bound of X, respectively.

Definition 3. Let $\mathbb{A} = (A, \rho)$ be a fuzzy poset and $X \in L^A$. An element $a \in A$ is said to be *supremum* (resp. *infimum*) of X if the following conditions hold:

1. $X^\rho(a) = 1$ (resp. $X_\rho(a) = 1$).
2. For all $x \in A$, $X^\rho(x) \leqslant \rho(a, x)$ (resp. $X_\rho(x) \leqslant \rho(x, a)$).

Theorem 1 (Ojeda-Hernández *et al.*, 2022a). *Let $\mathbb{A} = (A, \rho)$ be a fuzzy poset and $X \in L^A$. An element $a \in A$ is supremum (resp. infimum) of X if and only if*

$$\rho(a, x) = X^\rho(x) \quad \big(\textit{resp. } \rho(x, a) = X_\rho(x)\big).$$

It is not difficult to see that, if a supremum (resp. infimum) of X exists, it is unique. We will denote it by $\bigsqcup X$ (resp. $\bigsqcap X$). In addition, $a \sqcup b$ denotes $\bigsqcup\{a, b\}$ and $a \sqcap b$ denotes $\bigsqcap\{a, b\}$, for all $a, b \in A$.

Definition 4 (Bělohlávek, 2004). We say that a fuzzy poset (A, ρ) is a complete fuzzy lattice if every fuzzy subset $X \in L^A$ has supremum and infimum.

In this paper, we will extensively use the 1-cut of the fuzzy order ρ, hence we will denote by $a \trianglelefteq b$ the case $\rho(a, b) = 1$, and by $a \triangleleft b$ the case $\rho(a, b) = 1$ and $\rho(b, a) \neq 1$. Notice that, if (A, ρ) is a fuzzy poset (resp. complete fuzzy lattice), then $(A, \trianglelefteq)$ is a poset (resp. complete lattice).

Corollary 1 (Ojeda-Hernández *et al.*, 2022a). *Let (A, ρ) be a complete fuzzy lattice. For all $a, b, c \in A$,*

$$\rho(a \sqcup b, c) = \rho(a, c) \wedge \rho(b, c) \quad \text{and} \quad \rho(a, b \sqcap c) = \rho(a, b) \wedge \rho(a, c).$$

Proposition 1. *Let (A, ρ) be a complete fuzzy lattice. For all $X, Y \in L^A$,*

$$(X \cup Y)_\rho = X_\rho \cap Y_\rho \quad \text{and} \quad \bigsqcap(X \cup Y) = \bigsqcap X \sqcap \bigsqcap Y.$$

Proof. Let $X, Y \in L^U$,

$$\begin{aligned}(X \cup Y)_\rho(x) &= \bigwedge_{a\in A}(X \cup Y)(a) \to \rho(x, a) = \bigwedge_{a\in A}\big(\big(X(a) \vee Y(a)\big) \to \rho(x, a)\big)\\ &\overset{(i)}{=} \bigwedge_{a\in A}\big(\big(X(a) \to \rho(x, a)\big) \wedge \big(Y(a) \to \rho(x, a)\big)\big)\\ &= \bigwedge_{a\in A}\big(X(a) \to \rho(x, a)\big) \wedge \bigwedge_{a\in A}\big(Y(a) \to \rho(x, a)\big) = (X_\rho \cap Y_\rho)(x),\end{aligned}$$

where (i) holds by (2.52) in Bělohlávek (2002).

In order to prove the equality of infima we use Theorem 1. Consider the elements $x = \bigsqcap X$, $y = \bigsqcap Y$ and $z = \bigsqcap X \sqcap \bigsqcap Y$. Then, $z = \bigsqcap(X \cup Y)$ if and only if, for all $a \in A$, $(X \cup Y)_\rho(a) = \rho(a, z)$.

$$\begin{aligned}\rho(a, z) &= \rho(a, x \sqcap y) \overset{(ii)}{=} \rho(a, x) \wedge \rho(a, y)\\ &\overset{(iii)}{=} X_\rho(a) \wedge Y_\rho(a) = (X_\rho \cap Y_\rho)(a) = (X \cup Y)_\rho(a),\end{aligned}$$

where (ii) and (iii) hold by Corollary 1 and Theorem 1, respectively. □

We recall now the notions of closure operator and system that we are going to use throughout the paper.

DEFINITION 5. Given a fuzzy poset $\mathbb{A} = (A, \rho)$, a mapping $\mathtt{c}\colon A \to A$ is said to be a *closure operator* on $\mathbb{A}$ if the following conditions hold:

1. $\rho(a, b) \leqslant \rho(\mathtt{c}(a), \mathtt{c}(b))$, for all $a, b \in A$ (isotony).
2. $\rho(a, \mathtt{c}(a)) = 1$, for all $a \in A$ (inflationarity).
3. $\rho(\mathtt{c}(\mathtt{c}(a)), \mathtt{c}(a)) = 1$, for all $a \in A$ (idempotency).

An element $q \in A$ is said to be *closed* for $\mathtt{c}$ if $\rho(\mathtt{c}(q), q) = 1$.

Notice that, if q is a closed element then, $\rho(q, \mathtt{c}(q)) \otimes \rho(\mathtt{c}(q), q) = 1$, and, by anti-symmetry, $\mathtt{c}(q) = q$. In addition, as in the classical case, for all $a \in A$, the element $\mathtt{c}(a)$ is closed.

The counterpart of closure operators are the so-called closure systems, which were introduced in Ojeda-Hernández *et al.* (2022b) and are defined as follows.

DEFINITION 6. Let (A, ρ) be a complete fuzzy lattice. A crisp subset $\mathcal{F} \subseteq A$ is said to be a *closure system* if $\bigsqcap X \in \mathcal{F}$ for any fuzzy subset $X \in L^{\mathcal{F}}$.

Closure systems and operators are related in a one-to-one manner via the following theorem.

