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Some Generalized Aggregation Operators with Probabilistic Spherical Hesitant Fuzzy Information and Applications to Green Enterprise Credit Selection

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Abstract. In order to better solve the multi-attribute decision-making (MADM) issues in real life, this paper proposes the probabilistic spherical hesitant fuzzy set (PSHFS) theory based on spherical HFS (SHFS) and probabilistic HFS (PHFS). Firstly, PSHFS is developed, and its basic operations of PHSF element (PSHFE) are proposed. Secondly, generalized PSHF weighted averaging (GPSHFWA) and generalized PSHF weighted geometric (GPSHFWG) operators are constructed, and their excellent properties and some special forms are investigated. Thirdly, for MADM problems with different priorities of related evaluation criteria, we propose generalized PSHF prioritized weighted averaging (GPSHFPWA) and geometric (GPSHFPWG) operators, and investigate their excellent properties and some special operators. Fourthly, two new MADM techniques are constructed dependent on the proposed two types of operators in practical MADM problems. Finally, the effectiveness of the two MADM techniques constructed is tested through an application example of the green enterprise credit selection (GECS). The sensitivity analysis of parameter shows the influence on different values of parameter on the optimal alternatives by setting different parameter values, and shows the flexibility of the proposed MADM techniques. Meanwhile, the two MADM techniques are compared with several existing MADM techniques to prove the advantages of the two MADM techniques.

Key words: multi-attribute decision-making (MADM), probabilistic spherical hesitant fuzzy set, prioritized average operator, generalized probabilistic spherical hesitant fuzzy operator, green enterprise credit selection.

1. Introduction

MADM issue is a very common decision-making problem in real life (Arshad *et al.*, 2022; Mefgouda and Idoudi, 2022; Rahman *et al.*, 2023; Tang *et al.*, 2022; Wang *et al.*, 2024).

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The typical MADM task is the process of choosing the optimal among the limited alternatives according to certain evaluation criteria by using the decision-making approaches in the corresponding environment. The information of traditional decision problems is expressed by exact numbers. However, with the incompleteness of the data of practical problems and the uncertainty of the actual situation, it is often impossible to express the information by exact numbers. In order to better handle such MADM problems and better express the fuzziness of evaluation data, fuzzy set (FS) (Zadeh, 1965) is applied. In the past, a lot of scholars devoted themselves to the research of uncertain MADM issues. With the continuous deepening of research and the complicacy of the actual MADM tasks, how to effectively expand FS based on FS and solve the corresponding practical problems has always been a hot issue for academic scholars.

Although FS has shown strong advantages in expressing uncertain information, it is still unable to express some more complex cases in real life. In order to express evaluation information more effectively, some extended FSs dependent on FS were developed (Lima *et al.*, 2021; Zhao *et al.*, 2021). Several new types of FSs, such as the intuitionistic FS (IFS) (Atanassov, 1986), picture FS (PIFS) (De Miguel *et al.*, 2017), hesitant FS (HFS) (Torra, 2010), picture HFS (PIHFS) (Hosseinpour and Martynenko, 2021), were developed, and applied in many fields (Aktas *et al.*, 2022; Khatri *et al.*, 2022; Mahmood *et al.*, 2021; Senapati *et al.*, 2023; Yazdi *et al.*, 2022). In practical MADM problems, DMs often give more complex and fuzzy evaluation information when expressing decision information. For example, one class discussed the candidates for scholarships, 35 people agreed, 10 people disagreed, and the other 5 people chose to give up. It is obvious that FS can only express approval of such information, but cannot reflect disapproval and abstention. That is to say, FS cannot express the overall opinion of 50 students in this class. In order to better handle this situation, Atanassov (1986) proposed IFS that can reflect disapproval and waiver. It is obvious that the above situation can be well expressed in IFS. If the exact number in the membership degree (MD) and non-membership degree (NMD) is replaced by the interval number, the corresponding interval valued IFS (IVIFS) (Atanassov and Gargov, 1989) is developed. To deal with more complex practical situations, Ye (2015) proposed triangular IFSs and Liu and Yuan (2007) proposed trapezoidal IFSs. However, in some special cases, it is problematic to integrate decision attribute information with IFS and IVIFS. For example, when a class elects a monitor, generally speaking, the following four opinions will appear in the students' voting opinions: approval, abstention, objection and refusal to vote (Garg, 2017). In response to this sobriety, Atanassov (1986) built the concept of PIFS, which exactly reflects this situation, so that PIFS can well express the opinions of DMs in the actual cases we put forward above. Many scholars have conducted detailed research on it (Garg, 2017; Son, 2016; Wei, 2016). The SFS (Kahraman and Gündoğdu, 2018) and SHFS (Khan *et al.*, 2021) were proposed to tackle issues in some special and T-spherical fuzzy environments, and they have been utilized in various fields (Akram *et al.*, 2023; Khan *et al.*, 2021; Oezkan *et al.*, 2022; Rajput and Kumar, 2024).

On the other hand, sometimes FS is difficult to accurately describe the membership of decision attribute information. Obviously, FS has some defects. Torra (2010) defined HFS, and MD can be expressed by several possible exact numbers, it can better reflect

DMs' hesitant attitudes. But we also found that HFS cannot reflect the hesitation degree of DM, so it is necessary to use an appropriate variable to describe the hesitation degree of DM. Integrating probabilistic information into HFS, PHFS was developed (Xu and Zhou, 2017). But we also found that HFS cannot reflect the degree of decision-maker hesitation, so it is necessary to use an appropriate variable to describe the hesitation degree of DM. Then if the exact number in the membership degree is replaced by the interval number, the interval-valued HFS (IVHFS) (Chen *et al.*, 2013) was developed. On the basis of IFS and HFS and combining their respective advantages, Dual HFS (DHFS) (Zhu *et al.*, 2012) was developed. Then by replacing the exact number in the MD and NMD by the interval number, the dual IVHFS (DIVHFS) (Farhadinia, 2014) was constructed, and by combining the probability information to DHFS, PDHFS was built (Hao *et al.*, 2017), and it was utilized in many fields (Ning *et al.*, 2022a, 2022b; 2022c). The SHFS (Ning *et al.*, 2023) was proposed to tackle issues in some special fuzzy environments and utilized in various fields.

In MADM problem, we can generally aggregate multiple evaluation information into comprehensive evaluation information by aggregated operator according to the obtained assessment information of decision attributes given by a DM. Therefore, aggregation operator acts a very important role in aggregation information method. For this hot issue, many scholars have carried out detailed research on aggregation problems under various fuzzy information. Such as, Xu and Yager (2006) built the operation of IF number (IFN) and proposed IF geometric aggregation operator. Subsequently, Xu (2007a) proposed an IF weighted average aggregation operator to aggregate IFNs. Next, several IVIF aggregation operators are constructed to deal with MCDM issues (Xu, 2007b; Xu and Chen, 2007). Regarding the evaluation information of picture fuzziness, Wei (2017) defined the operation of PIF number (PIFN) dependent on the research of Xu (2007a), and built a PIF weighted aggregated operator. Moreover, some valued PIF aggregated operators are proposed depended on various operations (Wang *et al.*, 2017). In addition, a large number of HF aggregated operators and their generalized aggregated operators (Xia and Xu, 2011) were built, and some valued aggregated operators (Yu *et al.*, 2016; Zhao and Xu, 2018) under DHF and dual IVHF environments are developed. And some aggregation operators were developed in SF and SHF environments (Ashraf and Abdullah, 2019; Bonab *et al.*, 2023; Diao *et al.*, 2022; Donyatalab *et al.*, 2020; Rajput and Kumar, 2022; Yan *et al.*, 2024).

In practical decision-making problems, evaluation indicators often have different priorities. For example, when parents buy toys for their children, when considering both price and safety, it is obvious that safety is more important than price. However, for MADM problems with different priorities of decision attributes, the aggregation operators mentioned above cannot handle such MADM problems. In order to better handle such MADM problems, the well-known priority average (PA) operator was built by professor Yager (2008). Inspired by PA operator, Yager (2008); Yu (2013); Yu *et al.* (2012) constructed IF priority fuzzy and IVIF priority fuzzy aggregation operators. In addition, a HF priority aggregation operator was built (Wei, 2012).

Through years of practice and exploration, although FS can better express the evaluation information in MADM tasks, it still cannot handle some more complex MADM problems. The ultimate aim of this study is to act a novel type of FS, which can more accurately express the decision-making information. The SFS (Gündogdu and Kahraman, 2019) has good performance ability in expressing decision information, but faces some special MADM problems, there are still problems that cannot be solved. Therefore, combining the respective advantages of the two FSs (Mahmood *et al.*, 2019; Torra, 2010), the concept of SHFS (Ning *et al.*, 2023) was proposed as a generalized form of FS (Zadeh, 1965), IFS (Atanassov, 1986), HFS (Torra, 2010) and SFS (Gündogdu and Kahraman, 2019). Although both SFS and SHFS have played a very important role and achieved good results in the actual decision-making problem. For example, SFN (0.35, 0.5, 0.7), DMs or experts are skeptical that they can give such accurate values when making decisions. For the hesitation attitude of DMs or experts, it is obvious that SFS cannot cope with such a psychological situation. For such a situation, SHFS can cope well, such as SHFE $\langle\{0.36, 0.35\}, \{0.5, 0.51\}, \{0.6, 0.7\}\rangle$. However, there is a very realistic situation that although SHFE has solved the hesitation of DMs or experts, the DMs or experts are uncertain about the possibility of taking the exact value of the three membership degrees, that is, the probability of several values in SHFE $\langle\{0.36, 0.35\}, \{0.5, 0.51\}, \{0.6, 0.7\}\rangle$. Therefore, it is necessary to propose a new type of fuzzy set that includes both the advantages of SHFS and reflects probability information. This paper proposes a PSHFS that integrates the respective advantages of PHFS and SHFS, which is in line with such advantages, including both the advantages of SHFS and reflecting probability information. It is clear that the PSHFS proposed in this study can well solve this complex problem, this can be well reflected, such as PSHFE $\{\{0.36|0.4, 0.35|0.6\}, \{0.5|0.8, 0.51|0.2\}, \{0.6|0.3, 0.7|0.7\}\}$, which can more precisely indicate the indefinite decision-making information in MADM tasks, and better express the uncertainty, fuzziness and complexity in MADM problems. Furthermore, the possible positive-membership hesitant degree, neutral-membership hesitant degree, negative-membership hesitant degree and refusal membership hesitant degree can be represented by multiple values in [0, 1] according to the needs of DM. Due to the ripe applications of PA operator in various fuzzy settings, this paper effectively fuses PA operator with the newly proposed PSHFS. Finally, this paper develops PSHFS based on SHFS and PHFS. At the same time, inspired by the operation rules of the above two FSs, the operation rules of PSHFE are developed. Then, based on these operation rules, the GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators are proposed, and invest their excellent properties and some special operators. Finally, two new MADM techniques are built dependent on the two built types of aggregated operators to choose the best alternative in practical MADM issues. The effectiveness of the two MADM techniques built is tested through an application example of GECS.

In addition to the above important content, the rest sections are shown as: Section 2 reviews the SHFS, PHFS and PA operator. Section 3 puts forward the PSHFS, and develops the operations of PSHFE, scoring function, accurate function, and comparison method. Section 4 proposes the GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators. Two MADM techniques are developed based on the newly built aggregation operator in Section 5. The Section 6 implements the application of two algorithms to the GECS, the comparative analysis and sensitivity analysis are implemented to verify the effectiveness and flexibility of the two built MADM techniques. Finally, Section 7 summarizes all research work and prospects the future work.

2. Preliminaries

Before proposing PSHFS, we first review the PHFS (Xu and Zhou, 2017) and SHFS (Naeem *et al.*, 2021) that can be used later in this paper. The famous PHFS (Xu and Zhou, 2017) was proposed by integrating probability information into HFS, which not only reflects the hesitation of DMs in multiple choices, but also gives consideration to the degree of hesitation. The SHFS (Naeem *et al.*, 2021) combines the common advantages of SFS (Gündogdu and Kahraman, 2019) and HFS (Torra, 2010), and has been widely used in many fields.

Definition 1 (Xu and Zhou, 2017). A PHFS $\Im$ on X is recorded as: $\Im = \{(x, h_x|p_x) \mid x \in X\}$, where $h_x \in [0, 1]$ be MD values in $\Im$, and the probability is $p_x \in [0, 1]$. Generally, $\Im = \langle h(x)|p(x)\rangle$ is named as a PHFE and recorded as $\Im = \langle h|p\rangle$.

For the aggregation of decision information, (Xu and Zhou, 2017) proposes the operation rules of the PHFEs.

Definition 2 (Xu and Zhou, 2017). Let $\Im = \langle h|p\rangle$, $\Im_1 = \langle h_1|p_1\rangle$ and $\Im_2 = \langle h_2|p_2\rangle$ be three PHFEs, $\xi > 0$, then some operations of PHFEs are recorded as:

(1) $\Im_1 \oplus \Im_2 = \bigcup_{\mu_1\in h_1,\mu_1\in h_2} \{\langle \mu_1 + \mu_2 - \mu_1\mu_2|p_1p_2\rangle\}$;
(2) $\Im_1 \otimes \Im_2 = \bigcup_{\mu_1\in h_1,\mu_1\in h_2} \{\langle \mu_1\mu_2|p_1p_2\rangle\}$;
(3) $\xi\Im = \bigcup_{\mu\in h} \{\langle 1-(1-\mu)^{\xi}|p\rangle\}$;
(4) $\Im^{\xi} = \bigcup_{\mu\in h} \{\langle \mu^{\xi}|p\rangle\}$.

Definition 3 (Naeem *et al.*, 2021). Let the universe set be U. Then the set $A = \{\langle u, (M_A(u), N_A(u), K_A(u))|u \in U\rangle\}$ is said to be spherical hesitant fuzzy set, where $M_A(u) = \{\mu|\mu \in [0, 1]\}$, $N_A(u) = \{v|v \in [0, 1]\}$ and $M_A(u) = \{o|o \in [0, 1]\}$ are called as positive-membership degree of u in A, neutral-membership degree of u in A and negative-membership degree of u in A, respectively. Meanwhile, μ, v and o satisfy the following condition: $0 \leqslant (\mu^+)^2+(v^+)^2+(o^+)^2 \leqslant 1\ \forall u \in U$, and $\mu^+ = \bigcup_{\mu\in M_A(u)} \max\{\mu\}$, $v^+ = \bigcup_{v\in M_A(u)} \max\{v\}$, and $o^+ = \bigcup_{o\in M_A(u)} \max\{o\}$.

Generally, $a = \{M, N, K\}$ is called an SHFN. In order to facilitate the calculation between SHFNs, inspired by literature (Torra, 2010), Ren *et al.* (2017) gave the basic operation rules of SHFNs.

