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New Aggregation Multiple Attribute Methods Based on Indifference Threshold and Yearning Threshold Concepts

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Abstract. This paper focuses on the aggregation or scoring methods to evaluate the alternatives in Multiple Attribute Decision Making problems (MADM), e.g. Weighted Sum Model (WSM) and Weighted Product Model (WPM). The paper deals with the incorporation of the two concepts into the scoring methods, which has not been studied yet. These concepts are decision maker's Indifference Thresholds (IT) and Yearning Thresholds (YT) on the decision making criteria. Reviewing the related literature reveals that the existent scoring methods do not have a suitable structure to involve the IT, and there is no scoring method which addresses a way to take the YT into account. The paper shows that there is an important drawback to the famous Aspiration Level (AL) concept. Hence, the YT idea is given to resolve the AL limitation. Based on the IT and YT concepts, two new scoring methods are developed: Extended WPM (EWPM) and Extended WSM (EWSM). The EWPM and EWSM are compared with the other scoring methods using a set of simulation analysis. A real-world case extracted from Exploration and Production (E&P) companies in oil industry is examined.
Key words: multiple attribute decision making, EWPM, EWSM, IT, YT, oil exploration and production.

1. Introduction

A Multiple Attribute Decision Making problem (MADM) is a procedure to find the best alternative that has the highest degree of satisfaction from a finite set of alternatives characterized with multiple criteria (Biswas *et al.*, 2019). MADM problems have been established as an effective methodology for solving a wide variety of decision making issues. As shown in equation (1), a MADM matrix is generally displayed by m alternatives $A_1, \dots, A_m$, and n criteria $C_1, \dots, C_n$, such that entry x_{ij} indicates the performance measure of alternative A_i for criterion C_j. The weight attached to criterion C_j is represented by w_j in a way that $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix} \end{array}. \tag{1}$$

A criterion is either quantitative or qualitative. Quantitative criteria are stated in numerical performance measures, but qualitative criteria are expressed in ordinal qualitative data. There are many instructions for conversion of qualitative data into numerical values. Anyway, from now on, we assume that all of performance measures x_{ij} are numerical. Moreover, a criterion is either benefit type or cost type. The former receives high performance measures for preferable alternatives, and the latter receives low performance measures for desirable alternatives. Let's, $x_j^{\max} = \max_{i=1,\dots,m} x_{ij}$ and $x_j^{\min} = \min_{i=1,\dots,m} x_{ij}$, thus for a benefit criterion the former is the ideal performance measure, and the latter shows the anti-ideal performance measure. Absolutely, these terms are vice versa for a cost criterion.

In the literature, there are several methods to evaluate the alternatives in a MADM problem. These methods can be broadly classified into two major types: compensatory and non-compensatory (Hwang and Yoon, 1981). The former assumes compensation among the criteria, while the latter rejects this assumption. The compensatory methods are classified as scoring (or weighting or aggregating), compromising (or ideal point) and concordance (or outranking) (Hwang and Yoon, 1981). The concentration point of the current paper is the scoring methods. In the scoring methods, an aggregate score for each alternative is calculated (from now on, for short, we call it score), indicating total performance of that alternative. Generally, an alternative that has the largest score is selected. In Section 2, the existent scoring methods are reviewed. Let's explain two motivations of the current research, as follows:

- The first and major reason why the current study is conducted is relevant to the concept of Indifference Threshold (IT) (this concept is explained in Section 4). Notably, a few MADM tools involve the IT, i.e. Lexicographic semi-order (Tversky, 1969), Elimination and Choice Expressing the Reality III (ELECTRE III) (Roy, 1978), Indifference Threshold-based Attribute Ratio Analysis (ITARA) (Hatefi, 2019), and Block-wise Rating the Attribute Weights (BRAW) (Hatefi, 2021). None of the existent scoring methods apply the IT concept to their procedures. Thus a question arises: How can we build a scoring method containing the IT concept?
- The second motivation, but not less important, concerns the incorporation of the DM's Yearning Thresholds (YT) into the scoring methods (this concept is explained in Section 3). Hence, the current study is also to answer the question of how to develop a scoring method that involves the YT concept. Reviewing the respective literature shows that there is no scoring method which addresses a well-defined way to incorporate the YT concept.

The aim of the current paper is to extend scoring methods to evaluate the alternatives in an MADM problem, by taking the two above notions into consideration. The paper is organized as follows. Section 2 reviews the existent scoring methods in the related literature. Section 3 defines the YT concept, and justifies why any MADM evaluation method has to take this concept into account. Section 4 explicates the IT concept, and enhances it by giving a new definition. In sections 5 and 6, two proposed extended scoring methods are explained. Section 7 is to present the procedure and results of a simulation study carried out to compare the ranking structures of the alternatives by different scoring methods. Section 8 illustrates a real-world study case performed in the oil industry. Finally, discussion and conclusion are pointed out in the last section.

2. Overview of the Scoring Methods

The seminal and traditional scoring method is Weighted Sum Model (WSM) (Churchman and Ackoff, 1954). In this method, after normalization of the performance measures by the linear normalization (equations (2) and (3) respectively for benefit and cost criteria), the score for alternative A_i is calculated by equation (4).

$$r_{ij} = \frac{x_{ij}}{x_j^{\max}}, \quad i = 1, \dots, m, \ j = 1, \dots, n, \tag{2}$$

$$r_{ij} = \frac{x_j^{\min}}{x_{ij}}, \quad i = 1, \dots, m, \ j = 1, \dots, n, \tag{3}$$

$$S_i^{\text{WSM}} = \sum_{j=1}^{n} w_j r_{ij}, \quad i = 1, \dots, m. \tag{4}$$

The WSM formula is defined in additive form. The multiplicative pair for the WSM is called Weighted Product Model (WPM) (Miller and Starr, 1969). Equation (5) shows the WPM function to calculate the scores. The normalization technique of the WPM is just like the WSM.

$$S_i^{\text{WPM}} = \prod_{j=1}^{n} (r_{ij})^{w_j}, \quad i = 1, \dots, m. \tag{5}$$

In Additive Ratio Assessment (ARAS) (Zavadskas and Turskis, 2010a), firstly, a hypothetical alternative, namely, and ideal alternative including the ideal performance measures is added to the MADM matrix at row zero ($i = 0$). Then, the method benefits from the linear normalization sum-based, as follows from equations (6) and (7) respectively for benefit and cost criteria. Next, equation (8) is used to work the primary scores of the alternatives out. As it can be seen from this equation, the ARAS score calculation is like the WSM. At the end, final scores are determined by comparison of the primary scores with the primary score of the ideal alternative, as follows from equation (9).

$$r_{ij} = \frac{x_{ij}}{\sum_{k=0}^{m} x_{kj}}, \quad i = 0, \dots, m, \ j = 1, \dots, n, \tag{6}$$

$$r_{ij}=\frac{1/x_{ij}}{\sum_{k=0}^{m}(1/x_{kj})},\quad i=0,\dots,m,\ j=1,\dots,n, \tag{7}$$

$$S_i=\sum_{j=1}^{n}w_j r_{ij},\quad i=0,\dots,m, \tag{8}$$

$$S_i^{\text{ARAS}}=S_i/S_0,\quad i=1,\dots,m. \tag{9}$$

Measurement of Alternatives and Ranking according to Compromise Solution (MARCOS) was recently introduced by Stevic *et al.* (2020). This method is a generalized version of the ARAS. The MARCOS starts with adding two hypothetical alternatives[1] to the MADM matrix: ideal (denoted by AI) and anti-ideal (denoted by AAI). The ideal alternative contains the ideal performance measures, and vice versa. The MARCOS scores represent the position of the alternatives with regard to ideal and anti-ideal alternatives. This method employs the WSM formula (equation (4)), to calculate its scores, as follows from equation (10).

$$S_i^{\text{MARCOS}}=\frac{\big(S_i^{\text{WSM}}/S_{\text{AI}}^{\text{WSM}}\big)+\big(S_i^{\text{WSM}}/S_{\text{AAI}}^{\text{WSM}}\big)}{1+\frac{(S_i^{\text{WSM}}/S_{\text{AI}}^{\text{WSM}})}{(S_i^{\text{WSM}}/S_{\text{AAI}}^{\text{WSM}})}+\frac{(S_i^{\text{WSM}}/S_{\text{AAI}}^{\text{WSM}})}{(S_i^{\text{WSM}}/S_{\text{AI}}^{\text{WSM}})}},\quad i=1,\dots,m. \tag{10}$$

In the methods WSM, WPM, ARAS, and MARCOS, the relevant procedures to compute the scores do not distinguish between benefit criteria and cost criteria, and therefore, during the normalization step, cost criteria must be converted into benefit criteria. On the contrary, in the procedure of some other methods, benefit criteria are differentiated from cost criteria. The seminal method by such a procedure is Complex Proportional Assessment (COPRAS), proposed by Zavadskas and Kaklauskas (1996). Equation (14) depicts the COPRAS primary formula, where S_i^+ and S_i^- are determined by equations (12) and (13). Both these equations use only equation (11) to get the weighted normalized performance measures. Lastly, the COPRAS score is calculated by equation (15).

$$r_{ij}=\frac{w_j x_{ij}}{\sum_{k=1}^{m}x_{kj}},\quad i=1,\dots,m,\ j=1,\dots,n, \tag{11}$$

$$S_i^+=\sum_{j=1}^{n}r_{ij}\mid j \text{ is a benefit criterion},\quad i=1,\dots,m, \tag{12}$$

$$S_i^-=\sum_{j=1}^{n}r_{ij}\mid j \text{ is a cost criterion},\quad i=1,\dots,m, \tag{13}$$

$$S_i=S_i^+ +\frac{\min_{l=1,\dots,m}S_l^-\sum_{k=1}^{m}S_k^-}{S_i^-\sum_{k=1}^{m}\frac{\min_{l=1,\dots,m}S_l^-}{S_k^-}},\quad i=1,\dots,m, \tag{14}$$

$$S_i^{\text{COPRAS}}=\frac{S_i}{\max_{k=1,\dots,m}S_k},\quad i=1,\dots,m. \tag{15}$$

[1] Stevic *et al.* (2020), in their paper, used the terms 'ideal solution' and 'anti-ideal solution'. We know that the term 'solution' indicates a feasible point which can be chosen to apply. But the above points are actually artificial and cannot be selected. Hence, we prefer to use the term 'hypothetical alternative' instead of 'solution'.

Multi-Objective Optimization on the basis of Ratio Analysis (MOORA) was introduced by Brauers and Zavadskas (2006). In the ratio system part of this method, it makes use of equation (16), called vector normalization, to normalize all the MADM matrix entries. Then, equations (17) and (18) are applied to calculate the MOORA score as equation (19). A minor variant of the MOORA is called Multi-Objective Optimization on the basis of Simple Ratio Analysis (MOOSRA) (Das *et al.*, 2012) in which equation (20) is used instead of equation (19).