Theorem 2 (Ojeda-Hernández *et al.*, 2022b). *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice.*

1. *If $\mathcal{F}$ is a closure system on $\mathbb{A}$, then the mapping $\mathtt{c}_{\mathcal{F}}\colon A \to A$ defined as $\mathtt{c}_{\mathcal{F}}(x) = \bigsqcap(x^{\rho} \cap \mathcal{F})$ is a closure operator on $\mathbb{A}$.*
2. *If $\mathtt{c}\colon A \to A$ is a closure operator on $\mathbb{A}$, then $\mathcal{F}_{\mathtt{c}} = \{x \in A \mid \mathtt{c}(x) = x\}$ is a closure system on $\mathbb{A}$.*
3. *If $\mathcal{F}$ is a closure system on $\mathbb{A}$, then $\mathcal{F}_{\mathtt{c}_{\mathcal{F}}} = \mathcal{F}$.*
4. *If $\mathtt{c}\colon A \to A$ is a closure operator on $\mathbb{A}$, then $\mathtt{c}_{\mathcal{F}_{\mathtt{c}}} = \mathtt{c}$.*

Notice that this result ensures that for a closure operator $\mathtt{c}\colon A \to A$, its set of closed elements $\mathcal{F}_{\mathtt{c}}$ is a closure system. In addition, every closure system $\mathcal{F}$ is the set of closed elements of some closure operator, in particular of $\mathtt{c}_{\mathcal{F}}$. Thus, from now on, we will denote the sets of closed elements with $\mathcal{F}$, exactly as we do with closure systems.

The following result extends the first item of the previous one. More specifically, the mapping induced by any crisp set is always inflationary.

Proposition 2. *Let $\mathbb{A} = (A, \rho)$ be a complete fuzzy lattice. For all $X \subseteq A$, the mapping $\mathtt{c}_X\colon A \to A$ defined as $\mathtt{c}_X(x) = \bigsqcap(x^{\rho} \cap X)$ is inflationary.*

As mentioned in the introduction, this study is focused on defining quasi-closed elements in the fuzzy framework. This notion is tightly linked to closure structures, as these are elements which are not closed but give information about the closure operator. These elements appeared in the well-known paper (Guigues and Duquenne, 1986) and they are used for obtaining a minimal basis of attribute implications in FCA (Ganter and Wille, 1999).

The definition of quasi-closed element is well-known (Ganter, 2010; Grätzer and Wehrung, 2016), we give now an adapted version of the definition to our framework.

Definition 7. Consider a complete lattice $(A, \leqslant)$. Let $\mathsf{c}\colon A \to A$ be a closure operator and $\mathcal{F}$ be the set of closed elements. An element $q \in A$ is quasi-closed for c if and only if $\mathcal{F} \cup \{q\}$ is a crisp closure system.

One of the characterisations of quasi-closed element is the following Grätzer and Wehrung (2016).

Proposition 3. *Consider a complete lattice $(A, \leqslant)$. Let $\mathsf{c}\colon A \to A$ be a closure operator. An element $q \in A$ is quasi-closed for c if and only if $a < q$ implies $\mathsf{c}(a) \leqslant q$ or $\mathsf{c}(a) = \mathsf{c}(q)$, for all $a \in A$.*

In addition, quasi-closed elements have been characterised in a recursive manner (Kuznetsov and Obiedkov, 2008). Even though the original result makes no comment on the cardinality of the lattice, the proof makes it clear that it must be a finite set.

Proposition 4. *Let $(A, \leqslant)$ be a finite complete lattice, $\mathsf{c}\colon A \to A$ a closure operator and $q \in A$. Then the following two statements are equivalent:*

1. *q is quasi-closed.*
2. *For any quasi-closed $a < q$ one has $\mathsf{c}(a) \leqslant q$ or $\mathsf{c}(a) = \mathsf{c}(q)$.*

3. On the Definition of Quasi-Closed Elements in the Fuzzy Setting

The topic of this section is the extension of the notion of quasi-closed element to the fuzzy framework. Following the suit of Adaricheva and Nation (2016), this extension can be carried out in two stages: First, in the crisp style, where the result of adding a quasi-closed element to the set of closed elements is a classical closure system. Second, in the graded style, where the result of the addition is a closure system. Thus, candidates that arise quite naturally after taking all of the above into consideration are the following.

Definition 8. Let (A, ρ) be a complete fuzzy lattice, $\mathsf{c}\colon A \to A$ be a closure operator and $\mathcal{F}$ the set of closed elements. An element $q \in A$ is said to be:

1. Quasi-closed for c if $\mathcal{F} \cup \{q\}$ is a classical closure system.
2. Fuzzy quasi-closed for c if $\mathcal{F} \cup \{q\}$ is a closure system.

Notice that the main difference between Definitions 7 and 8 above is the setting. In the latter, the framework is a complete fuzzy lattice, c is isotone in the fuzzy sense, that is, $\rho(a, b) \leqslant \rho(\mathsf{c}(a), \mathsf{c}(b))$; and the set of closed elements $\mathcal{F}$, by Theorem 2, is a closure system in the sense of Definition 6, that is, $\bigsqcap X \in \mathcal{F}$ for all $X \in L^{\mathcal{F}}$.

Besides, Definition 8 is a proper extension of Definition 7, that is, every quasi-closed element q for c in (A, ρ) is a quasi-closed element for c in $(A, \trianglelefteq)$ in the classical sense. Recall that there are classical closure operators which are not closure operators, hence the notion of quasi-closed element in $(A, \trianglelefteq)$ differs from the one in (A, ρ).

Moreover, it is clear that, if (A, ρ) is a complete fuzzy lattice, $(A, \trianglelefteq)$ is a complete lattice and the suprema and infima coincide with those of (A, ρ).

Now, consider the two notions in Definition 8. It is clear by the definition that every fuzzy quasi-closed element is a quasi-closed element.

The definition of quasi-closed element in a complete fuzzy lattice has implicit properties due to $\mathcal{F}$ being a closure system, thus we wonder whether every quasi-closed element is fuzzy quasi-closed. The answer to this question is negative. There are examples of complete fuzzy lattices and closure operators where a quasi-closed element q does not satisfy that $\mathcal{F} \cup \{q\}$ is a closure system in the sense of Ojeda-Hernández *et al.* (2022b). The next example shows such a case.