Definition 4 (Ren *et al.*, 2017). For three SHFNs $\alpha = \langle M, N, K\rangle$, $\alpha_1 = \langle M_1, N_1, K_1\rangle$ and $\alpha_2 = \langle M_2, N_2, K_2\rangle$, and α^C represent the complementary set of α, the followings are valid under the condition $\lambda > 0$.

$$\alpha_1 \oplus \alpha_2 = \bigcup_{\substack{\mu_1 \in M_1, v_1 \in M_1, o_1 \in M_1 \\ \mu_2 \in M_2, v_2 \in M_2, o_2 \in M_2}} \left\{\left\langle \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}, v_1 v_2, o_1 o_2 \right\rangle\right\};$$

$$\alpha_1 \otimes \alpha_2 = \bigcup_{\substack{\mu_1 \in M_1, v_1 \in M_1, o_1 \in M_1 \\ \mu_2 \in M_2, v_2 \in M_2, o_2 \in M_2}} \left\{\left\langle \mu_1 \mu_2, v_1 v_2, \sqrt{o_1^2 + o_2^2 - o_1^2 o_2^2} \right\rangle\right\};$$

$$\lambda \cdot \alpha = \left\langle \sqrt{1 - \left(1 - \mu^2\right)^\lambda}, v^\lambda, o^\lambda \right\rangle; \quad \alpha^\lambda = \left\langle \mu^\lambda, v^\lambda, \sqrt{1 - \left(1 - \pi^2\right)^\lambda} \right\rangle;$$

$$\alpha^C = \langle K, N, M \rangle.$$

Definition 5 (Yager, 2008). Let $W = \{W_1, W_2, \dots, W_n\}$ be a group of evaluation criteria, which has different priority levels, i.e. when $s < t$, the priority of W_s is higher than W_t. The $W_j(x) \in [0, 1]$ is the value of x under W_j. Hence, the PA operator is recorded as $PA(W_1(x), W_2(x), \dots, W_n(x)) = \sum_{j=1}^n \zeta_j W_j(x)$, where $\zeta_j = S_j / \sum_{j=1}^n S_j$, $S_j = \prod_{r=1}^{j-1} W_r(x)$ and $S_1 = 1$.

3. PSHFS

According to the PHFS and SHFS, we define a PSHFS composed of positive, neutral, negative and refusal membership hesitant functions belonging to [0, 1]. The uncertainty and hesitation of DMs can be expressed better.

Definition 6. A PSHFS M on a non-empty and finite set X is defined by

$$M = \left\{\left\langle x, w(x) \middle| p(x), e(x) \middle| q(x), r(x) \middle| l(x) \right\rangle \middle| x \in X\right\}. \tag{1}$$

The components $w(x)|p(x)$, $e(x)|q(x)$ and $r(x)|l(x)$ are three sets of some possible elements, where $w(x)$, $e(x)$ and $r(x)$ represent the possible positive-membership hesitant degree, neutral-membership hesitant degree, and negative-membership hesitant degree to the set X of x, respectively. $p(x)$, $q(x)$ and $l(x)$ are the probabilities of $w(x)$, $e(x)$ and $r(x)$. There is: $0 \leqslant \mu, \nu, o \leqslant 1$, $0 \leqslant (\mu^+)^2 + (\nu^+)^2 + (o^+)^2 \leqslant 1$; $p_i \in [0, 1]$, $q_j \in [0, 1]$, $r_k \in [0, 1]$; $\sum_{i=1}^{\#w} p_i \leqslant 1$, $\sum_{j=1}^{\#e} q_j \leqslant 1$, $\sum_{k=1}^{\#r} l_k \leqslant 1$, where $\mu \in w(x)$, $\nu \in e(x)$, $o \in r(x)$. $\mu^+ \in w^+(x) = \bigcup_{\mu \in w(x)} \max \mu$, $\nu^+ \in e^+(x) = \bigcup_{\nu \in e(x)} \max \nu$, $o^+ \in r^+(x) = \bigcup_{o \in r(x)} \max o$, $p_i \in p$, $q_j \in q$, $l_k \in l$. The symbols $\#w$, $\#e$ and $\#r$ are total numbers of elements in $w(x)|p(x)$, $e(x)|q(x)$ and $r(x)|l(x)$.

We call $m = \langle w(x)|p(x), e(x)|q(x), r(x)|l(x)\rangle$ a probabilistic spherical hesitant fuzzy element (PSHFE) denoted by $m = \langle w|p, e|q, r|l\rangle$, and

$$\pi = \sqrt{1 - \sum_{i=1}^{\#w} \mu_i^2 p_i - \sum_{j=1}^{\#e} v_j^2 q_j - \sum_{k=1}^{\#r} o_k^2 l_k}$$

Table 1
A PSHFDM M_1 with respect to X_1.

Attributes	Supplier X_1
C_1	$\langle\{0.6\|0.2, 0.5\|0.1, 0.4\|0.7\}, \{0.3\|0.5, 0.4\|0.5\}, \{0.1\|0.2, 0.3\|0.3, 0.2\|0.5\}\rangle$
C_2	$\langle\{0.2\|0.4, 0.3\|0.6\}, \{0.7\|0.7, 0.6\|0.3\}, \{0.3\|0.1, 0.6\|0.1, 0.2\|0.8\}\rangle$
C_3	$\langle\{0.3\|0.5, 0.2\|0.5\}, \{0.4\|0.2, 0.3\|0.8\}, \{0.4\|0.2, 0.8\|0.1, 0.3\|0.7\}\rangle$
C_4	$\langle\{0.6\|1\}, \{0.2\|0.6, 0.1\|0.4\}, \{0.4\|0.7, 0.3\|0.3\}\rangle$

Table 2
PSHFDM M_2 with respect to X_2.

Attributes	Supplier X_2
C_1	$\langle\{0.6\|0.2, 0.1\|0.1, 0.2\|0.7\}, \{0.3\|0.5, 0.4\|0.5\}, \{0.4\|0.2, 0.4\|0.3, 0.3\|0.5\}\rangle$
C_2	$\langle\{0.7\|0.2, 0.3\|0.1, 0.2\|0.7\}, \{0.3\|0.5, 0.2\|0.5\}, \{0.4\|0.2, 0.5\|0.3, 0.2\|0.5\}\rangle$
C_3	$\langle\{0.6\|0.2, 0.3\|0.1, 0.4\|0.7\}, \{0.2\|0.5, 0.3\|0.5\}, \{0.3\|0.2, 0.4\|0.3, 0.2\|0.5\}\rangle$
C_4	$\langle\{0.1\|0.2, 0.3\|0.1, 0.8\|0.7\}, \{0.1\|0.5, 0.3\|0.5\}, \{0.3\|0.2, 0.4\|0.3, 0.5\|0.5\}\rangle$

Table 3
PSHFDM M_3 with respect to X_3.

Attributes	Supplier X_3
C_1	$\langle\{0.2\|0.2, 0.5\|0.1, 0.5\|0.7\}, \{0.6\|0.5, 0.2\|0.5\}, \{0.3\|0.2, 0.2\|0.3, 0.4\|0.5\}\rangle$
C_2	$\langle\{0.1\|0.2, 0.3\|0.1, 0.7\|0.7\}, \{0.3\|0.5, 0.2\|0.5\}, \{0.5\|0.2, 0.4\|0.3, 0.3\|0.5\}\rangle$
C_3	$\langle\{0.5\|0.2, 0.1\|0.1, 0.2\|0.7\}, \{0.3\|0.5, 0.4\|0.5\}, \{0.4\|0.2, 0.2\|0.3, 0.3\|0.5\}\rangle$
C_4	$\langle\{0.7\|0.2, 0.2\|0.8\}, \{0.3\|0.5, 0.4\|0.5\}, \{0.5\|0.2, 0.3\|0.3, 0.4\|0.5\}\rangle$

Table 4
PSHFDM M_4 with respect to X_4.

Attributes	Supplier X_4
C_1	$\langle\{0.6\|0.2, 0.5\|0.1, 0.7\|0.7\}, \{0.4\|0.5, 0.2\|0.5\}, \{0.5\|0.2, 0.4\|0.3, 0.1\|0.5\}\rangle$
C_2	$\langle\{0.5\|1\}, \{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.2, 0.1\|0.3, 0.2\|0.5\}\rangle$
C_3	$\langle\{0.5\|0.2, 0.3\|0.1, 0.4\|0.7\}, \{0.4\|0.5, 0.5\|0.5\}, \{0.6\|0.2, 0.2\|0.8\}\rangle$
C_4	$\langle\{0.6\|0.2, 0.5\|0.1, 0.2\|0.7\}, \{0.3\|0.5, 0.2\|0.5\}, \{0.6\|0.2, 0.5\|0.3, 0.2\|0.5\}\rangle$

is a refusal-membership hesitant degree of m. Next, an example of supplier selection is used to explain PSHFS: there are the following four alternatives X_a $(a = 1, 2, 3, 4)$ with four criteria: supply capacity (C_1), logistics speed (C_2), product quality (C_3), product price (C_4); all evaluation values are listed in Tables 1, 2, 3 and 4. Each table is named as probabilistic spherical hesitant fuzzy decision matrix (PSHFDM).

In the process of using PSHFE for actual MADM problems, we need to rank PSHFEs. Therefore, we will propose indicators for comparing two PSHFEs, namely score function and accurate function of PSHFE, and a theorem for determining the size relationship of two PSHFEs is given.

Definition 7. Let $m = \langle w|p, e|q, r|l\rangle$ be a PSHFE, then the score and accuracy functions are defined as

$$s(m) = \frac{2 + \sum_{i=1}^{\#w} \mu_i^2 p_i - \sum_{j=1}^{\#e} \nu_j^2 q_j - \sum_{k=1}^{\#r} o_k^2 l_k}{3}, \quad s(m) \in [0, 1], \tag{2}$$

and

$$h(m)=\frac{\sum_{i=1}^{\#w}\mu_i^2 p_i+\sum_{j=1}^{\#e}\nu_j^2 q_j+\sum_{k=1}^{\#r}o_k^2 l_k}{3},\quad h(m)\in[0,1]. \tag{3}$$

Theorem 1. *Let $m_1=\langle w_1|p_1,e_1|q_1,r_1|l_1\rangle$ and $m_2=\langle w_2|p_2,e_2|q_2,r_2|l_2\rangle$ be two PSHFEs, then*

(1) *If $s(m_1)>s(m_1)$, then $m_1>m_2$;*
(2) *If $s(m_1)=s(m_1)$, then*
  (a) *If $h(m_1)>h(m_1)$, then $m_1>m_2$;*
  (b) *If $h(m_1)=h(m_1)$, then $m_1=m_2$;*
  (c) *If $h(m_1)<h(m_1)$, then $m_1<m_2$.*

Example 1. Let $m_1=\langle\{0.4|1\},\{0.5|0.5,0.3|0.5\},\{0.5|0.5,0.2|0.5\}\rangle$ and $m_2=\langle\{0.6|0.5,0.6|0.5\},\{0.7|1\},\{0.2|0.5,0.1|0.5\}\rangle$ be two PSHFEs, according to the Theorem 1, we can get $s(m_1)=0.615$, $s(m_2)=0.615$, $h(m_1)=0.1583$ and $h(m_2)=0.2917$, then $m_1<m_2$.

Inspired by the operations of PIHFEs and PHFEs, we propose the operations of PSHFEs.

Definition 8. Let $m=\langle w|p,e|q,r|l\rangle$, $m_1=\langle w_1|p_1,e_1|q_1,r_1|l_1\rangle$ and $m_2=\langle w_2|p_2,e_2|q_2,r_2|l_2\rangle$ be three PSHFEs, $\xi>0$, and m^C represents the complementary set of m, and some operations of PSHFEs are defined as follows.

(1) $$m^C=\bigcup_{\mu\in w,\nu\in e,o\in r}\left\{\left\langle o|l_o,\nu|q_\nu,\mu|p_\mu\right\rangle\right\};$$

(2) $$\begin{aligned}m_1\oplus m_2&=\{w_1\oplus w_2,e_1\otimes e_2,r_1\otimes r_2\}\\&=\bigcup_{\substack{\mu_1\in w_1,\nu_1\in e_1,o_1\in r_1;\\ \mu_2\in w_2,\nu_2\in e_2,o_2\in r_2}}\left\{\left\langle\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}|p_1p_2,\nu_1\nu_2|q_1q_2,o_1o_2|l_1l_2\right\rangle\right\};\end{aligned}$$

(3) $$\begin{aligned}m_1\otimes m_2&=\{w_1\otimes w_2,e_1\oplus e_2,r_1\oplus r_2\}\\&=\bigcup_{\substack{\mu_1\in w_1,\nu_1\in e_1,o_1\in r_1;\\ \mu_2\in w_2,\nu_2\in e_2,o_2\in r_2}}\left\{\left\langle(\mu_1\mu_2)|p_1p_2,\sqrt{\nu_1^2+\nu_2^2-\nu_1^2\nu_2^2}|q_1q_2,\sqrt{o_1^2+o_2^2-o_1^2o_2^2}|l_1l_2\right\rangle\right\};\end{aligned}$$

(4) $$\xi m=\bigcup_{\mu\in w,\nu\in e,o\in r}\left\{\left\langle\sqrt{1-\left(1-\mu^2\right)^\xi}\,|p_\mu,\nu^\xi|q_\nu,o^\xi|l_o\right\rangle\right\};$$

(5) $$m^\xi=\bigcup_{\mu\in w,\nu\in e,o\in r}\left\{\left\langle\mu^\xi|p_\mu,\sqrt{1-\left(1-\nu^2\right)^\xi}\,|q_\nu,\sqrt{1-\left(1-o^2\right)^\xi}\,|l_o\right\rangle\right\}.$$

Example 2. Let $m_1=\langle\{0.5|1\},\{0.2|0.5,0.3|0.5\},\{0.1|0.4,0.3|0.6\}\rangle$ and $m_2=\langle\{0.5|0.5,0.4|0.5\},\{0.4|1\},\{0.2|0.3,0.3|0.7\}\rangle$ be two PSHFEs, $\xi=2$, then

(1) $m_1^C = \langle\{0.1|0.5, 0.3|0.5\}, \{0.2|0.5, 0.3|0.5\}, \{0.5|1\}\rangle$;

(2) $m_1 \oplus m_2 = \left\langle \begin{array}{l} \{0.6614|0.5, 0.6083|0.5\}, \{0.08|0.5, 0.12|0.5\}, \\ \{0.02|0.12, 0.03|0.28, 0.06|0.18, 0.09|0.42\} \end{array} \right\rangle$;

(3) $m_1 \otimes m_2 = \left\langle \begin{array}{l} \{0.25|0.5, 0.2|0.5\}, \{0.44|0.5, 0.4854|0.5\}, \\ \{0.2227|0.12, 0.3148|0.28, 0.3555|0.18, 0.4146|0.42\} \end{array} \right\rangle$;

(4) $2m_1 = \langle\{0.6614|1\}, \{0.04|0.5, 0.09|0.5\}, \{0.01|0.4, 0.09|0.6\}\rangle$;

(5) $m_1^2 = \langle\{0.25|1\}, \{0.28|0.5, 0.4146|0.5\}, \{0.1411|0.4, 0.4146|0.6\}\rangle$.

Based on the above operations of PSHFEs, it is obvious that the following theorem is true.

Theorem 2. *Let* $m = \langle w|p, e|q, r|l\rangle$, $m_1 = \langle w_1|p_1, e_1|q_1, r_1|l_1\rangle$ *and* $m_2 = \langle w_2|p_2, e_2|q_2, r_2|l_2\rangle$ *be three PSHFEs,* $\xi, \xi_1, \xi_2 > 0$, *then* (1) $m_1 \oplus m_2 = m_2 \oplus m_1$; (2) $m_1 \otimes m_2 = m_2 \otimes m_1$; (3) $\xi(m_1 \oplus m_2) = \xi m_2 \oplus \xi m_1$; (4) $(m_1 \otimes m_2)^\xi = m_1^\xi \otimes m_2^\xi$; (5) $\xi_1 m \oplus \xi_2 m = (\xi_1 + \xi_2)m$; (6) $m^{\xi_1} \oplus m^{\xi_2} = m^{(\xi_1+\xi_2)}$; (7) $(m^{\xi_1})^{\xi_2} = m^{\xi_1\xi_2}$.

