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CONTROL FOR DELAYED SYSTEMS (A SURVEY)

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Abstract. This paper investigates the progress made in the field of dynamic systems with delays over the last two decades. In particular, it is focused on the simulation and control techniques, which include also modelling, numerical solvability and stabilization procedures.

Key words: point and distributed delays, stabilizability, state-space.

1. Introduction

1.1. What is a delay system. Mathematical models with time delay constitute a natural way to represent a wide range of real systems. For example, it is not always the case that the time taken for measurement of some physical variable is negligible compared with the time constants of the system. Associated with this problem is the problem of sampling for purposes of measurement. This is not sampling in the random sense, but at fixed intervals of time.

Because of remote working there will be cases of delay in control. This obviously occurs in space technology where even the finiteness of the velocity of electromagnetic radiation causes significant delay. In radar this delay is used as the basis for distance measurement, and the corresponding electronic techniques need to be extremely accurate.

All systems include delays. Many industrial processes, particularly thermal processes and distillation processes may be represented by including time delay in the system model. It is well known that many man-made physical systems, including control systems, have these analogues in Nature. In fact many of the control techniques including computer control, adaptive and optimal, are to be found in human biology. Time delays are also important in the study of epidemics. It is also the case that time delays occur in economic systems, and the application of control theory to these areas has been attempted (see, for instance, Bateman, 1945; Tustin, 1953; Howarth and Parks, 1972; Howarth, 1973).

1.2. Different kinds of delays: point delay, distributed delay. From a purely mathematical viewpoint, there are mainly two ways a delay can be considered in a differential equation: point delay or distributed delay. Both types of delay can be applied on the state- or on the control-vector. A point delay applied on the state vector, for example, represents the dependence of such a vector with respect to the same vector in a previous moment, being the lapse of that previous moment and the current instant defined by the numerical value of the point delay at each time. More specifically, if one considers a point delay h in the state and a point delay k in the control, one has:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 u(t) + B_1 u(t-k). \tag{1.2.1}$$

In the general case, as the state- and control-matrices can be time-dependent, the delay dependence also varies dynamically.

On the other hand, a distributed delay (usually defined by using an integro-differential Volterra term) applied on the state vector, represents the dependence of such a vector with respect to the same vector during a previous (finite or infinite) interval of time, being the limits of such an interval defined by the integral of the Volterra term, provided that such a term is used to describe the distributed delay. More specifically, if one considers a system with one distributed delay in its state and one distributed delay in its control, one gets:

$$\dot{x}(t) = Ax(t) + \int_0^t A_0(\tau)x(t-\tau)d\tau + Bu(t) + \int_0^t B_0(\lambda)u(t-\lambda)d\lambda, \tag{1.2.2}$$

where the lower and upper limits of the integrals (i.e., 0 and t) mean that the delay term is distributed during the interval beginning at zero up to the current instant. Eq. (1.2.2) shows only a kind of distributed delay, but depending on the specific Volterra term being used, different types of distribution of delay can be established.

1.3. Influence of delays on the dynamic behaviour of a system. The presence of delays can incide dramatically on the dynamic behaviour of a system. For instance, a system with only a pole in an open-loop situation, when having a single point delay in cascade with the plant with unitary feedback, presents infinite poles, as was shown by many authors. From this fact it can be easily understood that the study of delays is of crucial importance when dealing with the dynamics, stability and observability of a system, among other characteristics.

2. Modelling, simulation and approximation for delayed systems

2.1. Modelling a delay system by matrix-state equations. One of the main motivations for utilizing matrix-state equations when dealing with delay systems is to exploit the digital computer as a design tool. In analogue computing, the reduction of nth-order differential equations to a set of simultaneous first-order differential equations as a preliminary step to systematic analogue computing follows a similar procedure. Once the nth-order differential equation corresponding to a delay system has been described in matrix terms, it is possible to apply matrix decomposition. From the several ways in which such decompositions may be performed, arose equivalent ways of describing the dynamics of the given system. Such methods are useful both for system simulation as well as for the analysis of system behaviour. A further extension which is necessary for the solution of dynamic problems by computing methods is that the time variable must be broken into discrete steps. Sampled-data techniques arising from this approach become a natural part of control theory where the computer is no longer just used as a design tool but as a part of the control scheme itself.

The counterpart for the advantages of the matrix formulation is that the variables to be used are not as directly related to physical quantities as in the transfer function description. The transfer-function approach preserves the system structure, which is an advantage. In a sense the structure of the original problem is mirrored in the "structure" of the state matrix A in the matrix description, as was pointed out by Marshall (1979). The problems of controllability and observability, arising specifically by using this approach, are inherent to matrix representation.

2.2. Exact solvability of delay-differential equations. Limitations. Although there are some results providing exact solutions for some limited number of differential equations with delays, in general there is no way to compute an exact explicit solution for equations of that kind. Even in those few cases for which an exact solution might be available, the provided solution is not explicit, but implicit, depending on an equation that contains the state vector - or other equivalent quantity, such as a solving or fundamental matrix – explicitly in the left side and under an integral in the right side of the equation. In practice, the given solution is not an explicit one: it involves a transcendent equation or a subsequent differential equation that is not directly solvable. In general, to obtain an exact and explicit solution for a delayed dynamical system is not possible, or up to now nobody knows a method for it. In the case of a general system with inner delays, it cannot be reduced to a delay-free dynamical expression; if the delay is inherent part of the system, some kind of delay always will appear in any subsequent transformation done on the original system. In the following, it is illustrated how the exact solvability of delay-differential equations is strongly limited by computational difficulties (Alastruey and González de Mendívil, 1994a).

Let us consider the following general MIMO linear system with multiple point delays in its state and in its control law.

$$\begin{aligned} \dot{x}(t) =& A(t)x(t) + \sum_{j=1}^{r} B_j(t)x(t-\tau_j) \\ &+ C(t)U(t) + \sum_{k=1}^{s} D_k(t)U(t-\delta_k), \\ x(t) =& F(t), \quad t \in [-\max\{\tau_j\}, 0)\,, \\ U(t) =& G(t), \quad t \in [-\max\{\delta_k\}, 0)\,. \end{aligned} \tag{2.2.1}$$

In principle, there are two approaches in order to compute an exact solution for system (2.2.1), depending on what one is defining as its associated free system.

First approach. The system composed by all the terms related to the state variables could be considered as the associated free system. It is called Delayed Free System:

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{r} B_j x(t-\tau_j). \tag{2.2.2}$$

In this case its forcing term is

$$CU(t) + \sum_{k=1}^{s} D_k U(t-\delta_k). \tag{2.2.3}$$

Second approach. The system composed by all the terms related to the state variables without delay also could be considered as the associated free system. It is called Non-Delayed Free System:

$$\dot{x}(t) = Ax(t). \tag{2.2.4}$$

In this second case, its forcing term is

$$\sum_{j=1}^{r} B_j x(t-\tau_j) + C\,U(t) + \sum_{k=1}^{s} D_k U(t-\delta_k). \tag{2.2.5}$$

In both cases the fundamental matrix $\Psi(t)$ must satisfy the equation of its respective free system.

Delayed free system approach. As it was pointed out, the fundamental matrix must satisfy the equation of the free system. In this case:

$$\dot{\Psi}(t) = A\Psi(t) + \sum_{j=1}^{r} B_j \Psi(t-\tau_j) \Rightarrow$$

$$\Psi(t) = e^{At}\left[I + \sum_{j=1}^{r}\left(\int_{\tau_j}^{t} e^{-A\tau} B_j \Psi(\tau-\tau_j) U(t-\delta_k) d\tau\right)\right]. \tag{2.2.6}$$

Eq. 2.2.6 involves a transcendental relation for the fundamental matrix. The solution for the state vector of system (2.2.1) by using this method, therefore, must be of the form:

$$\begin{aligned} x(t) =& \Psi(t)x_0 + \sum_{j=1}^{r} \int_0^{\tau_j} \Psi(t-\tau) B_j \varphi(\tau-\tau_j) d\tau \\ &+ \int_0^t \Psi(t-\tau)\left[C\,U(\tau) + \sum_{k=1}^{s} D_k\, U(\tau-\delta_k)\right] d\tau, \end{aligned} \tag{2.2.7}$$

where $\varphi(0) = x_0$ and the transition matrix $\Psi(t)$ has been used as it appears in (2.2.6). When one is trying to solve Eq. (2.2.7) several problems arise. There are mainly two procedures for solving Eq. (2.2.7).

First procedure. If one tries to deduce an associated matrix $\mathcal{A}$ (one of the traditional approaches) such that it satisfies the differential relation

$$\mathcal{A}e^{\mathcal{A}t} = \frac{d}{dt}\left(e^{\mathcal{A}t}\right), \tag{2.2.8}$$

and considering the simplest case – a unique point delay in the state – one gets:

$$\dot{\Psi}(t) = A\,\Psi(t) + B_1\,\Psi(t - \tau_1), \tag{2.2.9}$$

then

$$\mathcal{A}e^{\mathcal{A}t} = Ae^{\mathcal{A}t} + Be^{\mathcal{A}(t-\tau_1)} \Rightarrow \mathcal{A} = A + B_1 e^{-\mathcal{A}\tau_1}. \tag{2.2.10}$$

The main difficulty is that one cannot assert that the matrix solution for Eq. 2.2.10 has all its eigenvalues being real or complex by pairs. To say, dissaccopled complex eigenvalues could exist leading to the situation, therefore, that the computed solution is a formal one but not valid, as was pointed out by Fiagbedzi and Person (1990).

Second procedure. Another way to obtain an explicit solution would be to integrate by parts Eq. 2.2.6 in order to get an expression for the transition matrix. This expression would be substituted in (2.2.7) to get an explicit solution for the system dynamics, this is, the state vector. In order to illustrate the complexity of the problem, let us consider the scalar case; it would imply to solve the integral

$$I(\tau) = \int_h^t e^{a\tau} b_1 \Psi(\tau - \tau_1) d\tau. \tag{2.2.11}$$

If one takes

$$\begin{aligned} &e^{a\tau} d\tau = dv \Rightarrow v = \frac{1}{A} e^{a\tau}, \\ &u = b_1 \Psi(\tau - \tau_1) \Rightarrow du = b_1 \left(a\Psi(\tau - \tau_1) + b\Psi(\tau - 2\tau_1)\right), \end{aligned} \tag{2.2.12}$$

then

$$\begin{aligned} \int_{\tau_1}^t e^{at} b\Psi(\tau - \tau_1) d\tau &= \frac{1}{a} e^{at} b\Psi(\tau - \tau_1)\Big]_{\tau_1}^t \\ &- \int_{\tau_1}^t \frac{1}{a} e^{at} b \left(a\Psi(\tau - \tau_1) + b\Psi(\tau - 2\tau_1)\right) d\tau. \end{aligned} \tag{2.2.13}$$

It must be noted that Eq. 2.2.13 is now more complex than the original one (2.2.11). The integral term containing the transition function with delay is not eliminated and, in addition, it is accompained by a new integral term containing the transition function with the double of the delay value. To say, firstly one had a problem with a unique point delay, and now one has a problem with two point delays. Subsequent integrations would lead to the appearance of new delays.

Non-delayed free system approach. In this case, also, the fundamental matrix must satisfy the free system equation:

$$\dot{\Psi}(t) = A\Psi(t) \Rightarrow \Psi(t) = e^{At}. \tag{2.2.14}$$

The solution for the state vector of system (2.2.1) by using this method must be of the form:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}\left[\sum_{j=1}^{r} B_j x(\tau - \tau_j) + CU(\tau) + \sum_{k=1}^{s} D_k U(\tau - \delta_k)\right] d\tau. \tag{2.2.15}$$

The explicit and exact solution of (2.2.15) is not possible in the general case, and therefore approximate methods are needed.

2.3. Simulation of delay systems via polynomial and Taylor series

2.3.1. Introduction to simulation via Taylor series. Let's consider the following MIMO linear system with point and distributed delays:

$$\begin{aligned}\dot{x}(t) =& A(t)x(t) + \sum_{j=1}^{r} B_j(t)x(t-\tau_j) + \sum_{j=1}^{r}\int_0^{\tau_j} R_j(t,\tau)x(t-\tau)d\tau \\ &+ C(t)U(t) + \sum_{k=1}^{s} D_k(t)U(t-\delta_k)\end{aligned} \tag{2.3.1.1}$$

for all $t > 0$.

