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Strict Uncertainty Analysis with Fuzzy Payoffs and its Application to Portfolio Selection

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Abstract. Decision-making under strict uncertainty involves evaluating a set of alternatives without knowledge of the probability of scenarios using crisp evaluations. Our work reformulates traditional decision rules to a fuzzy environment, retaining the interpretability of classical principles while incorporating imprecision. Our methodological proposal provides a unified, flexible, and mathematically consistent framework for decision-making under imprecise payoffs. We adapt a total ordering mechanism for trapezoidal fuzzy numbers and admissible interval orders. Our application case study to portfolio selection under fuzzy strict uncertainty demonstrates how the proposed fuzzy generalization can handle financial imprecision and investor risk attitudes through ranking functions.

Key words: quantitative finance, fuzzy intervals, decision rules, moderate pessimism, portfolio selection.

1. Introduction

Decision-making can be categorized based on the availability of information about outcomes and their probabilities. In a deterministic environment, all relevant variables and consequences of actions are known with certainty, allowing decision-makers to predict outcomes precisely and choose the optimal course of action. In contrast, decision-making under risk involves situations where outcomes are uncertain but the probabilities of their occurrence are known or can be estimated, enabling the use of expected value or utility-based approaches to evaluate alternatives (Rapoport, 1998; Nakamori, 2025). Finally, decision-making under strict uncertainty arises when the decision-maker lacks sufficient information to assign probabilities to potential outcomes. This situation implies the need of applying different decision rules to guide choices (Ballestero, 2002).

In addition, instability, imprecision, and uncertainty are the rule rather than the exception in different decision-making contexts, and their impact is especially significant in economics and finance. When decision-making occurs in an environment with some degree of uncertainty (Figueira *et al.*, 2005; Pedrycz *et al.*, 2011; Miliauskaitė and Kalibatiene, 2025), we require special tools to find the appropriate solutions to economic and

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financial problems (Oderanti and De Wilde, 2010; Bu, 2024). Fuzzy optimization and decision-making describe the procedures and methods to deal with problems in which goals and constraints, but not necessarily the system under control, are vague (Bellman and Zadeh, 1970; Pedrycz *et al.*, 2011). Since Zadeh (1965) introduced fuzzy set theory, it has been extensively applied across various research areas dealing with uncertainty (Kimiagari and Keivanpour, 2019; Wang *et al.*, 2019). This approach is beneficial because, when experts provide approximate assessments using fuzzy values, achieving exact representations is unnecessary (Pedrycz, 1994; Delgado *et al.*, 1998). We here focus on using triangular fuzzy intervals as a suitable way to represent the payoffs or evaluations of a set of alternatives under different scenarios or future states of nature. The rationale behind this choice is that decision-makers can usually express payoffs using a maximum value, a minimum value, and a modal value as the most frequently occurring value.

In this paper, we address the problem of decision-making under strict uncertainty when the payoffs are imprecise or fuzzy. While some management and economic problems are characterized by deterministic equations, other related problems often involve uncertainty or stochastic processes (Gil-Aluja, 2004; Łyczkowska-Hanćkowiak and Piasecki, 2021). This complexity is further increased in scenarios with a complete lack of knowledge regarding potential outcomes, a circumstance referred to as strict uncertainty (Ballestero, 2002). Strict uncertainty is characterized by:

- A complete absence of knowledge concerning the probabilities associated with future states.
- A comprehensive information regarding the alternatives under consideration.
- An evaluation associated with each alternative in the event of a particular state.

To solve decision-making problems within the context of strict uncertainty, various decision rules have been devised integrating principles such as optimism, pessimism, moderate pessimism or regret minimization. Examples include the Laplace principle of insufficient reason (Laplace, 1825); the Wald maximin rule (Wald, 1950), the Hurwicz optimism-pessimism balance criterion (Hurwicz, 1951), the Savage minimax regret criterion (Savage, 1951), and the Ballestero moderate pessimism criterion (Ballestero, 2002). Other rules, such as the maximin joy criterion (Hayashi, 2008), the dominance joy criterion and the cumulative maximin joy criterion (Gaspars-Wieloch, 2014), and the criterion of extreme optimism (Shestakevych and Volkov, 2021), as summarized in Table 2, can also be considered. However, these rules operate only on crisp evaluations.

To solve this limitation, several works aimed to integrate the concepts of strict uncertainty and fuzzy sets. Nikolova *et al.* (2005) and Tenekedjiev *et al.* (2006) implemented the concept of fuzzy rationality in probabilities through the construction of ribbon distribution functions. Later on, Tenekedjiev and Nikolova (2008) presented methods to rank fuzzy-rational lotteries. The authors relied on decision rules under strict uncertainty to eliminate unquantified uncertainty by transforming fuzzy-rational lotteries into classical probability elicitation. Our work departs from this research line by respecting the complete absence of knowledge about probabilities of future states.

More precisely, we reformulate the classical decision rules to a fuzzy environment retaining the interpretability of principles while incorporating imprecision. We develop a methodological advancement that delivers a unified, adaptable, and mathematically consistent framework for decision-making under imprecise payoffs. To this end, we tackle the decision-making problem in strict uncertainty when a fuzzy interval describes the evaluation function for the combination of every alternative and future state. We adapt a total ordering mechanism for trapezoidal fuzzy numbers based on α-cuts and admissible interval orders. As an additional result, we examine the main properties of fuzzy decision rules under strict uncertainty. The main advantage of fuzzy strict uncertainty is the possibility of enriching decision-making processes in contexts where instability and imprecision add a new level of complexity.

We illustrate how our fuzzy strict uncertainty approach can facilitate solving financial problems. For instance, when considering alternative investments and possible future states, investors may consider all possible combinations of expected returns and volatility to elicit the payoffs of investments as described in Ballestero *et al.* (2007). Alternatively, they can integrate volatility as the membership function of a fuzzy number as an expression of the payoff for different scenarios: low, medium, and high return.

It is worth noting that the proposed fuzzy strict uncertainty framework can be compared and integrated with recent extensions of fuzzy set theory and fuzzy multi-criteria decision-making (MCDM) methods (Kahraman *et al.*, 2015). Specifically, our framework complements advanced fuzzy representations, including intuitionistic, type-2, hesitant fuzzy sets and their other variants (Bustince *et al.*, 2015), by providing a foundational structure in which classical strict uncertainty rules operate consistently within fuzzy environments. This compatibility allows meaningful comparison with contemporary fuzzy MCDM techniques such as fuzzy TOPSIS (Salih *et al.*, 2019), VIKOR (Jana *et al.*, 2023), WASPAS (Turskis *et al.*, 2015) and other extensions. Conceptually, the fuzzy strict uncertainty rules can be regarded as special or limiting cases of fuzzy MCDM aggregation, applicable when scenario probabilities or criteria weights are unknown. Consequently, our approach establishes a unifying layer that connects traditional decision rules (e.g. Laplace, Wald, Hurwicz, Ballestero, maximin joy) with modern fuzzy frameworks, facilitating coherent extensions to more complex forms of uncertainty.

In summary, the main contributions of this paper are the following:

1. We extend the concept of strict uncertainty to a fuzzy context. As a direct consequence of the inability to always compute exact values for payoffs in real-world problems, we propose using fuzzy intervals as a suitable approximation method.
2. We describe new fuzzy decision rules and their main properties. We provide a formal definition of fuzzy strict uncertainty to develop new fuzzy decision-making rules and study the properties of the new rules.
3. We illustrate the application of these rules to solve a quantitative finance problem. More precisely, we describe a portfolio selection case study as a specific financial scenario where our approach can be applied to deal with the inherent imprecision of future payoffs.

This paper is structured in the following sections. Section 2 contains preliminary definitions. Section 3 describes our fuzzy approach to decision-making under strict uncertainty. Section 4 illustrates the application of fuzzy strict uncertainty to solve a portfolio selection problem. Finally, Section 5 concludes by highlighting natural extensions of this work.

2. Preliminaries

This section briefly describes the basic concepts of strict uncertainty and classical decision rules for making decisions in such scenarios, along with their desirable properties. Afterward, the basic concepts of fuzzy sets and their arithmetic operations are illustrated.

2.1. *Strict Uncertainty and Decision Rules*

Definition 1 (*Strict uncertainty* (Ballestero, 2002)). A decision-making problem in strict uncertainty is characterized by:

1. A finite set of alternatives $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$.
2. A finite set of scenarios $\mathcal{C} = \{r_1, r_2, \dots, r_n\}$.
3. A scalar evaluation $V : \mathcal{A} \times \mathcal{C} \to \mathbb{R}$ of each alternative for each scenario.

As a result, a decision-making problem in strict uncertainty can be summarized in a decision table as shown in Table 1.

The decision-making literature proposes several criteria for solving decision problems from the information in a decision table. Table 2 summarizes the most relevant decision rules under strict uncertainty, including the underlying decision-making principle followed by each rule. The Laplace score function L_i (Laplace, 1825) aggregates evaluations for each alternative over the whole range of scenarios and selects the alternative with the maximum aggregated value. The Wald maximin rule (Wald, 1950) assumes that the worst scenario will occur. Then, the decision-maker focuses only on the minimum evaluation for each alternative and selects the alternative with the maximum W_i over the minimum evaluations. The Hurwicz criterion (Hurwicz, 1951) is characterized by parameter α that describes the attitude towards pessimism and optimism of a decision-maker who ultimately selects the alternative with maximum H_i. The Savage minimax regret criterion (Savage, 1951) requires the consideration of minimum regret S_i, defined as the difference

Table 1
Decision table.

Alternatives	Scenario					
	r_1	r_2	$\dots$	r_j	$\dots$	r_n
a_1	V_{11}	V_{12}	$\dots$	V_{1j}	$\dots$	V_{1n}
a_2	V_{21}	V_{22}	$\dots$	V_{2j}	$\dots$	V_{2n}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_i	V_{i1}	V_{i2}	$\dots$	V_{ij}	$\dots$	V_{in}
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
a_m	V_{m1}	V_{m1}	$\dots$	V_{m1}	$\dots$	V_{mn}

Table 2
Decision rules.