4. Generalized Probabilistic Spherical Hesitant Fuzzy Aggregation Operators

The GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators are built by combining with the definition and operations of PSHFE. Then, some valued properties of them are studied, and some special operators of the above built generalized operators under PSHF setting are presented.

4.1. *The GPSHFWA Operator*

Definition 9. Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, then the GPSHFWA operator is recorded as $GPSHFWA^\xi(m_1, m_2, \dots, m_n) = (\vartheta_1 m_1^\xi \oplus \vartheta_2 m_2^\xi \oplus \cdots \oplus \vartheta_n m_n^\xi)^{1/\xi} = (\bigoplus_{i=1}^n \vartheta_i m_i^\xi)^{1/\xi}$, where ϑ_i is the weight of each PSHFE m_i, and $\vartheta_i \geqslant 0$, $\sum_{i=1}^n \vartheta_i = 1$.

Some following meaningful theorems are proven real with the operations of PSHFEs in Definition 8.

Theorem 3. *Let* $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ *be a group of PSHFEs, then the aggregated result by utilizing the GPSHFWA operator is still a PSHFE, and*

$$GPSHFWA^\xi(m_1, m_2, \dots, m_n)$$
$$= \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \{(\sqrt{1 - \Pi_{i=1}^n (1 - \mu_i^{2\xi})^{\vartheta_i}})^{1/\xi} | \Pi_{i=1}^n p_{\mu_i}\}, \\ \{\sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1 - \nu_i^2)^\xi)^{\vartheta_i})^{1/\xi}} | \Pi_{i=1}^n q_{\nu_i}\}, \\ \{\sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1 - o_i^2)^\xi)^{\vartheta_i})^{1/\xi}} | \Pi_{i=1}^n l_{o_i}\} \end{array} \right\rangle \right\}.$$

Proof. According to the operations of PSHFEs, we can have

$$m_i^{\xi} = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \mu_i^{\xi} \middle| p_{\mu_i}, \sqrt{1-\left(1-\nu_i^2\right)^{\xi}} \middle| q_{\nu_i}, \sqrt{1-\left(1-o_i^2\right)^{\xi}} \middle| l_{o_i} \right\rangle \right\},$$

furthermore,

$$\vartheta_i m_i^{\xi} = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \sqrt{1-\left(1-\mu_i^{2\xi}\right)^{\vartheta_i}} \middle| p_{\mu_i}, \left(\sqrt{1-\left(1-\nu_i^2\right)^{\xi}}\right)^{\vartheta_i} \middle| q_{\nu_i}, \left(\sqrt{1-\left(1-o_i^2\right)^{\xi}}\right)^{\vartheta_i} \middle| l_{o_i} \right\rangle \right\},$$

thus,

$$\bigoplus_{i=1}^{n} \vartheta_i m_i^{\xi} = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \sqrt{1-\Pi_{i=1}^{n}(1-\mu_i^{2\xi})^{\vartheta_i}} \middle| \Pi_{i=1}^{n} p_{\mu_i}, \\ \Pi_{i=1}^{n}\left(\sqrt{1-(1-\nu_i^2)^{\xi}}\right)^{\vartheta_i} \middle| \Pi_{i=1}^{n} q_{\nu_i}, \\ \Pi_{i=1}^{n}\left(\sqrt{1-(1-o_i^2)^{\xi}}\right)^{\vartheta_i} \middle| \Pi_{i=1}^{n} l_{o_i} \end{array} \right\rangle \right\},$$

therefore,

$$\left(\bigoplus_{i=1}^{n} \vartheta_i m_i^{\xi}\right)^{1/\xi} = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \left(\sqrt{1-\Pi_{i=1}^{n}(1-\mu_i^{2\xi})^{\vartheta_i}}\right)^{1/\xi} \middle| \Pi_{i=1}^{n} p_{\mu_i}, \\ \sqrt{1-\left(1-\left(\Pi_{i=1}^{n}(1-(1-\nu_i^2)^{\xi})^{\vartheta_i}\right)\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} q_{\nu_i}, \\ \sqrt{1-\left(1-\left(\Pi_{i=1}^{n}(1-(1-o_i^2)^{\xi})^{\vartheta_i}\right)\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} l_{o_i} \end{array} \right\rangle \right\},$$

finally, we have

$$\begin{aligned} &GPSHFWA^{\xi}(m_1, m_2, \dots, m_n) \\ &= \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \left\{\left(\sqrt{1-\Pi_{i=1}^{n}(1-\mu_i^{2\xi})^{\vartheta_i}}\right)^{1/\xi} \middle| \Pi_{i=1}^{n} p_{\mu_i}\right\}, \\ \left\{\sqrt{1-\left(1-\Pi_{i=1}^{n}(1-(1-\nu_i^2)^{\xi})^{\vartheta_i}\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} q_{\nu_i}\right\}, \\ \left\{\sqrt{1-\left(1-\Pi_{i=1}^{n}(1-(1-o_i^2)^{\xi})^{\vartheta_i}\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} l_{o_i}\right\} \end{array} \right\rangle \right\}. \end{aligned}$$

□

Theorem 4 (Boundedness). *Let* $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ *be a group of PSHFEs, if* $m^+ = \langle\{\mu^+|p_{\mu^+}\}, \{\nu^-|q_{\nu^-}\}, \{o^-|l_{o^-}\}\rangle$ *and* $m^- = \langle\{\mu^-|p_{\mu^-}\}, \{\nu^+|q_{\nu^+}\}, \{o^+|l_{o^+}\}\rangle$*, where* $\mu^+ = \bigcup_{\mu_i \in w_i} \max\{\mu_i\}$*,* $\nu^+ = \bigcup_{\nu_j \in e_j} \max\{\nu_j\}$*,* $o^+ = \bigcup_{\mu_k \in w_k} \max\{o_k\}$*,* $\mu^- = \bigcup_{\mu_i \in w_i} \min\{\mu_i\}$*,* $\nu^- = \bigcup_{\nu_j \in e_j} \min\{\nu_j\}$*,* $o^- = \bigcup_{\mu_k \in w_k} \min\{o_k\}$*, then*

$$m^- \leqslant GPSHFWA^{\xi}(m_1, m_2, \dots, m_n) \leqslant m^+.$$

Proof. For $\xi \in (0, +\infty)$ and all i, we have $\mu^- \leqslant \mu_i \leqslant \mu^+$, then

$$\begin{aligned}
&\mu_i^{2\xi} \geqslant (\mu^-)^{2\xi}, \qquad 1-\mu_i^{2\xi} \leqslant 1-(\mu^-)^{2\xi}, \quad (1-\mu_i^{2\xi})^{\vartheta_i} \leqslant (1-(\mu^-)^{2\xi})^{\vartheta_i},\\
&\Pi_{i=1}^n(1-\mu_i^{2\xi})^{\vartheta_i} \leqslant \Pi_{i=1}^n(1-(\mu^-)^{2\xi})^{\vartheta_i},\\
&1-\Pi_{i=1}^n(1-\mu_i^{2\xi})^{\vartheta_i} \geqslant 1-\Pi_{i=1}^n(1-(\mu^-)^{2\xi})^{\vartheta_i},\\
&\left(\sqrt{1-\Pi_{i=1}^n(1-\mu_i^{2\xi})^{\vartheta_i}}\right)^{1/\xi} \geqslant \left(\sqrt{1-\Pi_{i=1}^n(1-(\mu^-)^{2\xi})^{\vartheta_i}}\right)^{1/\xi} = \mu^-,
\end{aligned}$$

similarly, we have

$$\left(\sqrt{1-\Pi_{i=1}^n(1-\mu_i^{2\xi})^{\vartheta_i}}\right)^{1/\xi} \leqslant \left(\sqrt{1-\Pi_{i=1}^n(1-(\mu^+)^{2\xi})^{\vartheta_i}}\right)^{1/\xi} = \mu^+.$$

Since $\nu^- \leqslant \nu_i \leqslant \nu^+$, then

$$\begin{aligned}
&1-\nu_i^2 \leqslant 1-(\nu^-)^2, \qquad (1-\nu_i^2)^{\xi} \leqslant (1-(\nu^-)^2)^{\xi},\\
&1-(1-\nu_i^2)^{\xi} \geqslant 1-(1-(\nu^-)^2)^{\xi},\\
&(1-(1-\mu_i^2)^{\xi})^{\vartheta_i} \geqslant (1-(1-(\mu^-)^2)^{\xi})^{\vartheta_i},\\
&\Pi_{i=1}^n(1-(1-\mu_i^2)^{\xi})^{\vartheta_i} \geqslant \Pi_{i=1}^n(1-(1-(\mu^-)^2)^{\xi})^{\vartheta_i},\\
&1-\Pi_{i=1}^n(1-(1-\mu_i^2)^{\xi})^{\vartheta_i} \leqslant 1-\Pi_{i=1}^n(1-(1-(\mu^-)^2)^{\xi})^{\vartheta_i},\\
&1-(1-\Pi_{i=1}^n(1-(1-\mu_i^2)^{\xi})^{\vartheta_i})^{1/\xi} \geqslant 1-(1-\Pi_{i=1}^n(1-(1-(\mu^-)^2)^{\xi})^{\vartheta_i})^{1/\xi},\\
&\sqrt{1-(1-\Pi_{i=1}^n(1-(1-\nu_i^2)^{\xi})^{\vartheta_i})^{1/\xi}} \geqslant \sqrt{1-(1-\Pi_{i=1}^n(1-(1-(\nu^-)^2)^{\xi})^{\vartheta_i})^{1/\xi}} = \nu^-.
\end{aligned}$$

Similarly, we have

$$\sqrt{1-(1-\Pi_{i=1}^n(1-(1-\nu_i^2)^{\xi})^{\vartheta_i})^{1/\xi}} \leqslant \sqrt{1-(1-\Pi_{i=1}^n(1-(1-(\nu^+)^2)^{\xi})^{\vartheta_i})^{1/\xi}} = \nu^+.$$

Again since $o^- \leqslant o_i \leqslant o^+$, then we have

$$o^- \leqslant \sqrt{1-(1-\Pi_{i=1}^n(1-(1-o_i^2)^{\xi})^{\vartheta_i})^{1/\xi}} \leqslant o^+.$$

Let $GSHFWA^{\xi}(m_1, m_2, \ldots, m_n) = m = \langle \mu, \nu, o\rangle$, then

$$\begin{aligned}
s(m) &= \frac{1}{2}\left(1+\sum_{i=1}^{\#W}(\mu_i)^2 p_{\mu_i} - \sum_{j=1}^{\#e}(\nu_j)^2 q_{\nu_i} - \sum_{k=1}^{\#r}(o_k)^2 l_{o_i}\right)\\
&\geqslant \frac{1}{2}\left(1+\sum_{i=1}^{\#W}(\mu^-)^2 p_{\mu^-} - \sum_{j=1}^{\#e}(\nu^-)^2 q_{\nu^-} - \sum_{k=1}^{\#r}(o^-)^2 l_{o^-}\right) = s(m^-),
\end{aligned}$$

and

$$s(m)=\frac{1}{2}\left(1+\sum_{i=1}^{\#W}(\mu_i)^2 p_{\mu_i}-\sum_{j=1}^{\#e}(\nu_j)^2 q_{\nu_i}-\sum_{k=1}^{\#r}(o_k)^2 l_{o_i}\right)$$
$$\leqslant \frac{1}{2}\left(1+\sum_{i=1}^{\#W}\left(\mu^+\right)^2 p_{\mu^+}-\sum_{j=1}^{\#e}\left(\nu^+\right)^2 q_{\nu^+}-\sum_{k=1}^{\#r}\left(o^+\right)^2 l_{o^+}\right)=s\left(m^+\right),$$

hence, we can obtain

$$m^- \leqslant GPSHFWA^{\xi}(m_1, m_2, \ldots, m_n) \leqslant m^+, \quad \xi \in (0, +\infty).$$

Similarly, we can also obtain

$$m^- \leqslant GPSHFWA^{\xi}(m_1, m_2, \cdots, m_n) \leqslant m^+, \quad \xi \in (-\infty, 0).$$

This property indicates that when we aggregate multiple PSHFEs, the resulting aggregation falls between the minimum PSHFE and maximum PSHFE among the aggregated PSHFEs. This property allows us to better estimate the minimum value and maximum value of the aggregated results. □

Theorem 5 (Monotonicity). *Let* $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \ldots, n)$ *and* $m_i^* = \langle w_i^*|p_{w_i^*}, e_i^*|q_{e_i^*}, r_i^*|l_{r_i^*}\rangle$ *be two groups of PSHFEs, if* $m_i \leqslant m_i^*$ *for all i, namely* $\mu_i \leqslant \mu_i^*$, $\upsilon_i \geqslant \upsilon_i^*$, $o_i \geqslant o_i^*$, $p_{\mu_i} \leqslant p_{\mu_i^*}$, $q_{\upsilon_i} \geqslant q_{\upsilon_i^*}$ *and* $l_{o_i} \geqslant l_{o_i^*}$ *for all i, then*

$$GPSHFWA^{\xi}(m_1, m_2, \ldots, m_n) \leqslant GPSHFWA^{\xi}\left(m_1^*, m_2^*, \cdots, m_n^*\right).$$

Proof. According to $\mu_i \leqslant \mu_i^*$, $\upsilon_i \geqslant \upsilon_i^*$, $o_i \geqslant o_i^*$, we have

$$\left(\sqrt{1-\Pi_{i=1}^n\left(1-\mu_i^{2\xi}\right)^{\vartheta_i}}\right)^{1/\xi} \leqslant \left(\sqrt{1-\Pi_{i=1}^n\left(1-\left(\mu_i^*\right)^{2\xi}\right)^{\vartheta_i}}\right)^{1/\xi},$$
$$\sqrt{1-\left(1-\Pi_{i=1}^n\left(1-\left(1-\nu_i^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} \geqslant \sqrt{1-\left(1-\Pi_{i=1}^n\left(1-\left(1-\left(\nu_i^*\right)^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}},$$

and

$$\sqrt{1-\left(1-\Pi_{i=1}^n\left(1-\left(1-o_i^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} \geqslant \sqrt{1-\left(1-\Pi_{i=1}^n\left(1-\left(1-\left(o_i^*\right)^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}}.$$

Again,

$$\sum_{i=1}^{\#w_i}\left(\sqrt{1-\Pi_{i=1}^n\left(1-\mu_i^{2\xi}\right)^{\vartheta_i}}\right)^{1/\xi} p_{w_i} \leqslant \sum_{i=1}^{\#w_i^*}\left(\sqrt{1-\Pi_{i=1}^n\left(1-\left(\mu_i^*\right)^{2\xi}\right)^{\vartheta_i}}\right)^{1/\xi} p_{\mu_i^*},$$

$$\sum_{i=1}^{\#\nu_i} \sqrt{1-\left(1-\Pi_{i=1}^{n}\left(1-\left(1-\nu_i^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} q_{\nu_i}$$
$$\geqslant \sum_{i=1}^{\#\nu_i^*} \sqrt{1-\left(1-\Pi_{i=1}^{n}\left(1-\left(1-\left(\nu_i^*\right)^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} q_{\nu_i^*},$$

and

$$\sum_{i=1}^{\#o_i} \sqrt{1-\left(1-\Pi_{i=1}^{n}\left(1-\left(1-o_i^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} l_{o_i}$$
$$\geqslant \sum_{i=1}^{\#o_i^*} \sqrt{1-\left(1-\Pi_{i=1}^{n}\left(1-\left(1-\left(o_i^*\right)^2\right)^{\xi}\right)^{\vartheta_i}\right)^{1/\xi}} l_{o_i^*}.$$

Hence, we can get $s(m) \leqslant s(m^*)$. Therefore, we have $GPSHFWA^{\xi}(m_1, m_2, \dots, m_n) \leqslant GPSHFWA^{\xi}(m_1^*, m_2^*, \dots, m_n^*)$. $\square$

This property tells us that when one group of PSHFEs is larger than another group of PSHFEs, the aggregated result is also larger, which helps us compare the size of the two groups of PSHFEs after aggregation. Some special valued operator of GPSHFWA operator can be obtained by adjusting the parameter ξ and weight ϑ though the detailed analysis.