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k=1}^{m}(x_{kj})^2}}, \quad i = 1, \dots, m, \; j = 1, \dots, n, \tag{16}$$

$$S_i^+ = \sum_{j=1}^{n} w_j r_{ij} \mid j \text{ is a benefit criterion}, \quad i = 1, \dots, m, \tag{17}$$

$$S_i^- = \sum_{j=1}^{n} w_j r_{ij} \mid j \text{ is a cost criterion}, \quad i = 1, \dots, m, \tag{18}$$

$$S_i^{\text{MOORA}} = S_i^+ - S_i^-, \quad i = 1, \dots, m, \tag{19}$$

$$S_i^{\text{MOOSRA}} = S_i^+ / S_i^-, \quad i = 1, \dots, m. \tag{20}$$

Brauers and Zavadskas (2010) suggested the multiplicative form of the MOORA (called MULTIMOORA or MMOORA). The normalization step is similar to the MOORA. Then, S_i^+ and S_i^+ are determined by equations (21) and (22). As it can be seen, these elementary functions act in a similar way to the WPM method. Finally, equation (23) shows the MMOORA function to reach the scores.

$$S_i^+ = \prod_{j=1}^{n} (r_{ij})^{w_j} \mid j \text{ is a benefit criterion}, \quad i = 1, \dots, m, \tag{21}$$

$$S_i^- = \prod_{j=1}^{n} (r_{ij})^{w_j} \mid j \text{ is a cost criterion}, \quad i = 1, \dots, m, \tag{22}$$

$$S_i^{\text{MMOORA}} = S_i^+ / S_i^-, \quad i = 1, \dots, m. \tag{23}$$

There are some methods which take a strategy to mix additive and multiplicative forms (often the WSM and the WPM). A convex linear combination of the WSM and WPM formulas is Weighted Aggregated Sum Product Assessment (WASPAS) proposed by Zavadskas *et al.* (2012). In the WASPAS function (equation (24)) μ is an adjusting parameter between 0 and 1. Usually $\mu = 0.5$ is chosen by the Decision Maker (DM).

$$S_i^{\text{WASPAS}} = \mu S_i^{\text{WSM}} + (1 - \mu) S_i^{\text{WPM}}, \quad i = 1, \dots, m. \tag{24}$$

Combined Compromise Solution (COCOSO) (Yazdani *et al.*, 2018) includes an idea to mix additive and multiplicative forms of the scores. Firstly, the COCOSO uses the linear max-min normalization (equations (25) and (26) respectively for benefit and cost criteria) to convert the performance measures into dimensionless values. Secondly, three appraisal formulas as equation (27) to equation (29) are used to get the mixtures. Equation (29) contains a parameter μ that similarly to the WASPAS is an adjusting parameter on interval [0, 1], with default value of 0.5. Finally, equation (30) is the proposed COCOSO model to determine the scores.

$$r_{ij} = \frac{x_{ij} - x_j^{\min}}{x_j^{\max} - x_j^{\min}}, \quad i = 1, \dots, m, \; j = 1, \dots, n, \tag{25}$$

$$r_{ij} = \frac{x_j^{\max} - x_{ij}}{x_j^{\max} - x_j^{\min}}, \quad i = 1, \dots, m, \; j = 1, \dots, n, \tag{26}$$

$$k_{ia} = \frac{\sum_{j=1}^{n} w_j r_{ij} + \prod_{j=1}^{n} (r_{ij})^{w_j}}{\sum_{k=1}^{m} (\sum_{j=1}^{n} w_j r_{kj} + \prod_{j=1}^{n} (r_{kj})^{w_j})}, \quad i = 1, \dots, m, \tag{27}$$

$$k_{ib} = \frac{\sum_{j=1}^{n} w_j r_{ij}}{\min_{k=1,\dots,m} \sum_{j=1}^{n} w_j r_{kj}} + \frac{\prod_{j=1}^{n} (r_{ij})^{w_j}}{\min_{k=1,\dots,m} \prod_{j=1}^{n} (r_{kj})^{w_j}}, \quad i = 1, \dots, m, \tag{28}$$

$$k_{ic} = \frac{\mu \sum_{j=1}^{n} w_j r_{ij} + (1-\mu) \prod_{j=1}^{n} (r_{ij})^{w_j}}{\mu \max_{k=1,\dots,m} \sum_{j=1}^{n} w_j r_{kj} + (1-\mu) \max_{k=1,\dots,m} \prod_{j=1}^{n} (r_{kj})^{w_j}},$$
$$i = 1, \dots, m, \tag{29}$$

$$S_i^{\text{COCOSO}} = (k_{ia} k_{ib} k_{ic})^{1/3} + \frac{1}{3}(k_{ia} + k_{ib} + k_{ic}), \quad i = 1, \dots, m. \tag{30}$$

Equation (31) was used by Keshavarz Ghorabaee *et al.* (2021) to determine the alternative scores. This formula was employed in a criteria weighting model called Method based on the Removal Effects of Criteria (MEREC), in which r_{ij} are dimensionless values of the performance measures obtained by linear normalization, i.e. equations (2) and (3). It seems equation (31) originates in the famous Entropy method (Hwang and Yoon, 1981) and a logarithmic normalization formula suggested by Zavadskas and Turskis (2008). Since the MEREC disregards the criteria weights, it may be inapplicable in many situations.

$$S_i^{\text{MEREC}} = \text{Ln}\left(1 + \left(\frac{1}{n} \sum_{j=1}^{n} \left|\text{Ln}(r_{ij})\right|\right)\right), \qquad i = 1, \dots, m. \tag{31}$$

Reviewing the scoring methods displays that different methods employ various techniques to remove the difference in scale of each criterion. For review of different kinds of normalization techniques, see Brauers and Zavadskas (2006), Zavadskas and Turskis (2008), and Jahan and Edwards (2015).

3. New Yearning Threshold (YT) Concept

The Aspiration Level (AL) idea was introduced by Lotfi *et al.* (1992). For a given criterion, the AL is a value that represents the minimum satisfactory outcomes for the DM. Lotfi *et* *al.* (1992) have stated that the AL for a criterion should not exceed the ideal performance measure of that criterion, and should be less than the anti-ideal performance measure of that criterion. For example, in a house selection MADM problem with three alternatives, if areas are 60, 40, and 80 square meters, the DM's AL for house area should be a number between 40 to 80. A drawback to such assumption is that the DM's willingness actually may be beyond this range, e.g. 200 square meters. In fact, confining the DM's desire to performance measure range is not reasonable. To overcome the AL limitation, this paper proposes the Yearning Threshold (YT) concept as follows.

Definition 1 (*Yearning Threshold or YT*). In MADM matrix, Yearning Threshold of criterion C_j is denoted by YT_j. The YT indicates a desired threshold for the performance measures of a criterion. In order to be meaningful, we suppose the YT can be any value regardless of the performance measures. Note that the YT is not a target level, but it is a lower/upper bound of the DM's desire for a benefit/cost criterion. For example, if a DM's YT for house area is 200 square meters, we interpret that the DM wishes to have a house with 200 square meters or more.

Now, having the above YT definition, let us discuss this fact that incorporation of the YT concept into the MADM evaluation models is not an option, but rather a requirement to ensure the accuracy of the decisions. Again, consider the example of house selection problem by the following MADM matrix. The criteria are (C_1:) area and (C_2:) the ratio of the area of windows to the house area. The YT is 200 for C_1, and 0.4 for C_2.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 60 & 0.2 \\ 40 & 0.4 \\ 80 & 0.3 \end{pmatrix} \end{array}.$$

Without loss of generality, we assume that the criteria have identical weights, and we use the linear normalization and the WSM to calculate the scores as follows:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 0.75 & 0.5 \\ 0.5 & 1 \\ 1 & 0.75 \end{pmatrix} \end{array}, \qquad \text{Score}_{A_1} = \frac{0.75 + 0.5}{2} = 0.625,$$

$$\text{Score}_{A_2} = \frac{0.5 + 1}{2} = 0.750, \qquad \text{Score}_{A_3} = \frac{1 + 0.75}{2} = 0.875.$$

The suggested solution is A_3. Is this decision accurate? Regarding C_1, all performance measures (60, 40, and 80) are very far from the DM's YT (200 square meters), and thus all the alternatives are rather undesirable for the DM. Regarding C_2, A_2 exactly meets the DM's YT. After applying any dimensionless rule, a reasonable expectation is that all "1" in the matrix must indicate the best preferred value for the DM, i.e. entries r_{31}

and r_{22} in the above matrix. Entry $r_{22} = 1$ is valid, because A_2 with 0.4 is completely welcome for the DM. Conversely, A_3 with 80 square meters is far from the minimum desired level of the DM, consequently, entry r_{31} must adjust a lot less than 1. In order to resolve this issue, we suggest normalizing the MADM matrix by using the YTs in spite of the ideal performance measures (e.g. in the following matrix, at column C_1 we would have $60/200 = 0.3$, $40/200 = 0.2$ and $80/200 = 0.4$):

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 0.3 & 0.5 \\ 0.2 & 1 \\ 0.4 & 0.75 \end{pmatrix} \end{array}, \qquad \text{Score}_{A_1} = \frac{0.3 + 0.5}{2} = 0.4,$$

$$\text{Score}_{A_2} = \frac{0.2 + 1}{2} = 0.6, \qquad \text{Score}_{A_3} = \frac{0.4 + 0.75}{2} = 0.575.$$

In the above matrix, we see that the normalized performance measures are scaled down in proportion to the relevant YTs, and the proper solution is A_2. We conclude that using the YTs to normalize the MADM matrix in any alternative evaluation method is a necessary condition to guarantee a proper decision. The YT is "null" when $\text{YT}_j \leqslant x_j^{\max}$ for benefit criteria, and when $\text{YT}_j \geqslant x_j^{\min}$ for cost criteria. Moreover, if the DM has no idea about the YT, it can be considered as $\text{YT}_j = x_j^{\max}$ for benefit criteria, and $\text{YT}_j = x_j^{\min}$ for cost criteria.

4. Enhancement of the Indifference Threshold (IT) Concept

We know that in mathematics, $|Y|$ shows the absolute value of Y, such that $|Y| = \max\{Y, -Y\}$. In fact, a number and its opposite have the same absolute value. This operator is defined in additive form of numbers. Now, let's generalize it into multiplicative form, by the following definition.

Definition 2 (*Minor value*). We define $\langle Y \rangle$ and call it minor value of Y, in a way that $\langle Y \rangle = \min\{Y, Y^{-1}\}$. Thus, a number and its reciprocal have the same minor value. For example, $\langle 0 \rangle = 0$, $\langle 1 \rangle = 1$, $\langle 2 \rangle = 0.5$, $\langle 0.5 \rangle = 0.5$, and $\langle 100 \rangle = 0.01$.

In what follows, two definitions are introduced. The former depicts the preliminary and traditional statement of the IT concept, and the latter is a novel statement for the IT concept that is firstly introduced in the current study.