Example 1. Let $\mathbb{L}$ be the three-valued Łukasiewicz residuated lattice and L^U be the lattice of the fuzzy subsets of $U = \{a, b\}$ with the order induced by the subsethood degree relation S. Let $\mathcal{F} = \{u, f\}$ the subset of L^U where $u = \{a/1, b/1\}$ and $f = \{a/1, b/0.5\}$. It can be checked that $\mathcal{F}$ is a closure system because for all fuzzy subset X of $\mathcal{F}$, it holds that $\bigsqcap X = f$ if $X(f) = 1$ and $\bigsqcap X = u$, otherwise.

The element $\{a/0, b/0\}$ is a quasi-closed element for $\mathsf{c}_{\mathcal{F}}$ since $\mathcal{F} \cup \{\{a/0, b/0\}\}$ is closed under classical infima. This can be easily seen since any subset $X \subseteq \mathcal{F} \cup \{\{a/0, b/0\}\}$ satisfies:

$$\bigcap X = \begin{cases} \{a/0, b/0\}, & \text{if } \{a/0, b/0\} \in X, \\ \{a/1, b/0.5\}, & \text{if } \{a/0, b/0\} \notin X \text{ and } \{a/1, b/0.5\} \in X, \\ \{a/1, b/1\}, & \text{if } X = \{\{a/1, b/1\}\} \text{ or } X = \varnothing. \end{cases}$$

However, $\mathcal{F} \cup \{\{a/0, b/0\}\}$ is not a fuzzy closure system since $\bigsqcap(\{a/0, b/0\}/0.5) = \{a/0.5, b/0.5\} \notin \mathcal{F} \cup \{\{a/0, b/0\}\}$.

In the crisp case there are several equivalent properties to being a quasi-closed element. The direct extensions of those properties were studied in Ojeda-Hernández *et al.* (2022a), and we highlight from that analysis the following one: for all $a \in A$,

$$\rho(a, q) \otimes \neg\rho(q, a) \otimes \rho\big(\mathsf{c}(a), \mathsf{c}(q)\big) \otimes \neg\rho\big(\mathsf{c}(q), \mathsf{c}(a)\big) \leqslant \rho\big(\mathsf{c}(a), q\big). \tag{1}$$

Indeed, this property is a proper extension of the notion of quasi-closed element in the classical setting. In addition, the following result shows that it is a necessary condition to be quasi-closed.

Proposition 5. *Let (A, ρ) be a complete fuzzy lattice endowed with a closure operator c. If an element $q \in A$ is quasi-closed for c then q satisfies (1).*

Proof. Let $q \in A$ be such that $\mathcal{F} \cup \{q\}$ is a closure system and let $x \in A$. Consider $\mathtt{c}(x) \sqcap q$, if $\mathtt{c}(x) \sqcap q = q$ then $1 = \rho(q, \mathtt{c}(x)) \leqslant \rho(\mathtt{c}(q), \mathtt{c}(x))$, hence $\neg\rho(\mathtt{c}(q), \mathtt{c}(x)) = 0$ and

$$\rho(x, q) \otimes \neg\rho(q, x) \otimes \rho\big(\mathtt{c}(x), \mathtt{c}(q)\big) \otimes \neg\rho\big(\mathtt{c}(q), \mathtt{c}(x)\big) = 0.$$

Otherwise, $\mathtt{c}(x) \sqcap q$ is closed and we get

$$\rho(x, q) = \rho\big(x, \mathtt{c}(x)\big) \wedge \rho(x, q) = \rho\big(x, \mathtt{c}(x) \sqcap q\big) \leqslant \rho\big(\mathtt{c}(x), \mathtt{c}(x) \sqcap q\big) = \rho\big(\mathtt{c}(x), q\big).$$

Thus, either $\rho(x, q) \leqslant \rho(\mathtt{c}(x), q)$ or $\neg\rho(\mathtt{c}(q), \mathtt{c}(x)) = 0$, which implies

$$\rho(x, q) \otimes \neg\rho(q, x) \otimes \rho\big(\mathtt{c}(x), \mathtt{c}(q)\big) \otimes \neg\rho\big(\mathtt{c}(q), \mathtt{c}(x)\big) \leqslant \rho\big(\mathtt{c}(x), q\big),$$

for all $x \in A$. □

Corollary 2. *Let (A, ρ) be a complete fuzzy lattice endowed with a closure operator $\mathtt{c}$. If an element $q \in A$ is fuzzy quasi-closed for $\mathtt{c}$ then q satisfies (1).*

Unfortunately, contrary to the well-known result in the crisp case, Condition (1) is not equivalent to being a quasi-closed element. Consider the next example.

Example 2. Let $\mathbb{L}$ be the three-valued Łukasiewicz residuated lattice, $A = \{\bot, a, b, c, d, e, \top\}$ and ρ the fuzzy order given by the following table:

ρ	$\bot$	a	b	c	d	e	$\top$
$\bot$	1	1	1	1	1	1	1
a	0.5	1	0.5	1	1	1	1
b	0.5	0.5	1	1	1	1	1
c	0.5	0.5	0.5	1	1	1	1
d	0	0.5	0	0.5	1	0.5	1
e	0	0	0.5	0.5	0.5	1	1
$\top$	0	0	0	0.5	0.5	0.5	1

Let $\mathtt{c}\colon A \to A$ be the closure operator defined as

$$\mathtt{c}(x) = \begin{cases} e, & \text{if } x = \bot, a, b, c, e, \\ \top, & \text{if } x = d, \top. \end{cases}$$