The initial conditions are:

$$\begin{aligned} x(t) &= F(t), \quad t \in [-\max\{\tau_j\}, 0), \\ U(t) &= G(t), \quad t \in [-\max\{\delta_j\}, 0), \end{aligned} \tag{2.3.1.2}$$

where τ_j, δ_k are real numbers in (0, 1), and, without loss of generality, the following order is assumed:

If $s < r$: $\tau_1 < \delta_1 < \tau_2 < \delta_2 < \ldots < \tau_s < \delta_s < \tau_{s+1} < \ldots < \tau_r$.

If $r \leqslant s$: $\tau_1 < \delta_1 < \tau_2 < \delta_2 < \ldots < \delta_{r-1} < \tau_r < \delta_r < \ldots < \delta_s$.

Furthermore: $x(t) \in \mathbf{R}^n$, $U(t) \in \mathbf{R}^p$, $\dim(A(t)) = n \times n$, $\dim(B_j(t)) = n \times n$, $\dim(C(t)) = n \times p$, $\dim(D_k(t)) = n \times p$, $\dim(R_j(t,\tau)) = n \times n$. And $x(0) = x_0$ is known.

As the function $F(t)$ is given, for every τ_j it is possible to expand $F(t-\tau_j)$ in a Taylor series and compute vectors $V_{j0}, \ldots, V_{j(m-1)}$ with dimension $n \times 1$:

$$F(t-\tau_j) = \widehat{T}^T(t)\left[V_{j0}^T\ V_{j1}^T \ldots V_{j(m-1)}^T\right]^T, \quad j = 1, \ldots, r. \qquad (2.3.1.3)$$

In the same way, as function $G(t)$ is given, for every δ_k it is possible to expand $G(t-\delta_k)$ in a Taylor series expansion and compute vectors $W_{k0}, \ldots, W_{k(m-1)}$ with dimension $p \times 1$, in such a way that:

$$G(t-\delta_k) = \widehat{T}^T(t)\left[W_{k0}^T\ W_{k1}^T \ldots W_{k(m-1)}^T\right]^T, \quad k = 1, \ldots, s. \qquad (2.3.1.4)$$

The first difficulty we find when we try to represent Eq. 2.3.1.1 by Taylor series is the presence of a double integral with two integration variables for the distributed delayed term. Note that if (2.3.1.1) is integrated over the interval (0, 1) the following expresion is obtained:

$$\begin{aligned} x(t) = &\int_0^t A(t')x(t')dt' + \int_0^t \sum_{j=1}^r B_j(t')x(t'-\tau_j)dt' \\ &+ \int_0^t \sum_{j=1}^r \int_0^{\tau_j} R_j(t',\tau)x(t'-\tau)d\tau dt' \\ &+ \int_0^t C(t')U(t')dt' + \int_0^t \sum_{k=1}^s D_k(t')U(t'-\delta_k)dt'. \qquad (2.3.1.5) \end{aligned}$$

By this reason in the algorithm it will be necessary to establish a double step in order to find the Taylor series expression for the double integral of the distributed delay. This question is not evident – because of the presence of the delay – and it involves a very interesting method of resolution. This situation is different with respect to the case with point delays only, because for any time t such that $0 < t < \tau_r$, the method cannot provide an expression for the solution of the state vector; only it can provide values in specific points. And only after instant $t = \tau_r$ the method can provide an approximate analytical expression. This is due to the fact that the integral corresponding to the distributed delay of the j term for $0 < t < \tau_j$ has two different terms for every instant of the interval. For example, for an instant t_1 such that $0 < t_1 < \tau_j$, the j-th term of the distributed delay in (2.3.1.1) presents the form:

$$\int_0^{\tau_j} R_j(t',\tau)x(t'-\tau)d\tau = \int_0^{t_1} R_j(t',\tau)x(t'-\tau)d\tau + \int_{t_1}^{\tau_j} R_j(t',\tau)F(t'-\tau)d\tau. \qquad (2.3.1.6)$$

Therefore in this case the situation is quite different from a situation with only point delays. When the instant is lesser than the value of the upper limit of the particular distributed delay, the contribution of that distributed delay to the global state vector only can be computed point by point. Nevertheless, when the instant is greater than such a limit, its contribution can be computed for an interval of values, to say, there will exist an approximate analytical expression, there will exist a constant Taylor expansion, valid for the whole interval.

2.3.2. Main results and algorithm structure. A method to represent (2.3.1.1) by Taylor series is introduced, by using the Taylor operationals of integration, delay and multiplication, that are common in the literature (Razzaghi and Razzaghi, 1989). This method consists in rewriting the differential Eq. 2.3.1.1 as an algebraic equation, by substituting the integrals by integration operational matrices, and the delayed terms by delay operational matrices multiplied by the terms without delays. In this case the increased difficulty consists in the fact that we need to do a double Taylor representation for the same functions: the first with respect to the inner variable of the distributed delay, and the second with respect to the general variable of time.

The initial conditions for state vector x must be expanded as follows:

$$F(t_i-\tau) = \widehat{T}^T(\tau)\left[V_{i0}^{\tau\,T}\; V_{i1}^{\tau\,T} \ldots V_{i(m-1)}^{\tau\,T}\right]^T. \qquad (2.3.2.1)$$

Remark: in this case subindex i denotes that the expansion depends upon the considered instant.

The matrix of distributed delay can be developed as follows:

$$R_j(t',\tau) = \left[R_{j0}(t')\ R_{j1}(t')\dots R_{j(m-1)}(t')\right]\widehat{T}(\tau). \tag{2.3.2.2}$$

Then for an instant t_i such that $0 < t_i < t_j$, the term of distributed delays in (2.3.1.1) becomes:

$$\int_0^{\tau_j} R_j(t',\tau)x(t'-\tau)d\tau = \int_0^{t_i} R_j(t',\tau)x(t'-\tau)d\tau + \int_{t_i}^{\tau_j} R_j(t',\tau)F(t_i-\tau)d\tau$$

$$= \int_0^{t_i}\left[R_{j0}(t')\ R_{j1}(t')\dots R_{j(m-1)}(t')\right]\widehat{T}(\tau)\widehat{T}^T(\tau)\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T d\tau$$

$$+ \int_{t_1}^{\tau_j}\left[R_{j0}(t')\ R_{j1}(t')\dots R_{j(m-1)}(t')\right]\widehat{T}(\tau)\widehat{T}^T(\tau)$$

$$\times\left[V_{i0}^{\tau T}\ V_{i1}^{\tau T}\dots V_{i(m-1)}^{\tau}{}^T\right]^T d\tau$$

$$= \int_0^{t_1}\widehat{T}^T(\tau)\widetilde{R}_j^T(t')\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T d\tau$$

$$+ \int_{t_1}^{\tau_j}\widehat{T}^T(\tau)\widetilde{R}_j^T(t')\left[V_{i0}^{\tau T}\ V_{i1}^{\tau T}\dots V_{i(m-1)}^{\tau}{}^T\right]^T d\tau$$

$$=\widehat{T}^T(\tau)\left[\widehat{P}^T(0)-\widehat{P}^T(t_1)\right]\widetilde{R}_j^T(t')\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T$$

$$+\widehat{T}^T(\tau)\left[\widehat{P}^T(t_1)-\widehat{P}^T(\tau_j)\right]\widetilde{R}_j^T(t')\left[V_{i0}^{\tau T}V_{i1}^{\tau T}\dots V_{i(m-1)}^{\tau}{}^T\right]^T, \tag{2.3.2.3}$$

where superindex τ denotes that the development has been made with respect the variable τ, $\widetilde{R}_j^T(t')$ is defined as in Razzaghi and Razzaghi (1989), and T is the Taylor series basis vector.

Let

$$K_j[\tau](t_i) = \widehat{T}^T(\tau)\left[\widehat{P}^T(0)-\widehat{P}^T(t_1)\right]\widetilde{R}_j^T(t')\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T$$

$$+\widehat{T}^T(\tau)\left[\widehat{P}^T(t_1)-\widehat{P}^T(\tau_j)\right]\widetilde{R}_j^T(t')\left[V_{i0}^{\tau T}\ V_{i1}^{\tau T}\dots V_{i(m-1)}^{\tau}{}^T\right]^T \tag{2.3.2.4}$$

Matrix $K_j[\tau](t_i)$ is the solution for instant $t_i < \tau_j$ of integral (2.3.1.6). In order to compute this matrix, an estimation for the state vector is required. Nevertheless it is possible to avoid using such an estimation by rewriting $K_j[\tau](t_i)$

as follows:

$$\begin{aligned} K_j[\tau](t_i) =& \widehat{T}^T(\tau)\left[\widehat{P}^T(0)-\widehat{P}^T(t_1)\right]\widetilde{R}_j^T(t')\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T \\ &+\widehat{T}^T(\tau)\left[\widehat{P}^T(t_1)-\widehat{P}^T(\tau_j)\right]\widetilde{R}_j^T(t')\left[V_{i0}^{\tau T}\ V_{i1}^{\tau T}\dots V_{i(m-1)}^{\tau}{}^T\right]^T \\ =& \widehat{T}^T(\tau)K1_j[\tau](t_i)\left[x_0^{\tau T}\ x_1^{\tau T}\dots x_{m-1}^{\tau}{}^T\right]^T \\ &+\widehat{T}^T(\tau)K2_j[\tau](t_i), \end{aligned} \tag{2.3.2.5}$$

where

$$\begin{aligned} K1_j[\tau](t_i) &= \left[\widehat{P}^T(0)-\widehat{P}^T(t_1)\right]\widetilde{R}_j^T(t'), \\ K2_j[\tau](t_i) &= \left[\widehat{P}^T(t_1)-\widehat{P}^T(\tau_j)\right]\widetilde{R}_j^T(t')\left[V_{i0}^{\tau T}\ V_{i1}^{\tau T}\right. \\ &\qquad\qquad \left.\dots V_{i(m-1)}^{\tau}{}^T\right]^T. \end{aligned} \tag{2.3.2.6}$$

Matrices (2.3.2.6) are expressed as Taylor series expansions with respect to an inner variable of distributed delay, this is, τ. Let $K1_j^*[t](t_i)$ and $K2_j^*[t](t_i)$ be those matrices but now expressed as Taylor series expansions with respect to the general variable of time t. Now one finds that:

$$\begin{aligned} \int_0^{\tau_j} R_j(t',\tau)x(t'-\tau)d\tau =& \widehat{T}^T(\tau)K1_j^*[t](t_i)\left[x_0^T\ x_1^T\dots x_{m-1}^T\right]^T \\ &+\widehat{T}^T(\tau)K2_j^*[t](t_i). \end{aligned} \tag{2.3.2.7}$$

As matrices $K1_j^*[t](t_i)$ and $K2_j^*[t](t_i)$ are computed by using expressions (2.3.2.6), for any instant t_i such that $0 < t_i < \tau_j$ it is possible to represent their corresponding distributed delay by using Eq. 2.3.2.7 in the general solution. This is the second step.

Now we will proceed to the resolution of the full state equation. We start with integral equation (2.3.1.5) and every element is substituted by its corresponding Taylor series expansion with respect to variable t. In the following paragraphs, for the sake of briefness, only solutions for different intervals are listed, but in the first one the mathematical development is specified.