Case 1. If $\xi = 1$, then the GPSHFWA operator reduces to the PSHF weighted averaging (PSHFWA) operator.

$$PSHFWA(m_1, m_2, \dots, m_n) = (\vartheta_1 m_1 \oplus \vartheta_2 m_2 \oplus \cdots \oplus \vartheta_n m_n) = \bigoplus_{i=1}^{n} (\vartheta_i m_i).$$

Case 2. If $\xi = 1$ and $\vartheta = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the GPSHFWA operator reduces to the PSHF arithmetic averaging (PSHFAA) operator.

$$PSHFAA(m_1, m_2, \dots, m_n) = \left(\frac{1}{n} m_1 \oplus \frac{1}{n} m_2 \oplus \cdots \oplus \frac{1}{n} m_n\right) = \frac{1}{n} \bigoplus_{i=1}^{n} m_i.$$

4.2. *The GPSHFWG Operator*

Definition 10. Let $m_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i} \rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, then the GPSHFWG operator is defined as $GPSHFWG^{\xi}(m_1, m_2, \dots, m_n) = \frac{1}{\xi}(\xi m_1^{\vartheta_1} \otimes \xi m_2^{\vartheta_2} \otimes \cdots \otimes \xi m_n^{\vartheta_n}) = \frac{1}{\xi} \bigotimes_{i=1}^{n} (\xi m_i)^{\vartheta_i}$, where ϑ_i is the weight of each PSHFE m_i, and $\vartheta_i \geqslant 0$, $\sum_{i=1}^{n} \vartheta_i = 1$.

Some following meaningful theorems are proven real with the operations of PSHFEs in Definition 8.

Theorem 6. *Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, then the aggregated result by utilizing the GPSHFWG operator is still a PSHFE, and*

$$\begin{aligned} &GPSHFWG^{\xi}(m_1, m_2, \dots, m_n) \\ &= \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \begin{array}{l} \Big\langle \sqrt{1-\big(1-\Pi_{i=1}^{n}\big(1-\big(1-\mu_i^2\big)^{\xi}\big)^{\vartheta_i}\big)^{1/\xi}} \,\big|\Pi_{i=1}^{n} p_{\mu_i}, \\ \big(\sqrt{1-\Pi_{i=1}^{n}\big(1-\nu_i^{2\xi}\big)^{\vartheta_i}}\big)^{1/\xi} \,\big|\Pi_{i=1}^{n} q_{\nu_i}, \\ \big(\sqrt{1-\Pi_{i=1}^{n}\big(1-o_i^{2\xi}\big)^{\vartheta_i}}\big)^{1/\xi} \,\big|\Pi_{i=1}^{n} l_{o_i} \Big\rangle \end{array} \right\}. \end{aligned}$$

Proof. The proof process of this theorem is similar to that of Theorem 3, and is omitted here. □

Theorem 7 (Boundedness). *Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, if $m^+ = \langle\{\mu^+|p_{\mu^+}\}, \{\nu^-|q_{\nu^-}\}, \{o^-|l_{o^-}\}\rangle$ and $m^- = \langle\{\mu^-|p_{\mu^-}\}, \{\nu^+|q_{\nu^+}\}, \{o^+|l_{o^+}\}\rangle$, where*

$$\begin{aligned} \mu^+ &= \bigcup_{\mu_i \in w_i} \max\{\mu_i\}, & \nu^+ &= \bigcup_{\nu_j \in e_j} \max\{\nu_j\}, & o^+ &= \bigcup_{\mu_k \in w_k} \max\{o_k\}, \\ \mu^- &= \bigcup_{\mu_i \in w_i} \min\{\mu_i\}, & \nu^- &= \bigcup_{\nu_j \in e_j} \min\{\nu_j\}, & o^- &= \bigcup_{\mu_k \in w_k} \min\{o_k\}, \end{aligned}$$

then

$$m^- \leqslant GPSHFWG^{\xi}(m_1, m_2, \dots, m_n) \leqslant m^+.$$

Proof. The proof process of this theorem is similar to that of Theorem 4, and is omitted here. □

Theorem 8 (Monotonicity). *Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ and $m_i^* = \langle w_i^*|p_{w_i^*}, e_i^*|q_{e_i^*}, r_i^*|l_{r_i^*}\rangle$ be two groups of PSHFEs, if $m_i \leqslant m_i^*$ for all i, then $GPSHFWG^{\xi}(m_1, m_2, \dots, m_n) \leqslant GPSHFWG^{\xi}(m_1^*, m_2^*, \dots, m_n^*)$.*

Proof. The proof process of this theorem is similar to that of Theorem 5, and is omitted here. □

Some special valued operator of GPSHFWG operator can be obtained by adjusting the parameters ξ and weight ϑ though the detailed analysis.

Case 3. If $\xi = 1$, then the GPSHFWG operator reduces to the PSHF weighted geometric (PSHFWG) operator.

$$PSHFWG(m_1, m_2, \dots, m_n) = \big(m_1^{\vartheta_1} \otimes m_2^{\vartheta_2} \otimes \cdots \otimes m_n^{\vartheta_n}\big) = \bigotimes_{i=1}^{n} m_i^{\vartheta_i}.$$

Case 4. If $\xi = 1$ and $\vartheta = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$, then the GPSHFWG operator reduces to the PSHF geometric averaging (PSHFGA) operator.

$$PSHFGA(m_1, m_2, \dots, m_n) = (m_1 \otimes m_2 \otimes \cdots \otimes m_n)^{1/n} = \bigotimes_{i=1}^{n} \left(m_i^{1/n}\right).$$

4.3. *The GPSHFPWA Operator*

In practical decision-making problems, evaluation indicators often have different priorities. For example, if parents buy toys for their children, when considering both price and safety, it is clear that safety is more important than price. It is obvious that the two operators GPSHFWA and GPSHFWG defined above and their special cases cannot tackle such decision-making problem. However, inspired by the PA operator built by Naeem *et al.* (2021), we extended it to the PSHF environment, developed GPSHFPWA and GPSHFPWG operators, and investigated several special meaningful cases.

Definition 11. Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, then the GPSHFPWA operator is defined as $GPSHFPWA^{\xi}(m_1, m_2, \dots, m_n) = (\frac{S_1}{\sum_{i=1}^{n} S_i} m_1^{\xi} \oplus \frac{S_2}{\sum_{i=1}^{n} S_i} m_2^{\xi} \oplus \cdots \oplus \frac{S_n}{\sum_{i=1}^{n} S_i} m_n^{\xi})^{1/\xi}$, where $S_i = \prod_{r=1}^{i-1} s(m_r)$ $(i = 1, 2, \dots, n)$, $S_1 = 1$, and $s(m_r)$ represents the score value of PSHFEs m_i.

Depending on the operations of PSHFEs in Definition 8, the following theorems can be obtained.

Theorem 9. *Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, then the aggregated result by utilizing the GPSHFPWA operator is still a PSHFE, and*

$$GPSHFPWA^{\xi}(m_1, m_2, \dots, m_n)$$
$$= \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \left\{ \left(\sqrt{1 - \Pi_{i=1}^{n} (1 - \mu_i^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}} \right)^{1/\xi} \middle| \Pi_{i=1}^{n} p_{\mu_i} \right\}, \\ \left\{ \sqrt{1 - \left(1 - \Pi_{i=1}^{n} (1 - (1 - \nu_i^2)^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} q_{\nu_i} \right\}, \\ \left\{ \sqrt{1 - \left(1 - \Pi_{i=1}^{n} (1 - (1 - o_i^2)^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}\right)^{1/\xi}} \middle| \Pi_{i=1}^{n} l_{o_i} \right\} \end{array} \right\rangle \right\}.$$

Proof. The proof process of this theorem is similar to that of Theorem 3, and is omitted here. □

Theorem 10 (Boundedness). *Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a group of PSHFEs, if $m^+ = \langle \{\mu^+|p_{\mu^+}\}, \{\nu^-|q_{\nu^-}\}, \{o^-|l_{o^-}\}\rangle$ and $m^- = \langle \{\mu^-|p_{\mu^-}\}, \{\nu^+|q_{\nu^+}\}, \{o^+|l_{o^+}\}\rangle$, where $\mu^+ = \bigcup_{\mu_i \in w_i} \max\{\mu_i\}$, $\nu^+ = \bigcup_{\nu_j \in e_j} \max\{\nu_j\}$, $o^+ = \bigcup_{\mu_k \in w_k} \max\{o_k\}$, $\mu^- = \bigcup_{\mu_i \in w_i} \min\{\mu_i\}$, $\nu^- = \bigcup_{\nu_j \in e_j} \min\{\nu_j\}$, $o^- = \bigcup_{\mu_k \in w_k} \min\{o_k\}$, then*

$$m^- \leqslant GPSHFPWA^{\xi}(m_1, m_2, \dots, m_n) \leqslant m^+.$$

Proof. The proof process of this theorem is similar to that of Theorem 4, and is omitted here. □

Theorem 11 (Monotonicity). *Let* $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ *and* $m_i^* = \langle w_i^*|p_{w_i^*}, e_i^*|q_{e_i^*}, r_i^*|l_{r_i^*}\rangle$ $(i = 1, 2, \dots, n)$ *be two groups of PSHFEs, if* $m_i \leqslant m_i^*$ *for all* i*, then*

$$GPSHFPWA^{\xi}(m_1, m_2, \dots, m_n) \leqslant GPSHFPWA^{\xi}\left(m_1^*, m_2^*, \dots, m_n^*\right).$$

Proof. The proof process of this theorem is similar to that of Theorem 5, and is omitted here. □

Some special valued operator of GPSHFPWA operator can be obtained by adjusting the parameters ξ, weight ϑ and priority of evaluation indicators though the detailed analysis.

Case 5. If $\xi = 1$, then the GPSHFPWA operator reduces to the PSHF prioritized weighted averaging (PSHFPWA) operator:

$$PSHFPWA(m_1, m_2, \dots, m_n) = \left(\frac{S_1}{\sum_{i=1}^{n} S_i} m_1 \oplus \frac{S_2}{\sum_{i=1}^{n} S_i} m_2 \oplus \dots \oplus \frac{S_n}{\sum_{i=1}^{n} S_i} m_n\right).$$

Case 6. If $\xi = 1$ and all evaluation indicators have the same priority, then the GPSHFPWA operator reduces to the PSHF weighted averaging (PSHFWA) operator: $PSHFWA(m_1, m_2, \dots, m_n) = (\vartheta_1 m_1 \oplus \vartheta_2 m_2 \oplus \dots \oplus \vartheta_n m_n)$.

Case 7. If $\xi = 1$, $\vartheta = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$ and all evaluation indicators have the same priority, then the GPSHFPWA operator reduces to the PSHFAA operator: $PSHFAA(m_1, m_2, \dots, m_n) = (\frac{1}{n} m_1 \oplus \frac{1}{n} m_2 \oplus \dots \oplus \frac{1}{n} m_n) = \frac{1}{n} \bigoplus_{i=1}^{n} m_i$.

4.4. *The GPSHFPWG Operator*

Definition 12. Let $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ be a collection of PSHFEs, then the GPSHFPWG operator is defined as $GPSHFPWG^{\xi}(m_1, m_2, \dots, m_n) = \frac{1}{\xi}\left((\xi m_1)^{\frac{S_1}{\sum_{i=1}^{n} S_i}} \otimes (\xi m_2)^{\frac{S_2}{\sum_{i=1}^{n} S_i}} \otimes \dots \otimes (\xi m_n)^{\frac{S_n}{\sum_{i=1}^{n} S_i}}\right)$, where $S_i = \prod_{r=1}^{i-1} s(m_r)$ $(i = 1, 2, \dots, n)$, $S_1 = 1$, and $s(m_r)$ represents the score value of PSHFEs m_i.

Depending on the operations of PSHFEs in Definition 8, the following theorems can be obtained.

Theorem 12. *Let* $m_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$ $(i = 1, 2, \dots, n)$ *be a collection of PSHFEs, then the aggregated result by utilizing the GPSHFPWG operator is still a PSHFE,* *and*

$$GPSHFPWG^{\xi}(m_1, m_2, \dots, m_n)$$
$$= \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \sqrt{1 - \left(1 - \Pi_{i=1}^{n}(1 - (1 - \mu_i^2)^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}\right)^{1/\xi}} \Big| \Pi_{i=1}^{n} p_{\mu_i}, \\ \left(\sqrt{1 - \Pi_{i=1}^{n}(1 - \nu_i^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}}\right)^{1/\xi} \Big| \Pi_{i=1}^{n} q_{\nu_i}, \\ \left(\sqrt{1 - \Pi_{i=1}^{n}(1 - o_i^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}}\right)^{1/\xi} \Big| \Pi_{i=1}^{n} l_{o_i} \end{array} \right\rangle \right\}.$$

Proof. The proof process of this theorem is similar to that of Theorem 3, and is omitted here. □

Theorem 13 (Boundedness). *Let* $m_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i} \rangle$ $(i = 1, 2, \dots, n)$ *be a group of PSHFEs, if* $m^+ = \langle \{\mu^+ | p_{\mu^+}\}, \{\nu^- | q_{\nu^-}\}, \{o^- | l_{o^-}\} \rangle$ *and* $m^- = \langle \{\mu^- | p_{\mu^-}\}, \{\nu^+ | q_{\nu^+}\}, \{o^+ | l_{o^+}\} \rangle$*, where* $\mu^+ = \bigcup_{\mu_i \in w_i} \max\{\mu_i\}$*,* $\nu^+ = \bigcup_{\nu_j \in e_j} \max\{\nu_j\}$*,* $o^+ = \bigcup_{\mu_k \in w_k} \max\{o_k\}$*,* $\mu^- = \bigcup_{\mu_i \in w_i} \min\{\mu_i\}$*,* $\nu^- = \bigcup_{\nu_j \in e_j} \min\{\nu_j\}$*,* $o^- = \bigcup_{\mu_k \in w_k} \min\{o_k\}$*, then*

$$m^- \leqslant GPSHFPWG^{\xi}(m_1, m_2, \dots, m_n) \leqslant m^+.$$

Proof. The proof process of this theorem is similar to that of Theorem 4, and is omitted here. □

Theorem 14 (Monotonicity). *Let* $m_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i} \rangle$ $(i = 1, 2, \dots, n)$ *and* $m_i^* = \langle w_i^* | p_{w_i^*}, e_i^* | q_{e_i^*}, r_i^* | l_{r_i^*} \rangle$ $(i = 1, 2, \dots, n)$ *be two groups of PSHFEs, if* $m_i \leqslant m_i^*$ *for all* i*, then*

$$GPSHFPWG^{\xi}(m_1, m_2, \dots, m_n) \leqslant GPSHFPWG^{\xi}\left(m_1^*, m_2^*, \dots, m_n^*\right).$$

Proof. The proof process of this theorem is similar to that of Theorem 5, and is omitted here. □

Some special valued operator of GPSHFPWG operator can be obtained by adjusting the parameters ξ, weight ϑ and priority of evaluation indicators though the detailed analysis.