By Theorem 2, we have that $\mathcal{F} = \{e, \top\}$ is a closure system. Then, d satisfies (1) for the closure operator $\mathtt{c}$ since

$$\begin{aligned} &\rho(\bot, d) \otimes \neg\rho(d, \bot) \otimes \rho\big(\mathtt{c}(\bot), \mathtt{c}(d)\big) \otimes \neg\rho\big(\mathtt{c}(d), \mathtt{c}(\bot)\big) \\ &\qquad = 1 \otimes \neg 0 \otimes 1 \otimes \neg 0.5 = 0.5 \leqslant 0.5 = \rho\big(\mathtt{c}(\bot), d\big), \end{aligned}$$

$$\begin{aligned}
\rho(a,d)\otimes\neg\rho(d,a)\otimes\rho\big(\mathsf{c}(a),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(a)\big)&\\
=1\otimes\neg 0.5\otimes 1\otimes\neg 0.5=0\leqslant 0.5=\rho\big(\mathsf{c}(a),d\big)&,\\
\rho(b,d)\otimes\neg\rho(d,b)\otimes\rho\big(\mathsf{c}(b),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(b)\big)&\\
=1\otimes\neg 0\otimes 1\otimes\neg 0.5=0.5\leqslant 0.5=\rho\big(\mathsf{c}(b),d\big)&,\\
\rho(c,d)\otimes\neg\rho(d,c)\otimes\rho\big(\mathsf{c}(c),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(c)\big)&\\
=1\otimes\neg 0.5\otimes 1\otimes\neg 0.5=0\leqslant 0.5=\rho\big(\mathsf{c}(c),d\big)&,\\
\rho(d,d)\otimes\neg\rho(d,d)\otimes\rho\big(\mathsf{c}(d),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(d)\big)&\\
=1\otimes\neg 1\otimes 1\otimes\neg 1=0\leqslant 0.5=\rho\big(\mathsf{c}(d),d\big)&,\\
\rho(e,d)\otimes\neg\rho(d,e)\otimes\rho\big(\mathsf{c}(e),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(e)\big)&\\
=0.5\otimes\neg 0.5\otimes 1\otimes\neg 0.5=0\leqslant 0.5=\rho\big(\mathsf{c}(e),d\big)&,\\
\rho(\top,d)\otimes\neg\rho(d,\top)\otimes\rho\big(\mathsf{c}(\top),\mathsf{c}(d)\big)\otimes\neg\rho\big(\mathsf{c}(d),\mathsf{c}(\top)\big)&\\
=0.5\otimes\neg 1\otimes 1\otimes\neg 1=0\leqslant 0.5=\rho\big(\mathsf{c}(\top),d\big)&.
\end{aligned}$$

However, the set $\mathcal{F}'=\mathcal{F}\cup\{d\}=\{d,e,\top\}$ is not a classical closure system, since the infimum of the subset $X=\{d,e\}$ is $c\notin\mathcal{F}'$.

4. Quasi-Closed Elements via Classical Closure Systems

In this section, we will look for characterisations of the notion of quasi-closed element and compare them with other approaches in the literature. The first set of equivalent statements is given in the next result.

Proposition 6. *Let (A,ρ) be a complete fuzzy lattice endowed with a closure operator $\mathsf{c}\colon A\to A$. The following statements are equivalent*:

1. *q is quasi-closed for c.*
2. *$a\lhd q$ implies $\mathsf{c}(a)\unlhd q$ or $\mathsf{c}(a)=\mathsf{c}(q)$, for all $a\in A$.*
3. *$(\rho(a,q)\to\rho(\mathsf{c}(a),q))=1$ or $\mathsf{c}(q)\unlhd\mathsf{c}(a)$, for all $a\in A$.*

Proof. It is clear that 3 implies 2. Since 2 is a translation of the classical characterisation, given in Proposition 3, 2 implies 1. Let us prove 1 implies 3.

Assume $q\in A$ satisfies $\mathcal{F}\cup\{q\}$ is a closure system. Consider $a\in A$ such that $\mathsf{c}(q)\not\trianglelefteq\mathsf{c}(a)$, the element $\mathsf{c}(a)\sqcap q$ is either q or closed. If $\mathsf{c}(a)\sqcap q=q$, we get $q\unlhd\mathsf{c}(a)$ which yields $\mathsf{c}(q)\unlhd\mathsf{c}(a)$, which is a contradiction. Thus, $\mathsf{c}(a)\sqcap q$ is closed. Then, we have

$$\begin{aligned}
\rho(a,q)&=\rho\big(a,\mathsf{c}(a)\sqcap q\big)\leqslant\rho\big(\mathsf{c}(a),\mathsf{c}\big(\mathsf{c}(a)\sqcap q\big)\big)\\
&=\rho\big(\mathsf{c}(a),\mathsf{c}\big(\mathsf{c}(a)\sqcap q\big)\big)\otimes\rho\big(\mathsf{c}\big(\mathsf{c}(a)\sqcap q\big),\mathsf{c}(a)\sqcap q\big)\otimes\rho\big(\mathsf{c}(a)\sqcap q,q\big)\\
&\leqslant\rho\big(\mathsf{c}(a),q\big).
\end{aligned}$$

Therefore, $(\rho(a,q)\to\rho(\mathsf{c}(a),q))=1$. $\square$

Observe that item 3 presents similarities with the approach to pseudointents by Vychodil and Bělohlávek (2005) since the formula $\rho(a, q) \to \rho(\mathsf{c}(a), q)$ is part of the definition. Notice as well that, if c is a closure operator in (A, ρ), then being a quasi-closed element for c in (A, ρ) is equivalent to being a quasi-closed element for c in $(A, \trianglelefteq)$. However, if $\gamma\colon A \to A$ is a classical closure operator which is not a closure operator, then the equivalence does not hold, as is shown in the next example.

Example 3. Let $L = \{0, 0.5, 1\}$ be the three-valued Łukasiewicz residuated lattice, $U = \{a, b\}$ and consider the complete fuzzy lattice (L^U, S).

Let $\gamma\colon L^U \to L^U$ defined by

$$\gamma\big(\{a/x, b/y\}\big) = \begin{cases} \{a/0, b/0\}, & \text{if } x = y = 0,\\ \{a/0, b/1\}, & \text{if } x = 0,\, y > 0,\\ \{a/1, b/0\}, & \text{if } x > 0,\, y = 0,\\ \{a/1, b/1\}, & \text{otherwise.} \end{cases}$$

This mapping is trivially a classical closure operator. However, it is not a closure operator since $S(\{a/0.5, b/0\}, \{a/0, b/0.5\}) \not\leq S(\{a/1, b/0\}, \{a/0, b/1\})$.