Definition 3 (*Distance-based Indifference Threshold or DIT*). In MADM matrix, for a given criterion C_j, two performance measures x_{ij} and x_{kj} are indifferent if and only if $|x_{ij} - x_{kj}| \leqslant \text{DIT}_j$, where DIT_j denotes Distance-based Indifference Threshold of criterion C_j. Using an example of a house selection problem, if DIT_j is 5 square meters area, we infer that two houses with 5 square meters difference or lesser in area are indifferent in view of the DM. Note that DIT_j is a non-negative value and homogeneous with dimensional unit of criterion C_j. A value of $\mathrm{DIT}_j = 0$ (i.e. the additive identity) is called null DIT.

Definition 4 (*Ratio-based Indifference Threshold or RIT*). In MADM matrix, for a given criterion C_j, two performance measures x_{ij} and x_{kj} are indifferent if and only if $\left\langle \frac{x_{ij}}{x_{kj}} \right\rangle \geqslant \mathrm{RIT}_j$, where RIT_j ($\in [0, 1]$) denotes Ratio-based Indifference Threshold of criterion C_j. Opposed to the DIT, the RIT is dimensionless. Using an example of a house area, if $\mathrm{RIT}_j = 0.9$, the DM believes that if the area of cheaper house divided by that of a more expensive house be 0.9 or more, he/she is indifferent about the house area. In fact, the DM is indifferent on this criterion, if ratio between areas be 90% to 100%. A null RIT refers $\mathrm{RIT}_j = 1$ (i.e. the multiplicative identity).

Notably, the DIT values are not homogeneous (because dimensional unit of DIT_j is the same as dimensional unit of criterion C_j); on the contrary, all the RIT values range from 0 to 1, thus the RIT has an advantage over the DIT from this point of view.

4.1. *The DIT and the RIT Relationship*

This section is to answer the questions of how to obtain the RIT value from the DIT value and how to obtain the DIT value from the RIT value. To the first question, let's draw up a guideline: Any distances between two given performance measures to which the DM is/isn't indifference, must be reflected in a RIT which causes the fact that the DM is/isn't indifferent to the relevant ratios. Based on this guideline, equations (32) and (33) can be formulated.

$$\mathrm{RIT}_j \leqslant \min_{i,k=1,\dots,m} \left\{ \left\langle \frac{x_{ij}}{x_{kj}} \right\rangle \middle| \, |x_{ij} - x_{kj}| \leqslant \mathrm{DIT}_j \right\}, \quad j = 1, \dots, n, \tag{32}$$

$$\mathrm{RIT}_j > \max_{i,k=1,\dots,m} \left\{ \left\langle \frac{x_{ij}}{x_{kj}} \right\rangle \middle| \, |x_{ij} - x_{kj}| > \mathrm{DIT}_j \right\}, \quad j = 1, \dots, n. \tag{33}$$

It should be noted that equations (32) and (33) relate to the concept of distance from the reference point (McCrimmon, 1968). Any RIT value that satisfies both the above equations is acceptable. Let's provide an example. For a given criterion, the performance measures are 2, 4, 6, and 15, and the DIT is 3. On one hand, the DM is indifferent to distances $|2-4|$ and $|4-6|$. The ratios related to these distances are $2/4$ and $4/6$, respectively. Logically, a RIT less than or equal to $2/4$ or 0.5 must be selected, because such a RIT causes the DM to become indifferent to both $2/4$ and $4/6$. On the other hand, the DM is not indifferent to four distances $|2-6|$, $|2-15|$, $|4-15|$, and $|6-15|$. The relevant ratios are $2/6$, $2/15$, $4/15$, and $6/15$, respectively. Clearly, the DM is not indifferent to all these four ratios, if a RIT greater than $6/15$ or 0.4 is chosen. Finally, a RIT from $(0.4, 0.5]$ can be selected.

It should be noted that there might be situations in which there is no intersection between equations (32) and (33). In such situations, there is no RIT value exactly compatible with a DIT, and vice versa. For example, such a situation happens for a criterion with the performance measures 2, 6, 8, and 20, and a DIT = 10. In this example, equation (32) results in RIT $\leqslant 0.25$, and equation (33) results in RIT > 0.4.

Logic like the above can be applied for the second question, i.e. how to get a DIT value using the RIT value. The guideline for this question is the contrary of that for the first question. The respective functions are shown in equations (34) and (35):

$$\mathrm{DIT}_j \geqslant \max_{i,k=1,\dots,m}\left\{|x_{ij}-x_{kj}| \,\middle|\, \left\langle\frac{x_{ij}}{x_{kj}}\right\rangle \geqslant \mathrm{RIT}_j\right\}, \quad j=1,\dots,n, \tag{34}$$

$$\mathrm{DIT}_j < \min_{i,k=1,\dots,m}\left\{|x_{ij}-x_{kj}| \,\middle|\, \left\langle\frac{x_{ij}}{x_{kj}}\right\rangle < \mathrm{RIT}_j\right\}, \quad j=1,\dots,n. \tag{35}$$

The above equations should hold true for any acceptable DIT. Again, consider the same above example, in which there are four performance measures 2, 4, 6, and 15, with a RIT of 0.45. On one hand, the DM is indifferent to the ratios $2/4$ and $4/6$, because $2/4 > 0.45$ and $4/6 > 0.45$. The relevant distances are $|2-4| = 2$ and $|4-6| = 2$, respectively. We need the DM to become indifferent to distance 2; consequently, a DIT greater than or equal to 2 must be chosen. On the other hand, the DM is not indifferent to ratios $2/6$, $2/15$, $4/15$, and $6/15$. The relevant distances are $|2-6| = 4$, $|2-15| = 13$, $|4-15| = 11$, and $|6-15| = 9$, respectively. In conclusion, a DIT less than 4 must be chosen, because by such a DIT the DM would not be indifferent to all the four distances. Finally, the DIT value from interval $[2, 4)$ should be selected.

5. The Extended WPM Method (EWPM)

The underlying idea of the proposed EWPM stands on the relative relations between the alternatives. This idea leads us to a pairwise comparison matrix whose rows and columns are the alternatives. Let's portray the notion using a simple numerical example as follows, including 3 alternatives A_1 A_2, and A_3, and two criteria C_1 (benefit type) and C_2 (cost type) weighted as 0.7 and 0.3, respectively.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 6 & 8 \\ 2 & 3 \\ 4 & 5 \end{pmatrix} \end{array}$$

If the analyst wishes to compare two alternatives A_1 and A_2 from the point of view of criterion C_1, then the fraction $6/2$ can be taken. This fraction is called individual importance of A_1 relative to A_2. Similarly, the individual relative importance of A_1 to A_2 in view of C_2 is $(\frac{1}{8})/(\frac{1}{3})$ or $3/8$ (Note that C_2 is a cost criterion). Now, we need to combine $6/2$ and $3/8$. These two numbers are ratios; therefore, to combine them the WPM form can be employed. Thus, the combined relative importance of A_1 to A_2 would be $(6/2)^{0.7} \times (3/8)^{0.3}$. This calculation is sometimes called dimensionless analysis because its mathematical structure removes any units of measure (Triantaphyllou, 2000). By performing similar calculations for the other pairs of the alternatives, the result is the following matrix called Objective Multiplicative Pairwise Comparison (OMPC):

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} \quad A_1 \quad & \quad\quad A_2 \quad\quad & \quad A_3 \quad \end{array} \\ \begin{pmatrix} (6/6)^{0.7}\times(8/8)^{0.3} & (6/2)^{0.7}\times(3/8)^{0.3} & (6/4)^{0.7}\times(5/8)^{0.3} \\ (2/6)^{0.7}\times(8/3)^{0.3} & (2/2)^{0.7}\times(3/3)^{0.3} & (2/4)^{0.7}\times(5/3)^{0.3} \\ (4/6)^{0.7}\times(8/5)^{0.3} & (4/2)^{0.7}\times(3/5)^{0.3} & (4/4)^{0.7}\times(5/5)^{0.3} \end{pmatrix} \end{array}.$$

The OMPC matrix is actually in a multiplicative preference relation format, but in spite of preference judgment by the DM, the objective relative importance is replaced. Thus, each entry a_{ik} $(i, k = 1, \dots, m)$ of the OMPC matrix represents the objective importance of alternative A_i relative to alternative A_k in terms of a multiplicative reasoning. Equation (36) shows a general formula to determine the entries of the OMPC matrix. In this function, t_j is determined by equation (37):

$$a_{ik} = \prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}\right)^{w_j}, \quad i, k = 1, \dots, m, \tag{36}$$

$$t_j = \begin{cases} +1, & \text{if } C_j \text{ is a benefit criterion,} \\ -1, & \text{if } C_j \text{ is a cost criterion,} \end{cases} \quad j = 1, \dots, n. \tag{37}$$

According to the following Proposition 1, the OMPC matrix is multiplicative inverse (i.e. reciprocity rule $a_{ik} \times a_{ki} = 1$) and full consistent (i.e. transitivity rule $a_{ik} \times a_{kl} = a_{il}$) for all entries of the matrix.

Proposition 1. *The OMPC matrix is multiplicative inverse and full consistent.*

Proof. There are straightforward ways to sketch the proofs:

$$a_{ik} \times a_{ki} = \prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}\right)^{w_j} \times \prod_{j=1}^{n}\left(\frac{x_{kj}^{t_j}}{x_{ij}^{t_j}}\right)^{w_j} = 1,$$

$$a_{ik} \times a_{kl} = \prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}\right)^{w_j} \times \prod_{j=1}^{n}\left(\frac{x_{kj}^{t_j}}{x_{lj}^{t_j}}\right)^{w_j} = \prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{lj}^{t_j}}\right)^{w_j} = a_{il}.$$

Thus, the proofs are easily established. □

Now we deal with the famous problem of deriving the scores from a pairwise comparison matrix. To extract the alternative scores from the OMPC matrix, with regard to the multiplicative form the entries, the Simple Geometric Mean (SGM) (Crawford and Williams, 1985) is utilized here. In the current situation, this method has been recommended by many researchers (Zhang *et al.*, 2004; Hovanov *et al.*, 2008). Therefore, equation (38) represents the formula for calculating the score of alternative A_i.

$$S_i = \sqrt[m]{\prod_{k=1}^{m}\prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}\right)^{w_j}}, \quad i, = 1, \dots, m. \tag{38}$$

The scores in the above example are 1.2286, 0.7642, and 1.0651, respectively.