Reference	Principles	Evaluation of alternatives	Selection
Laplace (1825)	Insufficient reason	$L_i = \sum_{j=1}^{n} V_{ij}$	$\max_i L_i$
Wald (1950)	Pessimism	$W_i = \min_j (V_{ij})$	$\max_i \ W_i$
Hurwicz (1951)	Optimism-pessimism	$H_i = \alpha \cdot \min_j (V_{ij}) + (1-\alpha) \max_i (V_{ij})$	$\max_i \ H_i$
Savage (1951)	Minimax regret	$S_i = \max_j (\max_i (V_{ij}) - V_{ij})$	$\min_i S_i$
Ballestero (2002)	Moderate pessimism	$B_i = \sum_{j=1}^{n} \frac{V_{ij}}{\max_i (V_{ij}) - \min\limits_i (V_{ij})}$	$\max_i \ B_i$
Hayashi (2008)	Maximin joy	$J_i = \min_j (V_{ij} - \min_i (V_{ij}))$	$\max_i \ J_i$
Gaspars-Wieloch (2014)	Dominance joy	$D_i = \sum_{j=1}^{n} (m - p_j (V_{ij}))$	$\max_i \ D_i$
Gaspars-Wieloch (2014)	Cumulative maximin joy	$C_i = \min_j (m \cdot V_{ij} - \sum_{i=1}^{m} V_{ij})$	$\max_i \ C_i$
Shestakevych and Volkov (2021)	Extreme optimism	$O_i = \max_j (V_{ij})$	$\max_i \ O_i$

Note: $p_j(V_{ij})$ is the position of evaluation V_{ij} in the non-increasing sequence of evaluations in scenario j. When V_{ij} is equal to at least one other payoff, the farthest position in the sequence is used.

between the best evaluation in the scenario and the particular evaluation of each alternative in this scenario. Finally, the Ballestero moderate pessimism rule (Ballestero, 2002) implies that the larger the range of evaluations for each scenario, the higher the distrust of the decision-maker towards the scenario. As a result, a set of weights inversely proportional to the range of evaluations for each scenario is used to adjust evaluations in score function B_i.

Hayashi (2008) proposed the maximin joy criterion, as a reciprocal of the Savage minimax regret criterion. As an extension of the maximin joy criterion, Gaspars-Wieloch (2014) introduced the dominance joy criterion and the cumulative maximin joy criterion. More recently, Shestakevych and Volkov (2021) described the criterion of extreme optimism in which the maximum of maximum payoffs is considered to select the best alternative. Other criteria such as the Bayesian criterion, the Hermeyer criterion and the Hodge-Lehman criterion, as defined in Shestakevych and Volkov (2021), imply the assumption of probabilities for scenarios. This assumption is out of the scope of this paper because we focus on strict uncertainty, characterized by a complete absence of knowledge concerning the probabilities associated with future states.

2.2. *Fuzzy Sets and Operations*

Since the pioneering work by Zadeh (1965), the theory of fuzzy sets has been developed in several research fields. In this section, we provide a brief introduction to some basic concepts in fuzzy set theory.

Definition 2. Let $X \subset \mathbb{R}$ be a non-empty reference set. A fuzzy set A is defined by its membership function $\mu_A : X \to [0, 1]$. For any $x \in X$, $\mu_A(x)$ is the degree of membership of x in fuzzy set A. Further,

- the support of A is defined as the set $supp(A) = \{x \in X \mid \mu_A(x) > 0\}$;
- the core of A is defined as the set $core(A) = \{x \in X \mid \mu_A(x) = 1\}$;
- the height of A is the largest membership degree such that $h(A) = \sup_{x \in X} \mu_A(x)$. A is said to be normal if $h(A) = 1$;
- for $\alpha \in (0, 1]$, the α-cut A is the set $A_\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$.

Definition 3 (*Fuzzy number*). A fuzzy set A over the real line $\mathbb{R}$, $\mu_A : \mathbb{R} \to [0, 1]$ is said to be a fuzzy number if the following properties hold:

- A is normal;
- for any $\alpha \in (0, 1]$, A_α is a closed interval;
- the $supp(A)$ is bounded.

Within the class of fuzzy numbers, the trapezoidal fuzzy number is most commonly used to quantify fuzzy evaluation in the decision-making process. The motivation behind their utilization comes from the simplicity of these membership functions (Delgado *et al.*, 1998), and their characterization requires reasonably limited information. Therefore, the definitions of the triangular and trapezoidal fuzzy numbers and operational laws (Klir and Yuan, 1995; Pedrycz *et al.*, 2011), are briefly provided below.

Definition 4. A trapezoidal fuzzy number (TrFN) $Tr = (a, b, c, d)$ with four parameters a, b, c, d ($a \leqslant b \leqslant c \leqslant d$) is a special fuzzy set on the real line $\mathbb{R}$ and described through piecewise linear membership function μ_{Tr} as follows:

$$\mu_{Tr}(x) = \begin{cases} \dfrac{(x-a)}{(b-a)}, & \text{for } a \leqslant x \leqslant b, \\ 1, & \text{for } b \leqslant x \leqslant c, \\ \dfrac{(d-x)}{(d-c)}, & \text{for } c \leqslant x \leqslant d, \\ 0, & \text{for otherwise.} \end{cases} \tag{1}$$

Let us denote the set of all trapezoidal fuzzy numbers on the real line $\mathbb{R}$ by $\mathcal{F}(\mathbb{R})$. The computation between the trapezoidal fuzzy numbers $Tr_1 = (a_1, b_1, c_1, d_1)$ and $Tr_2 = (a_2, b_2, c_2, d_2)$ from $\mathcal{F}(\mathbb{R})$ could be facilitated with the help of the following arithmetic operational laws:

- Addition: $Tr_1 \oplus Tr_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$.
- Subtraction: $Tr_1 \ominus Tr_2 = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2)$.
- Scalar multiplication: $r \odot Tr_1 = (ra_1, rb_1, rc_1, rd_1)$ for $r > 0$.

Note that the set of fuzzy numbers does not possess a natural order. Therefore, we require a mechanism to order or rank fuzzy numbers. Several methods have been proposed in the literature to rank or generate an order for fuzzy numbers. These methods can be classified into three main categories (Wang and Kerre, 2001; Yatsalo and Martínez, 2018):

1. Defuzzification-based ranking methods have been widely used in the literature for their simplicity. In such methods, fuzzy numbers are substituted for their corresponding crisp numbers, computed differently, with their subsequent ranking as in Yager (1981) or Gu and Xuan (2017). This class includes lexicographic methods whose order is established by an algorithm (Wang *et al.*, 2005; Farhadinia, 2009).
2. Ranking methods based on the distance to a reference set in which a reference set is defined and each fuzzy number is evaluated by comparing its distance to the reference set. One standard method of this class for ranking fuzzy numbers is based on the fuzzy maximum function (Wang and Kerre, 2001).
3. Ranking methods based on pairwise comparisons aim to order fuzzy quantities by pairwise comparisons and is the most extensively explored approach. These ranking methods construct a fuzzy preference relation for pairwise comparisons among the fuzzy numbers (Yatsalo and Martínez, 2018).

Though there are various methods to produce the ordering among the fuzzy numbers, most can generate only partial order. More precisely, these methods do not warrant the anti-symmetry property of the order relation. In this paper, we focus on the ordering mechanism that generates the total order of $\mathcal{F}(\mathbb{R})$.

In this regard, Zumelzu *et al.* (2022) introduced the total order of the fuzzy numbers based on the α-cut of the fuzzy numbers. It can overcome the drawback of defuzzification-based ranking methods. The key principle of this ordering mechanism is to compare the α-cuts of the fuzzy numbers, which are intervals. Therefore, it is necessary to introduce the concept of total ordering of intervals.

Let $\mathcal{I}(\mathbb{R}) = \{[a,b] : (a,b) \in \mathbb{R}^2, a \leqslant b\}$ be all the close and bounded sub-intervals of $\mathbb{R}$. Consider the usual lexicographic order on the $\mathbb{R}^2$ given by $(a,b) \geqq (c,d) \Leftrightarrow a \geqslant c \wedge b \geqslant d$. The order relation $\geqq$ is a partial order relation on $\mathbb{R}^2$. Further, the relation $\geqq$ induces the partial order of $\mathcal{I}(\mathbb{R})$. We denote this partial order relation by $\geqslant_2$ and interpret it as

$$[a,b] \geqslant_2 [c,d] \Leftrightarrow a \geqslant c \wedge b \geqslant d.$$

Bustince *et al.* (2013) introduced the notion of admissible order for intervals, which is linear and refined or encompasses a partial order.

Definition 5 (Bustince *et al.*, 2013). Let $\succcurlyeq$ be a relation on $\mathcal{I}(\mathbb{R})$. Then, the relation $\succcurlyeq$ is said to be an admissible order relation if it satisfies:

- $\succcurlyeq$ is a linear order on $\mathcal{I}(\mathbb{R})$; and
- for any $[a,b]$ and $[c,d]$ in $\mathcal{I}(\mathbb{R})$, $[a,b] \succcurlyeq [c,d]$ whenever $[a,b] \geqslant_2 [c,d]$.

Example 1. Some examples of the admissible order are:

1. $[a,b] \succcurlyeq_{Lex1} [c,d] \Leftrightarrow a > c \vee (a = c \wedge b \geqslant d)$.
2. Xu-Yager ordering: $[a,b] \succcurlyeq_{XY} [c,d] \Leftrightarrow a+b > c+d \vee (a+b = c+d \wedge b-a \geqslant d-c)$.

Another concept that enables us to compare two fuzzy numbers in terms of the α-cut is the representation of fuzzy numbers using α-cuts, specifically in terms of an upper dense sequence of α-cuts. Let $S = (\alpha_i)_{i\in\mathbb{N}}$ be a sequence in $[0, 1]$. Then, S is said to be an upper dense sequence if, for every $x \in [0, 1]$ and any $\epsilon > 0$, there exists $i \in \mathbb{N}$ such that $\alpha_i \in [x, x + \epsilon[$. It is noted that a fuzzy number can be represented via an upper dense sequence of α-cuts (Wang *et al.*, 2005):

$$Tr = \bigcup_{\alpha_i \in S} Tr_{\alpha_i}.$$

Based on the admissible order relation of intervals and upper dense sequence of α-cuts representation of fuzzy numbers, we can compare whether all α-cuts of two fuzzy numbers are equal or there exists an α-cut that dominates based on admissible order relation.