Indeed, the element $\{a/0, b/0.5\}$ is quasi-closed for γ in $(L^U, \subseteq)$, but it is not quasi-closed for γ in (L^U, S) since $S(\gamma(\{a/0, b/0.5\}), \gamma(\{a/0.5, b/0\})) = 0 < 1$ and $(S(\{a/0.5, b/0\}, \{a/0, b/0.5\}) \to S(\gamma(\{a/0.5, b/0\}), \{a/0, b/0.5\})) = 0.5 < 1$.

The following result characterises not being a quasi-closed element in terms of the existence of an element with remarkable properties.

Proposition 7. *Let (A, ρ) be a complete fuzzy lattice endowed with a closure operator $\mathsf{c}\colon A \to A$. The following conditions are equivalent*:

1. *$q \in A$ is not quasi-closed*;
2. *there exists $a \lhd q$ such that $a \lhd \mathsf{c}(a)$ and $\mathsf{c}(a) \sqcap q = a$.*

Proof. For the direct implication, assume $q \in A$ is not quasi-closed, by negating Definition 8, there exists $b \in A$ such that $\mathsf{c}(b) \sqcap q \notin \mathcal{F} \cup \{q\}$. Now, put $a = \mathsf{c}(b) \sqcap q$. Clearly, $a \lhd q$ because $a \trianglelefteq q$ and we cannot have $a = q$ since $a \notin \mathcal{F} \cup \{q\}$. Furthermore, $a \lhd \mathsf{c}(a)$ which is again a consequence of $a \notin \mathcal{F} \cup \{q\}$. Finally, observe that $\mathsf{c}(a) \sqcap q = \mathsf{c}(\mathsf{c}(b) \sqcap q) \sqcap q \trianglelefteq (\mathsf{c}(b) \sqcap \mathsf{c}(q)) \sqcap q = \mathsf{c}(b) \sqcap q = a$, where the isotonicity and associativity of $\sqcap$ and the idempotency of c are used. The converse inclusion follows from the facts that $a \lhd q$ and $a \lhd \mathsf{c}(a)$. Altogether, $a \trianglelefteq \mathsf{c}(a) \sqcap q = a$.

For 2 implies 1 assume there exists $a \lhd q$ such that $a \lhd \mathsf{c}(a)$ and $\mathsf{c}(a) \sqcap q = a$. Obviously, $\mathsf{c}(a) \sqcap q = a$ implies $\mathsf{c}(a) \sqcap q \notin \mathcal{F} \cup \{q\}$, i.e. q is not quasi-closed, the claim follows from Definition 8. □

We now look for another characterisation, trying to extend the recursive expression in Proposition 4.

Proposition 8. *Let (A, ρ) be a complete fuzzy lattice endowed with a closure operator $\mathtt{c}\colon A \to A$ and let the set of closed elements $\mathcal{F}$ satisfy the descending chain condition. Then, $q \in A$ is quasi-closed iff for each quasi-closed $a \lhd q$, we have $\mathtt{c}(a) \unlhd q$ or $\mathtt{c}(a) = \mathtt{c}(q)$.*

Proof. By Proposition 6, one of the implications holds, hence we will only prove the next statement, if $\mathtt{c}(a) \unlhd q$ or $\mathtt{c}(a) = \mathtt{c}(q)$ for each quasi-closed element $a \lhd q$ then q is quasi-closed.

Assume q is not quasi-closed, then we need to find a quasi-closed element $a \lhd q$ such that $\mathtt{c}(a) \ntrianglelefteq q$ and $\mathtt{c}(a) \lhd \mathtt{c}(q)$. Put $a_0 = q$. Using Proposition 7, there exists $a_1 \lhd a_0$ such that a_1 is not closed and $\mathtt{c}(a_1) \sqcap a_0 = a_1$. Observe we cannot have $\mathtt{c}(a_1) = \mathtt{c}(a_0)$ since it would yield $a_1 = \mathtt{c}(a_1) \sqcap a_0 = \mathtt{c}(a_0) \sqcap a_0 = a_0$, which is absurd. Hence $\mathtt{c}(a_1) \lhd \mathtt{c}(a_0)$. If a_1 is not quasi-closed, applying Proposition 7 again, we obtain a new element $a_2 \lhd a_1$ such that $a_2 \lhd \mathtt{c}(a_2)$, $\mathtt{c}(a_2) \sqcap a_1 = a_2$ and $\mathtt{c}(a_2) \lhd \mathtt{c}(a_1)$. In addition, $a_2 = \mathtt{c}(a_2) \sqcap a_1 = \mathtt{c}(a_2) \sqcap (\mathtt{c}(a_1) \sqcap a_0) = \mathtt{c}(a_2) \sqcap a_0 = \mathtt{c}(a_2) \sqcap q$.

Since $\mathcal{F}$ satisfies the descending chain condition, the strictly descending sequence $\mathtt{c}(a_0) \rhd \mathtt{c}(a_1) \rhd \mathtt{c}(a_2) \rhd \cdots$, whose existence is ensured by Proposition 7, eventually terminates, i.e. there exists a quasi-closed element $a_n \lhd \mathtt{c}(a_n)$ such that:

$$
\begin{aligned}
& a_n \lhd \cdots \lhd a_1 \lhd a_0, = q, \\
& \mathtt{c}(a_n) \lhd \cdots \lhd \mathtt{c}(a_1) \lhd \mathtt{c}(a_0) = \mathtt{c}(q), \\
& a_n = \mathtt{c}(a_n) \sqcap a_{n-1} = \cdots = \mathtt{c}(a_n) \sqcap a_1 = \mathtt{c}(a_n) \sqcap a_0 = \mathtt{c}(a_n) \sqcap q.
\end{aligned}
$$

It then suffices to show $\mathtt{c}(a_n) \ntrianglelefteq q$. By contradiction, assume $\mathtt{c}(a_n) \unlhd q$. Then, $a_n = \mathtt{c}(a_n) \sqcap q = \mathtt{c}(a_n)$, but a_n is not closed by construction. □

The following theorem summarises the distinct characterisations studied above, providing a unified view of the content of this section.