5.1. *Inclusion of the YT*

In order to incorporate the YT into the model, we use equation (39) instead of equation (38).

$$S_i = \sqrt[m]{\prod_{k=1}^{m}\prod_{j=1}^{n}\left(\frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}\right)^{\gamma_j w_j}}, \quad i, = 1, \dots, m, \tag{39}$$

where γ_j is named "YT-based weight reduction coefficient", and is calculated by equation (40).

$$\gamma_j = \begin{cases} \frac{x_j^{\max}}{\max\{\mathrm{YT}_j, x_j^{\max}\}}, & \text{if } C_j \text{ is a benefit criterion}, \\ \frac{\min\{\mathrm{YT}_j, x_j^{\min}\}}{x_j^{\min}}, & \text{if } C_j \text{ is a cost criterion}, \end{cases} \quad j = 1, \dots, n. \tag{40}$$

In fact, for a benefit criterion, if the YT is less than or equal to the ideal performance measure, as a result, γ coefficient would be neutral ($= 1$), but if the YT is greater than the ideal performance measure, a fraction coefficient $x_j^{\max}/\mathrm{YT}_j$ (i.e. less than 1) has a decreasing impact on weight w_j. For a cost criterion this decreasing impact acts while the YT is less than the ideal performance measure.

5.2. *Inclusion of the RIT*

Another aspect of the proposed method needs to be applied now, i.e. the IT concept. To incorporate this concept into the proposed method, firstly the Definition 4 helps us to define equation (41).

$$\alpha_{ikj} = \begin{cases} 1, & \left\langle \frac{x_{ij}}{x_{kj}} \right\rangle \geqslant \mathrm{RIT}_j, \\ \frac{x_{ij}^{t_j}}{x_{kj}^{t_j}}, & \left\langle \frac{x_{ij}}{x_{kj}} \right\rangle < \mathrm{RIT}_j, \end{cases} \quad i, k = 1, \dots, m, \; j = 1, \dots, n. \tag{41}$$

Now, we can write the final formula of the new method called Extended Weighted Product Model (EWPM) as equation (42).

$$S_i^{\mathrm{EWPM}} = \sqrt[m]{\prod_{k=1}^{m}\prod_{j=1}^{n}(\alpha_{ikj})^{\gamma_j w_j}}, \quad i = 1, \dots, m. \tag{42}$$

It should be declared that normalization action is skipped in the EWPM. Equation (38) is a simplified version of equation (42), if all RITs are null (i.e. $\mathrm{RIT}_j = 1$) and all YTs are null (i.e. $\mathrm{YT}_j \leqslant x_j^{\max}$ for benefit criteria, and $\mathrm{YT}_j \geqslant x_j^{\min}$ for cost criteria). Thus, we call equation (38) null EWPM formula.

5.3. *Characteristics of the EWPM Scores*

The EWPM, as an interpretation, gauges how much an alternative dominates the other alternatives. Interestingly, this method separates the alternatives into three classes, because the obtained scores may be less than 1 (weak), equal to 1 (moderate), or greater than 1 (strong). The analyst can define a constant π called "score clearance", in such a way that a score less than $1-\pi$ refers a weak alternative, in interval $[1-\pi, 1+\pi]$ indicates a moderate decision, and greater than $1+\pi$ stands for a strong option. The default value for score clearance is 0.05. A strong alternative has a high total dominance over the others. On the contrary, a weak alternative is dominated by the others in most situations. As a consequence, when selecting the best alternative or prioritizing the alternatives, the DM should drop out the weak candidates.

It is worth mentioning that the product of all the alternative scores (i.e. $\prod_{i=1}^{m} S_i^{\mathrm{EWPM}}$) is equal to 1 (see Proposition 2). In point of fact, the EWPM multiplicatively distributes a value of 1 among all the alternatives, depending on their relative importance.

Proposition 2. *In the EWPM, product of all the scores is equal to 1.*

Proof. Clearly, the product of the radicands in equation (38) equals the product of all the entries of the OMPC matrix. We know that in this matrix, for each entry there is a unique corresponding reciprocal entry. Furthermore, main diagonal of this matrix is 1. Thus, the result of multiplying the entries of the OMPC matrix is equal to 1, and we deduce that product of the scores equals 1. □

The following Proposition 3 proves that the alternative scores range between 0 and infinity.

Proposition 3. *In the EWPM, the scores may be any positive real number.*

Proof. The proof is trivial. Look at equation (38). There might exist values $x_{ij} = 0$ and $x_{kj} > 0$ in the MADM matrix, and also there might exist values $x_{ij} > 0$ and $x_{kj} = 0$. Thus, we have $S_i^{\mathrm{EWPM}} = 0$ for the former, and $S_i^{\mathrm{EWPM}} \to +\infty$ for the latter. □

Proposition 4. *Ratio of an alternative score to another alternative score in the null EWPM is equal to the same ratio in the WPM.*

Proof. Without loss of generality, assume that all the criteria are benefit type, thus we simply establish the proof as follows:

$$\frac{S_i^{\text{EWPM}}}{S_l^{\text{EWPM}}} = \frac{\sqrt[m]{\prod_{k=1}^{m}\prod_{j=1}^{n}\left(\frac{x_{ij}}{x_{kj}}\right)^{w_j}}}{\sqrt[m]{\prod_{k=1}^{m}\prod_{j=1}^{n}\left(\frac{x_{lj}}{x_{kj}}\right)^{w_j}}} = \frac{\sqrt[m]{\frac{(\prod_{j=1}^{n}x_{ij})^{mw_j}}{\prod_{k=1}^{m}\prod_{j=1}^{n}x_{kj}}}}{\sqrt[m]{\frac{(\prod_{j=1}^{n}x_{lj})^{mw_j}}{\prod_{k=1}^{m}\prod_{j=1}^{n}x_{kj}}}} = \frac{\prod_{j=1}^{n}(x_{ij})^{w_j}}{\prod_{j=1}^{n}(x_{lj})^{w_j}}.$$

Accordingly:

$$\frac{\prod_{j=1}^{n}(x_{ij})^{w_j}}{\prod_{j=1}^{n}(x_{lj})^{w_j}} = \frac{\prod_{j=1}^{n}\left(\frac{x_{ij}}{x_j^*}\right)^{w_j}}{\prod_{j=1}^{n}\left(\frac{x_{lj}}{x_j^*}\right)^{w_j}} = \frac{\prod_{j=1}^{n}(r_{ij})^{w_j}}{\prod_{j=1}^{n}(r_{lj})^{w_j}} = \frac{S_i^{\text{WPM}}}{S_l^{\text{WPM}}}.$$ □

Corollary 1. *This corollary is a major result of the Proposition 4, and says the overall rank of the alternatives by the null EWPM and by the WPM is the same.*

6. The Extended WSM Method (EWSM)

In the previous section, we saw that the EWPM is founded on a multiplicative structure. There is a pair for the EWPM, founded on the additive structure. Suppose the same numerical example of previous section:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_34 \end{array} \begin{array}{c} \begin{array}{cc} C_1 & C_2 \end{array} \\ \begin{pmatrix} 6 & 8 \\ 2 & 3 \\ 5 & \end{pmatrix} \end{array}.$$

In the additive form, we can say that in view of C_1, the importance of A_1 compared with A_2 is $6-2=4$. Indeed, regarding C_1, the distance between the importance of A_1 and A_2 is 4. In the same way, the relative importance of A_1 to A_2 from the point of view of C_2 is $(1/8)-(1/3)=-5/24$ (Note that C_2 is a cost criterion). The two numbers 4 and $-5/24$ are distances, thus the WSM rule directs the current method to additively combine the relative importance. Because the distances may differ in dimensional unit, distance $6-2$ is transformed to $(6-2)/6$ in which denominator 6 is the ideal performance measure of C_1. In the same way, distance $(1/8)-(1/3)$ is transformed to $((1/8)-(1/3))/(1/3)$ where denominator $1/3$ is the reciprocal of the ideal performance measure of C_2. The combined relative importance of A_1 and A_2 is written as $0.7\times(\frac{6-2}{6})+0.3\times(\frac{(1/8)-(1/3)}{(1/3)})$. By running the same calculations for the other pairs of the alternatives, the following matrix is obtained. We name it Objective Additive Pairwise Comparison matrix (OAPC):

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 0.7\times\left(\frac{6-6}{6}\right)+0.3\times\left(\frac{(1/8)-(1/8)}{(1/3)}\right) & 0.7\times\left(\frac{6-2}{6}\right)+0.3\times\left(\frac{(1/8)-(1/3)}{(1/3)}\right) & 0.7\times(\frac{6-4}{6})+0.3\times\left(\frac{(1/8)-(1/5)}{(1/3)}\right) \\ 0.7\times\left(\frac{2-6}{6}\right)+0.3\times\left(\frac{(1/3)-(1/8)}{(1/3)}\right) & 00.7\times\left(\frac{2-2}{6}\right)+0.3\times\left(\frac{(1/3)-(1/3)}{(1/3)}\right) & 0.7\times\left(\frac{2-4}{6}\right)+0.3\times\left(\frac{(1/3)-(1/5)}{(1/3)}\right) \\ 0.7\times\left(\frac{4-6}{6}\right)+0.3\times\left(\frac{(1/5)-(1/8)}{(1/3)}\right) & 0.7\times\left(\frac{4-2}{6}\right)+0.3\times\left(\frac{(1/5)-(1/3)}{(1/3)}\right) & 0.7\times\left(\frac{4-4}{6}\right)+0.3\times\left(\frac{(1/5)-(1/5)}{(1/3)}\right) \end{pmatrix}. \end{array}$$

The OAPC matrix is in additive preference relation format. In this matrix, each entry b_{ik} $(i, k = 1, \dots, m)$ is interpreted as the objective importance degree of alternative A_i over alternative A_k in terms of an additive reasoning. Equation (43) displays a general formula to obtain the entries of the OAPC matrix, in which t_j is determined like the EWPM (i.e. t_j is $+1$ for benefit criteria, and is -1 for cost criteria). In equation (43), x_j^* refers $x_j^{\max}$ for benefit criteria, and $x_j^{\min}$ for cost criteria.

$$b_{ik} = \sum_{j=1}^{n} w_j \left(\frac{x_{ij}^{t_j} - x_{kj}^{t_j}}{x_j^{*t_j}} \right), \quad i, k = 1, \dots, m. \tag{43}$$

It can be simply demonstrated that the OAPC matrix is additive inverse (opposition rule), i.e. $b_{ik} + b_{ki} = 0$, and full consistent (transitivity rule), i.e. usually $b_{ik} + b_{kl} = b_{il}$.

Proposition 5. *The OAPC matrix is additive inverse and full consistent.*

We omit the proof here, since the proof is straightforward and similar to that in Proposition 1 (In this point, the reader is advised to see Appendix A).

Because the entries of the OAPC matrix are in the additive form, the Simple Column Sum (SCS) method (Saaty, 1980) is a proper way to derive the scores. Equation (44) shows the relevant formula.

$$S_i = \frac{\sum_{k=1}^{m} \sum_{j=1}^{n} w_j \left(\frac{x_{ij}^{t_j} - x_{kj}^{t_j}}{x_j^{*t_j}} \right)}{m}, \quad i = 1, \dots, m. \tag{44}$$

The scores for alternatives A_1, A_2, and A_3 in the above example are 0.1483, −0.1308, and −0.0175, respectively, such that their sum is equal to 0.