<u>For $0 \leqslant t \leqslant \tau_1$:</u>

$$\left[x_0^T\ x_1^T\ \dots x_{m-1}^T\right]^T = \widehat{P}^T(0)\widetilde{A}^T\left[x_0^T\ x_1^T\dots x_{m-1}^T\right]^T$$

$$
\begin{aligned}
&+\sum_{j=1}^{r}\widehat{P}^T(0)\widetilde{B}_j^T\left[V_{j0}^T\;V_{j1}^T\ldots V_{j(m-1)}^T\right]^T\\
&+\sum_{j=1}^{r}\widehat{P}^T(0)K1_j^*[t](t_i)\left[x_0^T\;x_1^T\ldots x_{m-1}^T\right]^T\\
&+\sum_{j=1}^{r}\widehat{P}^T(0)K2_j^*[t](t_i)+\widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\;U_1^T\ldots U_{m-1}^T\right]^T\\
&+\sum_{k=1}^{s}\widehat{P}^T(0)\widetilde{D}_k^T\left[W_{k0}^T\;W_{k1}^T\ldots W_{k(m-1)}^T\right]^T\\
&+\left[x^T(0)\;\bar{0}^T\ldots\bar{0}^T\right]^T.
\end{aligned}
\tag{2.3.2.8}
$$

Define

$$
\begin{aligned}
\overset{\circ}{R}^{(1)} &=I_{mn}-\widehat{P}^T(0)\widetilde{A}^T-\sum_{j=1}^{r}\widehat{P}^T(0)K1_j^*[t](t_i),\\
\overset{\circ}{Q}^{(1,1)} &=\sum_{j=1}^{r}\widehat{P}^T(0)\left[V_{j0}^T\;V_{j1}^T\ldots V_{j(m-1)}^T\right]^T\\
&+\sum_{j=1}^{r}\widehat{P}^T(0)K2_j^*[t](t_i)+\widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\;U_1^T\ldots U_{m-1}^T\right]^T\\
&+\sum_{k=1}^{s}\widehat{P}^T(0)\widetilde{D}_k^T\left[W_{k0}^T\;W_{k1}^T\ldots W_{k(m-1)}^T\right]^T\\
&+\left[x^T(0)\;\bar{0}^T\ldots\bar{0}^T\right]^T.
\end{aligned}
\tag{2.3.2.9}
$$

Therefore (2.3.2.8) leads to

$$
\left[x_0^T\;x_1^T\ldots x_{m-1}^T\right]^T=\left[\overset{\circ}{R}^{(1)}\right]^{-1}\overset{\circ}{Q}^{(1,1)},
\tag{2.3.2.10}
$$

from what the coefficients for the state variables Taylor expansion corresponding to an instant of the interval comprised between zero and the first state delay, are computed. One must realize that Eq. 2.3.2.10 is not valid for the full interval, but for the considered instant (in this case t_i).

<u>For $\tau_1 \leqslant t \leqslant \delta_1$:</u>

$$
\left[x_0^T\;x_1^T\ldots x_{m-1}^T\right]^T=\left[\overset{\circ}{R}^{(2)}\right]^{-1}\overset{\circ}{Q}^{(2,1)},
\tag{2.3.2.11}
$$

where

$$\overset{\circ}{R}{}^{(2)} = I_{mn} - \widehat{P}^T(0)\widetilde{A}^T - \widehat{P}^T(0)K1_1^*[t](\tau_1) - \sum_{j=2}^{r}\widehat{P}^T(0)K1_j^*[t](t_i)$$
$$- \widehat{P}^T(\tau_1)\widetilde{B}_1^T\widehat{S}^T(\tau_1),$$
$$\overset{\circ}{Q}{}^{(2,1)} = \sum_{j=2}^{r}\widehat{P}^T(0)\widetilde{B}_j^T\left[V_{j0}^T\ V_{j1}^T\dots V_{j(m-1)}^T\right]^T$$
$$+ \sum_{j=2}^{r}\widehat{P}^T(0)K2_j^*[t](t_i) + \widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\ U_1^T\dots U_{m-1}^T\right]^T$$
$$+ \sum_{k=1}^{s}\widehat{P}^T(0)\widetilde{D}_k^T\left[W_{k0}^T\ W_{k1}^T\dots W_{k(m-1)}^T\right]^T$$
$$+ \left[x^T(0)\ \bar{0}^T\dots\bar{0}^T\right]^T + \left[z_1^T\ \bar{0}^T\dots\bar{0}^T\right]^T, \tag{2.3.2.12}$$

from what the coefficients for the state variables Taylor expansion corresponding to an instant of the interval comprised between the first state delay and the first control delay, are computed. One must realize that, as well as (2.3.2.10), (2.3.2.11) is not valid for the full interval, but for the considered instant (in this case t_i).

After having computed the first expressions for the approximate solution of the state equation, in the following the solution is established for intermediate general instants (Eq. 2.3.2.13 and Eq. 2.3.2.14) as well as for instants after the value of the greatest delay (Eq. 2.3.2.15).

<u>For $\tau_q \leqslant t \leqslant \delta_q$:</u>

$$\left[x_0^T\ x_1^T\dots x_{m-1}^T\right]^T = \left[\overset{\circ}{R}{}^{(q+1)}\right]^{-1}\overset{\circ}{Q}{}^{(q+1,1)},$$

$$\overset{\circ}{R}{}^{(q+1)} = I_{mn} - \widehat{P}^T(0)\widetilde{A}^T - \sum_{j=1}^{q}\widehat{P}^T(0)K1_j^*[t](\tau_j)$$
$$- \sum_{j=q+1}^{r}\widehat{P}^T(0)K1_j^*[t](t_i) - \sum_{j=1}^{q}\widehat{P}^T(\tau_j)\widetilde{B}_j^T\widehat{S}^T(\tau_j),$$
$$\overset{\circ}{Q}{}^{(q+1,1)} = \sum_{j=q+1}^{r}\widehat{P}^T(0)\widetilde{B}_j^T\left[V_{j0}^T\ V_{j1}^T\dots V_{j(m-1)}^T\right]^T$$

$$+\sum_{j=q+1}^{r}\widehat{P}^T(0)K2_j^*[t](t_i)+\widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\ U_1^T\ldots U_{m-1}^T\right]^T$$

$$+\sum_{k=1}^{q-1}\widehat{P}^T(\delta_k)\widetilde{D}_k^T\widehat{S}^T(\delta_k)\left[U_0^T\ U_1^T\ldots U_{m-1}^T\right]^T$$

$$+\sum_{k=q}^{s}\widehat{P}^T(0)\widetilde{D}_k^T\left[W_{k0}^T\ W_{k1}^T\ldots W_{k(m-1)}^T\right]^T$$

$$+\left[x^T(0)+\sum_{j=1}^{q}z_j^T+\sum_{k=1}^{q-1}w_k^T\ \bar{0}^T\ldots\bar{0}^T\right]^T. \qquad (2.3.2.13)$$

<u>For $\delta_q \leqslant t \leqslant \tau_{q+1}$:</u>

$$\left[x_0^T\ x_1^T\ldots x_{m-1}^T\right]^T=\left[\overset{\circ}{R}^{(q+1)}\right]^{-1}\overset{\circ}{Q}^{(q+1,2)},$$

$$\overset{\circ}{Q}^{(q+1,2)}=\sum_{j=q+1}^{r}\widehat{P}^T(0)\widetilde{B}_j^T\left[V_{j0}^T\ V_{j1}^T\ldots V_{j(m-1)}^T\right]^T$$

$$+\sum_{j=q+1}^{r}\widehat{P}^T(0)K2_j^*[t](t_i)+\widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\ U_1^T\ldots U_{m-1}^T\right]^T$$

$$+\sum_{k=1}^{q}\widehat{P}^T(\delta_k)\widetilde{D}_k^T\widehat{S}^T(\delta_k)\left[U_0^T\ U_1^T\ldots U_{m-1}^T\right]^T$$

$$+\sum_{k=q+1}^{s}\widehat{P}^T(0)\widetilde{D}_k^T\left[W_{k0}^T\ W_{k1}^T\ldots W_{k(m-1)}^T\right]^T$$

$$+\left[x^T(0)+\sum_{j=1}^{q}z_j^T+\sum_{k=1}^{q}w_k^T\ \bar{0}^T\ldots\bar{0}^T\right]^T. \qquad (2.3.2.14)$$

And finally, for any t greater than any delay appearing in the state equation one gets:

$$\left[x_0^T\ x_1^T\ldots x_{m-1}^T\right]^T=\left[\overset{\circ}{R}^{\text{fin}}\right]^{-1}\overset{\circ}{Q}^{\text{fin}},$$

$$\overset{\circ}{R}^{\text{fin}}=I_{mn}-\widehat{P}^T(0)\widetilde{A}^T-\sum_{j=1}^{r}\widehat{P}^T(0)K1_j^*[t](\tau_j)$$

$$
\begin{aligned}
&-\sum_{j=1}^{r}\widehat{P}^T(\tau_j)\widetilde{B}_j^T\widehat{S}^T(\tau_j),\\
\overset{\circ}{Q}^{\text{fin}} &=\widehat{P}^T(0)\widetilde{C}^T\left[U_0^T\;U_1^T\ldots U_{m-1}^T\right]^T\\
&+\sum_{k=1}^{r}\widehat{P}^T(\delta_k)\widetilde{D}_k^T\widehat{S}^T(\delta_k)\left[U_0^T\;U_1^T\ldots U_{m-1}^T\right]^T\\
&+\left[x^T(0)+\sum_{j=1}^{r}z_j^T+\sum_{k=1}^{r}w_k^T\;\bar{0}^T\ldots\bar{0}^T\right]^T, \qquad (2.3.2.15)
\end{aligned}
$$

being its fundamental characteristic that it is valid for all the interval, and not only for an specific instant.

3. Stability of delayed systems

3.1. General stability of delayed systems. Criteria for stability of linear delay systems can be classified in two main categories, according to their dependence on the delay's size: (a) Criteria non including information about delays are called free-delay criteria (Kamen, 1980; Bourl, 1987; Mori *et al.*, 1982; Lewis and Anderson, 1980; Wang *et al.*, 1992; Phoojaruenchanachai and Furuta, 1992). (b) Criteria containing information about delays are called delay criteria (Mori, 1985; Mori and Kokame, 1985; De la Sen, 1992; Alastruey and Etxebarria, 1992). These methods are of simple application. Chiasson (1986) developed a method to determine value ranges for delays without desestabilizing the system, but this method usually requires to solve trascendent equations.

Free-delay criteria are particularly suitable when delays are small with respect to some measures. Thus, it is reasonable firstly to apply free-delay criteria and, if they are not appropriate, apply delay criteria. In practice these two types of criteria are complementary.

3.2. Stability based on the delay measure

3.2.1. Matrix measure and delay measure. Matrix measure has been widely used in the literature when dealing with stability of delay-differential systems (see, for instance, Mori, 1986; Mori and Kokame, 1989). The matrix measure μ for matrix X is defined as follows:

$$
\mu(X)\equiv\lim_{\varepsilon\to 0}\frac{||I+\varepsilon X||-1}{\varepsilon}. \qquad (3.2.1.1)
$$

The following lemma provides some properties for the matrix measure $\mu(.)$.

Lemma 3.2.1.1 (Desoer and Vidyasagar, 1975). *For any matrices X, $Y \in C^{n\times n}$ the following inequalities hold:*

(i) $\mathrm{Re}\lambda_i(X) \leqslant \mu(X)$, (3.2.1.2)

(ii) $-\mu(jX) \leqslant Im\lambda_i(X) \leqslant \mu(-jX)$, (3.2.1.3)

(iii) $\mu(X+Y) \leqslant \mu(X) + \mu(Y)$, (3.2.1.4)

(iv) $\mu(X) \leqslant \|X\|$, (3.2.1.5)

(v) $\mu(\varepsilon X) = \varepsilon\mu(X)$, for any $\varepsilon \geqslant 0$. (3.2.1.6)

Consider the following class of linear delay-differential systems with a point delay in the state- and a point delay in the control-variables:

$$\dot{x}(t) = Ax(t) + A_0x(t-h) + Bu(t) + B_0u(t-q), \quad h, q \in R^+, \tag{3.2.1.7}$$

where $A,\ A_0,\ B,\ B_0 \in W \subset R^{n\times n}$, being W the set of n-matrices Q such that $\|Q\| < \infty$.

DEFINITION 3.2.1.1 (Alastruey and González de Mendívil, 1993). The Delay Measure for system (3.2.1.7) is defined as follows:

$$\xi(h,q) \equiv \frac{\|A_0\|h + \|B_0\|q}{\mu(A) + \mu(B)}. \tag{3.2.1.8}$$

REMARK 3.2.1.1. If there is no delay (i.e., $h,\ q = 0$, or A_0 and B_0 are null matrices), then the delay measure is zero. On the other hand, if the point delays h and q verify $0 < h < \infty$, $0 < q < \infty$, and there is not a delay-free term (i.e., $A,\ B$ are matrices of zeros) then the delay measure is infinite. Therefore, the delay measure can be considered, intuitively, as a way to evaluate the effect of delay terms in a system compared with its delay free terms.