Theorem 3. *Let (A, ρ) be a complete fuzzy lattice, $\mathtt{c}\colon A \to A$ be a closure operator, $\mathcal{F}$ be the set of closed elements and $q \in A$. Then, the following statements are equivalent*:

1. *q is quasi-closed for $\mathtt{c}$.*
2. *$a \lhd q$ implies $\mathtt{c}(a) \unlhd q$ or $\mathtt{c}(a) = \mathtt{c}(q)$, for all $a \in A$.*
3. *$(\rho(a, q) \to \rho(\mathtt{c}(a), q)) = 1$ or $\mathtt{c}(q) \unlhd \mathtt{c}(a)$, for all $a \in A$.*
4. *$a \lhd q$ implies $\mathtt{c}(a) = a$ or $\mathtt{c}(a) \sqcap q \neq a$, for all $a \in A$.*

In addition, if $\mathcal{F}$ satisfies the descending chain condition, then q is quasi-closed iff

5. *for each quasi-closed $a \lhd q$, we have $\mathtt{c}(a) \unlhd q$ or $\mathtt{c}(a) = \mathtt{c}(q)$.*

5. Quasi-Closed Elements via Closure Systems in a Fuzzy Framework

The aim of this section is to characterise the second item of Definition 8, Definition that is, we try to characterise the elements that satisfy that $\mathcal{F} \cup \{q\}$ is a closure system. As mentioned, being a quasi-closed element is a necessary condition to be a fuzzy quasi-closed element. Thus, the following result provides a stronger condition than the third statement in Theorem 3.

Proposition 9. *Let (A, ρ) be a complete fuzzy lattice and $\mathsf{c}\colon A \to A$ be a closure operator. An element $q \in A$ is a fuzzy quasi-closed element if and only if the following condition holds, for all $a \in A$:*

$$\big(\rho(a, q) \to \rho\big(\mathsf{c}(a), q\big)\big) = 1 \quad \text{or} \quad a \trianglelefteq q \trianglelefteq \mathsf{c}(a) = \mathsf{c}(q). \tag{2}$$

Proof. Let $\mathcal{F}$ be the closure system associated to c. Assume $q \in A$ is such that $\mathcal{F}' = \mathcal{F} \cup \{q\}$ is a closure system and let $\mathsf{c}_{\mathcal{F}'}\colon A \to A$ be its associated closure operator.

Notice that applying Proposition 1 we get

$$\mathsf{c}_{\mathcal{F}'}(a) = \bigsqcap\big(a^\rho \cap (\mathcal{F} \cup \{q\})\big) = \bigsqcap\big(a^\rho \cap \mathcal{F}\big) \sqcap \bigsqcap\big(a^\rho \cap \{q\}\big) = \mathsf{c}(a) \sqcap \bigsqcap\big\{q/\rho(a, q)\big\}.$$

Now, since $\mathsf{c}_{\mathcal{F}'}(a) \in \mathcal{F} \cup \{q\}$, we consider two possible situations:

1. Assume $\mathsf{c}_{\mathcal{F}'}(a) \in \mathcal{F}$, we have that $a \trianglelefteq \mathsf{c}_{\mathcal{F}'}(a)$ and by Definition 3, $\mathsf{c}(a)$ is the smallest element in $\mathcal{F}$ that is an upper bound of a. Thus, $\mathsf{c}(a) \trianglelefteq \mathsf{c}_{\mathcal{F}'}(a)$. Moreover, by the definition of $\mathsf{c}_{\mathcal{F}'}(a)$ we get $\mathsf{c}_{\mathcal{F}'}(a) \trianglelefteq \mathsf{c}(a)$. Therefore, $\mathsf{c}(a) = \mathsf{c}_{\mathcal{F}'}(a)$.

   Then, we have $\mathsf{c}(a) = \mathsf{c}_{\mathcal{F}'}(a) \trianglelefteq \bigsqcap\{q/\rho(a, q)\}$ and we can deduce, by Theorem 1,

$$\begin{aligned} 1 = \rho\Big(\mathsf{c}(a), \bigsqcap\big\{q/\rho(a, q)\big\}\Big) &= \bigwedge_{x \in A} \big(\big\{q/\rho(a, q)\big\}(x) \to \rho\big(\mathsf{c}(a), x\big)\big) \\ &= \rho(a, q) \to \rho\big(\mathsf{c}(a), q\big). \end{aligned}$$

2. Assume $\mathsf{c}_{\mathcal{F}'}(a) = \mathsf{c}(a) \sqcap \bigsqcap\{q/\rho(a, q)\} = q$, by Proposition 2, we have that $a \trianglelefteq \mathsf{c}_{\mathcal{F}'}(a) = q$ and we also have $q \trianglelefteq \mathsf{c}(a)$. By the isotonicity and idempotency of c we get $\mathsf{c}(a) \trianglelefteq \mathsf{c}(q)$ and $\mathsf{c}(q) \trianglelefteq \mathsf{c}(a)$. Therefore, we get the expected result $a \trianglelefteq q \trianglelefteq \mathsf{c}(a) = \mathsf{c}(q)$.

Conversely, assume q satisfies (2), we want to prove that $\mathcal{F}' = \mathcal{F} \cup \{q\}$ is a closure system. Let $\Phi \in L^{\mathcal{F}'}$, we will prove that $\bigsqcap \Phi \in \mathcal{F}'$. If we assume $q \notin \mathcal{F}$, we have that $\bigsqcap \Phi = \bigsqcap(\Phi \cap \mathcal{F}) \sqcap \bigsqcap\{q/\Phi(q)\}$, where $\bigsqcap(\Phi \cap \mathcal{F}) \in \mathcal{F}$ and will be denoted by f throughout the proof. We will also denote $\bigsqcap\{q/\Phi(q)\}$ by a. We will prove $f \sqcap a \in \mathcal{F}'$:

If $a = q$, we have two possible situations:

If $f \trianglelefteq q$ or $q \trianglelefteq f$, $f \sqcap q \in \mathcal{F} \cup \{q\}$.