Definition 6 (Zumelzu *et al.*, 2022). Let $Tr_1, Tr_2 \in \mathcal{F}(\mathbb{R})$ and $S = (\alpha_i)_{i\in\mathbb{N}}$ be an upper dense sequence in $(0, 1]$. For an admissible order $\succcurlyeq$ on $\mathcal{I}(\mathbb{R})$, we define an order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ as $Tr_1 \succeq Tr_2 \Leftrightarrow (Tr_1 = Tr_2) \ \vee \ (Tr_{1_{\alpha_{m_0}}} \succ Tr_{2_{\alpha_{m_0}}})$ where $m_0 = \min\{i : \alpha_i \in S, Tr_{1_{\alpha_i}} \neq Tr_{2_{\alpha_i}}\}$ if $Tr_1 \neq Tr_2$ and $m_0 = 0$, then $Tr_1 = Tr_2$.

It could be easily verified that the order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ induces a total order of the trapezoidal fuzzy numbers.

Note that although in Definition 6, we have mentioned that we need to consider an upper dense sequence of α-cuts to decide the order of any two trapezoidal fuzzy numbers. However, in practice, the trapezoidal fuzzy number could be fully characterized by its support, i.e. 0-cut, and core, i.e. 1-cut. Further, the two trapezoidal fuzzy numbers would be equal only when these two α-cuts are equal. Therefore, the condition of using an upper dense sequence of α-cuts for ordering the trapezoidal fuzzy numbers in Definition 6 could be reduced to the condition of using only two special α-cuts $\bar{S} = \{0, 1\}$.

Formally, two trapezoidal fuzzy numbers Tr_1 and Tr_2 are said to be equal $Tr_1 = Tr_2$ iff $supp(Tr_1) = supp(Tr_2)$ and $core(Tr_1) = core(Tr_2)$. Further, the condition for the order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ could be simplified as follows:

$$Tr_1 \succeq Tr_2 \Leftrightarrow (Tr_1 = Tr_2) \ \vee \ \big(core(Tr_1) \succ core(Tr_2)\big) \vee \big(supp(Tr_1) \succ supp(Tr_2)\big).$$

As the order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ induces a total order relation, any set $\mathcal{S} \subset \mathcal{F}(\mathbb{R})$ would be total ordered.

Definition 7 (*Total ordered set of fuzzy numbers*). Given set S of fuzzy numbers, S is a total ordered set if the following properties are satisfied:

1. Reflexivity. For all $\tilde{A} \in S$, $\tilde{A} \succeq \tilde{A}$.
2. Anti-symmetry. For all $\tilde{A}, \tilde{B} \in S$, if $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{A}$, then $\tilde{A} \sim \tilde{B}$.
3. Transitivity. For all $\tilde{A}, \tilde{B}, \tilde{C} \in S$, if $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$, then $\tilde{A} \succeq \tilde{C}$.
4. Comparability. For all $\tilde{A}, \tilde{B} \in S$, $\tilde{A} \succeq \tilde{B}$ or $\tilde{B} \succeq \tilde{A}$.

Now, we utilize the ordering relation $\succeq$ to define the minimum and maximum operation on a set of fuzzy numbers.

DEFINITION 8. Let $\mathcal{O} = (Tr_1, \dots, Tr_n) \in \mathcal{F}(\mathbb{R})^n$. The minimum operator of fuzzy numbers with respect to the order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ is a mapping $\tilde{\min} : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ such that: $\tilde{\min}(Tr_1, \dots, Tr_n) = Tr_k$, $Tr_i \succeq Tr_k$, $\forall\, i = 1, \dots, n$, $i \neq k$.

DEFINITION 9. Let $\mathcal{O} = (Tr_1, \dots, Tr_n) \in \mathcal{F}(\mathbb{R})^n$. The maximum operator of fuzzy numbers with respect to the order relation $\succeq$ on $\mathcal{F}(\mathbb{R})$ is a mapping $\tilde{\max} : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ such that: $\tilde{\max}(Tr_1, \dots, Tr_n) = Tr_k$, $Tr_i \preceq Tr_k$, $\forall\, i = 1, \dots, n$, $i \neq k$.

Aggregating fuzzy numbers to reach a final decision is critical in any decision-making process. Typically, the aggregation operations combine several fuzzy numbers and produce a single representative fuzzy number (Klir and Yuan, 1995). Formally, an aggregation function over trapezoidal fuzzy numbers of dimension n can be represented as a function, $\psi : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$.

DEFINITION 10. For a set of trapezoidal fuzzy numbers $(Tr_1, \dots, Tr_n) \in \mathcal{F}(\mathbb{R})^n$, the weighted fuzzy arithmetic mean operator $\widetilde{WA} : \mathcal{F}(\mathbb{R})^n \to \mathcal{F}(\mathbb{R})$ can be defined as: $\widetilde{WA}(Tr_1, \dots, Tr_n) = \bigoplus_{i=1}^n (w_i \odot Tr_i)$, where $w = (w_1, w_2, \dots, w_n) \in [0, 1]^n$ such that $w_i \geqslant 0$, $i = 1, \dots, n$ and $\sum_{i=1}^n w_i = 1$. Further, if $Tr_i = (a_i, b_i, c_i, d_i)$, $i = 1, \dots, n$, then it can be computed by utilizing the operational laws of trapezoidal fuzzy numbers and given by $\widetilde{WA}(Tr_1, \dots, Tr_n) = (\sum_{i=1}^n w_i a_i, \sum_{i=1}^n w_i b_i, \sum_{i=1}^n w_i c_i, \sum_{i=1}^n w_i d_i)$.

3. Strict Uncertainty with Fuzzy Payoffs

This section provides a formal definition of fuzzy strict uncertainty as an extension of the concept of strict uncertainty described in Section 2.1. We also develop new fuzzy decision-making rules and elaborate on the properties of these rules.

3.1. *Formal Definition*

DEFINITION 11 (*Strict uncertainty with fuzzy payoffs*). A decision-making problem in strict uncertainty with fuzzy payoffs is characterized by the following:

1. A finite set of alternatives $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$.
2. A finite set of scenarios $\mathcal{C} = \{r_1, r_2, \dots, r_n\}$.
3. A fuzzy payoff $\tilde{V} : \mathcal{A} \times \mathcal{C} \to \mathcal{F}(\mathbb{R})$ of each alternative for each scenario.

As a consequence of Definition 11, the combination of alternative a_i within scenario r_j results in fuzzy evaluation $\tilde{V}_{ij}$, following the same structure of Table 1. Again, for convenience, all fuzzy evaluations $\tilde{V}$ are assumed to be of the type the more the better.

3.2. *Fuzzy Decision Rules under Strict Uncertainty*

We next extend the decision rules described in Section 2.1 to a fuzzy environment. As a result of this extension, the Laplace (1825) criterion implies the use of fuzzy score function $\tilde{L}_i$:

$$\tilde{L}_i = \bigoplus_{j=1}^{n} \tilde{V}_{ij}. \tag{2}$$

The fuzzy maximin rule by Wald (1950) implies focusing on minimum evaluations for each alternative i over the n scenarios and selecting the alternative with maximum $\tilde{W}_i$:

$$\tilde{W}_i = \widetilde{\min}(\tilde{V}_{i1}, \ldots, \tilde{V}_{in}). \tag{3}$$

The fuzzy Hurwicz (1951) criterion uses parameter α to express the attitude towards pessimism and optimism to select the alternative with maximum $\tilde{H}_i$:

$$\tilde{H}_i = \alpha \odot \widetilde{\min}(\tilde{V}_{i1}, \ldots, \tilde{V}_{in}) \oplus (1-\alpha) \odot \widetilde{\max}(\tilde{V}_{i1}, \ldots, \tilde{V}_{in}). \tag{4}$$

The minimax regret fuzzy criterion by Savage (1951) computes the fuzzy regret for each combination and scenario to select the alternative with minimum $\tilde{S}_i$:

$$\tilde{S}_i = \widetilde{\max}(\tilde{V}_{i1}, \ldots, \tilde{V}_{in}) \ominus \tilde{V}_{ij}. \tag{5}$$

The extension of the Ballestero (2002) moderate pessimism rule to a fuzzy context requires the computation of fuzzy weights $\tilde{w}_j$:

$$\tilde{w}_j = \frac{1}{\max_j(\tilde{V}_{ij}) \ominus \min_j(\tilde{V}_{ij})} \tag{6}$$

provided that $supp(\max_j(\tilde{V}_{ij}) \ominus \min_j(\tilde{V}_{ij}))$ does not contain 0. However, this might be tedious to ensure in most of the cases, even when $\max_j(\tilde{V}_{ij})$ is not equal to $\min_j(\tilde{V}_{ij})$. To simplify this, we defuzzified the fuzzy values through the centre of gravity (COG) defuzzification method (Yager, 1978) and used these values to generate a weight proportional to the range of defuzzified values of the evaluations. Based on this, the weight could be computed as follows:

$$w_j = \frac{1}{COG(\max_j(\tilde{V}_{ij})) - COG(\min_j(\tilde{V}_{ij}))} \tag{7}$$

where, COG of a fuzzy number A with membership function $\mu_A : X \to [0, 1]$ is computed as $COG(A) = \int_X x\mu_A(x)dx / \int_X \mu_A(x)dx$. Finally, the decision maker aims to maximize score function $\tilde{B}_i$:

$$\tilde{B}_i = \bigoplus_{j=1}^{n} w_j \odot \tilde{V}_{ij}. \tag{8}$$

Note that all decision rules with fuzzy payoffs produce an output. When the fuzzy payoffs are represented by trapezoidal or triangular fuzzy numbers, the quantities $\tilde{L}_i$, $\tilde{W}_i$, $\tilde{H}_i$, $\tilde{S}_i$ and $\tilde{B}_i$ can be computed using the operational laws of fuzzy numbers described in Section 2. According to the previous decision rules, the decision-maker should choose the alternative with maximum $\tilde{L}_i$, $\tilde{W}_i$, $\tilde{H}_i$, $\tilde{B}_i$, and minimum $\tilde{S}_i$. The maximum and minimum values of fuzzy numbers can be obtained according to the ranking principle described in Section 2. Therefore, we obtain the desired ranking of the fuzzy numbers.