Some properties of the delay measure are outlined.

PROPERTY 3.2.1.1 (Alastruey and González de Mendívil, 1993). Lower bounds for the first derivatives of the delay measure:

$$\varrho_h \equiv \frac{\partial\xi(h,q)}{\partial h} = \frac{\|A_0\|}{\mu(A)+\mu(B)} \geqslant \frac{\|A_0\|}{\|A\|+\|B\|}, \tag{3.2.1.9}$$

$$\varrho_q \equiv \frac{\partial\xi(h,q)}{\partial q} = \frac{\|B_0\|}{\mu(A)+\mu(B)} \geqslant \frac{\|B_0\|}{\|A\|+\|B\|}, \tag{3.2.1.10}$$

PROPERTY 3.2.1.2 (Alastruey and González de Mendívil, 1993). Absolute lower bound for the delay measure (supposing $h,\ q$ variables):

$$\xi(h,q) = \frac{\|A_0\|h + \|B_0\|q}{\mu(A) + \mu(B)} \geqslant \frac{\|A_0\|\widehat{h} + \|B_0\|\widehat{q}}{\|(A)\| + \|(B)\|}, \qquad (3.2.1.11)$$
$$\text{with} \quad \widehat{h} = \min h \quad \text{and} \quad \widehat{q} = \min q.$$

REMARK 3.2.1.2. Observe that property 3.2.1.1 helps to estimate boundedness conditions for the variations in value of the delay measure. Property 3.2.1.2 gives absolute boundedness conditions for the delay measure, provided that n-matrices appearing in (3.2.1.7) belong to the class W.

3.2.2. Stability conditions under delay-measure notation. In this section, some stability results for a class of free linear differential systems with two point delays in the state vector are introduced under delay-measure notation, by using a result due to Schoen and Geering (1993).

Consider the free linear delay-differential system:

$$\dot{x}(t) = a_0 x(t) + a_1(t-h) + a_2 x(t-2h), \qquad (3.2.2.1)$$

where $a_0,\ a_1$ and a_2 are constant coefficients and $h > 0$. It is possible to extend the definition of delay measure (3.2.1.8) for system (3.2.2.1) as follows:

DEFINITION 3.2.2.1. The Delay Measure for system (3.2.2.1) is defined as follows:

$$\xi(h) \equiv \frac{|a_1| + 2|a_2|}{|a_0|}\, h. \qquad (3.2.2.2)$$

Now let's introduce a theorem that is used in the sequel to get the main results.

Theorem 3.2.2.1 (Schoen and Geering, 1993). *The time-delay system* (3.2.2.1) *with* $|a_2| < \pi/2h$ *is asymptotically stable if and only if the following three conditions hold for some* $y \in [0, \pi/h)$

(i) $a_0 + a_1 + a_2 < 0,$ (3.2.2.3)

(ii) $a_0 = \dfrac{y \cdot \cos(yh)}{\sin(yh)} + a_2,$ (3.2.2.4)

(iii) $a_1 > -\dfrac{y}{\sin(yh)} - 2a_2\cos(yh).$ (3.2.2.5)

The stability criteria introduced in Theorem 3.2.2.1 can be rewritten by using the delay measure as follows.

Theorem 3.2.2.2. *Suppose that $a_0 \in R^-$, $a_1, a_2 \in R^+$. System (3.2.2.1) is asymptotically stable if the following condition holds for some $y \in [0, \pi/h)$:*

$$\frac{\xi(h)}{h} + \frac{y \cdot \cos(yh)}{\sin(yh)} = -\frac{a_1}{a_0} - 2, \tag{3.2.2.6}$$

where the delay measure function is that defined in (3.2.2.2).

Proof. Condition (3.2.2.6) implies that

$$\frac{\xi(h)}{h} + \frac{y \cdot \cos(yh)}{\sin(yh)} = \frac{|a_1|}{|a_0|} - 2, \tag{3.2.2.7}$$

therefore

$$\begin{aligned}
&\left[\frac{|a_1|}{|a_0|} - 2 - \frac{y \cdot \cos(yh)}{\sin(yh)}\right] h = \xi(h) \Rightarrow \\
&|a_1|h - |a_0|2h - \frac{y \cdot \cos(yh)}{\sin(yh)}2h = |a_1|h + |a_2|2h \Rightarrow \\
&\frac{1}{2}|a_1| - |a_0| = \frac{y \cdot \cos(yh)}{\sin(yh)} + \frac{1}{2}|a_1| + |a_2| \Rightarrow \\
&-|a_0| = \frac{y \cdot \cos(yh)}{\sin(yh)} + |a_2|.
\end{aligned} \tag{3.2.2.8}$$

But expression (3.2.2.8) coincides with condition (ii) in Theorem 3.2.2.1.

The main utility of Theorem 3.2.2.2 is that it substitutes one of the system's parameters appearing in condition (ii), Theorem 3.2.2.1, (i.e., a_2) by another one (i.e., a_1). Thus, Theorem 3.2.2.2 can be useful for evaluating asymptotic stability of system (3.2.2.1) when a_2 is not available, or the use of a_1 is more suitable for some design reason.

Theorem 3.2.2.3. *Suppose that $a_0 \in R^-$, $a_1, a_2 \in R^+$. System (3.2.2.1) is asymptotically stable if the following condition holds for some $y \in [0, \pi/h)$:*

$$\frac{\xi(h)}{h} > [\cos(yh) - 1]\frac{2|a_2|}{|a_0|} - \frac{y}{|a_0|\sin(yh)}. \tag{3.2.2.9}$$

Proof. Condition (3.2.2.9) implies

$$\begin{aligned}
&\xi(h) > -\frac{yh}{|a_0|\sin(yh)} + [\cos(yh) - 1]\frac{|a_2|}{|a_0|}2h \Rightarrow \\
&\frac{|a_1|h + |a_2|2h}{|a_0|} + \frac{|a_2|2h}{|a_0|}[1 - \cos(yh)] > -\frac{yh}{|a_0|\sin(yh)} \Rightarrow \\
&|a_1| > -\frac{y}{\sin(yh)} - 2|a_2|\cos(yh).
\end{aligned} \tag{3.2.2.10}$$

But expression (3.2.2.10) coincides with condition (iii) in Theorem 3.2.2.1.

Similarly to Theorem 3.2.2.2, the main utility of Theorem 3.2.2.3 is that it substitutes one of the system's parameters appearing in condition (iii), Theorem 3.2.2.1, (i.e., a_1) by another one (i.e., a_0).

3.3. Stability equivalences for approximate delayed systems. In the present section is shown how to compute, in a simple way, some results about stability for a class of linear point-delayed systems. Sufficient conditions for asymptotic stability of linear delayed systems are greatly simplified and straightforwardly computed under a Taylor series representation of the state equations. Also, conditions under which asymptotic stability for the approximate system implies asymptotic stability for the real system are outlined.

Let's consider the following linear free system with point-delay in its state vector.

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t-\tau), \tag{3.3.1a}$$

$$A,\ B \in R^{n\times n}, \quad x(t) \in R^n, \quad \tau > 0. \tag{3.3.1b}$$

A condition for asymptotic stability in that system is given by the following result due to Mori (1985).

Theorem 3.3.1. *System* (3.3.1) *is asymptotically stable (AS), if the following conditions hold:*

$$\mu(A) + \max_{y\in\Delta}\left\{\mu\left(Be^{-\tau yj}\right)\right\} < 0 \tag{3.3.2}$$

$$\text{for} \quad \max_{y\in\Delta}\left\{\mu\left(Be^{-\tau yj}\right)\right\} \geqslant -\frac{1}{\tau},$$

$$1 + \tau\cdot\max_{y\in\Delta}\left\{\mu\left(Be^{-\tau yj}\right)e^{(1-\tau\mu(A))}\right\} < 0 \tag{3.3.3}$$

$$\text{for} \quad \max_{y\in\Delta}\left\{\mu\left(Be^{-\tau yj}\right)\right\} < -\frac{1}{\tau},$$

where Δ represents the range of values for the solution y of the following equations for all possible eigenvalues

$$y = \operatorname{Im}\left[\lambda_i\left(A + \left(Be^{-\tau uj}\cdot e^{-\operatorname{Re}[\tau s]}\right)\right)\right], \tag{3.3.4}$$

where $\operatorname{Re}[s] \geqslant 0,\ s \in C,\ \tau > 0$ *and the matrix measure μ of an arbitrary matrix X is defined in* (3.2.1.1).

The Lemma 3.2.1.1 gives some interesting properties of the matrix measure refered to in Theorem 3.3.1.

Note that conditions (3.3.2) and (3.3.3) in Theorem 3.3.1 are not easily computable and if an on-line stability test is required, a faster way of evaluation is needed. On the other hand, system (3.3.1) has not, in general, an explicit solution and this fact motivates the use of approximate methods for resolving the system. Taylor series representation allows to obtain an approximate solution for system (3.3.1) (Alastruey and Etxebarria, 1992) and it also provided a fast method for the analysis of the system. Therefore, a reformulation of Theorem 3.3.1 for an approximate system under Taylor series is required in order to define the stability criteria for such representation.

Theorem 3.3.2. *Under a Taylor series representation, system (3.3.1) is asymptotically stable within the definition interval if* $\mu(\widetilde{A}^T + \widetilde{B}^T \cdot \widehat{S}^T(\tau)) < 0$, *where* $\widetilde{A}^T$, $\widetilde{B}^T$ *are the product operators for the coefficient matrices of the Taylor expansions for* A *and* B, *and* $\widehat{S}^T(\tau)$ *is the Taylor delay operational matrix, defined by Razzaghi and Razzaghi (1989).*

Proof. Let's represent system (3.3.1) by using Taylor series expansions:

$$\frac{d}{dt}\left(\widehat{T}^T(t)\ \left[x_0^T\ x_1^T \ldots x_{m-1}^T\right]^T\right) = \widehat{T}^T(t)\widetilde{A}^T\ \left[x_0^T\ x_1^T \ldots x_{m-1}^T\right]^T$$
$$+ \widehat{T}^T(t)\widetilde{B}^T\widehat{S}^T(\tau)\left[x_0^T\ x_1^T \ldots x_{m-1}^T\right]^T. \tag{3.3.11}$$

Therefore, inequalities (3.3.2) and (3.3.3) in Theorem 3.3.1 take the new form

$$\mu\left(\widetilde{A}^T + \widetilde{B}^T \cdot \widehat{S}^T(\tau)\right) + \max_{y\in\Delta}\{\mu(0)\} < 0 \tag{3.3.12}$$

$$\text{for} \quad \max_{y\in\Delta}\{\mu(0)\} \geqslant -\frac{1}{\tau},$$

$$1 + \tau\cdot\max_{y\in\Delta}\left\{\mu(0)e^{\left(1-\tau\mu(\tilde{A}^T+\tilde{B}^T\cdot\hat{S}^T(\tau))\right)}\right\} < 0 \tag{3.3.13}$$

$$\text{for} \quad \max_{y\in\Delta}\{\mu(0)\} < -\frac{1}{\tau}.$$

Note that

$$\mu(0) = \lim_{\varepsilon\to 0+}\frac{\|I+\varepsilon\cdot 0\| - 1}{\varepsilon} = 0, \tag{3.3.14}$$

therefore

$$\max_{y\in\Delta}\{\mu(0)\} < -\frac{1}{\tau} \tag{3.3.15}$$

and only inequality (3.3.12) can exist. That condition becomes $\mu(\widetilde{A}^T + \widetilde{B}^T \cdot \cdot \widehat{S}^T(\tau)) < 0$ which is a sufficient condition for asymptotic stability of the Taylor series expansion of system (3.3.1).

It is evident that the usefulness of Theorem 3.3.2 occurs when stability in the approximate system implies stability in the real one. In what follows, some results concerned to this point are given. It is supposed that all the norms are 2-norms, and K is the condition number refered to that norm.