If $f \ntrianglelefteq q$ and $q \ntrianglelefteq f$, the element $f \sqcap q$ does not satisfy $f \sqcap q \trianglelefteq q \trianglelefteq \mathsf{c}(f \sqcap q) = \mathsf{c}(q)$, since should it satisfy this we would get

$$q \trianglelefteq \mathsf{c}(q) = \mathsf{c}(f \sqcap q) \trianglelefteq \mathsf{c}(f) \sqcap \mathsf{c}(q) = f \sqcap \mathsf{c}(q) \trianglelefteq f,$$

which is against the hypothesis. Thus, we have,

$$\begin{aligned} 1 &= \rho(f \sqcap q, q) \leqslant \rho\big(\mathsf{c}(f \sqcap q), q\big) && \text{by (2)}, \\ 1 &= \rho(f \sqcap q, f) \leqslant \rho\big(\mathsf{c}(f \sqcap q), f\big) && \text{since } f \in \mathcal{F}. \end{aligned}$$

And we conclude $\rho(\mathsf{c}(f \sqcap q), f \sqcap q) = 1$ and $f \sqcap q$ is closed.

Otherwise, since $q \trianglelefteq a$, we only have $q \triangleleft a$. We will prove that a is closed. Since $a \ntrianglelefteq q$, we have by (2), $\rho(a, q) \leqslant \rho(\mathsf{c}(a), q)$. Hence,

$$
\begin{aligned}
1 &= \rho\Big(\bigsqcap\{q/\Phi(q)\}, q\Big) \to \rho\big(\mathsf{c}(a), q\big) \\
&\overset{(i)}{\leqslant} \Phi(q) \to \rho\big(\mathsf{c}(a), q\big) \\
&= \bigwedge_{x \in A} \big(\{q/\Phi(q)\}(x) \to \rho\big(\mathsf{c}(a), x\big)\big) \\
&\overset{(ii)}{=} \rho\Big(\mathsf{c}(a), \bigsqcap\{q/\Phi(q)\}\Big)\big) = \rho\big(\mathsf{c}(a), a\big),
\end{aligned}
$$

where (i) holds by the definition of infimum and (ii) holds by Theorem 1. Thus, $a \in \mathcal{F}$ and consequently $f \sqcap a \in \mathcal{F}$. □

The condition on the right hand side of (2) is concise but the meaning is compound. The next result shows a characterisation of that condition as two joint properties.

Lemma 1. *Let (A, ρ) be a complete fuzzy lattice, $\mathsf{c}\colon A \to A$ be a closure operator and $a, q \in A$. The following conditions are equivalent*:

1. $a \trianglelefteq q \trianglelefteq \mathsf{c}(a) = \mathsf{c}(q)$.
2. $\mathsf{c}(\mathsf{c}(a) \sqcap q) \ntriangleleft \mathsf{c}(q)$ *and* $q \ntriangleleft a \sqcup q$.

Proof. First, assume $a \trianglelefteq q \trianglelefteq \mathsf{c}(a) = \mathsf{c}(q)$. Then, we have that

$$
\begin{aligned}
&\mathsf{c}\big(\mathsf{c}(a) \sqcap q\big) = \mathsf{c}(q), \text{ then } \mathsf{c}\big(\mathsf{c}(a) \sqcap q\big) \ntriangleleft \mathsf{c}(q). \\
&q = a \sqcup q, \text{ then } q \ntriangleleft a \sqcup q.
\end{aligned}
$$

Conversely, assume that $\mathsf{c}(\mathsf{c}(a) \sqcap q) \ntriangleleft \mathsf{c}(q)$ and $q \ntriangleleft a \sqcup q$. Since $q \ntriangleleft a \sqcup q$, we necessarily have $q = a \sqcup q$, which is equivalent to $a \trianglelefteq q$. Therefore, $\mathsf{c}(a) \trianglelefteq \mathsf{c}(q)$.

Since $a \trianglelefteq \mathsf{c}(a) \sqcap q \trianglelefteq \mathsf{c}(a)$, we have that $\mathsf{c}(a) \trianglelefteq \mathsf{c}(\mathsf{c}(a) \sqcap q) \trianglelefteq \mathsf{c}(a)$, thus $\mathsf{c}(a) = \mathsf{c}(\mathsf{c}(a) \sqcap q) \ntriangleleft \mathsf{c}(q)$. Therefore, $\mathsf{c}(a) = \mathsf{c}(q)$. □

The underlying idea of the following result is an analysis of the opposite of the condition of the right of (2). Rephrasing it as an implication and considering the distinct situations, we get to the two conditions shown below.

Proposition 10. *Let (A, ρ) be a complete fuzzy lattice and $\mathsf{c}\colon A \to A$ be a closure operator. An element $q \in A$ is a fuzzy quasi-closed element if and only if the following conditions hold*:

$$\mathsf{c}\big(\mathsf{c}(a) \sqcap q\big) \triangleleft \mathsf{c}(q) \textit{ implies } \rho(a, q) \leqslant \rho\big(\mathsf{c}(a), q\big), \quad \textit{for all } a \in A. \tag{3}$$

$$q \triangleleft a \sqcup q \textit{ implies } \rho(a, q) \leqslant \rho\big(\mathsf{c}(a), q\big), \quad \textit{for all } a \in A. \tag{4}$$

Proof. To prove this we will use the *material implication*. Thus, (2) is equivalent to

$$\text{Not } a \trianglelefteq q \trianglelefteq \mathtt{c}(a) = \mathtt{c}(q) \text{ implies } \rho(a,q) \leqslant \rho\big(\mathtt{c}(a), q\big), \quad \text{for all } a \in A.$$

By Lemma 1, Condition (2) is equivalent to

$$\mathtt{c}\big(\mathtt{c}(a) \sqcap q\big) \vartriangleleft \mathtt{c}(q) \text{ or } q \vartriangleleft a \sqcup q \text{ implies } \rho(a,q) \leqslant \rho\big(\mathtt{c}(a), q\big), \quad \text{for all } a \in A,$$

which is also equivalent to (3) and (4) holding simultaneously. ☐

The following result shows a characterisation of the premises in both (3) and (4) in terms of elements strictly above or below q. This way the use of infima and suprema is avoided.