6.1. *Inclusion of the YT*

For the sake of incorporating the YT values into the model, equation (44) is enhanced to equation (45), where δ_j is a YT-based weight reduction coefficient in the current method, and is calculated by equation (46).

$$S_i = \frac{\sum_{k=1}^{m} \sum_{j=1}^{n} \delta_j w_j (x_{ij}^{t_j} - x_{kj}^{t_j})}{m}, \quad i = 1, \dots, m, \tag{45}$$

$$\delta_j = \begin{cases} \max\{\mathrm{YT}_j, x_j^{\max}\}^{-1}, & \text{if} C_j \text{ is a benefit criterion,} \\ \min\{\mathrm{YT}_j, x_j^{\min}\}, & \text{if } C_j \text{ is a cost criterion,} \end{cases} \quad j = 1, \dots, n. \tag{46}$$

Note that in equation (45) the multiplier δ_j plays two roles simultaneously, normalization and inclusion of the YT. As a straightforward interpretation, for benefit/cost criterion, if the YT is greater/less than the ideal performance measure, then the ideal performance measure is replaced by the YT to normalize the performance measures.

6.2. *Inclusion of the DIT*

To incorporate the IT concept into equation (45), the Definition 3 is our major guideline. Firstly, based on this definition we can write equation (47). Note that β_{ikj} is homogeneous with normalized distances.

$$\beta_{ikj} = \begin{cases} 0, & |x_{ij} - x_{kj}| \leqslant \text{DIT}_j, \\ x_{ij}^{t_j} - x_{kj}^{t_j}, & |x_{ij} - x_{kj}| > \text{DIT}_j, \end{cases} \quad i, k = 1, \dots, m, \ \ j = 1, \dots, n. \tag{47}$$

Therefore, equation (45) is reformulated as equation (48). The term EWSM refers to Extended Weighted Sum Model (EWSM).

$$S_i^{\text{EWSM}} = \frac{\sum_{k=1}^{m} \sum_{j=1}^{n} \gamma_j w_j \beta_{ikj}}{m}, \quad i = 1, \dots, m. \tag{48}$$

Obviously, equation (48) is simplified as equation (44), provided that all the criteria have null DIT (i.e. zero) and null YT. In such a way, equation (44) is called null EWSM formula.

6.3. *Characteristics of the EWSM Scores*

Let's put an interpretation on the EWSM scores. Like the EWPM, the EWSM assigns a rate to each alternative to make measurement of the alternative dominance over the other alternatives. The EWSM segregates the alternatives into three classes: weak, moderate, and strong. Like the EWPM, by defining a score clearance π (default value: 0.05), an alternative by a score less than $-\pi$, between or equal to $-\pi$ and $+\pi$, or greater than $+\pi$, is considered as a weak, moderate, or strong decision, respectively. In such a way, the weak alternatives should be removed from the selection list. It is also striking that the sum of all the EWSM scores is equal to 0. Proposition 6 is to prove this claim. We conclude that the proposed method assigns the positive or negative scores to the alternatives depending on their relative importance, in such a way that the sum of the scores be equal to additive identity.

Proposition 6. *Sum of the EWSM scores is equal to 0.*

Proof. Obviously, the sum of the numerators in equation (44) equals the sum of all the entries of the OAPC matrix. Each entry of this matrix has an opposite entry (additive inverse) in the matrix. By the way, main diagonal of the OAPC matrix consists of zeros. In such a way, the result of summation of the entries equals zero, and thus, the sum of the EWSM scores would be zero. □

The following Proposition 7 is to prove that the EWSM scores range from -1 to $+1$.

Proposition 7. *The EWSM score ranges from* -1 *to* $+1$.

Proof. Look at equation (44). Without loss of generality, assume that all the criteria are of benefit type. There are two extreme cases. At the worst case, for all the pairs of the performance measures, $x_{ij}/x_j^{\max}$ equals zero and $x_{kj}/x_j^{\max}$ equals one. Consequently, S_i would be equal to -1. The best case is counter of the worst case, i.e. $x_{ij}/x_j^{\max} = 1$ and $x_{kj}/x_j^{\max} = 0$, thus S_i would be equal to $+1$. □

Proposition 8. *Subtraction of an alternative score from another alternative score in the null EWSM equals the same subtraction in the WSM.*

Proof. The proof is similar to that in Proposition 4. Without loss of generality, let us assume that all the criteria are benefit criteria, thus we follow the relations to get the result:

$$
\begin{aligned}
S_i^{\text{EWSM}} - S_l^{\text{EWSM}} &= \frac{\sum_{k=1}^{m}\sum_{j=1}^{n} w_j\left(\frac{x_{ij}-x_{kj}}{x_j^{\max}}\right)}{m} - \frac{\sum_{k=1}^{m}\sum_{j=1}^{n} w_j\left(\frac{x_{lj}-x_{kj}}{x_j^{\max}}\right)}{m} \\
&= \frac{m\sum_{j=1}^{n} w_j\left(\frac{x_{ij}}{x_j^{\max}}\right) + \sum_{k=1}^{m}\sum_{j=1}^{n} w_j\left(\frac{-x_{kj}}{x_j^{\max}}\right)}{m} \\
&\quad - \frac{m\sum_{j=1}^{n} w_j\left(\frac{x_{lj}}{x_j^{\max}}\right) + \sum_{k=1}^{m}\sum_{j=1}^{n} w_j\left(\frac{-x_{kj}}{x_j^{\max}}\right)}{m} \\
&= \frac{m\sum_{j=1}^{n} w_j\left(\frac{x_{ij}}{x_j^{\max}}\right) - m\sum_{j=1}^{n} w_j\left(\frac{x_{lj}}{x_j^{\max}}\right)}{m} \\
&= \sum_{j=1}^{n} w_j r_{ij} - \sum_{j=1}^{n} w_j r_{lj} = S_i^{\text{WSM}} - S_l^{\text{WSM}}.
\end{aligned}
$$

□

Corollary 2. *As a result of the Proposition 8, we deduce that the null EWSM and the WSM produce the same overall rank of the alternatives.*

6.4. *A Numerical Example*

This section provides a numerical example which is extracted from Fan *et al.* (2002). Through this example, we have an aim to compare the EWSM with the WSM. A DM wishes to buy one of four houses A_1 to A_4. His/her criteria are C_1: house price (in thousand dollars, cost criterion), C_2: house area (in square meters, benefit criterion), C_3: distance from house to work location (in kilometers, cost criterion), and environmental characteristics (in ordinal numbers, benefit criterion). The criteria have identical weights. The

MADM matrix for this problem is:

$$\begin{pmatrix} 30 & 100 & 10 & 7 \\ 25 & 80 & 8 & 5 \\ 18 & 50 & 20 & 11 \\ 22 & 70 & 12 & 9 \end{pmatrix}.$$

Let's add new data to this example. The DM would prefer to give ITs of the criteria in the DIT format as 1 thousand dollars for C_1, 10 square meters for C_2, 10 kilometers for C_3, and 4 for C_4. Additionally, the DM's YT is 5 thousand dollars for C_1, and 80 square meters for C_2. The DM has no idea about the YT for C_3 and C_4, thus we adjust $\mathrm{YT}_3 = x_3^{\min} = 8$ and $\mathrm{YT}_4 = x_4^{\max} = 11$. The ranking result of solving the current problem by the WSM is $A_1 > A_3 > A_4 > A_2$, while the result for the EWSM is $A_1 > A_2 > A_4 > A_3$. These rankings show that the ranking structure of the alternatives by the EWSM is different from that of conventional methods such as the WSM. The reason is clear. While the EWSM is sensitive to the DIT and YT values, the others neglect these concepts. When we adjust the DITs and YTs to null values, the results by the null EWSM become similar to that by the WSM.

7. Simulation Analysis

This section is to answer two questions. Whether a given scoring method selects the same best alternative as another given scoring method? And, how much is similarity between the overall rank structures of the alternatives obtained by two given scoring methods? To these, we examine a kind of simulation process that is a broadly accepted framework to compare the performance of the methods (Ahn, 2017; Sarabando and Dias, 2010; Ahn and Park, 2008; Barron and Barrett, 1996). The idea is to repeatedly (N times, namely simulation run number) generate a random MADM matrix and random criteria weights, and investigate how well the result of a given method matches another one, in terms of an efficacy measure. To answer the above first question, a measure called Hit Rate (HR) is used as equation (49).

$$\mathrm{HR} = \frac{B}{N}, \tag{49}$$

where N is a simulation run number, and B is the number of runs in which a method selects the same best alternative as another method does. The HR value ranges from 0 to 1, such that 1 indicates two methods select identical best alternative through all simulation runs. To answer the second question, Rank order Correlation (RC) is employed here. The RC shows the similarity of the overall rank structures of the alternatives constructed by two given methods. The RC is calculated using Kendall's formula (Winkler and Hays, 1985) as equation (50).

$$\mathrm{RC} = 1 - \frac{2 \times V}{A}, \tag{50}$$

where A is the sum of alternative numbers through all simulation runs, and V indicates the number of pairwise preference violations in all simulation runs. The RC value ranges from -1 to $+1$, where $+1$ means perfect correspondence between the two rank orders in all simulation runs.

The simulation is designed with two levels of the alternatives: Low ($3 \leqslant m \leqslant 6$) and High ($9 \leqslant m \leqslant 12$), and two levels of the criteria: Low ($3 \leqslant n \leqslant 6$) and High ($9 \leqslant n \leqslant 12$). For each combination of the levels (i.e. combination 1: low level of the alternatives and low level of the criteria, combination 2: low level of the alternatives and high level of the criteria, combination 3: high level of the alternatives and low level of the criteria, and combination 4: high level of the alternatives and high level of the criteria), the following 10-step procedure is performed.

Step 1: Set $N = 10000$ (N shows simulation run number), $k = 0$, and $c = 1$ (c indicates combination type, i.e. c could be 1, 2, 3, or 4).

Step 2: Set $k = k + 1$, $A = 0$, $B_{pq} = 0$ and $V_{pq} = 0$ ($p, q = 1, \ldots, 12$). Where "12" is the number of the methods to be compared (the WSM, WPM, ARAS, MARCOS, COPRAS, MOORA, MOOSRA, MMOORA, WASPAS, COCOSO, null EWPM, and null EWSM). Thus, $p = 1$ indicates the WSM, $p = 2$ indicates the WPM method, and so on.

Step 3: Based on the combination type, generate a random integer number for the number of alternatives (m), and an independent random integer number for the number of criteria (n). Set $A = A + m$.

Step 4: Generate a random MADM matrix from the independent uniform distribution on (0, 1].

Step 5: Transform the MADM matrix to different normalized forms, depending on different methods mentioned in Step 2.

Step 6: Generate a random weight vector from the independent uniform distribution on (0, 1]. Then normalize the weights to add up to 1.

Step 7: Determine the ranking structure of the alternatives by employing each of the 12 methods.

Step 8: Make HR pairwise comparison between each two methods. For two given methods p and q, if method p selects the same best alternative as method q, then set $B_{pq} = B_{pq}+1$.