Lemma 3.3.1. *Define*

$$D \equiv A + B \cdot e^{-\tau y j} \tag{3.3.16}$$

being $y = \mathrm{Im}\left[\lambda_i\left(A + (Be^{-\tau y j} \cdot e^{-\mathrm{Re}[\tau s]})\right)\right]$, *where* $\mathrm{Re}[s] \geqslant 0,\ s \in C,\ \tau > 0,$ *and*

$$\widetilde{D} \equiv \widetilde{A}^T + \widetilde{B}^T \widehat{S}^T(\tau). \tag{3.3.17}$$

By construction, coefficient matrix dimension is $\dim(\widetilde{D}) = m \cdot n \times m \cdot n$. *In order to compare it with real values matrix* D, *let's construct*

$$\widehat{D} \equiv \left[\sum_{i=0}^{m-1} \widetilde{d}_{n \cdot i + k, l} t^i\right]_{k=0,\dots,n-1;\ l=0,\dots,n-1}, \tag{3.3.18}$$

where now this last matrix has dimension $n \times n$. *Define*

$$\Delta D \equiv D - \widehat{D}. \tag{3.3.19}$$

Then, if there exist $\mu(D)$ *and* $\mu(\widehat{D})$, *and*

$$||\Delta D|| < \lim_{\varepsilon \to 0+} \frac{1 - ||I + \varepsilon \widehat{D}||}{\varepsilon} \tag{3.3.20}$$

is satisfied, the following implication holds

$$\mu(\widehat{D}) < 0 \Rightarrow \mu(D) < 0. \tag{3.3.21}$$

Proof. Suppose $\mu(\widehat{D}) < 0$, then

$$\begin{aligned}
\mu(D) &= \lim_{\varepsilon \to 0+} \frac{||I + \varepsilon D|| - 1}{\varepsilon} = \lim_{\varepsilon \to 0+} \frac{||I + \varepsilon(\widehat{D} + \Delta D)|| - 1}{\varepsilon} \\
&< \lim_{\varepsilon \to 0+} \left[\frac{||I + \varepsilon \widehat{D}|| - 1}{\varepsilon} + ||\Delta D||\right] \\
&= \lim_{\varepsilon \to 0+} \left[\frac{||I + \varepsilon \widehat{D}|| - 1}{\varepsilon}\right] + ||\Delta D|| \\
&= \mu(\widehat{D}) + ||\Delta D|| \Rightarrow \mu(D) < \mu(\widehat{D}) + ||\Delta D||.
\end{aligned} \tag{3.3.22}$$

Then if $||\Delta D|| < -\mu(\widehat{D}) = \lim_{\varepsilon \to 0+} \left[\frac{1 - ||I + \varepsilon \widehat{D}||}{\varepsilon}\right]$ it follows that $\mu(D) < 0$.

Lemma 3.3.1 leads to the following result.

Theorem 3.3.3. *Let $\Sigma \subseteq R$ be the set of real values taken by variable t, such that*

$$\operatorname{sign}\left(\mu(\widehat{D})\right) = \operatorname{sign}\left(\mu(\widetilde{D})\right). \tag{3.3.23}$$

If $\mu(\widehat{D}^T) < 0$ and $||\Delta D|| < -\mu(\widehat{D}) = \lim_{\varepsilon \to 0+} \left[\frac{1 - ||I + \varepsilon \widehat{D}||}{\varepsilon}\right]$ it follows that the real system (3.3.1) is asymptotically stable if $\exists k_0 \in R$ such that $(k_0, \infty) \subset \Sigma$.

Proof. The proof is immediate from Lemma 3.3.1.

$$\begin{aligned} &\forall t > k_0,\ t \in \Sigma \quad \text{and, therefore,} \quad t > k_0 \Rightarrow \\ &\qquad \operatorname{sign}\left(\mu(\widehat{D}^T)\right) = \operatorname{sign}\left(\mu(\widetilde{D}^T)\right). \end{aligned} \tag{3.3.24}$$

Note that $\mu(X) = \mu(X^T)$ for an arbitrary matrix X.

By hypothesis $\mu(\widehat{D}^T) < 0$ and $||\Delta D|| < -\mu(\widehat{D}) = \lim_{\varepsilon \to 0+} \left[\frac{1 - ||I + \varepsilon \widehat{D}||}{\varepsilon}\right]$. Therefore, by Lemma 3.3.1 and Eq. 3.3.24 it follows that $\mu(\widetilde{D}^T) < 0 \Rightarrow \mu(\widehat{D}^T) < 0 \Rightarrow \mu(D) < 0$. Thus, by Theorem 3.3.1, the real system (3.3.1) is asymptotically stable.

The following results complement the above mentioned, in the sense that they give the conditions to determine the bound for stability.

Lemma 3.3.2. *Define:*

$$\sigma'_\varepsilon(D) \equiv \frac{||I + \varepsilon D|| - 1}{\varepsilon}, \quad \forall \varepsilon \in R^+, \tag{3.3.25}$$

$$\sigma_\varepsilon(D) \equiv \sigma'_\varepsilon(D) \left\| \left(I + \varepsilon \widehat{D}\right)^{-1} \right\|. \tag{3.3.26}$$

Then, if $\exists \mu(D)$ the following implications hold

$$\begin{aligned} &\text{(i)} && \mu(D) = 0 \ \Rightarrow\ \mu(\widehat{D}) = 0 \quad \forall D \neq 0, \\ &\text{(ii)} && \mu(\widehat{D}) = 0 \ \Rightarrow\ \mu(D) = 0 \quad \forall D. \end{aligned} \tag{3.3.27}$$

Proof. Let's consider definitions (3.3.25) and (3.3.26). Then

$$\begin{aligned}\sigma_\varepsilon(D) \equiv&\sigma'_\varepsilon(D)\left\|\left(I+\varepsilon\widehat{D}\right)^{-1}\right\| \\ =&\frac{\|I+\varepsilon D\|\ \|(I+\varepsilon D)^{-1}\| - \|(I+\varepsilon\widehat{D})^{-1}\|}{\varepsilon} \\ =&\frac{K(I+\varepsilon D) - \|(I+\varepsilon\widehat{D})^{-1}\|}{\varepsilon}.\end{aligned} \tag{3.3.28}$$

By hypothesis $\exists\mu(D) = \lim_{\varepsilon\to 0+} \sigma_\varepsilon(D)$. As $\sigma_\varepsilon(D) = \sigma_\varepsilon(D)\left\|(I+\varepsilon\widehat{D})^{-1}\right\|$ one gets

$$\lim_{\varepsilon\to 0+} \sigma_\varepsilon(D) = \lim_{\varepsilon\to 0+} \sigma_\varepsilon(D)\cdot 1. \tag{3.3.29}$$

Then

$$\exists \lim_{\varepsilon\to 0+} \sigma_\varepsilon(D) = \lim_{\varepsilon\to 0+} \frac{K(I+\varepsilon D) - \|(I+\varepsilon\widehat{D})^{-1}\|}{\varepsilon}, \tag{3.3.30}$$

and therefore

$$\exists\mu(\widehat{D}) = \lim_{\varepsilon\to 0+} \frac{K(I+\varepsilon D) - \|(I+\varepsilon\widehat{D})^{-1}\|}{\varepsilon}. \tag{3.3.31}$$

Hence $\mu(D) = 0 \ \Rightarrow\ \mu(\widehat{D}) = 0\ \forall D \neq 0$ and $\mu(\widehat{D}) = 0 \ \Rightarrow\ \mu(D) = 0\ \forall D$.

COROLLARY 3.3.1.

$$\forall D \neq 0,\ \mu(D) = 0 \iff \mu(\widehat{D}) = 0. \tag{3.3.32}$$

Theorem 3.3.4. *$\forall t \in \Sigma \subseteq R^+$, being Σ the set defined in Theorem 3.3.3, the following implications hold*

$$\begin{aligned}&\text{(i)} \qquad \forall D \neq 0,\ \mu(D) = 0 \Rightarrow \mu(\widetilde{D}) = 0,\\ &\text{(ii)} \qquad \forall D,\ \mu(\widetilde{D}) = 0 \Rightarrow \mu(D) = 0.\end{aligned} \tag{3.3.33}$$

COROLLARY 3.3.2.

$$\forall D \neq 0,\ \forall t \in \Sigma \subseteq \mathbf{R}^+,\ \mu(D) = 0 \iff \mu(\widetilde{D}) = 0. \tag{3.3.34}$$

REMARK 3.3.1. Let's see that

$$\mu(\widehat{D}) = \lim_{\varepsilon\to 0+} \left[\frac{K(I+\varepsilon D)}{\varepsilon} - \left\|(\varepsilon I + \varepsilon^2 D)^{-1}\right\|\right], \tag{3.3.35}$$

therefore, if $\operatorname{rank}(I+\varepsilon D)$ decreases, the limit (3.3.35) tends to ∞.

Theorem 3.3.5. *If the approximate system is asymptotically stable for all t, then a necessary condition for stability in the real system is*

$$\mu(\widehat{D}) \geqslant \lim_{\varepsilon\to 0+} \left[-\frac{1}{1-\varepsilon\|D\|} + K(I+\varepsilon D)\right]. \tag{3.3.36}$$

Proof. The following inequality holds

$$\begin{aligned} -\mu(\widehat{D}) &= \lim_{\varepsilon\to 0+} \left[(\varepsilon I + \varepsilon^2 D)^{-1} - \frac{K(I+\varepsilon D)}{\varepsilon}\right] \\ &\leqslant \lim_{\varepsilon\to 0+} \left[\frac{\frac{1}{\varepsilon}\|I\|}{1-\frac{1}{\varepsilon}\|I\|\varepsilon^2\|D\|} - \frac{K(I+\varepsilon D)}{\varepsilon}\right], \end{aligned} \tag{3.3.37}$$

and

$$\varepsilon\|I\|\cdot\|D\| < 1 \quad \text{for} \quad \varepsilon \text{ small} \Rightarrow \|D\| < \frac{1}{\varepsilon\|I\|} = \frac{1}{\varepsilon}. \tag{3.3.38}$$

Therefore

$$\begin{aligned} -\mu(\widehat{D}) &\leqslant \lim_{\varepsilon\to 0+} \left[\frac{1}{\varepsilon(1-\varepsilon\|D\|)} - \frac{K(I+\varepsilon D)}{\varepsilon}\right] \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\left[\frac{1}{1-\varepsilon\|D\|} - K(I+\varepsilon D)\right]. \end{aligned} \tag{3.3.39}$$

And hence a necessary condition for real system to be asymptotically stable is

$$-\mu(\widehat{D}) \leqslant \lim_{\varepsilon\to 0+} \left[\frac{1}{1-\varepsilon\|D\|} - K(I+\varepsilon D)\right]. \tag{3.3.40}$$

But condition (3.3.40) coincides with (3.3.38).

REMARK 3.3.2. Let's point out that one of the simplest conditions for stability in system (3.3.1) is (Mori and Kokame, 1989):

$$\mu(A) + \|B\| < 0. \tag{3.3.41}$$

Obviously (3.3.41) is a delay-independent criterion and it assures stability for any value of τ.

EXAMPLE. Now an illustrative example (Mori, 1985) is considered. The stability region for the system is studied by using Taylor series representation and stability criteria above developed. It is shown that the obtained results are coherent with those obtained by Mori (1985). Let's consider the delayed scalar system described by

$$\dot{x}(t) = ax(t) + bx(t-\tau), \quad \tau > 0, \; a, b \text{ reals.} \tag{3.3.42}$$

Under a series representation, the system is expressed as follows:

$$\dot{x}^{exT} = \widehat{A}^T x^{exT} + \widehat{B}^T S^T(\tau) x^{exT}, \tag{3.3.43}$$

where $\widehat{A}^T$, $\widehat{B}^T$ represent operators $\widetilde{A}^T$, $\widetilde{B}^T$ for the scalar case, x^{ex} represents the Taylor coefficient vector for the scalar function $x(t)$ and $\dot{x}^{ex}$ represents the Taylor coefficient vector for the scalar function $\dot{x}^T$.