Proposition 11. *Let* (A, ρ) *be a complete fuzzy lattice and* $\mathtt{c}\colon A \to A$ *be a closure operator. Then, the following statements hold*:

1. *An element* $q \in A$ *satisfies* (3) *if and only if satisfies*

$$a \vartriangleleft q \text{ and } \mathtt{c}(a) \neq \mathtt{c}(q) \text{ implies } \mathtt{c}(a) \trianglelefteq q, \quad \text{for all } a \in A. \tag{5}$$

2. *An element* $q \in A$ *satisfies* (4) *if and only if satisfies*

$$q \vartriangleleft a \text{ implies } \rho(a,q) \leqslant \rho\big(\mathtt{c}(a), q\big), \quad \text{for all } a \in A. \tag{6}$$

Proof. Assume $q \in A$ satisfies (3) and let $a \vartriangleleft q$ and $\mathtt{c}(a) \neq \mathtt{c}(q)$. This implies, by isotonicity, that $\mathtt{c}(a) \trianglelefteq \mathtt{c}(q)$, which together with $\mathtt{c}(a) \neq \mathtt{c}(q)$, gives $\mathtt{c}(a) \vartriangleleft \mathtt{c}(q)$. Therefore, $\mathtt{c}(\mathtt{c}(a) \sqcap q) \trianglelefteq \mathtt{c}(\mathtt{c}(a)) = \mathtt{c}(a) \vartriangleleft \mathtt{c}(q)$, and we can apply (3) to ensure $\rho(a,q) \leqslant \rho(\mathtt{c}(a), q)$. Since $a \vartriangleleft q$, we get $\mathtt{c}(a) \trianglelefteq q$.

Conversely, assume $q \in A$ satisfies (5) and let $a \in A$ be an element such that $\mathtt{c}(\mathtt{c}(a) \sqcap q) \vartriangleleft \mathtt{c}(q)$. Notice that $\mathtt{c}(a) \sqcap q$ satisfies $\mathtt{c}(a) \sqcap q \vartriangleleft q$ and $\mathtt{c}(\mathtt{c}(a) \sqcap q) \neq \mathtt{c}(q)$, hence by applying (5) to $\mathtt{c}(a) \sqcap q$, we have $\rho(\mathtt{c}(\mathtt{c}(a) \sqcap q), q) = 1$. We trivially had $\rho(\mathtt{c}(\mathtt{c}(a) \sqcap q), \mathtt{c}(a)) = 1$, thus, by Corollary 1, we have that $\mathtt{c}(a) \sqcap q$ is closed. Since $\mathtt{c}(a) \sqcap q$ is closed we have that $\rho(a, \mathtt{c}(a) \sqcap q) \leqslant \rho(\mathtt{c}(a), \mathtt{c}(a) \sqcap q)$. By Corollary 1 we have that $\rho(a, \mathtt{c}(a)) \wedge \rho(a,q) \leqslant \rho(\mathtt{c}(a), \mathtt{c}(a)) \wedge \rho(\mathtt{c}(a), q)$ and using inflationarity and reflexivity we deduce $\rho(a,q) \leqslant \rho(\mathtt{c}(a), q)$.

Assume $q \in A$ satisfies (4) and let $a \in A$ such that $q \vartriangleleft a$. Then, $a = a \sqcup q$, so we can apply (4) to get $\rho(a,q) \leqslant \rho(\mathtt{c}(a), q)$.

Conversely, assume $q \in A$ satisfies (6) and let $a \in A$ such that $q \vartriangleleft a \sqcup q$. In particular, considering the element $a \sqcup q$, we get $\rho(a \sqcup q, q) \leqslant \rho(\mathtt{c}(a \sqcup q), q)$, thus, $\rho(a,q) = \rho(a \sqcup q, q) \leqslant \rho(\mathtt{c}(a \sqcup q), q) \otimes \rho(\mathtt{c}(a), \mathtt{c}(a \sqcup q)) \leqslant \rho(\mathtt{c}(a), q)$. ☐

It is clear that fuzzy quasi-closed elements are quasi-closed elements and the converse does not hold. Nevertheless, the following proposition shows that by adding a new condition we can get the converse.

Proposition 12. *Let* (A, ρ) *be a complete fuzzy lattice endowed with a closure operator* $\mathtt{c}\colon A \to A$. *An element* $q \in A$ *is a fuzzy quasi-closed for* $\mathtt{c}$ *if and only if it is a quasi-closed element and satisfies* (6).

Proof. This is a direct conclusion of item 2 in Theorem 3 and Propositions 11 and 10. □

The following theorem summarises the distinct characterisations studied above, providing a unified view of the content of this section.

Theorem 4. *Let* (A, ρ) *be a complete fuzzy lattice,* $\mathtt{c}\colon A \to A$ *be a closure operator and* $q \in A$. *Then, the following statements are equivalent*:

1. *The element* q *is fuzzy quasi-closed for* $\mathtt{c}$.
2. *The element* q *satisfies* (2).
3. *The element* q *satisfies* (3) *and* (4).
4. *The element* q *satisfies* (5) *and* (6).
5. *The element* q *is quasi-closed for* $\mathtt{c}$ *and satisfies* (6).

6. Conclusions and Further Works

In this paper, we have delved into the notion of quasi-closed element in the fuzzy setting. To do this, we have extended the well-established characterisation of quasi-closed sets in terms of closure systems. Thus, our approach to solving this problem has been two-fold. Firstly, we considered classical closure systems. The approach in Ojeda-Hernández *et al.* (2022a) turned out to be a necessary condition to be a quasi-closed element. We have obtained a characterisation that resembles the crisp case. In addition, following the suit of Kuznetsov and Obiedkov (2008), we achieved a recursive definition of the notion of quasi-closed element. Secondly, we considered the case of closure systems in the fuzzy sense. These so-called fuzzy quasi-closed elements have also been characterised in several ways. Remarkably, we proved that being a fuzzy quasi-closed element is equivalent to being a quasi-closed element and satisfying one extra condition.

As a prospect of future work, the study of quasi-closed elements in this work will be continued to define pseudo-closed elements in complete fuzzy lattices and we will examine the possibility of defining minimal, complete and non-redundant sets of implications using this notion.

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