Step 9: Draw RC pairwise comparison between each two methods. For two given methods p and q, and for each alternative, if the methods p and q do not assign a similar rank to that alternative, then set $V_{pq} = V_{pq} + 1$.

Step 10: If $k = c * N/4$ then calculate the average HR for each two given methods p and q by $\mathrm{HR}^c_{pq} = \mathrm{B}_{pq}/N$, calculate the average RC for these two methods by $\mathrm{RC}^c_{pq} = 1 - (2V_{pq}/A)$, set $c = c + 1$, and set $k = 0$. In this point, if $k = N$, calculate the overall HR and overall RC by $\mathrm{HR}_{pq} = \sum_{c=1}^{4} \mathrm{HR}^c_{pq}/4$ and $\mathrm{RC}_{pq} = \sum_{c=1}^{4} \mathrm{RC}^c_{pq}/4$, and stop the process, otherwise go to Step 2.

Table 1
Simulation results of the average HR, for different number of the alternatives and criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.7754	0.8105	1.0000	0.4361	0.7165	0.6356	0.7754	0.8737	0.5748	0.7754	1.0000
WPM		0.8009	0.7754	0.3707	0.6957	0.6546	1.0000	0.8935	0.5808	1.0000	0.7754
ARAS			0.8105	0.3498	0.6226	0.6097	0.8009	0.8335	0.4951	0.8009	0.8105
MARCOS				0.4361	0.7165	0.6356	0.7754	0.8737	0.5748	0.7754	1.0000
COPRAS					0.4985	0.3715	0.3707	0.4039	0.4214	0.3707	0.4361
MOORA						0.7266	0.6957	0.7220	0.7175	0.6957	0.7165
MOOSRA							0.6546	0.6568	0.6023	0.6546	0.6356
MMOORA								0.8935	0.5808	1.0000	0.7754
WASPAS									0.5879	0.8935	0.8737
COCOSO										0.5808	0.5748
Null EWPM											0.7754

Table 2
Simulation results of the average RC, for different number of the alternatives and criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null-EWPM	Null-EWSM
WSM	−0.0701	0.2829	1.0000	−0.3616	−0.0927	−0.2210	−0.0701	0.3534	−0.3476	−0.0701	1.0000
WPM		−0.0191	−0.0701	−0.4504	−0.0362	−0.1644	1.0000	0.2340	−0.2830	1.0000	−0.0701
ARAS			0.2829	−0.4494	−0.1292	−0.2365	−0.0191	0.2756	−0.3876	−0.0191	0.2829
MARCOS				−0.3616	−0.0927	−0.2210	−0.0701	0.3534	−0.3476	−0.0701	1.0000
COPRAS					−0.4022	−0.4149	−0.4504	−0.3904	−0.4996	−0.4504	−0.3616
MOORA						0.0753	−0.0362	−0.0386	−0.1401	−0.0362	−0.0927
MOOSRA							−0.1644	−0.1639	−0.2869	−0.1644	−0.2210
MMOORA								0.2340	−0.2830	1.0000	−0.0701
WASPAS									−0.3050	0.2340	0.3534
COCOSO										−0.2830	−0.3476
Null EWPM											−0.0701

The above simulation experiment is conducted using the Visual Basic for application in the Excel's programming language on a personal computer. The simulation runs are made in 5 rounds. Finally, the mean of 5 rounds are considered to compare the results. The individual results of 4 combinations are presented in Appendix B and Appendix C. In a broader sense, 5 values (for the HR and RC and for each combination) are obtained from 5 rounds. Each of these 5 values is the mean of 10.000 values resulted from 10.000 simulation runs. The averages of the presented values in Appendix B and Appendix C are shown in Table 1 and Table 2.

Looking at the above tables, some considerable facts are found out. The null EWSM and the MARCOS generate ranking structures of the alternatives similar to the WSM. Moreover, ranking structures of the alternatives by the null EWPM and the MMOORA are similar to that by the WPM. We name these kinds of correlations full HR (i.e. HR $= 1$) and full RC (i.e. RC $= 1$). By assuming high HR values as $1 >$ HR $\geqslant 0.85$, the following pairs fall into this category: the WASPAS and WSM, the WASPAS and WPM, the WASPAS and MARCOS, the WASPAS and MMOORA, the null EWPM and COCOSO, and the null EWSM and WASPAS. Similarly, by assuming high RC by $1 >$ RC $\geqslant 0.35$, the following pairs fall into this category: the WASPAS and WSM, the WASPAS and MARCOS, and the WASPAS and null EWSM. Taking together, we conclude that the WASPAS results are the most apt to be similar to the other methods.

By reviewing Tables 7–14 in Appendix B and Appendix C, an important fact is also clarified. As a remarkable outcome, for most pair of the methods, as the number of alternatives/criteria increases, both the HR and RC values decrease. In this regard, the impact of the number of alternatives is more than that of the number of criteria. As an example, for pair of the WSM and null EWPM, for low level of the alternatives and criteria, the HR equals 0.8589, for low level of the alternatives and high level of the criteria, the HR becomes 0.7953, for high level of the alternatives and low level of the criteria, the HR decreases to 0.7649, and for high low level of the alternatives and criteria, the HR decreases to 0.6825. Such a downward trend is obviously visible in most analogous cells in sequence Tables 7 > Table 8 > Table 9 > Table 10 and, in most cells, in sequence Table 11 > Table 12 > Table 13 > Table 14, except the cells indicating full HR or full RC.

8. A Real-World Study Case

One of the most fundamental challenges in oil industries is the need of Exploration and Production (E&P) companies for selecting and performing portfolios which have enough capacity from the view of technical, financial, and economic aspects. A portfolio is a collection of projects, programmes, and operations handled as a pack to achieve strategic objectives. In a real-world problem, the EWPM was used to select the best portfolio under Iranian financial regimes. In this case, a Delphi analysis was done to identify key criteria, with participation of sixteen experts. The participants for this analysis were chosen among a variety of project managers and subject matter experts of E&P companies, with more than 12 years of working experience on average. Fifteen criteria were identified through Delphi method:

(C_1) *Size of reservoir*: The larger the reservoir, the higher potential for oil recovery, thus this is a benefit criterion.

(C_2) *Average distance from resources/infrastructures*: Oil field that is close to pipelines, export systems, storage capacities, refineries, power stations, processing facilities, water resources, and so on, is preferred as it needs less investment to dispose of its oil and gas.

(C_3) *Net return on investment*: This economic indicator is a function of development cost analysis and benchmarking international oil prices. Obviously, the higher this criterion results in, the better the development portfolio.

(C_4) *Security and political stability*: A politically stable and safe region and country of investment is more favourable in the oil field.

(C_5) *Quality of upstream oil field development plan*: Clearly, a development plan that accurately addresses key issues is considered as a favourable plan. The key items refer to a variety of things, including, but not limited to, project management guidelines, production challenges, separation systems, logistics, fiscal policy, budget constraints, water supplies, gas capturing and the number of wells to be drilled.

(C_6) *Cooperation opportunities*: It indicates capabilities for making benefits from local communities, local businesses and subcontractors.

(C_7) *Reservoir characteristics*: It refers properties such as oil/water level, oil movement behaviour, reservoir pressure, etc. The relevant data includes a variety of data acquired by logging, coring, and testing for each well drilled.

(C_8) *Plateau production rate*: This is the major field output, and refers to the level of production rate at the plateau phase of oil production.

(C_9) *Plateau production period*: This criterion depicts the fact that longer duration of plateau production justifies employment of installed capacities, decreases capital cost per unit of production, and causes minimal spare capacity maintenance cost.

(C_{10}) *Subsurface complexity*: This criterion stands for several rock properties such as porosity, permeability, hydrocarbon layer continuity, and water saturation. A less subsurface complexity means optimal rock properties.

(C_{11}) *Human resources*: Some zones have qualified human resources in abundance, while they are scarce in others. Oilfields that are located in areas where qualified human resources are available could be considered preferably.

(C_{12}) *Safe location*: The safer the oilfield is during natural disasters, such as flooding and quakes or from damages to existing protective structures such as dams and bunds, the better.

(C_{13}) *Type of contract*: The common types of current E&P agreements include Service Contract (SC) and Production Sharing Agreement (PSA). Preferred type would be based on whether the cash flow will ensure highest returns for the investor.

(C_{14}) *Environmental protection*: A project with potential for optimal reduction of footprint is preferred. Footprint reduction indicators include (1) protecting surface and subsurface sources of fresh water, (2) minimizing effluents, (3) treating and re-injecting produced water, and (4) using land that is already degraded rather than agricultural land.

(C_{15}) *Management capacity*: An upstream development program with existing capacity to efficiently and effectively supervise implementation, track commitments and avoid delays and penalties could be considered more favourably.

Later, the criteria were assigned numerical weights using Improved Rank Order Centroid or IROC method (Hatefi, 2023). For employing the IROC method, firstly, each expert was asked to individually rank the $n = 15$ criteria. The ranks are integer number ranges from 1 to 15. Then, the Kendall's coefficient of concordance (Kendall and Gibbons, 1990) was used in order to measure the degree of agreement among the expert' judgments. Equation (51) represents this coefficient, i.e. a ratio between 0 (no agreement) and 1 (full agreement), where $E = 16$ shows the number of experts, and R_j displays the sum of given ranks by the experts for criterion C_j.

$$T = 12\frac{\sum_{j=1}^{n}(R_j - E(n+1)/2)^2}{E^2(n^3 - n)}. \tag{51}$$

An interpretation of T is that the consensus is very week, weak, fair, strong, very strong, and complete, if it is in intervals [0.0, 0.1), [0.1, 0.3), [0.3, 0.5), [0.5, 0.7), [0.7, 0.9), and [0.9, 1.0], respectively. If T indicates a strong or higher consensus (i.e. not less than 0.5), then we can accept the ranks as valid data. In the study case, $T = 0.607$ was obtained,

Table 3
The criteria ranks and weights.

Criterion:	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
Rank	3	4	1	5	7	9	6	8	2	15	13	11	14	12	10
Weight	0.1265	0.1043	0.2057	0.0866	0.0599	0.0401	0.0722	0.0493	0.1573	0.0044	0.0133	0.0249	0.0048	0.0188	0.0319

indicating a strong agreement. After that, we prioritized the criteria, such that the bigger value of R_j indicates the lower concordant rank of criterion C_j. After ranking, the IROC formula (equation (52)) was used to estimate the criteria weights. In this formula φ_{yn} are constant parameters of the IROC model, reported in Hatefi (2023). These constants for $n = 15$ are 0.04916, 0.06070, 0.06690, 0.07044, 0.07264, 0.07378, 0.07352, 0.07305, 0.07161, 0.06962, 0.06781, 0.06597, 0.06366, 0.06155, and 0.05959. Table 3 represents the obtained ranks and weights for the 15 criteria.

$$w_j = \sum_{y=j}^{n} (\varphi_{yn}/y). \tag{52}$$

Checking the 15 weights, the most crucial criterion weight is 0.2057, while the least important equals 0.0040. This means that criteria with weights less than 0.1028 ($\cong 0.2057/2$) are not important, thus we eliminate them from the model. As a matter of fact, the fast and frugal concept (Katsikopoulos and Fasolo, 2006) leads us to concentrate on just the important criteria. Consequently, only four criteria are essential, i.e. C_3: Net return on investment (in percents), C_9: Plateau production period (in years), C_1: Size of reservoir (in billion barrels), and C_2: Average distance from resources/infrastructures (in kilometers).