If five terms in the Taylor expansions ($m = 5$) are taken, one obtains:

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & b \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\tau & 1 & 0 & 0 & 0 \\ \tau^2 & -2\tau & 1 & 0 & 0 \\ -\tau^3 & 3\tau^2 & -3\tau & 1 & 0 \\ \tau^4 & -4\tau^3 & 6\tau^2 & -4\tau & 1 \end{bmatrix}^T \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{3.3.44}$$

Let's construct now matrix $\widehat{X}^T = \widehat{A}^T + \widehat{B}^T \widehat{S}^T(\tau)$:

$$\begin{aligned} \widehat{X}^T &= \widehat{A}^T + \widehat{B}^T \widehat{S}^T(\tau) \\ &= \begin{bmatrix} a+b & 0 & 0 & 0 & 0 \\ -b\tau & a+b & 0 & 0 & 0 \\ b\tau^2 & -2b\tau & a+b & 0 & 0 \\ -b\tau^3 & 3b\tau^2 & -3b\tau & a+b & 0 \\ b\tau^4 & -4b\tau^3 & 6b\tau^2 & -4b\tau & a+b \end{bmatrix}^T. \end{aligned} \tag{3.3.45}$$

The system will be stable if $\mu(\widehat{X}^T) < 0$ or equivalently if

$$\lim_{\varepsilon \to 0+} \frac{\|I + \varepsilon \widehat{X}^T\| - 1}{\varepsilon} < 0. \tag{3.3.46}$$

In order to check this condition let's construct matrix $M^T(\tau, \varepsilon, a, b)$ as follows:

$$M^T(\tau, \varepsilon, a, b) = I + \varepsilon \widehat{X}^T \tag{3.3.47}$$

$$= \begin{bmatrix} 1+\varepsilon(a+b) & 0 & 0 & 0 & 0 \\ -\varepsilon b\tau & 1+\varepsilon(a+b) & 0 & 0 & 0 \\ \varepsilon b\tau^2 & -2\varepsilon b\tau & 1+\varepsilon(a+b) & 0 & 0 \\ -\varepsilon b\tau^3 & 3\varepsilon b\tau^2 & -3\varepsilon b\tau & 1+\varepsilon(a+b) & 0 \\ b\varepsilon\tau^4 & -4\varepsilon b\tau^3 & 6\varepsilon b\tau^2 & -4\varepsilon b\tau & 1+\varepsilon(a+b) \end{bmatrix}^T.$$

This permits us to rewrite the stability condition as

$$\lim_{\varepsilon \to 0+} \frac{\|M(\tau, \varepsilon, a, b)^T\| - 1}{\varepsilon} < 0. \tag{3.3.48}$$

Note firstly that if there is no delay ($b = 0$) then matrix (3.3.47) becomes

$$M(\varepsilon, a) = (1 + \varepsilon a)I. \tag{3.3.49}$$

In this case the condition is

$$\lim_{\varepsilon \to 0+} \frac{\|(1 + \varepsilon a)I\| - 1}{\varepsilon} < 0 \ \Rightarrow \ a < 0. \tag{3.3.50}$$

and this is in agreement with the results obtained by Mori (1985).

4. Controlling plants with delays

4.1. Control and stabilization of delayed systems. Methods appropriate to delay-free systems have found suitable extensions to the delay case. However, it is essential that the delay-free part be very well modelled. In fact, an accurate model of the delay-free system is part of the control scheme (Marshall, 1979). Systems for which the delay may be classified as small may be controlled by using delay-free methods with the usual iterative tuning on the plant. On the other hand, systems for which there is non-negligible phase shift at input frequencies or in the region of delay-free unity-gain frequency must be controlled by methods which are different in kind from the small delay case. The presence of delays strongly limits the open-loop gain. The performance in the delay-free case is after categorised in terms of overshoot to step input, flatness of frequency response, and system time constants. Any stabilizing or controlling method which preserves these performance criteria is to be recommended (Marshall, 1979) so that the intuitive and practical importance of these criteria may be maintained.

4.2. Stabilizability by using the delay measure. Firstly, some stability results (Alastruey and González de Mendívil, 1993) for a class of free linear delay-differential systems are introduced under delay-measure notation. This representation will be useful in order to deduce the main stabilizability results.

Consider the free linear delay-differential system:

$$\dot{x}(t) = Ax(t) + A_0 x(t-h), \text{ with } A, A_0 \in W. \tag{4.2.1}$$

Lemma 4.2.1. *Provided* $h \geqslant 1$, a *sufficient condition for system* (4.2.1) *to be stable is*

$$\xi(h) < -1. \tag{4.2.2}$$

Proof. Observe that for system (4.2.1) the delay measure is reduced to

$$\xi(h,q) = \xi(h) = \frac{||A_0||h}{\mu(A)}. \tag{4.2.3}$$

Suppose $\xi(h) < -1$, then

$$\frac{||A_0||h}{\mu(A)} < -1 \Rightarrow -\mu(A) > ||A_0||h \Rightarrow \mu(A) < -||A_0||h. \tag{4.2.4}$$

As $h \geqslant 1$ then

$$\mu(A) < -||A_0|| \Rightarrow \mu(A) + ||A_0|| < 0, \tag{4.2.5}$$

that is one of the simplest conditions for stability in system (4.2.1) (Mori *et al.*, 1982).

Lemma 4.2.2 (Mori and Kokame, 1989). *Consider system* (4.2.1). *Assume that* $L1 := \mu(A) + ||A_0|| \geqslant 0$ *(otherwise system* (4.2.1) *is stable* *because of* (4.2.5)) *and* $L2 := \mu(-jA) + ||A_0||$ $(j^2 = -1)$. *If no solutions of the characteristic equation of* (4.2.1)

$$\det\left(sI - A - A_0 \exp(-hs)\right) = 0 \tag{4.2.6}$$

lie in the rectangular region Σ *shown in Fig.*4.2.1, *then system* (4.2.1) *is asymptotically stable.*

An equivalence of Lemma 4.2.2 under Delay-Measure notation can be established as follows.

Lemma 4.2.3. *Consider system* (4.2.1) *and suppose* $-h \leqslant \xi(h)$. *Consider the auxiliary complex system:*

$$\dot{x}(t) = -jAx(t) + A_0 x(t-h). \tag{4.2.7}$$

Assume

$$\begin{aligned} M1 &:= ||A_0|| \left[\frac{h}{\xi(h)} + 1\right], \\ M2 &:= ||A_0|| \left[\frac{h}{\xi_{complex}(h)} + 1\right]. \end{aligned} \tag{4.2.8}$$

Then, if no solutions of the characteristic equation of (4.2.1) *lie in the rectangular region* Λ *shown in Fig.* 4.2.2, *system* (4.2.1) *is asymptotically stable.*

Proof. Firstly, observe that condition $-h \leqslant \xi(h)$ implies that:

$$-h \leqslant \frac{h||A_0||}{\mu(A)} \Rightarrow -1 \leqslant \frac{||A_0||}{\mu(A)} \Rightarrow 1 \geqslant \frac{-||A_0||}{\mu(A)}. \tag{4.2.9}$$

Then one gets: $\mu(A) \geqslant -||A_0|| \Rightarrow \mu(A) + ||A_0|| \geqslant 0$, that is the same precondition than in Lemma 4.2.2. Furthermore, quantities $M1$ and $M2$ verify:

$$\begin{aligned} M1 &= ||A_0|| \left[\frac{h}{\xi(h)} + 1\right] \\ &= ||A_0|| \left[\frac{\mu(A)\cdot h}{||A_0||h} + 1\right] = \mu(A) + ||A_0|| \geqslant 0, \qquad (4.2.10) \\ M2 &= ||A_0|| \left[\frac{h}{\xi_{complex}(h)} + 1\right] \\ &= ||A_0|| \left[\frac{\mu(-jA)\cdot h}{||A_0||h} + 1\right] = \mu(-jA) + ||A_0||. \qquad (4.2.11) \end{aligned}$$

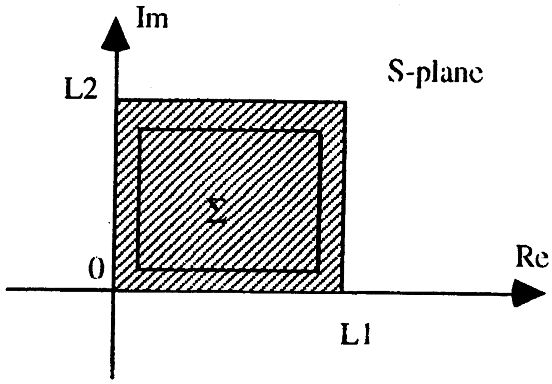

Fig. 4.2.1. Existence region of unstable characteristic roots in the S-plane for Lemma 4.2.2.

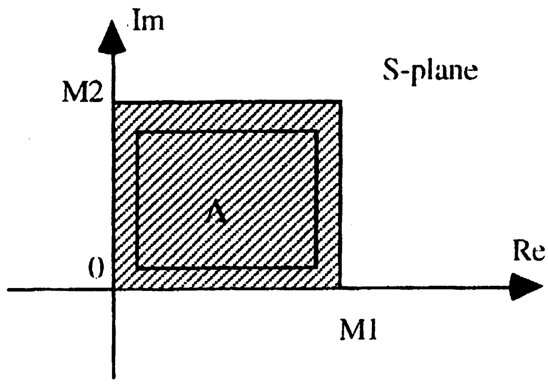

Fig. 4.2.2. Existence region of unstable characteristic roots in the S-plane for Lemma 4.2.3.

But $M1 = L1 \geqslant 0$ and $M2 = L2$.

Now, conditions for a control law to stabilize a linear system with delayed state will be discussed and several results are to be introduced. Consider the linear delayed system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0x(t-h) + Bu(t) + B_0u(t-h), \\ x(t) &= \varphi(t), \quad t \in [-h, 0], \end{aligned} \tag{4.2.12}$$

where

$$x \in R^n, \ u \in R^m, \ A_0, A \in R^{nxn}, \ B, B_0 \in R^{n\times m}, \ h \geqslant 0,$$

and $\varphi(t)$ is a continuous vector-valued initial function.

The following result refers to stabilizability for system (4.2.12) by using a control law $u(t)$ defined through a delay-differential equation.

Result 4.2.4. *Consider a control law $u(t)$ defined by the delay-differential equation*

$$\dot{u}(t) = Dx(t) + Eu(t) + D_0x(t-h) + E_0u(t-h). \tag{4.2.13}$$

A sufficient condition for control law (4.2.13) to stabilize system (4.2.12) is given by

$$\xi(h) < -h, \tag{4.2.14}$$

where the delay measure is refered to the extended system

$$\dot{z} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} z(t) + \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix} z(t-h). \tag{4.2.15}$$

Proof. Firstly observe that the two delay-differential equations (4.2.12) and (4.2.13) can be rewritten as one single delay-differential equation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix} \begin{bmatrix} x(t-h) \\ u(t-h) \end{bmatrix}. \tag{4.2.16}$$

Define

$$\begin{gathered} z(t) = \left[x^T(t)\ u^T(t)\right]^T, \\ \widetilde{A} = \begin{bmatrix} A & B \\ D & E \end{bmatrix}, \quad \widetilde{A}_0 = \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix}. \end{gathered} \tag{4.2.17}$$

Then (4.2.16) can be rewritten as

$$\dot{z} = \widetilde{A}z(t) + \widetilde{A}_0 z(t-h). \tag{4.2.18}$$

If, by hypothesis, condition (4.2.14) holds for system (4.2.18) then

$$\frac{\|\widetilde{A}_0\| \cdot h}{\mu(\widetilde{A})} < -h \Rightarrow \|\widetilde{A}_0\| < -\mu(\widetilde{A}) \Rightarrow \|\widetilde{A}_0\| + \mu(\widetilde{A}) < 0. \tag{4.2.19}$$

Therefore, system (4.2.18) is stable, which implies that control law (4.2.13) stabilizes system (4.2.12).

The two following corollaries are immediatly deduced from Result 4.2.4.

Corollary 4.2.1. *System (4.2.12) is stabilizable by control law (4.2.13) if*

$$\xi(h) < -1. \tag{4.2.20}$$

Corollary 4.2.2. *System (4.2.12) is stabilizable by control law (4.2.13) if*

$$||\widetilde{A}_0|| < \frac{1}{h} \lim_{\varepsilon \to 0} \frac{1 - ||I + \varepsilon \widetilde{A}||}{\varepsilon}, \tag{4.2.21}$$

provided $h \geqslant 1$.

Proof of Corollary 4.2.2. By hypothesis, condition (4.2.21) holds. Then

$$||\widetilde{A}_0|| < -\frac{1}{h} \cdot \mu(\widetilde{A}) \;\Rightarrow\; \xi(h) = \frac{||\widetilde{A}_0||h}{\mu(\widetilde{A})} < -1. \tag{4.2.22}$$

But, by Lemma 4.2.2, provided $h \geqslant 1$, (4.2.22) is a sufficient condition for stability in a system like (4.2.18).