3.3. *Properties of Fuzzy Decision Rules under Strict Uncertainty*

Given a fuzzy decision table with fuzzy payoffs $\tilde{V}_{ij}$, with alternatives indexed by $i = \{1, 2, \ldots, m\}$ and scenarios indexed by $j = \{1, 2, \ldots, n\}$, a set of operational laws $\{\oplus, \ominus, \odot\}$ defined for trapezoidal fuzzy numbers on $\mathcal{F}(\mathbb{R})$, a set of minimum and maximum operators $\{\tilde{\min}, \tilde{\max}\}$, and an aggregation operator $\widetilde{WA}$ defined on $\mathcal{F}(\mathbb{R})^n$, we derive the following properties:

1. **Complete ranking**. One and only one numerical index is assigned to each alternative. By computing quantities $\tilde{L}_i$, $\tilde{W}_i$, $\tilde{H}_i$, $\tilde{S}_i$ and $\tilde{B}_i$ from $\tilde{V}_{ij}$, the application of the minimum and maximum operators $\{\tilde{\min}, \tilde{\max}\}$ induces a total order set of fuzzy numbers and a complete ranking.
2. **Independence of labelling**. Straightforwardly derived from the ranking procedure.
3. **Independence of the value scale**. It is sufficient to show that the order relation $\succeq$ over $\mathcal{F}(\mathbb{R})$ induced by admissible order $\succcurlyeq$ on intervals is independent of the scale.

   Let $\succcurlyeq$ be an admissible order over the set of intervals $\mathcal{I}(\mathbb{R})$. For an $\alpha > 0$ and $\beta \in \mathbb{R}$, it is said to be invariant to value scale,

   $$[a, b] \succcurlyeq [c, d] \iff \alpha[a, b] + \beta \succcurlyeq \alpha[c, d] + \beta,$$

   where scalar multiplication and addition operations on intervals will be performed according to Moore's interval arithmetic.

   Following this notion of independent of scale on admissible order, it is easy to show that $\succcurlyeq_{Lex1}$ and $\succcurlyeq_{XY}$ are independent of the scale.

   Let $Tr_1 = (a_1, b_1, c_1, d_1)$ and $Tr_2 = (a_2, b_2, c_2, d_2)$ be the two trapezoidal fuzzy numbers such that $Tr_1 \succeq Tr_2$ and $\succcurlyeq$ is the associated scale independence admissible order on $\mathcal{I}(\mathbb{R})$. This implies that

   $$(Tr_1 = Tr_2) \ \vee \ \big(core(Tr_1) \succ core(Tr_2)\big) \vee \big(supp(Tr_1) \succ supp(Tr_2)\big).$$

   Now, we consider the transformation of scale, specifically, the transformation of the discourse of fuzzy numbers through linear map $x \mapsto \alpha x + \beta$. Such transformation impacted the support of trapezoidal fuzzy numbers Tr_1 and Tr_2. Further, the transformed fuzzy numbers could be obtained via arithmetic operations on trapezoidal fuzzy numbers as $\alpha \odot Tr_1 + \beta = (\alpha a_1 + \beta, \alpha b_1 + \beta, \alpha c_1 + \beta, \alpha d_1 + \beta)$ and $\alpha \odot Tr_2 + \beta = (\alpha a_2 + \beta, \alpha b_2 + \beta, \alpha c_2 + \beta, \alpha d_2 + \beta)$. Thus, the core of translated fuzzy numbers $\alpha \odot Tr_1 + \beta$

and $\alpha \odot Tr_1 + \beta$ transformed into $[\alpha b_1 + \beta, \alpha c_1 + \beta] = \alpha[b_1, c_1] + \beta = \alpha Core(Tr_1) + \beta$ and $[\alpha b_2 + \beta, \alpha c_2 + \beta] = \alpha[b_2, c_2] + \beta = \alpha Core(Tr_2) + \beta$, which are nothing but the translation of the original cores of $[b_1, c_1]$ and $[b_2, c_2]$. The same is true for the supports of Tr_1 and Tr_2. Since the admissible order $\succcurlyeq$ on $\mathcal{I}(\mathbb{R})$ is independent of value scale,

$$[b_1, c_1] \succcurlyeq [b_2, c_2] \implies \alpha[b_1, c_1] + \beta \succcurlyeq \alpha[b_2, c_2] + \beta,$$

and so,

$$Core(Tr_1) \succcurlyeq Core(Tr_1) \implies \alpha Core(Tr_1) + \beta \succcurlyeq \alpha Core(Tr_1) + \beta$$

and

$$[a_1, d_1] \succcurlyeq [a_2, d_2] \implies \alpha[a_1, d_1] + \beta \succcurlyeq \alpha[a_2, d_2] + \beta.$$

Similarly, we obtain the relationship for support of translated trapezoidal fuzzy numbers

$$supp(Tr_1) \succcurlyeq supp(Tr_1) \implies \alpha supp(Tr_1) + \beta \succcurlyeq \alpha supp(Tr_1) + \beta.$$

From this, we can infer that

$$Tr_1 \succeq Tr_2 \implies \alpha \odot Tr_1 + \beta \succeq \alpha \odot Tr_2 + \beta.$$

As the ordering relation does not change with the transformation of the value scale, the ranking of the alternatives remains unaltered.

4. **Domination**. An alternative a_k is dominated by the $(\varphi_1, \ldots, \varphi_m)$ convex combination of alternatives if:

$$\bigoplus_{i=1}^{m} (\varphi_j \odot \tilde{V}_{ij}) \succeq \tilde{V}_{kj}, \quad \forall j = 1, 2, \ldots, n.$$

5. **Independence of irrelevant alternatives**. Straightforwardly derived from the ranking procedure.
6. **Independence of addition of a constant to a column**. Straightforwardly derived from the ranking procedure.
7. **Independence of row permutation**. Straightforwardly derived from the ranking procedure.
8. **Independence of column duplication**. Straightforwardly derived from the ranking procedure.

We leave more involved properties for subsequent work.

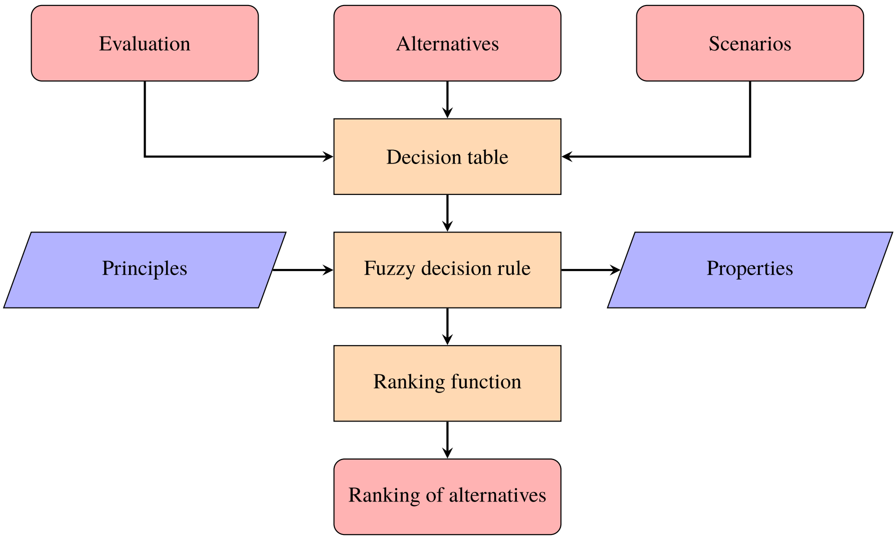

Fig. 1. A graphical representation of fuzzy decision-making under strict uncertainty.

3.4. *Integrating Fuzzy Strict Uncertainty and Decision-Making*

The decision-making process under strict uncertainty is graphically summarized in Fig. 1. The process begins by gathering information about evaluating alternatives under different scenarios. The main assumption is that no information about the probability of occurrence of any scenario is available. According to Table 2, decision-makers can deploy different guiding principles by selecting one of the rules in the table. For instance, the Wald rule is considered appropriate for a pessimistic decision-maker because it is assumed that the worst scenario will occur, while the Ballestero rule fits a moderately pessimistic attitude. In addition, selecting any of the rules (Laplace, Wald, Hurwicz, Savage, Ballestero, and possibly others) implies a set of reasonable properties as described in Section 2.1.

By extending the concept of strict uncertainty to a fuzzy context, we also extend decision-making principles and properties to provide a more general decision-making framework. For instance, the moderate pessimism principle remains the same but requires adaptation through a fuzzy rule. Similarly, we can study new properties of fuzzy decision rules under strict uncertainty as proposed in Section 3.3. Finally, using a fuzzy decision rule implies using a ranking function to derive an admissible order of fuzzy numbers, pointing out which alternative is best. A wide range of ranking functions can provide the desired order of fuzzy numbers. However, decision-makers can also integrate their attitude by selecting a ranking function. For instance, decision-makers with risk aversion to low values in an interval fuzzy evaluation may select a ranking function focusing first on the left end-points of fuzzy numbers rather than modal values.

To summarize, our decision-making framework for fuzzy strict uncertainty combines the principles and properties derived from strict uncertainty. Uncertainty and imprecision are inherent to many decision-making problems in economics and finance. Indeed, forecasting the results (payoffs) of a set of alternatives under different future scenarios is crucial. Because of the difficulty of computing exact values for payoffs in economics and finance problems, we propose using fuzzy intervals as a suitable approximation method. Consider, for instance, the problem investors face when evaluating a set of investing alternatives under different future scenarios. They will probably have problems establishing a precise evaluation of future payoffs. A suitable way to solve this problem is by approximating these payoffs through interval fuzzy numbers. As a result, they must select a fuzzy decision rule adapted to the context of strict uncertainty. To this end, they can rely on the decision-making principle underlying each decision rule described in this paper to make a choice. In addition, their risk attitudes can also be integrated into the decision-making process by establishing the ranking function that ultimately produces an admissible order of fuzzy numbers.