Table 4 presents the MADM matrix and some additional data for the study case. All the performance measures (x_{ij}) for all the criteria (C_1, C_2, C_3 and C_9) are received from the relevant technical documents and reports. As mentioned above, the criteria weights are determined using the IROC method. In Table 4, the IROC weights for the four criteria (0.1265, 0.1043, 0.2057, and 0.1573) are normalized to add up to one. Moreover, the RIT and YT values are given by the E&P company' owners.

The EWPM scores are obtained as 0.8553, 1.0992, 0.9700, 1.0391, 0.9711, and 1.0866 for A_1 to A_6, respectively. Correspondingly, by setting $\pi = 0.05$, the portfolios A_2 and A_6 are identified as strong alternatives; among them, A_2 is proposed as the best portfolio to be taken.

8.1. *A Comparison*

In this subsection, the result of solving the study case by the proposed method is compared to those of the various MADM scoring methods. The MADM matrix, presented in Table 4, was solved by the EWPM, WPM, Null EWPM, WSM, ARAS, MARCOS, COPRAS, MOORA, MOOSRA, MMOORA, WASPAS, COCOSO, and MEREC. The final results are exhibited in Table 5.

Notably, the IT and YT data were considered only in the procedure of the EWPM. As mentioned before, the other methods internally do not consider the IT and YT concepts.

Table 4
The MADM matrix in the study case.

Criteria	Code:	C_1	C_2	C_3	C_9
	Title:	Size of reservoir	Average distance from resources/infrastructures	Net return on investment	Plateau production period
	Type:	Benefit	Cost	Benefit	Benefit
	Dimension:	billion barrels	Kilometers	Percents	Years
	Weight:	0.2130	0.1756	0.3465	0.2649
	RIT:	0.8	0.2	0.9	0.85
	YT:	50	20	80	60
Alternatives	A_1	37.5	47	22.5	25
	A_2	32.4	45	48.5	40
	A_3	25.5	19	37.5	35
	A_4	43.1	65	43.0	35
	A_5	33.7	80	33.7	45
	A_6	15.5	10	53.1	40

Table 5
The ranks of the alternatives by various MADM methods.

Row	Method	Scores of alternatives A_1 to A_6 respectively	The obtained ranking of the 6 alternatives
1	EWPM	0.8553 1.0992 0.9700 1.0391 0.9711 1.0866	$A_2 > A_6 > A_4 > A_5 > A_3 > A_1$
2	WPM	0.4702 0.6787 0.6626 0.6260 0.5626 0.7795	$A_6 > A_2 > A_3 > A_4 > A_5 > A_1$
3	Null EWPM	0.7556 1.0907 1.0648 1.0060 0.9042 1.2528	$A_6 > A_2 > A_3 > A_4 > A_5 > A_1$
4	WSM	0.5167 0.7511 0.6692 0.7266 0.6733 0.8341	$A_6 > A_2 > A_4 > A_5 > A_3 > A_1$
5	ARAS	0.4843 0.6907 0.6513 0.6642 0.6083 0.8492	$A_6 > A_2 > A_4 > A_3 > A_5 > A_1$
6	MARCOS	0.4652 0.6762 0.6025 0.6542 0.6062 0.7510	$A_6 > A_2 > A_4 > A_5 > A_3 > A_1$
7	COPRAS	0.5514 0.7809 0.7538 0.7483 0.6832 1.0000	$A_6 > A_2 > A_3 > A_4 > A_5 > A_1$
8	MOORA	0.0035 0.0035 0.0033 0.0030 0.0028 0.0023	$A_2 > A_6 > A_4 > A_3 > A_5 > A_1$
9	MOOSRA	31.6113 14.9838 7.8187 5.4273 5.2556 4.0645	$A_6 > A_3 > A_2 > A_4 > A_1 > A_5$
10	MMOORA	0.0398 0.0347 0.0339 0.0320 0.0287 0.0240	$A_6 > A_2 > A_3 > A_4 > A_5 > A_1$
11	WASPAS	0.8068 0.7149 0.6659 0.6763 0.6179 0.4934	$A_6 > A_2 > A_3 > A_4 > A_5 > A_1$
12	COCOSO	2.2775 1.9320 1.7556 1.6965 1.2963 0.6153	$A_2 > A_4 > A_6 > A_3 > A_5 > A_1$
13	MEREC	0.5785 0.5277 0.4597 0.4051 0.3657 0.2508	$A_1 > A_5 > A_4 > A_2 > A_3 > A_6$

The obtained ranking of the alternatives by the EWPM is $A_2 > A_6 > A_4 > A_5 > A_3 > A_1$; but if we remove the thresholds (the YTs, and the RITs), the overall rank structure of the portfolios will obtained as $A_6 > A_2 > A_3 > A_4 > A_5 > A_1$. In fact, considering the thresholds affects not only the overall ranking of the alternatives, but also the best alternative to be selected. This demonstrates the remarkable impact of the thresholds on the decisions. Another point to be borne in mind is that there is a considerable difference between the MEREC result and the other ranks. It is for this reason that the criteria weights are ignored in the MEREC formula. Let's point another considerable remark out. All the methods generate an alternative score between 0 and 1, except the EWPM (including the Null EWPM), the MOOSRA, and the COCOSO. Among these three methods, only the EWPM provide an inference guideline on the scores, i.e. categorizing the alternatives in weak, moderate, and strong classes.

9. Discussion and Conclusion

This paper started by discussing the fact that, in evaluation of the alternatives in MADM, the existent scoring methods do not take the two momentous concepts into account, i.e.

Table 6
A comparison between the EWPM and EWSM.

Feature		EWPM	EWSM
MADM matrix	Transformation of cost criterion to benefit criterion	Reciprocal	Reciprocal
	Conversion of the performance measures into dimensionless numbers	Unnecessary	Linear normalization, indirectly
Thresholds	IT type	Ratio-based Indifference Threshold (RIT)	Distance-based Indifference Threshold (DIT)
	YT inclusion	A weight reduction coefficient	A weight reduction coefficient
Pairwise comparison matrix	Type of the matrix	OMPC matrix	OAPC matrix
	Individual relative importance of the alternatives (as for a given criterion)	Subtracting the performance measures	Dividing the performance measures
	The entries of the matrix	Weighted multiplication	Weighted addition
	The inverse feature of the matrix	Multiplicative inverse ($a_{uv} \times a_{vu} = 1$)	Additive inverse ($b_{uv} + b_{vu} = 0$)
	The consistency of the matrix	Full consistent ($a_{uv} \times a_{vk} = a_{uk}$)	Full consistent ($b_{uv} + b_{vk} = b_{uk}$)
	Mathematical model to derive the scores from the matrix	Geometric mean	Arithmetic mean
Alternative Score	Range of the scores	$[0, +\infty)$	$[-1, +1]$
	Product of all the scores	Multiplicative identity, 1	N.A.
	Sum of all the scores	N.A.	Additive identity, 0
	Classify the alternatives?	Yes	Yes
	Strong alternatives	Score $> 1 + \pi$	Score $> +\pi$
	Weak alternatives	Score $< 1 - \pi$	Score $< -\pi$
	Moderate alternatives	Score in interval $[1 - \pi, 1 + \pi]$	Score in interval $[-\pi, +\pi]$

the Indifference Threshold (IT), and the Yearning Threshold (YT). The paper, firstly, enhanced the IT and the YT concepts. As a matter of fact, along the traditional IT definition (Distance-based IT or DIT), a new definition called Ratio-based IT or RIT was developed. Additionally, the traditional idea of Aspiration Level (AL) was revised and redefined to develop the new YT concept. Accordingly, to incorporate the IT and YT concepts into the scoring methods, the paper proposed two extended methods called Extended Weighted Product Model (EWPM) and Extended Weighted Sum Model (EWSM). Both the methods try to establish a pairwise comparison matrix extracted from the MADM matrix. As a remarkable advantage of the proposed methods over the existent methods, the calculation relations in both the methods allow the Decision Maker (DM) to incorporate his/her ITs and YTs into the models. Table 6 draws a brief comparison between the EWPM and EWSM methods in view of their structures and scores.

Several advantages of the proposed methods over the existent methods in the literature (reviewed in Section 2) are worthwhile emphasizing as follows:

1. The existent scoring methods calculate the score of a given alternative by using its performance measures, without considering the relative relations between the perfor-

mance measures of the alternative being evaluated and those of other alternatives. This issue is resolved in the EWPM and EWSM.

2. The above note 1 causes a major superiority of the EWPM and EWSM over the existent methods. This refers to the capability of the incorporation of the IT values into the score calculations, while the other methods do not have such a capability.
3. In the proposed models of the EWPM and EWSM, the impact of the YTs is taken into account.
4. Most of the existent scoring methods generally generate alternative scores, without any additional interpretation or post guideline. The EWPM and EWSM take a novel approach about the score values. Both the proposed methods, regarding the values of the scores, separate the alternatives into three classes, i.e. weak decision, moderate decision, and strong decision. Owing to this characteristic, by using the EWPM or EWSM, the DM has an opportunity to focus his/her attention on the strong alternatives.
5. In both the EWPM and the EWSM, the mathematical relations are based on the extraction of the scores from full consistent pairwise comparison matrices. This property confirms that these methods have well-founded platforms.

The paper carried out a simulation experiment to compare the outputs of different scoring methods. Both the simulation experiment and theoretical analysis presented that if the ITs and YTs are null, the WPM and the EWPM gave the same results. This fact is true of the WSM and the EWSM. Nevertheless, the DM is advised to use the EWPM and the EWSM, because these extended methods can classify the alternatives, while the WPM and the WSM do not have such an advantage. When there are effective IT values or YT values, application of the EWPM or the EWSM is confidently the decisive and main recommendation, because the other methods disregard the IT and YT values.

Further works can be undertaken in some areas. The first idea is mixing the proposed EWPM and EWSM. In this case, approaches of the mix methods such as the WASPAS and the COCOSO can be applied. The major focus point of the current paper was incorporation of the IT and YT concepts into the scoring methods. In such a way, future studies can be performed to address the question of how the IT and YT values can be incorporated into the compromising and concordance methods. Another idea, for future studies, is to extend the EWPM and EWSM in uncertain environments, e.g. grey numbers (Deng, 1982; Zavadskas and Turskis, 2010b) and fuzzy numbers (Zadeh, 1965; Aydogdu *et al.*, 2023).