Case 8. If $\xi = 1$, then the GPSHFPWG operator reduces to the PSHF prioritized weighted averaging (PSHFPWG) operator:

$$GPSHFPWG(m_1, m_2, \dots, m_n) = m_1^{\frac{S_1}{\sum_{i=1}^{n} S_i}} \otimes m_2^{\frac{S_2}{\sum_{i=1}^{n} S_i}} \otimes \cdots \otimes m_n^{\frac{S_n}{\sum_{i=1}^{n} S_i}}.$$

Case 9. If $\xi = 1$ and all evaluation indicators have the same priority, then the GPSHFPWG operator reduces to the PSHFWG operator: $PSHFWG(m_1, m_2, \dots, m_n) = (m_1^{\vartheta_1} \otimes m_2^{\vartheta_2} \otimes \cdots \otimes m_n^{\vartheta_n}) = \otimes_{i=1}^{n} m_i^{\vartheta_i}$.

Case 10. If $\xi = 1$, $\vartheta = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$ and all evaluation indicators have the same priority, then the GPSHFPWG operator reduces to the PSHFGA operator: $PSHFGA(m_1, m_2, \ldots, m_n) = (m_1 \otimes m_2 \otimes \cdots \otimes m_n)^{1/n} = \bigotimes_{i=1}^{n}(m_i^{1/n})$.

5. MADM Methods under PSHF Setting

In this subsection, we will use the GPSHFWA, GPSHFWG, GPSHFPWA and GPSHF-PWG operators developed in this paper to deal with different MADM problems in PSHF environment. Let alternatives set is $AL = \{AL_1, AL_2, \ldots, AL_m\}$ with decision attributes set $AT = \{AT_1, AT_2, \ldots, AT_n\}$. Decision maker utilizes the PSHFEs to express the evaluation information under m alternatives with n decision attributes. Hence, a PSHF evaluation decision matrix $M = (m_{ij})_{m\times n}$ and $m_{ij} = \langle w_{ij}|p_{w_{ij}}, e_{ij}|q_{e_{ij}}, r_{ij}|l_{r_{ij}}\rangle$ is evaluation information when alternative AL_i under decision attribute AT_j. In general, decision attributes can be divided into benefit (B) and cost (C) decision attributes. We convert cost decision attributes into benefit decision attributes to obtain a normalized PSHF evaluation decision matrix $\overline{M} = (\overline{m}_{ij})_{m\times n}$ by Eq. (4), m_{ij}^{C} is based on Eq. (1) in Definition 8.

$$\overline{m}_{ij} = \begin{cases} m_{ij}, & \text{if } m_{ij} \in B, \\ m_{ij}^{C}, & \text{if } m_{ij} \in C. \end{cases} \tag{4}$$

Based on the above assumptions, we first consider the case where the MADM decision attributes have the same priority. ϑ_j is the weight of AT_j. We design a decision Algorithm 1 based on GPSHFWA or GPSHFWG operator. Figure 1 shows Algorithm 1 more intuitively, so that readers can understand Algorithm 1 more clearly.

Based on the above assumptions, we first consider the case where the MADM decision attributes have the different priority. ϑ_j is the weight of AT_j. We design a decision Algorithm 2 based on GPSHFWA or GPSHFWG operator. Figure 2 shows Algorithm 2 more intuitively, so that readers can understand Algorithm 2 more clearly.

6. Numerical Cases

China's industrial development has promoted the strong rise of the economy. The undertaking of a large number of global manufacturing industries has brought an unbearable burden on the ecological environment. Low-carbon sustainable green development has become a global consensus. In 2007, the Chinese government officially put forward the concept of "green credit" for the first time, hoping that financial institutions can adjust the demand and flow of social funds by carrying out green credit business to promote energy conservation and low carbon in green industries; In 2015, "green development" was proposed as one of the five development concepts. As the core financial institution in China's social financing system, the banking industry is an important driver and active practitioner of green credit business. Therefore, building a scientific and reasonable evaluation model

Algorithm 1 MADM technique based on the GPSHFWA or the GPSHFWG operator.

***Step* 1.** Convert the original PSHFDM M to the normalized PSHFDM $\overline{M}$ by employing the Eq. (4).

***Step* 2.** Utilize the GPSHFWA operator

$$GPSHFWA^{\xi}(\overline{m}_{i1},\overline{m}_{i2},\ldots,\overline{m}_{in}) = (\vartheta_1\overline{m}_{i1}^{\xi}\oplus\vartheta_2\overline{m}_{i2}^{\xi}\oplus\cdots\oplus\vartheta_n\overline{m}_{in}^{\xi})^{1/\xi} = \bigoplus_{j=1}^{n}(\vartheta_j\overline{m}_{ij}^{\xi})^{1/\xi}$$

$$= \bigcup_{\substack{\overline{\mu}_{i1}\in\overline{w}_{i1},\overline{\mu}_{i2}\in\overline{w}_{i2},\ldots,\overline{\mu}_{in}\in\overline{w}_{in};\\ \overline{\nu}_{i1}\in\overline{e}_{i1},\overline{\nu}_{i2}\in\overline{e}_{i2},\ldots,\overline{\nu}_{in}\in\overline{e}_{in};\\ \overline{o}_{i1}\in\overline{r}_{i1},\overline{o}_{i2}\in\overline{r}_{i2},\ldots,\overline{o}_{in}\in\overline{r}_{in}}} \left\{\left\langle \begin{array}{l} \left\{\left(\sqrt{1-\Pi_{i=1}^{n}(1-\mu^{2\xi}_{\overline{w}_{ij}})^{\vartheta_i}}\right)^{1/\xi} \middle| \Pi_{i=1}^{n}p_{\overline{w}_{ij}}\right\}, \\ \left\{\sqrt{1-(1-\Pi_{i=1}^{n}(1-(1-\nu^{2}_{\overline{e}_{ij}})^{\xi})^{\vartheta_i})^{1/\xi}} \middle| \Pi_{i=1}^{n}q_{\overline{e}_{ij}}\right\}, \\ \left\{\sqrt{1-(1-\Pi_{i=1}^{n}(1-(1-o^{2}_{\overline{r}_{ij}})^{\xi})^{\vartheta_i})^{1/\xi}} \middle| \Pi_{i=1}^{n}l_{\overline{r}_{ij}}\right\} \end{array}\right\rangle\right\}$$

or the GPSHFWG operator

$$GPSHFWG^{\xi}(m_1,m_2,\ldots,m_n) = \frac{1}{\xi}(\xi\overline{m}_{i1}^{\vartheta_1}\otimes\xi\overline{m}_{i2}^{\vartheta_2}\otimes\cdots\otimes\xi\overline{m}_{in}^{\vartheta_n}) = \frac{1}{\xi}\bigoplus_{j=1}^{n}(\xi\overline{m}_{ij}^{\vartheta_j})$$

$$= \bigcup_{\substack{\overline{\mu}_{i1}\in\overline{w}_{i1},\overline{\mu}_{i2}\in\overline{w}_{i2},\ldots,\overline{\mu}_{in}\in\overline{w}_{in};\\ \overline{\nu}_{i1}\in\overline{e}_{i1},\overline{\nu}_{i2}\in\overline{e}_{i2},\ldots,\overline{\nu}_{in}\in\overline{e}_{in};\\ \overline{o}_{i1}\in\overline{r}_{i1},\overline{o}_{i2}\in\overline{r}_{i2},\ldots,\overline{o}_{in}\in\overline{r}_{in}}} \left\{\left\langle \begin{array}{l} \sqrt{1-\left(1-\Pi_{i=1}^{n}(1-(1-\mu^{2}_{\overline{w}_{ij}})^{\xi})^{\vartheta_i}\right)^{1/\xi}} \middle| \Pi_{i=1}^{n}p_{\overline{w}_{ij}}, \\ \left(\sqrt{1-\Pi_{i=1}^{n}(1-\nu^{2\xi}_{\overline{e}_{ij}})^{\vartheta_i}}\right)^{1/\xi} \middle| \Pi_{i=1}^{n}q_{\overline{e}_{ij}}, \\ \left(\sqrt{1-\Pi_{i=1}^{n}(1-o^{2\xi}_{\overline{r}_{ij}})^{\vartheta_i}}\right)^{1/\xi} \middle| \Pi_{i=1}^{n}l_{\overline{r}_{ij}} \end{array}\right\rangle\right\}$$

to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$.

***Step* 3.** Calculate the $s(\tilde{m}_i)$ and $h(\tilde{m}_i)$ values of AL_i by employing Eqs. (2) and (3), and obtain the ranking of all alternatives by the comparison method of PSHFEs.

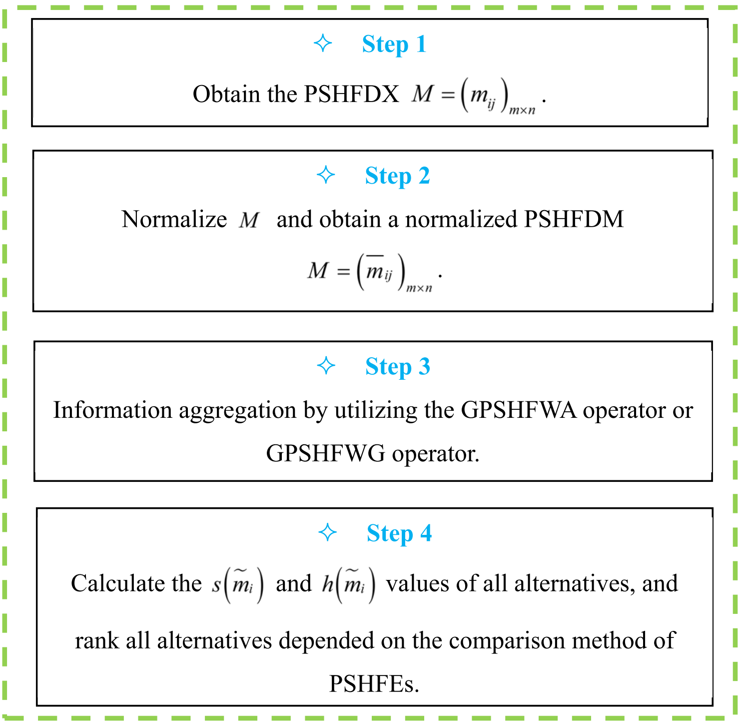

Fig. 1. The flowchart of the built Algorithm 1.

Algorithm 2 MADM technique based on the GPSHFPWA or the GPSHFPWG operator.

***Step* 1.** Convert the PSHFDM M to the normalized PSHFDM $\overline{M}$ by employing the Eq. (4).
***Step* 2.** Calculate S_{ij} value by employing the following equation.
$S_{ij} = \prod_{r=1}^{j-1} s(\overline{m}_{ir})$, $S_{i1} = 1$
***Step* 3.** Utilize the GPSHFPWA operator

$$GPPHFPWA^{\xi}(\overline{m}_{i1}, \overline{m}_{i2}, \dots, \overline{m}_{in}) = \Big(\frac{S_{i1}}{\sum_{j=1}^{n} S_{ij}}\overline{m}_{i1}^{\xi} \oplus \frac{S_{i2}}{\sum_{j=1}^{n} S_{ij}}\overline{m}_{i2}^{\xi} \oplus \dots \oplus \frac{S_{in}}{\sum_{j=1}^{n} S_{ij}}\overline{m}_{in}^{\xi}\Big)^{1/\xi}$$

$$= \bigcup_{\substack{\overline{\mu}_{i1}\in\overline{w}_{i1},\overline{\mu}_{i2}\in\overline{w}_{i2},\dots,\overline{\mu}_{in}\in\overline{w}_{in};\\ \overline{v}_{i1}\in\overline{e}_{i1},\overline{v}_{i2}\in\overline{e}_{i2},\dots,\overline{v}_{in}\in\overline{e}_{in};\\ \overline{o}_{i1}\in\overline{r}_{i1},\overline{o}_{i2}\in\overline{r}_{i2},\dots,\overline{o}_{in}\in\overline{r}_{in}}} \left\{\left\langle \begin{array}{l} \Big\{\Big(\sqrt{1-\Pi_{i=1}^{n}(1-\mu_{\overline{w}_{ij}}^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}}\Big)^{1/\xi} \Big| \Pi_{i=1}^{n} p_{\overline{w}_{ij}}\Big\}, \\ \Big\{\sqrt{1-(1-\Pi_{i=1}^{n}(1-(1-\nu_{\overline{e}_{ij}}^{2})^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}})^{1/\xi}} \Big| \Pi_{i=1}^{n} q_{\overline{e}_{ij}}\Big\}, \\ \Big\{\sqrt{1-(1-\Pi_{i=1}^{n}(1-(1-o_{\overline{r}_{ij}}^{2})^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}})^{1/\xi}} \Big| \Pi_{i=1}^{n} l_{\overline{r}_{ij}}\Big\} \end{array}\right\rangle\right\}$$

or the GPSHFPWG operator

$$GPSHFPWG^{\xi}(m_1, m_2, \dots, m_n) = \frac{1}{\xi}\Big(\xi\overline{m}_{i1}^{\frac{S_{i1}}{\sum_{j=1}^{n} S_{ij}}} \otimes \xi\overline{m}_{i2}^{\frac{S_{i1}}{\sum_{j=1}^{n} S_{ij}}} \otimes \dots \otimes \xi\overline{m}_{in}^{\frac{S_{i1}}{\sum_{j=1}^{n} S_{ij}}}\Big)$$

$$= \bigcup_{\substack{\overline{\mu}_{i1}\in\overline{w}_{i1},\overline{\mu}_{i2}\in\overline{w}_{i2},\dots,\overline{\mu}_{in}\in\overline{w}_{in};\\ \overline{v}_{i1}\in\overline{e}_{i1},\overline{v}_{i2}\in\overline{e}_{i2},\dots,\overline{v}_{in}\in\overline{e}_{in};\\ \overline{o}_{i1}\in\overline{r}_{i1},\overline{o}_{i2}\in\overline{r}_{i2},\dots,\overline{o}_{in}\in\overline{r}_{in}}} \left\{\left\langle \begin{array}{l} \sqrt{1-(1-\Pi_{i=1}^{n}(1-1-\mu_{\overline{w}_{ij}}^{2})^{\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}})^{1/\xi}} \Big| \Pi_{i=1}^{n} p_{\overline{w}_{ij}}, \\ \Big(\sqrt{1-\Pi_{i=1}^{n}(1-\nu_{\overline{e}_{ij}}^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}}\Big)^{1/\xi} \Big| \Pi_{i=1}^{n} q_{\overline{e}_{ij}}, \\ \Big(\sqrt{1-\Pi_{i=1}^{n}(1-o_{\overline{r}_{ij}}^{2\xi})^{\frac{S_i}{\sum_{i=1}^{n} S_i}}}\Big)^{1/\xi} \Big| \Pi_{i=1}^{n} l_{\overline{r}_{ij}} \end{array}\right\rangle\right\}$$

to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i}\rangle$.
***Step* 4.** Calculate the $s(\tilde{m}_i)$ and $h(\tilde{m}_i)$ values of AL_i by employing Eqs. (2) and (3), and obtain the ranking of all alternatives by the comparison method of PSHFEs.

to help banks evaluate many green enterprises and select appropriate green enterprises from many green enterprises has become an important topic. The effectiveness of the two MADM techniques constructed is tested through an application example of the GECS.