A further detailed formalization of our methodology for decision-making under strict uncertainty is shown in Algorithm 1.

Algorithm 1 Pseudo-code for fuzzy decision-making process under strict uncertainty

```
Input 1: A finite set of alternatives 𝒜 = {a_1, a_2, ..., a_m}
Input 2: A set of scenarios 𝒞 = {r_1, r_2, ..., r_n}
Output: An admissible ordered set S of fuzzy numbers
for a_i ∈ 𝒜 do
    for r_j ∈ 𝒞 do
        Compute Ṽ_ij                     ▷ Fuzzy payoff for every alternative and scenario
    end for
end for
Select score function R̃_i = f(Ṽ_ij) ∈ ℛ from {L̃_i, W̃_i, H̃_i, S̃_i, B̃_i}     ▷ Or any other proposal
Obtain set of fuzzy numbers ℛ = {R̃_1, ..., R̃_m}     ▷ Evaluation using a guiding rule
Define a ranking function g : ℛ × ℛ → S              ▷ To rank fuzzy numbers
Derive ordered set S from set ℛ and ranking function g
return S                                             ▷ Implicitly an ordered set of alternatives
```

4. Portfolio Selection under Fuzzy Strict Uncertainty

In this section, we first illustrate our approach through an application case study in a portfolio selection problem. Later, we discuss and analyse the implications derived from it.

4.1. *An Application in Portfolio Selection*

Portfolio selection is a critical task of financial decision-making that involves the allocation of a given budget to a set of assets to optimize returns while managing risks according to investors' preferences. This process consists of constructing a diversified portfolio that balances profitability and risk. Investors typically seek to maximize returns while minimizing the inherent uncertainties associated with financial markets. As a result, the fundamental principle of portfolio selection is rooted in the trade-off between risk and return (Markowitz, 1952). The concept of diversification implies the selection of an optimal portfolio that balances the expected returns of individual assets with their corresponding risks.

Furthermore, the inherent uncertainty of expected returns poses a significant challenge in portfolio selection. Traditional models often rely on historical data and assumptions about future returns, which may not accurately capture the dynamic nature of financial markets. As a result, including fuzzy payoff decision-making methods (Wang and Zhu, 2002; Bilbao-Terol *et al.*, 2006; Jalota *et al.*, 2023), such as the one described in this paper, represents a reliable way to account for the uncertainty surrounding expected returns. This paper delves into portfolio selection through the strict uncertainty procedure proposed in Ballestero *et al.* (2007). The proposed methodology comprises two main steps:

1. Computing the mean-variance frontier provides several pre-selected efficient portfolios (PEP).
2. Simulating the future performance of the PEPs as a way to rank them under uncertainty. In this paper, the second phase differs from the approach described in Ballestero *et al.* (2007) because we use a decision table with fuzzy payoffs.

Given the close price for a set of available assets included in the Dow Jones Industrial Average index from 2006-10-60 to 2020-12-31 as an input, we follow the next procedure to construct a decision table with fuzzy payoffs:

1. Obtain monthly returns for securities from historical data.
2. Calculate mean returns and the covariance matrix.
3. Compute the mean-variance efficient frontier and identify the pre-selected efficient portfolios (PEP).
4. Define several uncertain future scenarios for the market index.
5. Derive the returns on each stock using beta Sharpe's regression equation (Sharpe, 1964).
6. For each scenario, simulate the monthly market returns for the next year by assuming the normal distribution of mean and standard deviation defined in Step 4.
7. For the i-th PEP and the j-th scenario, compute the return performance mean value μ_{ij}, minimum value $\min_{ij}$, and maximum value $\max_{ij}$ over the monthly stock returns derived from the fitted Sharpe's equation. By introducing a triangular fuzzy performance, the dispersion from the mean is evaluated through the min and max values. Then, the fuzzy triangular payoff is defined as follows:

$$\tilde{V}_{ij} = \left(\min_{ij}, \mu_{ij}, \max_{ij}\right). \tag{9}$$

Once we have built a decision table with fuzzy payoffs summarized in Table 3, we can select the best portfolio according to the fuzzy strict uncertainty procedure described in Section 3.

Table 3
Decision table with fuzzy payoffs for 12 pre-selected efficient portfolios (PEP) and 9 scenarios.

PEP	Scenario 1	Scenario 2	Scenario 3	...	Scenario 9
1	$(-0.020, 0.003, 0.022)$	$(-0.052, 0.027, 0.096)$	$(-0.069, 0.040, 0.135)$	...	$(-0.049, 0.060, 0.155)$
2	$(-0.021, 0.003, 0.024)$	$(-0.055, 0.029, 0.101)$	$(-0.072, 0.042, 0.142)$	...	$(-0.051, 0.063, 0.163)$
3	$(-0.022, 0.003, 0.025)$	$(-0.057, 0.030, 0.105)$	$(-0.076, 0.044, 0.148)$	...	$(-0.054, 0.066, 0.170)$
4	$(-0.023, 0.003, 0.026)$	$(-0.060, 0.031, 0.110)$	$(-0.079, 0.046, 0.155)$	...	$(-0.056, 0.069, 0.178)$
5	$(-0.024, 0.003, 0.027)$	$(-0.062, 0.033, 0.115)$	$(-0.082, 0.048, 0.161)$	...	$(-0.058, 0.072, 0.185)$
6	$(-0.025, 0.004, 0.028)$	$(-0.065, 0.034, 0.120)$	$(-0.086, 0.050, 0.168)$	...	$(-0.061, 0.075, 0.193)$
7	$(-0.026, 0.004, 0.029)$	$(-0.067, 0.036, 0.124)$	$(-0.089, 0.052, 0.175)$	...	$(-0.063, 0.078, 0.200)$
8	$(-0.027, 0.004, 0.030)$	$(-0.070, 0.037, 0.129)$	$(-0.092, 0.054, 0.181)$	...	$(-0.066, 0.081, 0.208)$
9	$(-0.027, 0.004, 0.031)$	$(-0.072, 0.038, 0.134)$	$(-0.096, 0.056, 0.188)$	...	$(-0.068, 0.084, 0.216)$
10	$(-0.028, 0.004, 0.033)$	$(-0.075, 0.040, 0.139)$	$(-0.099, 0.058, 0.195)$	...	$(-0.070, 0.087, 0.223)$
11	$(-0.029, 0.004, 0.034)$	$(-0.077, 0.041, 0.143)$	$(-0.102, 0.060, 0.201)$	...	$(-0.073, 0.090, 0.231)$
12	$(-0.030, 0.005, 0.035)$	$(-0.000, 0.042, 0.148)$	$(-0.106, 0.062, 0.208)$	...	$(-0.075, 0.093, 0.239)$

To this end, we must select an admissible order described in Section 2.2. Investors are usually concerned about expected returns (represented here as the core of a triangular fuzzy number) and the volatility of returns (represented here as the support of a triangular fuzzy number). Moreover, a risky investor prioritizes returns over volatility (core over support), and a conservative investor prioritizes volatility over returns (support over core). Finally, investors are usually concerned with downside risk because they are more averse to returns below the mean value than to deviations above the mean value. As a result, given $Tr_1 = (a_1, b_1, c_1)$ and $Tr_2 = (a_2, b_2, c_2)$, and changing notation for economy of space ($C(Tr_1) = core(Tr_1)$ and $S(Tr_1) = supp(Tr_1)$), we consider the following ranking functions.

Ranking 1. For risky investors. Given $Tr_1, Tr_2 \in \mathcal{R}$, where $\mathcal{R}$ is a set of fuzzy numbers, focus first on the core and second on the support of alternative fuzzy payoffs.

$$Tr_1 \succeq Tr_2 \Leftrightarrow \big(C(Tr_1) \succcurlyeq C(Tr_2)\big) \vee \big(C(Tr_1) = C(Tr_2)\big) \wedge \big(S(Tr_1) \succcurlyeq S(Tr_2)\big)$$

interpreted within the context of investing and assuming the interval order $\succcurlyeq$ as $\succcurlyeq_{Lex1}$ the ordering conditions could be simplified as follows:

$$Tr_1 \succeq Tr_2 \Leftrightarrow (b_1 > b_2) \vee (b_1 = b_2) \wedge (a_1 > a_2) \vee (b_1 = b_2 \wedge a_1 = a_2) \wedge (c_1 \geqslant c_2).$$

Ranking 2. For conservative investors. Given $Tr_1, Tr_2 \in \mathcal{R}$, where $\mathcal{R}$ is a set of fuzzy numbers, focus first on the support and second on the core of alternative fuzzy payoffs.

$$Tr_1 \succeq Tr_2 \Leftrightarrow \big(S(Tr_1) \succcurlyeq S(Tr_2)\big) \vee \big(S(Tr_1) = S(Tr_2)\big) \wedge \big(C(Tr_1) \succcurlyeq C(Tr_2)\big)$$

interpreted within the context of investing and assuming the interval order $\succcurlyeq$ as $\succcurlyeq_{Lex1}$ the ordering conditions could be simplified as follows:

$$Tr_1 \succeq Tr_2 \Leftrightarrow (a_1 > a_2) \vee (a_1 = a_2) \wedge (c_1 > c_2) \vee (a_1 = a_2 \wedge c_1 = c_2) \wedge (b_1 \geqslant b_2).$$

Table 4
Laplace, Wald and Hurwicz fuzzy payoffs and rankings.