Consider now a system defined by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0 x(t-h) + Bu(t), \\ x(t) &= \varphi(t), \quad t \in [-h, 0]. \end{aligned} \tag{4.2.23}$$

The following result refers to stabilizability for system (4.2.23) by using a control law $u(t)$ which is proportional to the state vector $x(t)$.

Result 4.2.5. *Consider a control law $u(t)$ defined as*

$$u(t) = kx(t), \quad k \text{ real}. \tag{4.2.24}$$

Then system (4.2.25) is stabilizable by control law (4.2.26) if

$$\xi(h) > \frac{||A_0||h}{(k-1) \cdot \mu(b) - ||A_0||}. \tag{4.2.25}$$

Proof. The following implications hold:

$$\begin{aligned} &\frac{||A_0||h}{\mu(A) + \mu(B)} > \frac{||A_0||h}{(k-1) \cdot \mu(B) - ||A_0||} \;\Rightarrow \\ &\mu(A) + \mu(B) < (k-1) \cdot \mu(B) - ||A_0|| \;\Rightarrow \\ &\mu(A) + k \cdot \mu(B) + ||A_0|| < 0. \end{aligned} \tag{4.2.26}$$

By using property (v) of Lemma 3.2.1.1 one gets

$$\mu(A) + \mu(kB) + \|A_0\| < 0, \tag{4.2.27}$$

but property (iii), Lemma 3.2.1.1, leads to

$$\mu(A + kB) + \|A_0\| \leqslant \mu(A) + \mu(kB) + \|A_0\| < 0. \tag{4.2.28}$$

Finally, provided $u(t)$ defined in Eq. 4.2.24, let's see that system (4.2.23) can be rewritten as

$$\begin{aligned} &\dot{x}(t) = (A + kB)x(t) + A_0 x(t-h) = \overline{A}x(t) + A_0 x(t-h), \\ &x(t) = \varphi(t), \quad t \in [-h, 0], \end{aligned} \tag{4.2.29}$$

where $\overline{A} = A + kB$. But (4.2.28) becomes $\mu(\overline{A}) + \|A_0\| < 0$, which is a sufficient condition for stability in system 4.2.29 (Mori *et al.*, 1982).

5. Large and nonlinear delay systems

5.1. Interconnected systems with delays. The problem of the interconnection of systems has been widely dealt with in the literature (Fessas, 1986; Cheung and Yurkovich, 1992), including the problems of stability and stabilizability of interconnected systems (Saberi and Khalil, 1985; Feliachi, 1986; de la Sen, 1986; Fessas, 1987; Lee and Radovic, 1988; Abdul-Wahab and Zohdy, 1992) and also the design of controllers for such systems (Özgüner and Hemami, 1985; Hovd and Skogestad, 1992; Shi and Singh, 1992). Nevertheless, no method was known in order to deduce the dynamics of the global system by using the knowledge of the dynamics of its subsystems and data about their interconnections, until a recent work (Alastruey and González de Mendívil, 1994b) that included the question of delays. In that paper we dealt with such a description by using the Taylor series representation (Razzaghi and Razzaghi, 1989).

Let's consider two subsystems 1 and 2, with dimensions n_1 and n_2 respectively, without any interconnection between them. Let $x(t)$ be the state vector for subsystem 1 and $w(t)$ the state vector for subsystem 2. Let S-0 be the global system composed by these two subsystems, with its state vector being

$$X(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \tag{5.1.1}$$

where we represent the column vector composed by the n_1 components of $x(t)$ and the n_2 components of $w(t)$.

Let's consider now another situation, where the subsystems 1 and 2 are interconnected. Due to this interconnection, for equal initial conditions, and for the same instant t, the state vectors for subsystems 1 and 2 corresponding to the new situation will be different than in the former case. Let's denote them as $x^*(t)$ and $w^*(t)$. Let S-I be the global system now composed, being its state vector

$$X^*(t) = \begin{bmatrix} x^*(t) \\ w^*(t) \end{bmatrix}. \tag{5.1.2}$$

A basic tool will be the interconnector operator which once applied to state vectors for systems S-0, and depending on existing interconnections, will give as a result the state vector for S-I. In practice, the resolution of the state equation for complex systems can be impossible. Even the simple fact of describe the state equation can be very complex. In the paper two results are introduced to describe state vectors for a class of complex large systems, by using the knowledge about their subsystems and their respective interconnections.

In the following, several concepts (interconnection, interconnection operator) – that are useful in the sequel – will be introduced.

DEFINITION 5.1.1 (Alastruey and González de Mendívil, 1994b). Let S1 and S2 be two subsystems, with state variables $x_1^1, \ldots, x_{n1}^1$ and $x_1^2, \ldots, x_{n2}^2$, respectively. We will define *interconnection* from the state variable x_i^1 of S1 to the state variable x_j^2 of S2, with operation Q, the creation of a line going out from x_i^1, being modified by the operation Q and arriving to the input of the integrator with output x_j^2. The variable x_i^1 is denoted *starting variable* and the variable x_j^2 *ending variable*. The interconnection can be represented in a short way as $< x_i^1 Q x_j^2 >$.

DEFINITION 5.1.2 (Alastruey and González de Mendívil, 1994b). We will define S-0 with respect to a set of N subsystems, with state vectors $[x_1^1 \ldots x_{n1}^1]^T, \ldots, [x_1^N \ldots x_{nN}^N]^T$, respectively, as the system with state vector being $[x_1^1 \ldots x_{n1}^1 \ldots x_1^N \ldots x_{nN}^N]^T$.

DEFINITION 5.1.3 (Alastruey and González de Mendívil, 1994b). Let I be a set of r interconnections between the variables of S-0. The system obtained by applying all the interconnections belonging to the set I on the system S-0 will be defined as the interconnected system, and will be denoted by S-I.

These two previous definitions greatly simplify the definition of interconnector, provided as follows.

DEFINITION 5.1.4 (Alastruey and González de Mendívil, 1994b). Denote $[x_1^{1*}\dots x_{n1}^{1*}\dots x_1^{N*}\dots x_{nN}^{N*}]^T$ the state vector of S-I. We will define as interconnector (or interconnection operator) with respect to the set I over S-0, the operator that when it is applied on the state vector of S-0 gives as a result the state vector for S-I. To say, if one denotes Ξ_I the interconnection operator thus

$$\Xi_I \left[x_1^1\, x_2^1 \dots x_{n_N}^N\right]^T = \left[x_1^{1*}\, x_2^{1*} \dots x_{n_N}^{N*}\right]^T . \tag{5.1.3}$$

DEFINITION 5.1.5. Let us define (Razzaghi and Razzaghi, 1989) the following operators which are matrices containing coefficients of Taylor series expansions:

$$T(t) = \left[1\; t\; t^2 \dots t^{m-1}\right]^T ,$$
$$\widehat{T}^T(t) = I_n \otimes T^T(t),$$

where $\otimes$ stands for the Kronecker product (Bellman, 1970). The integration operator for the Taylor series can be computed from the equation (Razzaghi and Razzaghi, 1989)

$$\int_\alpha^t T(t)dt \cong P(\alpha)T(t),$$

and then

$$P(\alpha) = \begin{bmatrix} -\alpha & 1 & 0 & \dots & 0 & 0 \\ -\frac{\alpha^2}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ -\frac{\alpha^3}{3} & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{\alpha^{m-1}}{m-1} & 0 & 0 & \dots & 0 & \frac{1}{m-1} \\ -\frac{\alpha^m}{m} & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

$$\widehat{P}^T(\alpha) = I_n \otimes P^T(\alpha).$$

The delay operator for the Taylor series satisfies the relation

$$T(t-\tau) = S(\tau)T(t)$$

and can be computed for $i, j = 0, 1, \ldots, m-1$ by using the formula

$$S_{ij} = \binom{i}{j}(-\tau)^{i-j}, \quad \text{if} \quad j \leqslant i,$$
$$S_{ij} = 0, \quad \text{if} \quad j > i.$$

Thus, for $m = 4$,

$$S(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\tau & 1 & 0 & 0 \\ \tau^2 & -2\tau & 1 & 0 \\ -\tau^3 & 3\tau^2 & -3\tau & 1 \end{bmatrix}.$$

Similarly, $\widehat{S}^T(\tau) = I_n \otimes S^T(\tau)$.

In the following, two basic results for the deduction of an interconnector are introduced. The first one is referred to a delay-free interconnection; the second one applies when there is a point-delay interconnection. Both apply for the case of two subsystems, but their generalization for the case of N subsystems is straightforward.

Lemma 5.1.1 (Free-delay interconnection) (Alastruey and González de Mendívil, 1994b). *Let us consider the following free subsystems with multiple state delays:*

$$\dot{x}(t) = A(t)x(t) + \sum_{j=1}^{r} B_j(t)x(t-\tau_j), \tag{5.1.4}$$

$$\dot{w}(t) = H(t)w(t) + \sum_{j=1}^{r} K_j(t)w(t-\tau_j), \tag{5.1.5}$$

where both subsystem (5.1.4) *as well as subsystem* (5.1.5) *have N state variables. Without loss of generality, we will assume that the point delays have the following order: $\tau_1 < \tau_2 < \ldots < \tau_r$. In addition, the initial conditions are known:*

$$x(t) = F(t), \quad t \in [-\max\{\tau_j\}, 0), \tag{5.1.6}$$

$$w(t) = G(t), \quad t \in [-\max\{\tau_j\}, 0). \tag{5.1.7}$$

The matrices in (5.1.4) *and* (5.1.5) *have appropriate dimensions, then:*

$$\dim(A(t)) = \dim(B_j(t)) = \dim(H(t)) = \dim(K_j(t)) = N \times N. \tag{5.1.8}$$

Let $< x_i q(t) w_j >$ *be a free delay interconnection. Let S-0 be the global system without interconnections, with its state vector being* $X(t) = [x(t)^T \; w(t)^T]^T$ *and let S-I be the global interconnected system with state vector* $X^*(t) = [x(t)^{*T} \; w(t)^{*T}]^T$*, with dimension* $N + N$*. Let the Tayor series representation of this state vector be:*

$$X^{*T} = \left[x_0^{*T} \; x_1^{*T} \ldots x_{m-1}^{*T} \; w_0^{*T} \ldots w_{m-1}^{*T}\right]^T, \tag{5.1.9a}$$

where

$$X^*(t) = \Big[\widehat{T}^T(t)\left[x_0^{*T} \; x_1^{*T} \ldots x_{m-1}^{*T}\right] \\ \widehat{T}^T(t)\left[w_0^{*T} \; w_1^{*T} \ldots w_{m-1}^{*T}\right]\Big]^T. \tag{5.1.9b}$$

Let the Tayor series representation of $X(t)$*, the state vector for S-0 be:*

$$X^T = \left[x_0^T \; x_1^T \ldots x_{m-1}^T \; w_0^T \ldots w_{m-1}^T\right]^T, \tag{5.1.10a}$$

where

$$X(t) = \Big[\widehat{T}^T(t)\left[x_0^T \; x_1^T \ldots x_{m-1}^T\right] \; \widehat{T}^T(t)\left[w_0^T \; w_1^T \ldots w_{m-1}^T\right]\Big]^T, \tag{5.1.10b}$$

and where, for instance, x_0 *is a coefficient vector containing the* N *first coefficients in the Taylor series expansions of the* N *state variables of the first subsystem.*

Let $\widetilde{A}$, $\widetilde{B}_j$, $\widetilde{H}$, $\widetilde{K}_j$ *be the product operational matrices (Razzaghi and Razzaghi, 1989) for the matrices* A, B_j, H *and* K_j*. Let* q *be a natural number such that* $0 < q \leqslant r$*.*

If submatrix

$$L = I_{mN} - \widehat{P}^T(0)\,\widetilde{H}^T - \sum_{j=1}^{q} \widehat{P}^T(\tau_j)\widetilde{K}_j^T\,\widehat{S}^T(\tau_j) \tag{5.1.11a}$$

is invertible, let's consider the operator (with elements being blocks mN*):*

$$\Xi = \begin{bmatrix} I_{mN} & \bar{0} \\ \left[I_{mN} - \widehat{P}^T(0)\,\widetilde{H}^T - \sum_{j=1}^{q} \widehat{P}^T(\tau_j)\widetilde{K}_j^T\,\widehat{S}^T(\tau_j)\right]^{-1}\widehat{P}^T(0)\,\widetilde{Q}^T & I_{mN} \end{bmatrix}, \tag{5.1.11b}$$

where $\widetilde{Q}$ is the product operational matrix for the function

$$\begin{bmatrix} 0 \ldots\ldots\ldots 0 \\ \ldots . q(t) \ldots . \\ 0 \ldots\ldots\ldots 0 \end{bmatrix},$$

where $q(t)$ occupies the position j, i.