6.1. *Decision Process*

Example 3. This numerical case is adapted from literature (Naeem *et al.*, 2021; Khan *et al.*, 2021). A bank selects an eligible green enterprise for credit. The bank preliminarily selected four green enterprises, namely AL_i $(i = 1, 2, 3, 4)$. DMs use PSHFEs to express the following decision attributes: the operating conditions (AT_1), the applicant's credit (AT_2), the asset scale (AT_3), and the resource consumption (AT_4) to evaluate four GEs. The weight vector of the decision attributes is $\vartheta = (0.2, 0.4, 0.1, 0.3)^T$, there are the following priority relationships among all decision attributes $AT_1 \succ AT_2 \succ AT_3 \succ AT_4$. The original evaluation decision matrix $M = (m_{ij})_{4\times4}$ is listed in Table 5. The process of data acquisition: The bank invites industry experts to rate the four participating bidding companies AL_1, AL_2, AL_3, and AL_4, based on the four decision attributes of the operating conditions (AT_1), the application's credit (AT_2), the asset scale (AT_3), and the resource consumption (AT_4). The following is the scoring process of the first enterprise AL_1 in Table 5 under the decision attribute AT_1: the expert scores the first enterprise, who

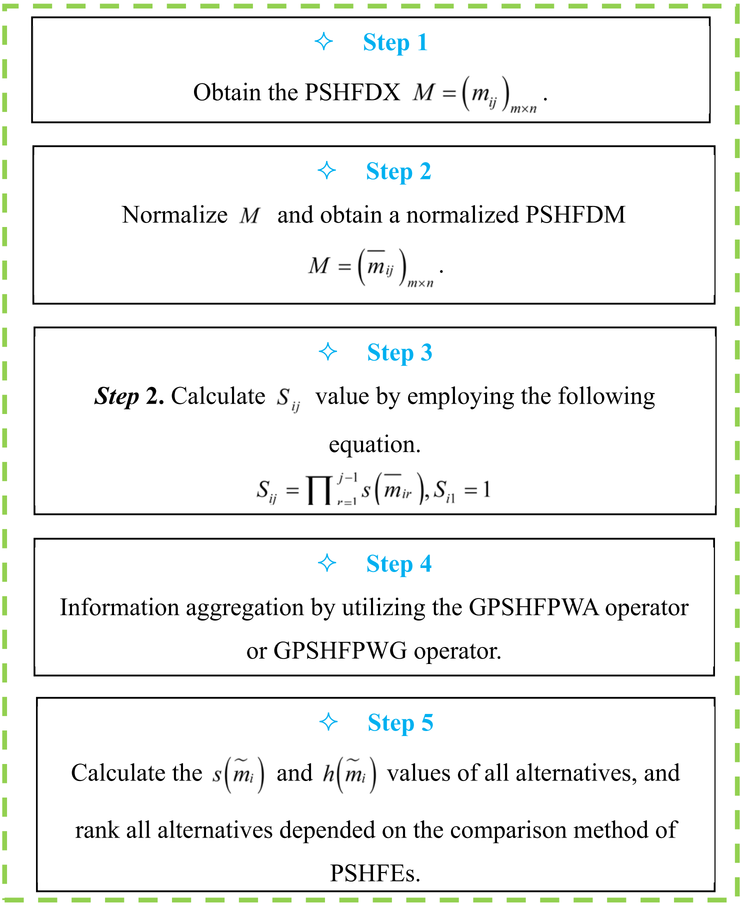

Fig. 2. The flowchart of the built Algorithm 2.

believes that the operating conditions AT_1 of this enterprise are good at a level of 0.4 and the probability of being at this level is 1. The bad level is 0.3, and the probability of being at this level is considered to be 1. The general level is 0.2, and the probability of being at this level is considered to be 1. Therefore, the following data $\langle \{0.4|1\}, \{0.2|1\}, \{0.3|1\} \rangle$ are formed, and other situations are also generated in this way. This will not be listed again.

Next, we utilize Algorithm 1 to rank four potential GEs, the detailed process is shown below.

***Step* 1.** Since all attributes in the numerical case are benefit, except for AT_4, we only normalize AT_4 by employing Eq. (4); m_{ij}^{C} is based on Eq. (1) in Definition 8; the normalized evaluation decision matrix $\overline{M}$ is obtained.

***Step* 2.** Utilize the GPSHFPWA ($\xi = 1$) operator to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i} \rangle$.

Table 5
The original PSHFDX M.

Alternatives	AT_1	AT_2	AT_3	AT_4
AL_1	$\langle\{0.4\|1\}, \{0.2\|1\}, \{0.3\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.6\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.5\|1\}, \{0.1\|1\}\rangle$	$\left\langle \begin{array}{l}\{0.1\|0.5, 0.3\|0.5\},\\ \{0.5\|0.4, 0.4\|0.6\},\\ \{0.6\|0.5, 0.5\|0.5\}\end{array}\right\rangle$
AL_2	$\langle\{0.2\|1\}, \{0.5\|1\}, \{0.1\|1\}\rangle$	$\langle\{0.4\|1\}, \{0.3\|1\}, \{0.2\|1\}\rangle$	$\left\langle \begin{array}{l}\{0.6\|0.5, 0.4\|0.5\},\\ \{0.1\|0.6, 0.1\|0.4\},\\ \{0.1\|0.5, 0.3\|0.5\}\end{array}\right\rangle$	$\langle\{0.1\|1\}, \{0.4\|1\}, \{0.2\|1\}\rangle$
AL_3	$\langle\{0.5\|1\}, \{0.1\|1\}, \{0.4\|1\}\rangle$	$\left\langle \begin{array}{l}\{0.6\|0.5, 0.4\|0.5\},\\ \{0.1\|0.6, 0.2\|0.4\},\\ \{0.1\|0.5, 0.3\|0.5\}\end{array}\right\rangle$	$\langle\{0.5\|1\}, \{0.2\|1\}, \{0.4\|1\}\rangle$	$\langle\{0.4\|1\}, \{0.2\|1\}, \{0.5\|1\}\rangle$
AL_4	$\langle\{0.3\|1\}, \{0.2\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.2\|1\}, \{0.1\|1\}\rangle$	$\left\langle \begin{array}{l}\{0.3\|0.5, 0.5\|0.5\},\\ \{0.4\|0.4, 0.4\|0.6\},\\ \{0.2\|0.5, 0.4\|0.5\}\end{array}\right\rangle$	$\langle\{0.2\|1\}, \{0.4\|1\}, \{0.3\|1\}\rangle$

$$\begin{aligned}
\tilde{m}_1 &= \big\langle\{0.5042|0.5, 0.4667|0.5\}, \{0.4478|0.4, 0.4188|0.6\}, \{0.1644|0.5, 0.2285|0.5\}\big\rangle,\\
\tilde{m}_2 &= \big\langle\{0.3551|0.5, 0.2814|0.5\}, \{0.3245|0.6, 0.3245|0.4\}, \{0.1320|0.5, 0.1473|0.5\}\big\rangle,\\
\tilde{m}_3 &= \big\langle\{0.5442|0.5, 0.4639|0.5\}, \{0.1320|0.6, 0.1741|0.4\}, \{0.2297|0.5, 0.3565|0.5\}\big\rangle,\\
\tilde{m}_4 &= \big\langle\{0.3000|0.5, 0.2860|0.5\}, \{0.2639|0.4, 0.2639|0.6\}, \{0.1516|0.5, 0.1625|0.5\}\big\rangle.
\end{aligned}$$

***Step* 3.** Calculate $s(\tilde{m}_i)$ values of AL_i by employing Eq. (2). We can get $s(\tilde{m}_1) = 0.6195$, $s(\tilde{m}_2) = 0.6180$, $s(\tilde{m}_3) = 0.6874$, $s(\tilde{m}_4) = 0.6240$.

***Step* 4.** According to $s(\tilde{m}_i)$ of each GE AL_i, the ranking of four GEs is $AL_3 \succ AL_4 \succ AL_1 \succ AL_2$.

If GPSHFWA operator is replaced by GPSHFWG operator in the above calculation process, the calculation process is as follows.

***Step* 1'.** The process is same with ***Step* 1**.

***Step* 2'.** Utilize the GPSHFWG ($\xi = 1$) operator to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i|p_{w_i}, e_i|q_{e_i}, r_i|l_{r_i}\rangle$.

$$\begin{aligned}
\tilde{m}_1 &= \big\langle\{0.4799|0.5, 0.4544|0.5\}, \{0.5104|0.4, 0.4847|0.6\}, \{0.1961|0.5, 0.2499|0.5\}\big\rangle,\\
\tilde{m}_2 &= \big\langle\{0.2945|0.5, 0.2828|0.5\}, \{0.3716|0.6, 0.3716|0.4\}, \{0.1487|0.5, 0.1741|0.5\}\big\rangle,\\
\tilde{m}_3 &= \big\langle\{0.5378|0.5, 0.4573|0.5\}, \{0.1487|0.6, 0.1846|0.4\}, \{0.3208|0.5, 0.3642|0.5\}\big\rangle,\\
\tilde{m}_4 &= \big\langle\{0.3000|0.5, 0.3157|0.5\}, \{0.2999|0.4, 0.2999|0.6\}, \{0.1677|0.5, 0.2025|0.5\}\big\rangle.
\end{aligned}$$

***Step* 3'.** Calculate $s(\tilde{m}_i)$ values of AL_i by employing Eq. (2). We can get $s(\tilde{m}_1) = 0.5831$, $s(\tilde{m}_2) = 0.5852$, $s(\tilde{m}_3) = 0.6640$, $s(\tilde{m}_4) = 0.6076$.

***Step* 4'.** According to $s(\tilde{m}_i)$ of each GE AL_i, the ranking of four GEs is $AL_3 \succ AL_4 \succ AL_2 \succ AL_1$.

Next, we utilize Algorithm 2 to rank four potential GEs, the detailed process is shown below.

Table 6
The normalized PSHFDX $\overline{M}$.

Alternatives	AT_1	AT_2	AT_3	AT_4
AL_1	$\langle\{0.4\|1\}, \{0.2\|1\}, \{0.3\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.6\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.5\|1\}, \{0.1\|1\}\rangle$	$\langle\{0.6\|0.5, 0.5\|0.5\}, \{0.5\|0.4, 0.4\|0.6\}, \{0.1\|0.5, 0.3\|0.5\}\rangle$
AL_2	$\langle\{0.2\|1\}, \{0.5\|1\}, \{0.1\|1\}\rangle$	$\langle\{0.4\|1\}, \{0.3\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.6\|0.5, 0.4\|0.5\}, \{0.1\|0.6, 0.1\|0.4\}, \{0.1\|0.5, 0.3\|0.5\}\rangle$	$\langle\{0.2\|1\}, \{0.4\|1\}, \{0.1\|1\}\rangle$
AL_3	$\langle\{0.5\|1\}, \{0.1\|1\}, \{0.4\|1\}\rangle$	$\langle\{0.6\|0.5, 0.4\|0.5\}, \{0.1\|0.6, 0.2\|0.4\}, \{0.1\|0.5, 0.3\|0.5\}\rangle$	$\langle\{0.5\|1\}, \{0.2\|1\}, \{0.4\|1\}\rangle$	$\langle\{0.5\|1\}, \{0.2\|1\}, \{0.4\|1\}\rangle$
AL_4	$\langle\{0.3\|1\}, \{0.2\|1\}, \{0.2\|1\}\rangle$	$\langle\{0.3\|1\}, \{0.2\|1\}, \{0.1\|1\}\rangle$	$\langle\{0.3\|0.5, 0.5\|0.5\}, \{0.4\|0.4, 0.4\|0.6\}, \{0.2\|0.5, 0.4\|0.5\}\rangle$	$\langle\{0.3\|1\}, \{0.4\|1\}, \{0.2\|1\}\rangle$

Table 7
All S_{ij} values.

S_{ij}	1.0000	0.4500	0.1575	0.0551
	1.0000	0.3000	0.1350	0.0810
	1.0000	0.5000	0.2900	0.1305
	1.0000	0.4500	0.2250	0.0788

***Step* 1.** Since all attributes in the numerical case are benefit, except for AT_4, we only normalize AT_4 by employing Eq. (4), the normalized evaluation decision matrix $\overline{M}$ is obtained and shown in Table 6.

***Step* 2.** Calculate all S_{ij} values by employing the Eq. (4); all S_{ij} values are listed in Table 7.

***Step* 3.** Utilize the GPSHFPWA ($\xi = 1$) operator to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i}\rangle$.

$$\begin{aligned}
\tilde{m}_1 &= \big\langle\{0.5390|0.5, 0.5332|0.5\}, \{0.1371|0.4, 0.1355|0.6\}, \{0.0891|0.5, 0.0947|0.5\}\big\rangle,\\
\tilde{m}_2 &= \big\langle\{0.3808|0.5, 0.3362|0.5\}, \{0.2371|0.6, 0.2371|0.4\}, \{0.0375|0.5, 0.0357|0.5\}\big\rangle,\\
\tilde{m}_3 &= \big\langle\{0.6844|0.5, 0.6252|0.5\}, \{0.0161|0.6, 0.0227|0.4\}, \{0.0860|0.5, 0.1490|0.5\}\big\rangle,\\
\tilde{m}_4 &= \big\langle\{0.3904|0.5, 0.4342|0.5\}, \{0.0691|0.4, 0.0734|0.6\}, \{0.0435|0.5, 0.0509|0.5\}\big\rangle.
\end{aligned}$$

***Step* 4.** Calculate $s(\tilde{m}_i)$ values of all alternatives AL_i by employing Eq. (2). We can get $s(\tilde{m}_1) = 0.6300$, $s(\tilde{m}_2) = 0.5960$, $s(\tilde{m}_3) = 0.6842$, $s(\tilde{m}_4) = 0.6396$.

***Step* 5.** According to the score value of each GE AL_i, the ranking of four GEs is $AL_3 \succ AL_4 \succ AL_1 \succ AL_2$.

If GPSHFPWA operator is replaced by GPSHFPWG operator in the above calculation process, the calculation process is as follows.