PEP	Laplace	Rank 1	Rank 2	Wald	Rank 1	Rank 2	Hurwicz	Rank 1	Rank 2
1	$(-0.333, 0.300, 0.849)$	12	1	$(-0.020, 0.003, 0.022)$	8	1	$(-0.035, 0.032, 0.089)$	12	1
2	$(-0.349, 0.316, 0.893)$	11	2	$(-0.021, 0.003, 0.024)$	9	2	$(-0.036, 0.033, 0.094)$	11	2
3	$(-0.366, 0.330, 0.933)$	10	3	$(-0.022, 0.003, 0.025)$	10	3	$(-0.038, 0.035, 0.098)$	10	3
4	$(-0.382, 0.345, 0.976)$	9	4	$(-0.023, 0.003, 0.026)$	11	4	$(-0.040, 0.036, 0.102)$	9	4
5	$(-0.396, 0.360, 1.017)$	8	5	$(-0.024, 0.003, 0.027)$	12	5	$(-0.041, 0.038, 0.106)$	8	5
6	$(-0.414, 0.376, 1.058)$	7	6	$(-0.025, 0.004, 0.028)$	2	6	$(-0.043, 0.040, 0.111)$	7	6
7	$(-0.429, 0.391, 1.100)$	6	7	$(-0.026, 0.004, 0.029)$	3	7	$(-0.045, 0.041, 0.115)$	6	7
8	$(-0.446, 0.406, 1.142)$	5	8	$(-0.027, 0.004, 0.030)$	5	9	$(-0.047, 0.043, 0.119)$	5	8
9	$(-0.461, 0.420, 1.185)$	4	9	$(-0.027, 0.004, 0.031)$	4	8	$(-0.048, 0.044, 0.124)$	4	9
10	$(-0.477, 0.436, 1.227)$	3	10	$(-0.028, 0.004, 0.033)$	6	10	$(-0.049, 0.046, 0.128)$	3	10
11	$(-0.492, 0.450, 1.268)$	2	11	$(-0.029, 0.004, 0.034)$	7	11	$(-0.051, 0.047, 0.133)$	2	11
12	$(-0.509, 0.466, 1.310)$	1	12	$(-0.030, 0.005, 0.035)$	1	12	$(-0.053, 0.049, 0.137)$	1	12

As a result, applying the Laplace rule with a fuzzy payoff as described in Section 3, we obtain the rankings summarized in Table 4 for risky (Ranking 1) and conservative (Ranking 2) investors. As expected, both rankings lead to an opposite evaluation of alternatives because the highest core of aggregated triangular payoffs comes in conjunction with the lowest endpoint of the support.

To apply the Wald rule with fuzzy payoffs, we need to find the minimum return for each PEP and scenario and select the PEP with the maximum return. For illustrative purposes and economy of space, we use Ranking 1 when applying the minimum operator from Definition 8 to find the minimum return, and then we use Rankings 1 and 2 when applying the maximum operator from Definition 9 to compare the ranking results for risky and conservative investors as shown in Table 4. The use of any other ranking operator is straightforward. In this case, we find ties when comparing the core of triangular payoffs, implying the use of the left and right support endpoints to obtain Ranking 1.

Similarly, using the minimum and maximum operators and setting $\alpha = 0.5$ to represent a decision-maker characterized by neutrality concerning optimism and pessimism, we use the Hurwicz fuzzy decision rule under strict uncertainty to derive the rankings. In the case of the Savage fuzzy rule, we first build a new decision table with fuzzy regrets by using the maximum fuzzy operator and the initial decision table. Next, we follow the minimax criterion to select the minimum fuzzy regret among the maximum regrets for each PEP and scenario in an opposite way of the maximin criterion by Wald. Finally, to apply the fuzzy Ballestero rule, we find the minimum and maximum fuzzy payoffs for each scenario to compute their respective weights by computing the COG of a triangular fuzzy number. Then, we apply these weights to obtain the score function in equation (8) and derive rankings. The Savage, Ballestero and the extreme optimism criterion, described in Shestakevych and Volkov (2021), ranking results are summarized in Table 5. Furthermore, the results derived from the maximin joy criterion by Hayashi (2008), the dominance joy and the cumulative maximin joy criteria by Gaspars-Wieloch (2014) are shown in Table 6.

To facilitate the comparison of alternative fuzzy decision rules, Figs. 2 and 3 show the resulting rankings using a heat map in which darker colours correspond to preferred PEP alternatives. Some insights that can be derived from the observation of these heat maps are: 1) Savage rankings are the opposite of the Hurwicz and Ballestero rankings;

Table 5
Savage, Ballestero and Extreme Optimism fuzzy payoffs and rankings.

PEP	Savage	Rank 1	Rank 2	Ballestero	Rank 1	Rank 2	Extreme Optimism	Rank 1	Rank 2
1	(−0.071, 0.057, 0.175)	1	12	(−26.5, 16.8, 53.9)	12	1	(−0.049, 0.060, 0.155)	12	1
2	(−0.075, 0.060, 0.184)	2	11	(−27.8, 17.6, 57.2)	11	2	(−0.051, 0.063, 0.163)	11	2
3	(−0.079, 0.063, 0.192)	3	10	(−29.2, 18.3, 59.8)	10	3	(−0.054, 0.066, 0.170)	10	3
4	(−0.082, 0.066, 0.201)	4	9	(−30.4, 19.1, 62.4)	9	4	(−0.056, 0.069, 0.178)	9	4
5	(−0.085, 0.069, 0.209)	5	8	(−31.7, 19.8, 65.0)	8	5	(−0.059, 0.073, 0.187)	8	5
6	(−0.089, 0.071, 0.218)	6	7	(−33.0, 21.2, 67.6)	7	6	(−0.061, 0.076, 0.194)	7	6
7	(−0.092, 0.074, 0.226)	7	6	(−34.3, 22.0, 70.2)	6	7	(−0.063, 0.078, 0.201)	6	7
8	(−0.096, 0.077, 0.235)	8	5	(−35.6, 22.7, 72.9)	5	8	(−0.066, 0.082, 0.209)	5	8
9	(−0.099, 0.080, 0.243)	9	4	(−36.4, 23.5, 75.5)	4	9	(−0.068, 0.084, 0.217)	4	9
10	(−0.103, 0.083, 0.251)	10	3	(−37.7, 24.3, 78.7)	3	10	(−0.070, 0.087, 0.224)	3	10
11	(−0.107, 0.086, 0.260)	11	2	(−38.9, 25.0, 81.3)	2	11	(−0.073, 0.090, 0.231)	2	11
12	(−0.110, 0.088, 0.269)	12	1	(−40.3, 26.3, 83.8)	1	12	(−0.075, 0.093, 0.239)	1	12

Table 6
Maximin Joy, Dominance Joy and Cumulative Maximin Joy fuzzy payoffs and rankings.

PEP	Maximin Joy	Rank 1	Rank 2	Dominance Joy	Rank 1	Rank 2	Cumulative Maximin Joy	Rank 1	Rank 2
1	(−0.042, 0.000, 0.042)	1	1	–	12	1	(−2.956, −0.201, 2.605)	12	8
2	(−0.044, 0.000, 0.044)	2	2	–	11	2	(−2.980, −0.165, 2.701)	11	9
3	(−0.047, 0.000, 0.047)	3	3	–	10	3	(−3.016, −0.129, 2.785)	10	10
4	(−0.049, 0.000, 0.049)	4	4	–	9	4	(−3.040, −0.093, 2.881)	9	11
5	(−0.051, 0.000, 0.051)	5	5	–	8	5	(−3.076, −0.045, 2.989)	8	12
6	(−0.053, 0.000, 0.053)	6	6	–	7	6	(−2.255, −0.016, 2.239)	7	7
7	(−0.055, 0.000, 0.055)	7	7	–	6	7	(−0.656, 0.004, 0.651)	2	1
8	(−0.057, 0.000, 0.057)	8	8	–	5	8	(−0.668, 0.004, 0.663)	3	2
9	(−0.060, 0.000, 0.060)	9	9	–	4	9	(−0.680, 0.004, 0.687)	4	4
10	(−0.061, 0.000, 0.061)	10	10	–	3	10	(−0.680, 0.004, 0.699)	5	3
11	(−0.063, 0.000, 0.063)	11	11	–	2	11	(−0.692, 0.004, 0.711)	6	5
12	(−0.065, 0.000, 0.065)	12	12	–	1	12	(−0.704, 0.016, 0.723)	1	6

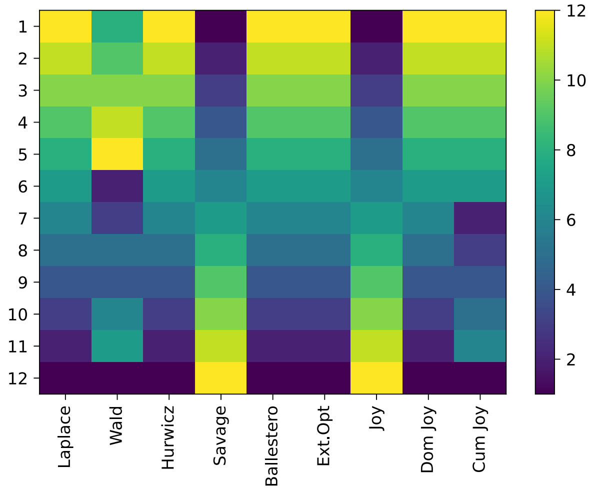

Fig. 2. Heat map of fuzzy rules using Ranking 1.

2) Laplace and Savage produce the same rankings; and 3) changing the ranking function implies obtaining the opposite results for all rules except Wald.

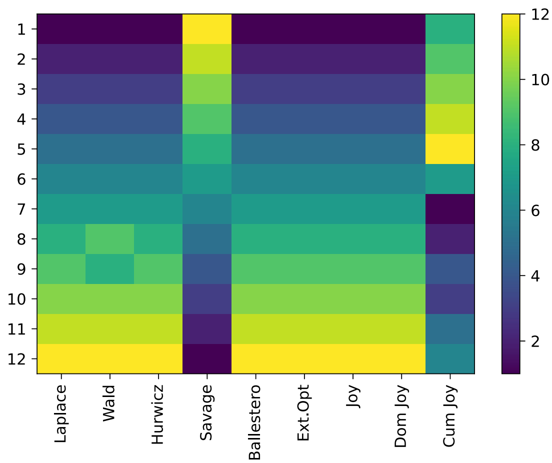

Fig. 3. Heatmap of fuzzy rules using Ranking 2.

4.2. *Sensitivity Analysis*

This section provides a sensitivity analysis to assess how changes in key input parameters or assumptions affect the decision outcomes. We mainly focus on three elements: the underlying decision-making principles for each rule, the pessimism-optimism preference parameter in the Hurwicz rule, and the impact of risk attitudes implemented in ranking functions for fuzzy numbers.