Then, for any t such that $\tau_q < t < \tau_{q+1}$ the following equality holds:

$$\begin{aligned} &\Xi \left[x_0^T \ x_1^T \ldots x_{m-1}^T \ w_0^T \ldots w_{m-1}^T\right]^T \\ &= \left[x_0^{*T} \ x_1^{*T} \ldots x_{m-1}^{*T} \ w_0^{*T} \ldots w_{m-1}^{*T}\right]^T . \end{aligned} \tag{5.1.12}$$

To say $\Xi X^T = X^{*T}$.

Lemma 5.1.2 (Point-delay interconnection) (Alastruey and González de Mendívil, 1994b). *Let us consider the same conditions than in Lemma 5.1.1, including the invertibility of submatrix L (Eq. 5.1.11a), but in this case the interconnection being $< x_i(t-\tau_k)q(t)w_j(t) >$, to say, taking the inner signal x_i with delay τ_k time units, multiplied by amplitude $q(t)$ and applied as input for an integrator with output being $w_j(t)$. Let us consider the operator*

$$\begin{aligned} &\Xi = \\ &= \begin{bmatrix} I_{mN} & \bar{0} \\ \left[I_{mN} - \widehat{P}^T(0)\,\widetilde{H}^T - \sum_{j=1}^{q} \widehat{P}^T(\tau_j)\widetilde{K}_j^T\,\widehat{S}^T(\tau_j)\right]^{-1} \widehat{P}^T(\tau_k)\,\widetilde{Q}^T\,\widehat{S}^T(\tau_k) & I_{mN} \end{bmatrix} . \end{aligned} \tag{5.1.13}$$

Then, for any t such that $\tau_q < t < \tau_{q+1}$ the operator (5.1.13) is an interconnector in Taylor series representation for the two subsystems (5.1.4) and (5.1.5), with the interconnection $< x_i(t-\tau_k)q(t)w_j >$.

Note that, under its present form, the method is applicable to the case of multiple interconnections only if these interconnections do not create a loop between the subsystems affected by them. For instance, suppose the global system being composed of three subsystems

$$\begin{aligned} x &= \left\{x^1(t) \ldots x^N(t)\right\}, \\ w &= \left\{w^1(t) \ldots w^N(t)\right\}, \\ y &= \left\{y^1(t) \ldots y^N(t)\right\}, \end{aligned} \tag{5.1.14}$$

defined by state equations (5.1.4), (5.1.5) and

$$\dot{y}(t) = M(t)y(t) + \sum_{j=1}^{r} N_j(t)y(t-\tau_j). \tag{5.1.15}$$

It can be easily proven that the method is still valid considering the connections $< x_i q_k(t) w_j > < w_j q_h(t) y_l >$. In this case the interconnector would be computed as the product of two interconnectors corresponding to the two proposed connections, i.e, $\Xi = \Xi_h \Xi_k$. These two interconnectors will be computed in a similar way than that shown in Lemma 5.1.1, but now taking into account the existence of three subsystems, and thus considering the following state vector for the global system

$$X^T(t) = \left[x^1(t)\ldots x^N(t)\ w^1(t)\ldots w^N(t)\ y^1(t)\ldots y^N(t)\right]^T. \tag{5.1.16}$$

More specifically (Alastruey and González de Mendívil, 1994b):

$$\Xi_h = \tag{5.1.17}$$

$$= \begin{bmatrix} I_{mN} & \bar{0} & \bar{0} \\ \bar{0} & I_{mN} & \bar{0} \\ \bar{0} & \left[I_{mN} - \widehat{P}^T(0)\,\widetilde{M}^T - \sum_{j=1}^{q} \widehat{P}^T(\tau_j)\,\widetilde{N}_j^T\,\widehat{S}^T(\tau_j)\right]^{-1} \widehat{P}^T(0)\,\widetilde{Q}_h^T & I_{mN} \end{bmatrix},$$

$$\Xi_k = \tag{5.1.18}$$

$$= \begin{bmatrix} I_{mN} & \bar{0} & \bar{0} \\ \left[I_{mN} - \widehat{P}^T(0)\,\widetilde{H}^T - \sum_{j=1}^{q} \widehat{P}^T(\tau_j)\,\widetilde{K}_j^T\,\widehat{S}^T(\tau_j)\right]^{-1} \widehat{P}^T(0)\,\widetilde{Q}_k^T & I_{mN} & \bar{0} \\ \bar{0} & \bar{0} & I_{mN} \end{bmatrix},$$

where $\widetilde{M}$, $\widetilde{N}_j$, $\widetilde{Q}_h$, $\widetilde{Q}_k$ are respectively the Taylor product operational matrices for the functions

$$M(t),\ N_j(t),\ \begin{bmatrix} 0 \ldots\ldots\ldots 0 \\ \ldots . q_h(t) \ldots . \\ 0 \ldots\ldots\ldots 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \ldots\ldots\ldots 0 \\ \ldots . q_k(t) \ldots . \\ 0 \ldots\ldots\ldots 0 \end{bmatrix}, \tag{5.1.19}$$

where $q_h(t)$ occupies the position l, j and $q_k(t)$ occupies the position j, i. A similar way can be followed when there are two delayed interconnections (by using Lemma 5.1.2) or one delayed interconection and one free-delay interconnection, or a greater number of subsystems with more connections without a loop.

However, if one considers several connections which create a loop between two or more subsystems, Lemmas 5.1.1 and 5.1.2 are not applicable. For instance, if one considers a loop $< x_i q_k(t) w_j > < w_j q_k(t) x_j >$, it is obvious that it changes the dynamic properties of the whole system. For the cases of a set of interconnections creating such a loop, the difficulty can be overcome in part by considering all the subsystems affected by such loop as a new greater subsystem, and by studying the dynamics of this new subsystem separately.

For systems that can be modelled by using polynomial matrices containing only positive powers of t or can be reduced to such models (which is the case in many practical examples) the method is applicable in its present form for any $t > 0$ by considering m (the number for terms in the Taylor series expansions) equal to $pmax + 1$, where $pmax$ is the maximum power of t appearing in the model. This is obvious, since $a_{m-1}t^{m-1}$ is the term with maximum power appearing in a troncated Taylor expansion. In these cases, several results have been established about under which conditions stability in the approximate system implies stability in the exact one (see, for instance, Alastruey *et al.*, 1992). In the general case, however, the troncation of Taylor series for $t > 1$ is not possible. Then arises the question of how to use the method for stability study or control design.

The present method, as was previously pointed out, is mainly based on the results given by Razzaghi and Razzaghi (1989) by using Taylor series. Similar research on time-delay systems has been done by many authors using polynomial series. In particular, Kung and Lee (1983) developed a similar formulation by using Laguerre polynomials, and Shyu (1984) did it also by using Hermite polynomials. Laguerre polynomials are applicable to the interval 0 to ∞, and Hermite polynomials are applicable to the interval $-\infty$ to ∞. It is known that the finite troncation of these two kinds of polynomial expansions is possible in all the interval if enough terms are considered in the expansions, depending on the original time-functions.

The method proposed here can be rewritten by using Laguerre or Hermite formulation (including Laguerre or Hermite operators of product, integration and delay) and therefore conditions can be found to relate the number of terms in the polynomial expansions of the time functions with the validity of the approximations with respect to stability implications or control design strategies. Furthermore, in a work due to Chen and Yang (1987), polynomial delay matrices (including those of Hermite and Laguerre polynomials) were derived via a transformation technique which involved the transformation of the time-delay matrix of the Taylor series into time-delay matrices of polynomial series. Therefore, two results similar to Lemmas 1 and 2 can be formulated for Laguerre or Hermite operators of product, integration and delay, and consequently interconnectors can be computed giving an approximation of the state vector for the interconnected global system. The approximation would be valid – under certain circumstances – within the intervals 0 to ∞ or $-\infty$ to ∞, respectively.

To choose m is a crucial question when applying the method to a particular system. In practice there are cases when an addition of terms in the Taylor expansions does not improve the accuracy degree of the approximations, a circumstance to avoid when dealing with computer-processing-speed restrictions or on-line applications; on the other hand, if one chooses m lower than the particular needs of the problem, the approximate solution is very poor. When the dominant time-functions appearing in the state matrices can be reduced to equivalent polynomials of positive powers of t, then the best choice is m equal to $pmax + 1$, where $pmax$ is the maximum power of t appearing in the equivalent polynomials. On the contrary, if not all the dominant time-functions appearing in the state matrices are reducible to equivalent polynomials of positive powers of t then a more detailed study is required. The approach undertaken by the authors comprises several steps (Alastruey and González de Mendívil, 1994b), that can be easily implemented in software:

1. Determine M, the maximum value for m in terms of computer- or on-line- constraints.
2. Determine which are the dominant time-functions in the state matrices.
3. Compute the Taylor expansions for the dominant time-functions up to the M-th term.
4. Evaluate separately (by powers) the terms of every expansion, for a fixed value (i.e., 0.9).
5. For every expansion, compute the ratios of the values computed in step 4, between a term and the following, starting with the term with power zero, for every element of the state vector.
6. Choose $m - 1$ equal to the maximum term-power for which the ratios computed in step 5 are under the value k, where k is a design constant.

5.2. Non-linear delayed systems. One of the difficulties in dealing with non-linear systems is the lack of unified mathematical theory for representing various non-linear-system characteristics. It is therefore necessary to specify the system representation before carrying out the analysis and identification of non-linear systems, as was pointed out by Kung and Shih (1986). Those authors considered the Hammerstein model (Narenda and Gallman, 1966), which consists of a zero-memory non-linear element followed by a linear delay plant. For system analysis, the variables of the non-linear model were expanded into a finite-dimensional block-pulse series so that a non-linear time-delayed state equation was reduced to a set of linear algebraic equations. For system identification, through the block-pulse expansions of the measured input-output data, the unknown parameters of the linear part and coefficients of the polynomial representation of the non-linear element can be estimated using the least-squares method.

Controllability of non-linear delayed systems has been studied by many authors by using the Schauder's fixed-point theorem (Balachandran and Dauer, 1987). The approximate controllability of on-linear systems with delays in their states and control has been examined by Sinha (1986). If the uncontrolled system is asymptotically stable and if the linear part of the control system is controllable, then the non-linear delay system is approximately controllable. An upper bound on the magnitude of retardation for the system to be controllable was estimated in that work. Conditions were placed on the delay value and on a perturbation function, which represented the nonlinearity. A general method to derive Lyapunov functions for non-linear systems was developed by Chin (1986), a method that could be applied, under certain circumstances, to non-linear delayed systems.

Finally, the optimal control for a class of non-linear multiple-delay systems was undertaken by Balachandran (1989). This author proved the existence theorems for the optimal control of non-linear multiple-delay systems having an implicit derivative with quadratic performance criteria by suitably adopting the techniques of Dacka (1980).

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SISTEMŲ SU VĖLINIMU VALDYMAS

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Straipsnyje pateikiama dinaminių sistemų su vėlinimu apžvalga. Aptariami šias sistemas aprašančių skirtuminių lygčių sprendimo metodai, nagrinėjami jų ribojimai ir galimybės. Pateikiamas gana bendro pavidalo tiesinės sitemos su vėlinimu lygties skleidimo Teiloro eilute metodas, nagrinėjamas sistemų su vėlinimu stabilumas. Taip pat pateikiama tarpusavyje surištų sistemų su vėlinimu interpretacija, apžvelgiami darbai, nagrinėjantys netiesines sistemas su vėlinimu. Apžvalgoje nurodomos pagrindinės lygčių sprendimo problemos remiantis paskutinių dviejų dešimtmečių žinomais darbais šioje srityje.