***Step* 1'.** The process is same with ***Step* 1**.

***Step* 2'.** The calculation results are shown in ***Step* 2**.

***Step* 3'.** Utilize the GPSHFPWG ($\xi = 1$) operator to aggregate the normalized PSHFDM $\overline{M}$ to obtain evaluation information of all alternatives AL_i, i.e. $\tilde{m}_i = \langle w_i | p_{w_i}, e_i | q_{e_i}, r_i | l_{r_i} \rangle$.

$$\tilde{m}_1 = \big\langle \{0.2355|0.5, 0.2332|0.5\}, \{0.5111|0.4, 0.5066|0.6\}, \{0.3293|0.5, 0.3356|0.5\} \big\rangle,$$
$$\tilde{m}_2 = \big\langle \{0.1245|0.5, 0.1178|0.5\}, \{0.5312|0.6, 0.5312|0.4\}, \{0.5722|0.5, 0.5788|0.5\} \big\rangle,$$
$$\tilde{m}_3 = \big\langle \{0.2894|0.5, 0.2363|0.5\}, \{0.1781|0.6, 0.2157|0.4\}, \{0.6119|0.5, 0.6326|0.5\} \big\rangle,$$
$$\tilde{m}_4 = \big\langle \{0.1211|0.5, 0.1358|0.5\}, \{0.3257|0.4, 0.3257|0.6\}, \{0.5149|0.5, 0.5356|0.5\} \big\rangle.$$

***Step* 4'.** Calculate $s(\tilde{m}_i)$ values of all alternatives AL_i by employing Eq. (2).

$$s(\tilde{m}_1) = 0.5838, \qquad s(\tilde{m}_2) = 0.4408, \qquad s(\tilde{m}_3) = 0.6278, \qquad s(\tilde{m}_4) = 0.5504.$$

***Step* 5'.** According to the score value of each GE AL_i, the ranking of four GEs is $AL_3 \succ AL_1 \succ AL_4 \succ AL_2$.

We apply the two MADM methods proposed in this article without considering priority relationships to GECS, and the optimal GE obtained is always AL_3. The four GEs obtained using the GPSHFWA operator are sorted as $AL_3 \succ AL_4 \succ AL_1 \succ AL_2$, and the four GEs obtained using the GPSHFWG operator are sorted as $AL_3 \succ AL_4 \succ AL_2 \succ AL_1$. This is because the GPSHFWA operator leans more towards the average of the overall data, while the GPSHFWG operator leans more towards the advantages of individual large or small data. Therefore, the focus of these two operators is different, resulting in different aggregation results. We apply the two MADM methods that consider priority relationships proposed in this article to GECS, and the optimal GE obtained is always AL_3. The four GEs obtained by using the GPSHFPWA operator are sorted as $AL_3 \succ AL_4 \succ AL_1 \succ AL_2$, and the four GEs obtained by using the GPSHFPWG operator are sorted as $AL_3 \succ AL_1 \succ AL_4 \succ AL_2$. This is because the GPSHFPWA operator not only considers the priority relationship between decision attributes but also tends to average the overall data, while the GPSHFPWG operator not only considers the priority relationship between decision attributes but also tends to have the advantage of individual large or small data. Therefore, the focus of these two operators is different, resulting in different aggregation results.

6.2. *Sensitivity Analysis of Parameter*

Because the generalized aggregation operator contains parameter ξ, we think that parameter ξ will affect the decision ranking results in MADM problems, so we have carried out sensitivity analysis on parameter ξ, such as parameter ξ takes some discrete values in closed interval [0,50], respectively. Then, for the proposed MADM technique based on GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators, Tables 8–11 discuss the ranking results of the four alternatives when the parameter ξ takes the above different values. According to the ranking in Tables 8 and 9, we can clearly see that we apply the GPSHFWA operator to the GECS problem in Example 3, and the optimal GE has some differences, and there are also some differences in the ranking of the four alternatives with different values of parameter ξ, so we can clearly see that the optimal GE is $AL_3(0.5 \leqslant \xi \leqslant 11)$, the optimal GE is $AL_2(12 \leqslant \xi \leqslant 50)$. Meanwhile, we can clearly see that if we apply the GPSHFWG operator to the GECS problem in Example 3, the optimal GE is always $AL_3(0.5 \leqslant \xi \leqslant 50)$. According to the ranking in Tables 10 and 11, if we apply the GPSHFPWA operator to in Example 3, we can clearly see that the optimal GE is $AL_3(0.5 \leqslant \xi \leqslant 35)$, the optimal GE is $AL_2(36 \leqslant \xi \leqslant 50)$; if we apply the GPSHFPWG operator in Example 3, the optimal GE is $AL_3(0.5 \leqslant \xi \leqslant 5)$, the optimal GE $AL_1(6 \leqslant \xi \leqslant 11)$, the optimal GE $AL_3(12 \leqslant \xi \leqslant 15)$, the optimal GE $AL_1(16 \leqslant \xi \leqslant 50)$. Meanwhile, there are some differences in the ranking of the four alternatives with different values of parameter ξ. At the same time, we also see that the ranking of the four alternatives changes with the different values of parameter ξ. Through parameter sensitivity analysis, we found that the process of information aggregation using the proposed generalized aggregation operator will change with the variation of parameters.

For example, when $\xi = 2$, GPSHFWA operator degenerates into PSHF weighted quadratic average (PSHFWQA) operator, namely

$$
\begin{aligned}
&GPSHFWA^2(m_1, m_2, \ldots, m_n)\\
&\quad = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \left\{ \left(\sqrt{1 - \Pi_{i=1}^n (1-\mu_i^4)^{\vartheta_i}} \right)^{1/2} \middle| \Pi_{i=1}^n p_{\mu_i} \right\}, \\ \left\{ \sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1-\nu_i^2)^2)^{\vartheta_i})^{1/2}} \middle| \Pi_{i=1}^n q_{\nu_i} \right\}, \\ \left\{ \sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1-o_i^2)^2)^{\vartheta_i})^{1/2}} \middle| \Pi_{i=1}^n l_{o_i} \right\} \end{array} \right\rangle \right\}.
\end{aligned}
$$

When $\xi = 3$, GPSHFWA operator degenerates into PSHF weighted cubic average (PSHFWCA) operator, namely

$$
\begin{aligned}
&GPSHFWA^3(m_1, m_2, \ldots, m_n)\\
&\quad = \bigcup_{\mu_i \in W_i, \nu_i \in E_i, o_i \in R_i} \left\{ \left\langle \begin{array}{l} \left\{ \left(\sqrt{1 - \Pi_{i=1}^n (1-\mu_i^6)^{\vartheta_i}} \right)^{1/3} \middle| \Pi_{i=1}^n p_{\mu_i} \right\}, \\ \left\{ \sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1-\nu_i^2)^3)^{\vartheta_i})^{1/3}} \middle| \Pi_{i=1}^n q_{\nu_i} \right\}, \\ \left\{ \sqrt{1 - (1 - \Pi_{i=1}^n (1 - (1-o_i^2)^3)^{\vartheta_i})^{1/3}} \middle| \Pi_{i=1}^n l_{o_i} \right\} \end{array} \right\rangle \right\}.
\end{aligned}
$$

In addition, we can also get some enlightenment from the results in Tables 8–11:

(1) In numerical Example 3, we applied Algorithm 1 based on GPSHFWA and GPSHFWG operator to GECS. We found that the score values of the four alternatives aggregated by GPSHFWA operator and the score values of the four alternatives aggregated by GPSHFWG operator were significantly higher than those of the four alternatives aggregated by GPSHFWG operator. The greater the value of ξ, the greater the difference between them. This shows that GPSHFWA operator is more suitable for aggregating the decision information given by the DMs with a more optimistic

Table 8
Sensitivity analysis of parameter ξ in GPSHFWA operator.

Values of parameter ξ	$s(\tilde{m}_1)$	$s(\tilde{m}_2)$	$s(\tilde{m}_3)$	$s(\tilde{m}_4)$	Ranking
0.5	0.6168	0.6047	0.6865	0.6172	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
1	0.6195	0.6180	0.6874	0.6240	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
2	0.6251	0.6394	0.6892	0.6313	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
3	0.6303	0.6553	0.6912	0.6367	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
4	0.6351	0.6669	0.6931	0.6414	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
5	0.6395	0.6757	0.6949	0.6451	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
10	0.6569	0.6997	0.7019	0.6558	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$
11	0.6597	0.7026	0.7030	0.6571	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$
12	0.6623	0.7053	0.7040	0.6583	$AL_2 \succ AL_3 \succ AL_1 \succ AL_4$
15	0.6690	0.7118	0.7067	0.6614	$AL_2 \succ AL_3 \succ AL_1 \succ AL_4$
20	0.6778	0.7202	0.7102	0.6151	$AL_2 \succ AL_3 \succ AL_1 \succ AL_4$
50	0.5222	0.5517	0.5403	0.5454	$AL_2 \succ AL_4 \succ AL_3 \succ AL_1$

Table 9
Sensitivity analysis of parameter ξ in GPSHFWG operator.

Values of parameter ξ	$s(\tilde{m}_1)$	$s(\tilde{m}_2)$	$s(\tilde{m}_3)$	$s(\tilde{m}_4)$	Ranking
0.5	0.5915	0.5928	0.6709	0.6159	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
1	0.5831	0.5852	0.6640	0.6076	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
2	0.5689	0.5715	0.6540	0.5906	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
3	0.5586	0.5606	0.6481	0.5775	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
4	0.5511	0.5521	0.6443	0.5686	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
5	0.5453	0.5454	0.6417	0.5626	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$
10	0.5283	0.5263	0.6348	0.5492	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
15	0.5658	0.5958	0.6964	0.5771	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
20	0.6086	0.5913	0.8259	0.7679	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
50	0.7870	0.7431	0.8210	0.7672	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$

decision attitude, while GPSHFWG operator is more suitable for aggregating the decision information given by the DMs with a more pessimistic decision attitude, and the optimistic and pessimistic attitude will increase with the increase of ξ value.

(2) In numerical Example 3, when the parameter ξ takes different values, when we use GPSHFPWA and GPSHFPWG operators to aggregate the decision information, the score of each alternative changes relatively little; unlike GPSHFWA and GPSHFWG operators, parameter ξ can reflect the decision attitude of DMs. When we take different values of ξ, the best alternative is different, especially when ξ is relatively large, the best alternative changes greatly. Such results also show that parameter ξ has a greater impact on the ranking of alternatives by using GPSHFPWA and GPSHFPWG operators than by using GPSHFWA and GPSHFWG operators. In the sensitivity analysis of parameter, we also show that parameter ξ plays a key role in the MADM problem, especially the impact is greater when it is larger. DMs can choose different parameters ξ according to the needs of the actual MADM problem. It can be seen that the technique proposed in this study has advantages in dealing with a variety of practical MADM problems.

Table 10
Sensitivity analysis of parameter ξ in GPSHFPWA operator.

Values of parameter ξ	$s(\tilde{m}_1)$	$s(\tilde{m}_2)$	$s(\tilde{m}_3)$	$s(\tilde{m}_4)$	Ranking
0.5	0.6279	0.5891	0.6835	0.6387	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
1	0.6300	0.5960	0.6842	0.6396	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
2	0.6347	0.6115	0.6859	0.6424	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
3	0.6394	0.6249	0.6877	0.6461	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
4	0.6441	0.6349	0.6895	0.6498	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
5	0.6485	0.6426	0.6912	0.6528	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$
10	0.66577	0.66581	0.6987	0.6614	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$
15	0.6768	0.6795	0.7043	0.6656	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$
20	0.6842	0.6247	0.7087	0.6181	$AL_3 \succ AL_1 \succ AL_2 \succ AL_4$
35	0.5204	0.5455	0.6352	0.5422	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$
36	0.5209	0.5463	0.5370	0.5423	$AL_2 \succ AL_4 \succ AL_3 \succ AL_1$
40	0.5228	0.5491	0.5385	0.5430	$AL_2 \succ AL_4 \succ AL_3 \succ AL_1$
50	0.5268	0.5548	0.5417	0.5446	$AL_2 \succ AL_4 \succ AL_3 \succ AL_1$

Table 11
Sensitivity analysis of parameter ξ in GPSHFPWG operator.

Values of parameter ξ	$s(\tilde{m}_1)$	$s(\tilde{m}_2)$	$s(\tilde{m}_3)$	$s(\tilde{m}_4)$	Ranking
0.5	0.5971	0.4923	0.6432	0.5859	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
1	0.5838	0.4408	0.6278	0.5504	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
2	0.5615	0.3759	0.5953	0.4858	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
3	0.5474	0.3436	0.5675	0.4448	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
4	0.5385	0.3246	0.5475	0.4187	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
5	0.5323	0.3120	0.5333	0.4010	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
6	0.5278	0.3029	0.5228	0.3883	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$
10	0.5171	0.2827	0.4991	0.3605	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$
11	0.5154	0.2796	0.4949	0.3564	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$
12	0.5139	0.2770	0.5556	0.3529	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
15	0.5106	0.2708	0.5489	0.3448	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$
16	0.6076	0.2692	0.5471	0.3428	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$
20	0.6043	0.2640	0.5413	0.4640	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$
50	0.7868	0.4148	0.5245	0.4505	$AL_1 \succ AL_3 \succ AL_4 \succ AL_2$

6.3. *Comparative Analysis with Other Existing Methods*

In order to fully explain the effectiveness of the MADM techniques proposed in this study, we have fully compared the Example 3 implemented in this paper with several existing MADM techniques in Table 12, such as $WA_{ST\text{-}TSHF}$ and $WG_{ST\text{-}TSHF}$ operators in Naeem *et al.* (2021), $WA^{(A)}_{SHF}$, $WA^{(E)}_{SHF}$, $WA^{(H)}_{SHF}$, $WG^{(A)}_{SHF}$, $WG^{(E)}_{SHF}$, and $WG^{(H)}_{SHF}$ operators in Khan *et al.* (2021), SHFYWA operator in Naeem *et al.* (2022).

(1) Comparison with aggregation operators in Naeem *et al.* (2021).

(a) Comparison with $WA_{ST\text{-}TSHF}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WA_{ST\text{-}TSHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \big\langle\{0.3189, 0.3595, 0.2562\}, \{0.3306, 0.3515, 0.2515\}\big\rangle,$$
$$\tilde{m}_2 = \big\langle\{0.3839, 02724, 0.1741\}, \{0.2737, 0.2724, 0.1967\}\big\rangle,$$
$$\tilde{m}_3 = \big\langle\{0.4621, 0.1320, 0.3544\}, \{0.4118, 0.1515, 0.4415\}\big\rangle,$$
$$\tilde{m}_4 = \big\langle\{0.2354, 0.2639, 0.2312\}, \{0.3040, 0.2639, 0.2847\}\big\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.4769, \quad s(\tilde{m}_2) = 0.5806, \quad s(\tilde{m}_3) = 0.5981, \quad s(\tilde{m}_4) = 0.4985.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_2 \succ AL_4 \succ AL_1$, the optimal GE is AL_3.

(b) Comparison with $WG_{ST\text{-}TSHF}$ operator: We brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WG_{ST\text{-}TSHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \big\langle\{0.2781, 0.3595, 0.3647\}, \{0.3104, 0.3515, 0.3453\}\big\rangle,$$
$$\tilde{m}_2 = \big\langle\{0.2594, 0.2724, 0.2121\}, \{0.2297, 0.2724, 0.2623\}\big\rangle,$$
$$\tilde{m}_3 = \big\langle\{0.4435, 0.1320, 0.4452\}, \{0.4090, 0.1515, 0.4600\}\big\rangle,$$
$$\tilde{m}_4 = \big\langle\{0.2259, 0.2639, 0.2636\}, \{0.2781, 0.2639, 0.4034\}\big\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.2533, \quad s(\tilde{m}_2) = 0.4364, \quad s(\tilde{m}_3) = 0.5546, \quad s(\tilde{m}_4) = 0.3891.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_2 \succ AL_4 \succ AL_1$, the optimal GE is AL_3.