4.2.1. *The Impact of Decision-Making Principles*

Recall from Table 2 that selecting any rule implies the assumption of a decision-making principle. For instance, the Wald rule implies the principle of pessimism in decision-making because it assumes that the worst scenario will occur. Similarly, the Hurwicz rule implies a mix between pessimism and optimism through parameter α.

Focusing on Ranking 1 for risky investors, the principles of insufficient reason by Laplace and the minimax regret by Savage produced the same order of preferred portfolios. On the contrary, the equally weighted mix of pessimism and optimism by Hurwicz and the moderate pessimism principle by Ballestero produced the opposite order. In this case, the pessimism principle by Wald produced a different order when compared to the rest of the principles. However, we can reasonably state that the Wald order with Ranking 1 was closer to the resulting order by Hurwicz and Ballestero.

This pattern is even more apparent when considering Ranking 2 for conservative investors. The principles of insufficient reason by Laplace and the minimax regret by Savage produced the same order of preferred portfolios. On the contrary, the pessimism principle by Wald, the equally weighted mix of pessimism and optimism by Hurwicz, and the moderate pessimism principle by Ballestero produced the opposite order.

As a result, we find an important degree of similar behaviour in the principles of insufficient reason and the minimax regret. Similarly, there is also a parallel behaviour when deploying the principles of pessimism, an equally weighted mix of pessimism and optimism, and moderate pessimism. From a decision-maker perspective, the equally weighted mix of pessimism and optimism and the moderate pessimism principle will not make any difference within the context of our portfolio selection results.

These results lead us to explore further the impact of ranking functions and the pessimism-optimism parameter α in the Hurwicz rule.

4.2.2. *The Impact of Risk Attitudes in Ranking Functions*

Within the context of our case study in portfolio selection, investors are usually concerned about expected returns and the volatility of returns. Moreover, a risky investor prioritizes returns over volatility, and a conservative investor prioritizes volatility over returns. Investors are usually concerned with downside risk because they are more averse to returns below the mean value than to deviations above the mean value. This contextual knowledge allowed us to propose two ranking functions showcasing investors' attitudes toward risk: Ranking 1 prioritizing the core over the support of fuzzy numbers for risky investors, and Ranking 2 prioritizing the support over the core of fuzzy for conservative investors.

The results presented in Section 4.1 show that the ranking method is a crucial selection when applying fuzzy decision rules in a strict uncertainty context. Except for the Wald rule, the rest of the decision rules led to an opposite evaluation of alternatives because of the ranking method. This result is not surprising because, in our application, we find that the highest core of aggregated triangular payoffs comes in conjunction with the lowest endpoint of the support. In the case of the Wald rule, the modal values of the triangular fuzzy payoffs are very similar, allowing the ranking methods proposed in our approach to express their potential to discriminate among fuzzy numbers.

In summary, our fuzzy strict uncertainty approach allows for considering various ranking options, accommodating different perspectives on attitudes toward risk. In addition to the underlying decision-making principle, investors and other decision-makers may have different preferences regarding how alternative evaluations are managed to produce an admissible order. As a suitable way to achieve a more flexible and nuanced selection process, decision-makers can now incorporate these diverse preferences in the decision-making process through ranking functions.

4.2.3. *The Impact of Pessimism Parameter α in Hurwicz Rule*

Interestingly, the Hurwicz and the Ballestero rules produced results opposite to the Laplace and Savage rules when comparing risky and conservative rankings. We argue that this result is a direct consequence of parameter α used in the case of the Hurwicz rule to represent the position of a decision-maker concerning optimism and pessimism and the moderate pessimism implied by the Ballestero rule. In other words, the Hurwicz and Ballestero rules can represent intermediate points from optimism to pessimism, as in the strict uncertainty approach with crisp evaluations. However, the only parametric rule is the Hurwicz rule. Then, by varying input parameter α within plausible ranges, decision-makers can understand the impact of this parameter on their decisions.

We summarize the results of this sensitivity analysis in Table 7 for values of pessimism parameter α ranging from zero (full optimism) to one (full pessimism, equivalent to the

Table 7
Hurwicz fuzzy rankings for different pessimism parameter α values.

	$\alpha = 0$		$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1$	
PEP	Rank 1	Rank 2	Rank 1	Rank 2	Rank 1	Rank 2	Rank 1	Rank 2	Rank 1	Rank 2
1	12	1	12	1	12	1	12	1	8	1
2	11	2	11	2	11	2	11	2	9	2
3	10	3	10	3	10	3	10	3	10	3
4	9	4	9	4	9	4	8	4	11	4
5	8	5	8	5	8	5	9	5	12	5
6	7	6	7	6	7	6	7	6	2	6
7	6	7	6	7	6	7	5	7	3	7
8	5	8	5	8	5	8	6	9	5	9
9	4	9	4	9	4	9	4	8	4	8
10	3	10	3	10	3	10	3	10	6	10
11	2	11	2	11	2	11	2	11	7	11
12	1	12	1	12	1	12	1	12	1	12

Wald rule). We found no difference in the ranking results for $\alpha < 0.5$. As a result, investors applying the Hurwicz rule should not bother too much about parameter α if they feel optimistic, namely, if they think that the value of parameter α that best represents them is below 0.5. However, we find some differences when parameter α moves close to the pessimism zone. For parameter $\alpha = 0.75$, we find small changes in PEP 7 and 8 ranking positions for Ranking 1 and PEP 9 and 10 for Ranking 2. These changes become more apparent when setting $\alpha = 1$, equivalent to applying the Wald rule. This behaviour is because differences in PEP fuzzy payoffs become smaller as long as parameter α increases. For instance, the core value range (maximum value minus minimum value) for Hurwicz fuzzy triangular payoffs in Table 5 is 0.017, while the core value range for Wald fuzzy payoffs in Table 4 is 0.002. As a result, the ranking methods become more relevant to discriminate among similar fuzzy numbers because of the similarity of fuzzy payoffs.

4.3. *Comparative Analysis*

To further validate the proposed fuzzy strict uncertainty framework, a comparative analysis with well-established fuzzy MCDM techniques is conducted. Initially, fuzzy TOPSIS (Salih *et al.*, 2019) was selected for comparison due to its extensive adoption in portfolio selection and decision analysis problems, its intuitive interpretation, and its solid geometric foundation based on distances to ideal and anti-ideal solutions. Because of these characteristics, fuzzy TOPSIS is frequently employed as a benchmark in the fuzzy decision-making literature and thus represents a natural first choice for validation. However, fuzzy MCDM encompasses a variety of aggregation paradigms, and relying on a single comparative technique may limit the generality of the conclusions. In response to this concern, this section extends the comparative analysis by incorporating two additional state-of-the-art fuzzy MCDM methods:

- Fuzzy VIKOR (Jana *et al.*, 2023), representing compromise-based decision-making through the joint consideration of group utility and individual regret;

Table 8
Comparisons between fuzzy MCDM methods (TOPSIS, VIKOR, WASPAS) and fuzzy decision rules using Ranking 1.

PEP	TOPSIS	VIKOR	WASPAS	Laplace	Wald	Hurwicz	Savage	Ballestero	Ext. Opt.	Joy	Dom. Joy	Cum. Joy
1	12	12	12	12	8	12	1	12	12	1	12	12
2	11	11	11	11	9	11	2	11	11	2	11	11
3	10	10	10	10	10	10	3	10	10	3	10	10
4	9	9	9	9	11	9	4	9	9	4	9	9
5	8	8	8	8	12	8	5	8	8	5	8	8
6	7	7	7	7	2	7	6	7	7	6	7	7
7	6	6	6	6	3	6	7	6	6	7	6	2
8	5	5	5	5	5	5	8	5	5	8	5	3
9	4	4	4	4	4	4	9	4	4	9	4	4
10	3	3	3	3	6	3	10	3	3	10	3	5
11	2	2	2	2	7	2	11	2	2	11	2	6
12	1	1	1	1	1	1	12	1	1	12	1	1

- Fuzzy WASPAS (Turskis *et al.*, 2015), integrating weighted sum and weighted product aggregation mechanisms.

These methods represent three complementary decision-making philosophies: distance-based compromise (TOPSIS), regret-based compromise programming (VIKOR) and hybrid utility aggregation (WASPAS), allowing for a more comprehensive and balanced validation of the proposed framework.

To conduct the comparison, possible PEPs are designated as alternatives, while uncertain scenarios are treated as evaluation criteria, assuming equal importance across criteria. Under identical fuzzy representations and evaluation conditions, fuzzy TOPSIS, fuzzy VIKOR, and fuzzy WASPAS were applied to the dataset presented in Table 3. The resulting rankings were then compared with those obtained from the fuzzy decision rules using Ranking 1. The comparative results are reported in Table 8.

Notably, the rankings produced by fuzzy TOPSIS, VIKOR, and WASPAS are fully consistent with the fuzzy Laplace, Hurwicz, Ballestero, Extreme Optimism, and Dominance Joy decision rules, while they contradict the rankings derived from the fuzzy Savage and Joy decision rules. In the case of Wald and Cumulative Joy fuzzy decision rules, partial agreement is observed, particularly in identifying the worst-performing PEP.

The consistency between fuzzy TOPSIS, VIKOR, WASPAS and the Laplace, Hurwicz, Ballestero, Extreme Optimism, and Dominance Joy fuzzy rules can be attributed to their inherent compromise-oriented nature. TOPSIS seeks a balance between ideal and anti-ideal solutions; VIKOR explicitly formalizes compromise through regret minimization; and WASPAS blends additive and multiplicative utilities to achieve moderated aggregation. These characteristics resonate strongly with the optimism-pessimism trade-off embedded in the fuzzy Hurwicz rule and the moderate-pessimism of the fuzzy Ballestero rule. Conversely, the non-compromise principles underlying the fuzzy Savage and Joy decision rules explain their divergence from the MCDM-based rankings. These rules focus exclusively on extreme-case regret and joy, which contrasts with the balanced evaluation mechanisms employed by fuzzy TOPSIS, VIKOR, and WASPAS.

From a methodological perspective, fuzzy TOPSIS relies on the definition of a distance function for fuzzy numbers, which can be challenging and may introduce subjectivity. In addition, decision-makers cannot explicitly express their risk attitudes or preferences through alternative fuzzy ranking functions. In this sense, fuzzy TOPSIS, as well as fuzzy VIKOR and fuzzy WASPAS, can be viewed as special cases within fuzzy strict uncertainty, whereas the proposed fuzzy decision rules offer greater flexibility by directly modelling different behavioural attitudes toward uncertainty.

To objectively quantify the degree of agreement and dissimilarity between the rankings obtained from the proposed fuzzy strict uncertainty rules and the fuzzy MCDM methods, two complementary ranking distance measures are employed: the Euclidean distance (E_d) (Serra and Arcos, 2014) and the normalized Kendall's tau rank distance (K_n) (Kendall, 1938). For that purpose, we first introduce these distance measures below.

Given vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ of size n with elements set to the rank between 1 and n achieved by each of the n alternatives under evaluation, the Euclidean distance between the ordered alternatives following two decision-making rules is given by:

$$E_d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{10}$$

The normalized Kendall tau rank distance is defined as:

$$K_n(\boldsymbol{x}, \boldsymbol{y}) = \frac{\sum_{(i,j)\in\mathcal{P}, i<j} \bar{K}_{ij}(\boldsymbol{x}, \boldsymbol{y})}{\frac{1}{2}n(n-1)}, \tag{11}$$

where $\mathcal{P}$ is the set of unordered pairs of distinct elements in ranks $\boldsymbol{x}$ and $\boldsymbol{y}$, $\bar{K}_{ij}(\boldsymbol{x}, \boldsymbol{y}) = 0$ if i and j are in the same order in $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\bar{K}_{ij}(\boldsymbol{x}, \boldsymbol{y}) = 1$ if i and j are in the opposite order in $\boldsymbol{x}$ and $\boldsymbol{y}$.

These two metrics provide a quantitative evaluation of the distance between two sets of ordered alternatives, hence measuring the discrepancy of rules differently. While the Euclidean distance (E_d) quantifies the absolute degree of discrepancy, the normalized Kendall tau rank distance (K_n) indicates the percentage of pairs that differ in ordering between the two ranks because it lies in the interval [0, 1]. To objectively measure the differences in the rankings obtained by different rules, we compute dissimilarity metrics E_d and K_n for each pair of rankings given in Table 8 and present them in Table 9. As expected, both metrics provide similar results with the highest dissimilarity when the rules provide opposite orders, as in the case of the Fuzzy TOPSIS and Laplace rules. Small dissimilarity occurs when the rules provide similar orders, as in the Wald and Hurwicz rules case.

These findings confirm that, although the proposed approach is fundamentally grounded in fuzzy strict uncertainty rather than weighted multi-criteria aggregation, it produces rankings that are fully compatible with widely used fuzzy MCDM techniques. This demonstrates that fuzzy strict uncertainty rules constitute a complementary and theoretically consistent decision framework, particularly suited to decision environments where scenario probabilities and criterion weights are unavailable or unreliable.

Table 9
Euclidean distances (E_d) (value above) and normalized Kendall tau distances (K_n) (value below) as dissimilarity metrics for ranks in Table 8.

	TOPSIS	VIKOR	WASPAS	Laplace	Wald	Hurwicz	Savage	Ballestero	Ext. Opt	Joy	Dom. Joy	Cum. Joy
TOPSIS	0.00	0.00	0.00	0.00	10.39	0.00	23.92	0.00	0.00	23.92	0.00	6.32
	0.00	0.00	0.00	0.00	0.36	0.00	1.00	0.00	0.00	1.00	0.00	0.15
VIKOR		0.00	0.00	0.00	10.39	0.00	23.92	0.00	0.00	23.92	0.00	6.32
		0.00	0.00	0.00	0.36	0.00	1.00	0.00	0.00	1.00	0.00	0.15
WASPAS			0.00	0.00	10.39	0.00	23.92	0.00	0.00	23.92	0.00	6.32
			0.00	0.00	0.36	0.00	1.00	0.00	0.00	1.00	0.00	0.15
Laplace				0.00	10.39	0.00	23.92	0.00	0.00	23.92	0.00	6.32
				0.00	0.36	0.00	1.00	0.00	0.00	1.00	0.00	0.15
Wald					0.00	10.39	21.54	10.39	10.39	21.54	10.39	8.49
					0.00	0.36	0.64	0.36	0.36	0.64	0.36	0.24
Hurwicz						0.00	23.92	0.00	0.00	23.92	0.00	6.32
						0.00	1.00	0.00	0.00	1.00	0.00	0.15
Savage							0.00	23.92	23.92	0.00	23.92	23.07
							0.00	1.00	1.00	0.00	1.00	0.85
Ballestero								0.00	0.00	23.92	0.00	6.32
								0.00	0.00	1.00	0.00	0.15
Ext. Opt.									0.00	23.92	0.00	6.32
									0.00	1.00	0.00	0.15
Joy										0.00	23.92	23.07
										0.00	1.00	0.85
Dom. Joy											0.00	6.32
											0.00	0.15
Cum. Joy												0.00
												0.00

Overall, the comparison study demonstrates that the proposed fuzzy strict uncertainty framework is not only theoretically well-grounded but also empirically robust, yielding stable and interpretable results across a diverse family of fuzzy MCDM methods. Consequently, the framework can be regarded as a complementary and consistent decision-making approach alongside established fuzzy MCDM techniques.

4.4. *Discussion*

From our portfolio selection case study, including the sensitivity and comparative analysis, we conclude that a fuzzy strict uncertainty approach in decision-making offers a more nuanced and flexible framework, accommodating imprecision, multiple scenarios, subjective opinions, and diverse decision criteria and rankings. This leads to a more robust and adaptive decision-making process, better aligned with complex and uncertain problems such as portfolio selection. In direct comparison to fuzzy TOPSIS, we claim that our fuzzy strict uncertainty approach is more flexible because decision-makers can select not only among decision-making principles but also the ranking functions that best reflect their risk attitudes or other preferences concerning the specific decision-making context.

From the sensitivity analysis, we learn that decision-makers can integrate their principles and preferences by combining fuzzy rules, ranking functions, and parameters (if any). On the one hand, the underlying decision-making principles of insufficient reason, optimism, pessimism, and different degrees of optimism and pessimism can guide decision-makers to select one of the proposed rules. On the other hand, selecting the fuzzy ranking function refines the decision-making principle by integrating risk attitudes or preferences when comparing alternatives evaluated using fuzzy numbers. Indeed, the range of fuzzy rules and ranking functions is not limited to those used in this paper. Practitioners and researchers can design new approaches that best fit their specific decision-making contexts.

In summary, a fuzzy approach in portfolio selection can quantify imprecision by using fuzzy numbers representing the continuous distribution possibilities for investor payoffs. Moreover, the expert's knowledge or the investor's subjective opinions can be better integrated into a portfolio selection model. The results described in this section on the application of a fuzzy strict uncertainty approach in portfolio selection offer the following advantages:

1. *Incorporation of imprecision and subjectivity about future states*. Fuzzy numbers allow for the representation of imprecision in the knowledge about future payoffs. In addition, subjective opinions and expert knowledge can be effectively integrated into the model, facilitating a more realistic representation of uncertainty.
2. *Consideration of multiple scenarios*. Fuzzy strict uncertainty allows for the consideration of multiple but imprecise future scenarios. This is important in portfolio management, where diverse economic conditions, market fluctuations, and other variables can significantly impact the performance of investments.
3. *Flexibility in fuzzy decision rules*. Fuzzy strict uncertainty provides flexibility in decision-making by incorporating different decision criteria and rules that may better align with the decision-maker. The classical decision rules summarized in Table 2 now apply in a fuzzy context.
4. *Different ranking options*. Fuzzy's strict uncertainty allows for considering various ranking options, accommodating different perspectives on the importance of criteria and preferences. Investors may have different risk attitudes, and fuzzy portfolio selection enables the incorporation of these diverse preferences. For instance, risky investors prioritizing returns over volatility would select a different ranking method than conservative investors prioritizing volatility over returns.

5. Concluding Remarks

This paper has introduced a novel decision-making methodology that extends the concept of strict uncertainty to a fuzzy environment with fuzzy payoffs. By integrating strict uncertainty and fuzzy sets theory, we have developed a set of decision rules tailored for individuals and social groups. These rules cover a wide range of decision profiles in a fuzzy context. For instance, a pessimistic decision-maker would likely feel comfortable with the fuzzy Wald decision rule. At the same time, an optimistic one would select the fuzzy Hurwicz decision rule with the appropriate weight system.

Our work contributes significantly to fuzzy optimization and decision-making by expanding the understanding of strict uncertainty and providing practical tools to solve decision problems with fuzzy payoffs. The integration with fuzzy sets theory allows for a more realistic representation of the uncertainty inherent in financial scenarios, such as investment, budgeting, and alternative ranking. The key contributions of this research include the extension of strict uncertainty to a fuzzy context, the introduction of new fuzzy decision rules, and the illustration of their application in solving a portfolio selection problem. By doing so, we aim to provide a valuable framework for decision-makers facing complex decisions in which deterministic models may need to capture the inherent uncertainty.

In direct comparison to fuzzy TOPSIS, we enhance flexibility, allowing decision-makers to select among a set of decision-making principles and ranking functions that best reflect their risk attitudes or other preferences. The underlying decision-making principles of insufficient reason, optimism, pessimism, and different degrees of optimism and pessimism can guide decision-makers in selecting one of the proposed rules. Moreover, selecting the fuzzy ranking function refines the decision-making principle by integrating risk attitudes or preferences when comparing alternatives evaluated using fuzzy numbers. As a result, practitioners and researchers can build on our approach to propose new rules and ranking functions adapted to specific decision-making contexts.

In conclusion, this research opens avenues for further exploration and refinement of decision-making methodologies under fuzzy strict uncertainty, encouraging future studies to delve deeper into specific applications and refine the rules to accommodate diverse decision contexts. Moreover, the formal definition of fuzzy strict uncertainty represents a first step in studying their properties that may lead to a better understanding of fuzzy decision-making. Ultimately, we anticipate that our proposed approach will contribute to a more realistic approach to several decision-making processes, especially in economics and finance, due to the inherent uncertainty, offering valuable insights for both theoretical advancements and practical applications.

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