(2) Comparison with aggregation operators in (Khan *et al.*, 2021).

(a) Comparison with $WA^{(A)}_{SHF}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WA^{(A)}_{SHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \big\langle\{0.4259, 0.4478, 0.1643\}, \{0.4104, 0.4380, 0.1834\}\big\rangle,$$
$$\tilde{m}_2 = \big\langle\{0.4151, 0.2460, 0.2460\}, \{0.3195, 0.1319, 0.1835\}\big\rangle,$$
$$\tilde{m}_3 = \big\langle\{0.5229, 0.1320, 0.2297\}, \{0.4826, 0.1741, 0.3566\}\big\rangle,$$
$$\tilde{m}_4 = \big\langle\{0.2853, 0.5639, 0.3000\}, \{0.4164, 0.5639, 0.3694\}\big\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.5343, \quad s(\tilde{m}_2) = 0.6424, \quad s(\tilde{m}_3) = 0.7044, \quad s(\tilde{m}_4) = 0.3018.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_2 \succ AL_1 \succ AL_4$, the optimal GE is AL_3.

(b) Comparison with $WA^{(E)}_{SHF}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WA^{(E)}_{SHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \left\langle \{0.4223, 0.3318, 0.1676\}, \{0.4086, 0.3243, 0.1859\} \right\rangle,$$
$$\tilde{m}_2 = \left\langle \{0.4047, 0.2556, 0.1094\}, \{0.3166, 0.2556, 0.1507\} \right\rangle,$$
$$\tilde{m}_3 = \left\langle \{0.5220, 0.1249, 0.2789\}, \{0.4822, 0.1426, 0.3382\} \right\rangle,$$
$$\tilde{m}_4 = \left\langle \{0.3000, 0.2484, 0.1625\}, \{0.4177, 0.2484, 0.1996\} \right\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.6710, \quad s(\tilde{m}_2) = 0.6500, \quad s(\tilde{m}_3) = 0.7915, \quad s(\tilde{m}_4) = 0.6196.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_1 \succ AL_2 \succ AL_4$, the optimal GE is AL_3.

(c) Comparison with $WA^{(H)}_{SHF}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WA^{(H)}_{SHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \left\langle \{0.4223, 0.4133, 0.1651\}, \{0.4086, 0.4100, 0.1844\} \right\rangle,$$
$$\tilde{m}_2 = \left\langle \{0.4047, 0.3695, 0.1166\}, \{0.3166, 0.4243, 0.1838\} \right\rangle,$$
$$\tilde{m}_3 = \left\langle \{0.5220, 0.5475, 0.2342\}, \{0.4822, 0.4672, 0.3582\} \right\rangle,$$
$$\tilde{m}_4 = \left\langle \{0.3000, 0.3000, 0.5117\}, \{0.4177, 0.3652, 0.1882\} \right\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.5527, \quad s(\tilde{m}_2) = 0.5423, \quad s(\tilde{m}_3) = 0.7048, \quad s(\tilde{m}_4) = 0.4508.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_1 \succ AL_2 \succ AL_4$, the optimal GE is AL_3.

(d) Comparison with $WG^{(A)}_{SHF}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WG^{(A)}_{SHF}$ operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \left\langle \{0.3595, 0.3595, 0.1992\}, \{0.3922, 0.3515, 0.2184\} \right\rangle,$$
$$\tilde{m}_2 = \left\langle \{0.3194, 0.2724, 0.1229\}, \{0.2829, 0.2724, 0.1992\} \right\rangle,$$
$$\tilde{m}_3 = \left\langle \{0.5187, 0.1320, 0.3416\}, \{0.4783, 0.1515, 0.3626\} \right\rangle,$$
$$\tilde{m}_4 = \left\langle \{0.3000, 0.3000, 0.1738\}, \{0.3694, 0.3694, 0.2613\} \right\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.5410, \quad s(\tilde{m}_2) = 0.5784, \quad s(\tilde{m}_3) = 0.6697, \quad s(\tilde{m}_4) = 0.5215.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_2 \succ AL_1 \succ AL_4$, the optimal GE is AL_3.

(e) Comparison with $WG_{SHF}^{(E)}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WG_{SHF}^{(E)}$ operator and obtained the aggregated results of four GEs as follows:

$$\begin{aligned}
\tilde{m}_1 &= \langle\{0.4014, 0.3652, 0.2188\}, \{0.3909, 0.3567, 0.2364\}\rangle,\\
\tilde{m}_2 &= \langle\{0.3243, 0.2769, 0.1266\}, \{0.2842, 0.2769, 0.1997\}\rangle,\\
\tilde{m}_3 &= \langle\{0.5188, 0.1399, 0.3615\}, \{0.4783, 0.1515, 0.3824\}\rangle,\\
\tilde{m}_4 &= \langle\{0.3000, 0.2651, 0.1846\}, \{0.3729, 0.2651, 0.2658\}\rangle.
\end{aligned}$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.5326, \quad s(\tilde{m}_2) = 0.4778, \quad s(\tilde{m}_3) = 0.6386, \quad s(\tilde{m}_4) = 0.5681.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_4 \succ AL_1 \succ AL_2$, the optimal GE is AL_3.

(f) Comparison with $WG_{SHF}^{(H)}$ operator: we brought $\vartheta = (0.4, 0.2, 0.3, 0.1)^T$ and the data from Table 6 into the $WG_{SHF}^{(H)}$ operator and obtained the aggregated results of four GEs as follows:

$$\begin{aligned}
\tilde{m}_1 &= \langle\{0.4014, 0.3652, 0.2188\}, \{0.3909, 0.3567, 0.2364\}\rangle,\\
\tilde{m}_2 &= \langle\{0.3243, 0.2769, 0.1266\}, \{0.2842, 0.2769, 0.1997\}\rangle,\\
\tilde{m}_3 &= \langle\{0.5188, 0.1399, 0.3615\}, \{0.4783, 0.1515, 0.3824\}\rangle,\\
\tilde{m}_4 &= \langle\{0.3000, 0.2651, 0.1846\}, \{0.3729, 0.2651, 0.2658\}\rangle.
\end{aligned}$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = 0.5384, \quad s(\tilde{m}_2) = 0.5761, \quad s(\tilde{m}_3) = 0.6539, \quad s(\tilde{m}_4) = 0.5641.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_2 \succ AL_4 \succ AL_1$, the optimal GE is AL_3.

(3) Comparison with SHFYWA operator in (Naeem *et al.*, 2022).

We brought $\vartheta = (0.2, 0.4, 0.1, 0.3)^T$ and the data from Table 6 into the SHFYWA operator and obtained the aggregated results of four GEs as follows:

$$\tilde{m}_1 = \langle\{0.5140, 0.4928, 0.1939\}, \{0.4721, 0.4638, 0.2482\}\rangle,$$

Table 12
Comparison results of numerical Example 3.

MADM technique	Ranking	Optimal alternative
The $WA_{ST\text{-}TSHF}$ operator	$AL_3 \succ AL_1 \succ AL_2 \succ AL_4$	AL_3
The $WG_{ST\text{-}TSHF}$ operator	$AL_3 \succ AL_1 \succ AL_2 \succ AL_4$	AL_3
The $WA^{(A)}_{SHF}$ operator	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$	AL_3
The $WA^{(E)}_{SHF}$ operator	$AL_3 \succ AL_1 \succ AL_2 \succ AL_4$	AL_3
The $WA^{(H)}_{SHF}$ operator	$AL_3 \succ AL_1 \succ AL_2 \succ AL_4$	AL_3
The $WG^{(A)}_{SHF}$ operator	$AL_3 \succ AL_2 \succ AL_1 \succ AL_4$	AL_3
The $WG^{(E)}_{SHF}$ operator	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$	AL_3
The $WG^{(H)}_{SHF}$ operator	$AL_3 \succ AL_2 \succ AL_4 \succ AL_1$	AL_3
The SHFYWA operator	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$	AL_3
The GPSHFWA operator in our paper ($\xi = 1$)	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$	AL_3
The GPSHFWG operator in our paper ($\xi = 1$)	$AL_3 \succ AL_4 \succ AL_2 \succ AL_1$	AL_3
The GPSHFPWA operator in our paper ($\xi = 1$)	$AL_3 \succ AL_4 \succ AL_1 \succ AL_2$	AL_3
The GPSHFPWG operator in our paper ($\xi = 1$)	$AL_3 \succ AL_1 \succ AL_4 \succ AL_2$	AL_3

$$\tilde{m}_2 = \big\langle \{0.3936, 0.3633, 0.1480\}, \{0.3415, 0.3633, 0.1723\} \big\rangle,$$
$$\tilde{m}_3 = \big\langle \{0.5467, 0.1480, 0.3114\}, \{0.4674, 0.1841, 0.3624\} \big\rangle,$$
$$\tilde{m}_4 = \big\langle \{0.3000, 0.2934, 0.1670\}, \{0.3772, 0.2934, 0.1977\} \big\rangle.$$

Therefore, according to the score calculation formula, we obtain the scores of the four GEs as follows:

$$s(\tilde{m}_1) = -0.2063, \quad s(\tilde{m}_2) = -0.1559, \quad s(\tilde{m}_3) = 0.0041, \quad s(\tilde{m}_4) = -0.1371.$$

Therefore, we obtained the ranking of the four GEs as $AL_3 \succ AL_4 \succ AL_2 \succ AL_1$, the optimal GE is AL_3.

Table 12 shows that the best alternative of the MADM method developed based on GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators in Example 3 is AL_3 when parameter $\xi = 1$, which is consistent with the results in the existing literature. Such results also show the feasibility and effectiveness of the MADM method proposed in this study. Compared with the existing SFS and SHFS, the PSHFS proposed in this study can more accurately express human opinions including yes, waiver, no and rejection. For example, SFN (0.35, 0.5, 0.7), DMs or experts are skeptical that they can give such accurate values when making decisions. For the hesitation attitude of DMs or experts, it is obvious that SFS cannot cope with such a psychological situation. For such a situation, SHFS can cope well, such as SHFE $\langle \{0.36, 0.35\}, \{0.5, 0.51\}, \{0.6, 0.7\} \rangle$. However, there is a very realistic situation that although SHFE has solved the hesitation of DMs or experts, the DMs or experts are uncertain about the possibility of taking the exact value of the three membership degrees, that is, the probability of several values in SHFE $\langle \{0.36, 0.35\}, \{0.5, 0.51\}, \{0.6, 0.7\} \rangle$. However, for such problems, it is clear that the PSHFS proposed in this study can well solve this complex problem, this can be well reflected, such as PSHFE $\langle\{0.36|0.4, 0.35|0.6\}, \{0.5|0.8, 0.51|0.2\}, \{0.6|0.3, 0.7|0.7\}\rangle$, which can also help DMs solve more complex MADM problems in real life. However, the PSHFS proposed in this study also has its own disadvantages. When the amount of data is large, it will lead to a particularly large amount of calculation, which is also the limitation of the technique.

Furthermore, we can find that the aggregation process of the aggregation operator proposed in this paper has some advantages as follows.

(1) PSHFS is more effective in expressing fuzzy information. GSPHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators can deal well with the MADM problem in the PSHF environment. The PSHFS proposed in this study can better express the MADM problems and people's views in real life, and can also express the decision information with several possible values of four MDs. Therefore, PSHFS can better reflect people's hesitation in making decisions, as well as the possibility of hesitation, which is impossible for SFS and SHFS. Therefore, when the DM cannot accurately evaluate the possibility of the alternatives for each degree of membership, it is very reasonable to use the MADM method to solve such MADM problems. In addition, as a more general form of FS, IFS, SFS and SHFS, the MADM method of PSHFS can be converted into the MADM method in these fuzzy environments, so as to better solve the MADM problem in these fuzzy environments.

(2) The aggregation operators with different value of ξ are more flexible. Through the sensitivity analysis of parameter ξ in Section 6.2, we can find that by selecting some special ξ, GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators can degenerate into some existing operators. Therefore, the generalized operators proposed in this study are more flexible and can deal with MADM problems in different situations. In addition, the parameter ξ in the proposed GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators can better reflect the different psychological levels of DMs, and DMs can choose different values of parameter ξ according to the needs of actual MADM problems.

(3) More advantages in dealing with MADM problems with different standards. In the MADM problem, the decision attributes have different importance in principle, that is, they have different weights, and affect the decision results to a certain extent. In general, decision attributes can be divided into two categories with the same priority and different priorities. In the GPSHFWA, GPSHFWG, GPSHFPWA and GPSHFPWG operators proposed in this study, when the priority of the decision attribute is the same, we can choose GPSHFWA and GPSHFWG operators when solving the MADM problem; when the priority of the decision attribute is different, we can choose GPSHFPWA and GPSHFPWG operators when solving the MADM problem. It shows that the aggregation operator proposed in this study is more flexible and can handle different MADM problems.

7. Conclusions

Combining the advantages of the existing SHFS and PHFS, this paper develops PSHFS. From the definition of PSHFS, it is a more general form of FSs such as FS, IFS, SFS, SHFS, etc., which has better advantages than the above FSs. The positive, neutral, negative and refusal membership hesitant degree in the PSHFS can be represented by multiple values in [0, 1] according to the needs of DM, and the four membership parts have their own probability information. The main contributions of this paper are as follows: (1) This paper proposes a new type of fuzzy set named probabilistic spherical hesitation fuzzy set. This fuzzy set, especially in its own application environment, can better handle real-world decision-making problems. This plays a key role in the development of fuzzy sets and can also make decision-making in practical decision-making problems more scientific; (2) The basic operation rules of PSHFEs and the comparison method of two PSHFEs are developed; (3) In order to better solve the MADM problem in the PSHF environment, some new aggregation operators with excellent properties have been proposed in the newly proposed PSHF environment, such as GPSHFWA and GPSHFWG operators. These operators have better properties and can provide better decision-making methods for practical decision-making problems; (4) Considering the different priorities of decision attributes in practical MADM problems, the GPSHFPWA and GPSHFPWG operators with excellent properties are proposed; (5) We have proposed two new MADM methods for GECS, which provide a very important theoretical basis for scientific decision-making in GECS; (6) We have provided a more scientific decision-making model for GECS, which will make the selection of GEs more scientific and reasonable.

Although the MADM method proposed in this paper has certain advantages, there may also be some limitations as follows:

(1) Although the proposed MADM method can handle MADM problems in PSHF environments, it may be powerless for future development of more complex MADM problems in fuzzy environments; (2) The MADM method proposed in this paper was ultimately applied to GECS, but it did not take into account group decision-making problems involving multiple experts, which is also a limitation of this paper.

In future research, we will study other operators of PSHFEs and develop different aggregation operators to aggregate PSHFEs. In addition, we will integrate the existing decision-making methods with the PSHFS, propose some new MADM methods to solve different types of MADM problems, and apply the MADM method proposed in this study to supplier selection (Nguyen *et al.*, 2022; Wang *et al.*, 2022), station selection (Ayyildiz and Taskin, 2022), industrial robot selection (Garg and Sharaf, 2022), strategies evaluation (Ghoushchi *et al.*, 2023), face mask selection (Gul *et al.*, 2022), best variety of maize selection (Gurmani *et al.*, 2022), location selection (Zhang and Wei, 2023).

Compliance with Ethical Standards

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included within the article.

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