Compliance with Ethical Statements

Author contributions: The author confirms sole responsibility for study conception and design, data collection, analysis and interpretation of results, and manuscript preparation.
Conflict of interest: The author declares that there is no conflict of interests regarding the publication of this paper.
Human participants and/or animals: This article does not contain any studies with human participants or animals performed by the author.
Ethical approval: There is no ethical approval, due to this study is not with human or animal subjects, or where private, protected, or culturally significant research locations were used.

A. Appendix

In the relevant literature, there is another format called fuzzy preference relation (Kacprzyk, 1986). In this format, the DMs' preference relation is defined by a binary fuzzy relation $p_{ik} \in [0, 1]$ that denotes the preference degree of alternative A_i over alternative A_k, where the range of p_{ik} is between 0 and 1 (equal importance $= 0.5$, absolute importance $= 1$, etc.). The obtained pairwise comparison matrix is additive inverse (i.e. $p_{ik} + p_{ki} = 1$). To derive the scores from a fuzzy preference relation matrix, this matrix should be transformed into a multiplicative preference relation using $\mathrm{q}_{ik} = p_{ik}/p_{ki}$ (Zhang *et al.*, 2004; Chiclana *et al.*, 1998).

If we want to transform the entries as equation (43) into fuzzy preference relation, regarding the Proposition 7, a 1 is added to each entry b_{ik} and the result is divided by 2, so we have $p_{ik} = \frac{1+b_{ik}}{2}$. Consequently, the obtained matrix is transformed to multiplicative form by $\mathrm{q}_{ik} = (1 + b_{ik})/(1 + b_{ki})$. It can be easily proved that this kind of multiplicative preference relation matrix is multiplicative inverse but is not full consistent.

B. Appendix

Individual HRs for combinations 1 to 4:

Table 7
Simulation results of the average HR, for low number of the alternatives and low number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.8589	0.9103	1.0000	0.5029	0.8266	0.7202	0.8589	0.9170	0.6691	0.8589	1.0000
WPM		0.8851	0.8589	0.4630	0.8101	0.7276	1.0000	0.9386	0.6700	1.0000	0.8589
ARAS			0.9103	0.4727	0.7958	0.7150	0.8851	0.9190	0.6304	0.8851	0.9103
MARCOS				0.5029	0.8266	0.7202	0.8589	0.9170	0.6691	0.8589	1.0000
COPRAS					0.5578	0.4510	0.4630	0.4846	0.4859	0.4630	0.5029
MOORA						0.7470	0.8101	0.8324	0.7439	0.8101	0.8266
MOOSRA							0.7276	0.7308	0.6535	0.7276	0.7202
MMOORA								0.9386	0.6700	1.0000	0.8589
WASPAS									0.6758	0.9386	0.9170
COCOSO										0.6700	0.6691
Null EWPM											0.8589

Table 8
Simulation results of the average HR, for low number of the alternatives and high number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.7953	0.8754	1.0000	0.5212	0.7982	0.7324	0.7953	0.8850	0.6098	0.7953	1.0000
WPM		0.8306	0.7953	0.4752	0.7547	0.7095	1.0000	0.9054	0.5933	1.0000	0.7953
ARAS			0.8754	0.4719	0.7424	0.7073	0.8306	0.8826	0.5580	0.8306	0.8754
MARCOS				0.5212	0.7982	0.7324	0.7953	0.8850	0.6098	0.7953	1.0000
COPRAS					0.5760	0.4978	0.4752	0.5016	0.4738	0.4752	0.5212
MOORA						0.8145	0.7547	0.7914	0.6706	0.7547	0.7982
MOOSRA							0.7095	0.7317	0.6282	0.7095	0.7324
MMOORA								0.9054	0.5933	1.0000	0.7953
WASPAS									0.6089	0.9054	0.8850
COCOSO										0.5933	0.6098
Null EWPM											0.7953

Table 9
Simulation results of the average HR, for high number of the alternatives and low number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.7649	0.7718	1.0000	0.3519	0.6345	0.5609	0.7649	0.8664	0.5551	0.7649	1.0000
WPM		0.7850	0.7649	0.2652	0.6239	0.6132	1.0000	0.8892	0.5717	1.0000	0.7649
ARAS			0.7718	0.2376	0.5114	0.5584	0.7850	0.8047	0.4458	0.7850	0.7718
MARCOS				0.3519	0.6345	0.5609	0.7649	0.8664	0.5551	0.7649	1.0000
COPRAS					0.4245	0.2336	0.2652	0.3022	0.3706	0.2652	0.3519
MOORA						0.6233	0.6239	0.6411	0.7990	0.6239	0.6345
MOOSRA							0.6132	0.5991	0.5719	0.6132	0.5609
MMOORA								0.8892	0.5717	1.0000	0.7649
WASPAS									0.5726	0.8892	0.8664
COCOSO										0.5717	0.5551
Null EWPM											0.7649

Table 10
Simulation results of the average HR, for high number of the alternatives and high number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.6825	0.6844	1.0000	0.3686	0.6068	0.5290	0.6825	0.8266	0.4653	0.6825	1.0000
WPM		0.7030	0.6825	0.2794	0.5942	0.5679	1.0000	0.8408	0.4884	1.0000	0.6825
ARAS			0.6844	0.2169	0.4406	0.4582	0.7030	0.7278	0.3462	0.7030	0.6844
MARCOS				0.3686	0.6068	0.5290	0.6825	0.8266	0.4653	0.6825	1.0000
COPRAS					0.4358	0.3034	0.2794	0.3270	0.3553	0.2794	0.3686
MOORA						0.7217	0.5942	0.6233	0.6565	0.5942	0.6068
MOOSRA							0.5679	0.5658	0.5557	0.5679	0.5290
MMOORA								0.8408	0.4884	1.0000	0.6825
WASPAS									0.4943	0.8408	0.8266
COCOSO										0.4884	0.4653
Null EWPM											0.6825

C. Appendix

Individual RCs for combinations 1 to 4:

Table 11
Simulation results of the average RC, for low number of the alternatives and low number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.3909	0.7286	1.0000	−0.0958	0.3871	0.1619	0.3909	0.6650	0.0589	0.3909	1.0000
WPM		0.4486	0.3909	−0.1323	0.4198	0.2100	1.0000	0.6423	0.0750	1.0000	0.3909
ARAS			0.7286	−0.1317	0.3701	0.1632	0.4486	0.6838	0.0204	0.4486	0.7286
MARCOS				−0.0958	0.3871	0.1619	0.3909	0.6650	0.0589	0.3909	1.0000
COPRAS					−0.1057	−0.1414	−0.1323	−0.0994	−0.2242	−0.1323	−0.0958
MOORA						0.2957	0.4198	0.4294	0.2073	0.4198	0.3871
MOOSRA							0.2100	0.2047	0.0534	0.2100	0.1619
MMOORA								0.6423	0.0750	1.0000	0.3909
WASPAS									0.0805	0.6423	0.6650
COCOSO										0.0750	0.0589
Null EWPM											0.3909

Table 12
Simulation results of the average RC, for low number of the alternatives and high number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	0.2777	0.6085	1.0000	−0.1488	0.3199	0.1683	0.2777	0.5882	−0.0590	0.2777	1.0000
WPM		0.3334	0.2777	−0.1929	0.2683	0.1561	1.0000	0.5829	−0.0651	1.0000	0.2777
ARAS			0.6085	−0.2105	0.2462	0.1263	0.3334	0.5624	−0.1185	0.3334	0.6085
MARCOS				−0.1488	0.3199	0.1683	0.2777	0.5882	−0.0590	0.2777	1.0000
COPRAS					−0.1275	−0.1359	−0.1929	−0.1572	−0.2629	−0.1929	−0.1488
MOORA						0.4382	0.2683	0.3357	0.0572	0.2683	0.3199
MOOSRA							0.1561	0.1948	−0.0138	0.1561	0.1683
MMOORA								0.5829	−0.0651	1.0000	0.2777
WASPAS									−0.0430	0.5829	0.5882
COCOSO										−0.0651	−0.0590
Null EWPM											0.2777

Table 13
Simulation results of the average RC, for high number of the alternatives and low number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	−0.1909	0.2267	1.0000	−0.4123	−0.2478	−0.3668	−0.1909	0.2814	−0.4466	−0.1909	1.0000
WPM		−0.1175	−0.1909	−0.5463	−0.1226	−0.2634	1.0000	0.1196	−0.3371	1.0000	−0.1909
ARAS			0.2267	−0.5135	−0.2435	−0.3516	−0.1175	0.2585	−0.4740	−0.1175	0.2267
MARCOS				−0.4123	−0.2478	−0.3668	−0.1909	0.2814	−0.4466	−0.1909	1.0000
COPRAS					−0.5074	−0.5291	−0.5463	−0.4657	−0.5941	−0.5463	−0.4123
MOORA						−0.1400	−0.1226	−0.1768	−0.1496	−0.1226	−0.2478
MOOSRA							−0.2634	−0.2915	−0.3853	−0.2634	−0.3668
MMOORA								0.1196	−0.3371	1.0000	−0.1909
WASPAS									−0.3864	0.1196	0.2814
COCOSO										−0.3371	−0.4466
Null EWPM											−0.1909

Table 14
Simulation results of the average RC, for high number of the alternatives and high number of the criteria.

	WPM	ARAS	MARCOS	COPRAS	MOORA	MOOSRA	MMOORA	WASPAS	COCOSO	Null EWPM	Null EWSM
WSM	−0.2979	0.0067	1.0000	−0.5171	−0.3221	−0.4079	−0.2979	0.1901	−0.5481	−0.2979	1.0000
WPM		−0.2743	−0.2979	−0.6026	−0.2774	−0.3649	1.0000	0.0221	−0.4772	1.0000	−0.2979
ARAS			0.0067	−0.6253	−0.3919	−0.4500	−0.2743	−0.0068	−0.5931	−0.2743	0.0067
MARCOS				−0.5171	−0.3221	−0.4079	−0.2979	0.1901	−0.5481	−0.2979	1.0000
COPRAS					−0.5433	−0.5389	−0.6026	−0.5412	−0.6259	−0.6026	−0.5171
MOORA						0.0391	−0.2774	−0.2633	−0.3652	−0.2774	−0.3221
MOOSRA							−0.3649	−0.3496	−0.4527	−0.3649	−0.4079
MMOORA								0.0221	−0.4772	1.0000	−0.2979
WASPAS									−0.5027	0.0221	0.1901
COCOSO										−0.4772	−0.5481
Null EWPM											−0.2979

References

Ahn, B.S. (2017). Approximate weighting method for multi-attribute decision problems with imprecise parameters. *Omega*, 72, 87–95.

Ahn, B.S., Park, K.S. (2008). Comparing methods for multi attribute decision making with ordinal weights. *Computers and Operations Research*, 35(5), 1660–1670.

Aydogdu, E., Aldemir, B., Guner, E., Aygun, H. (2023). A novel entropy measure with its application to the COPRAS method in complex spherical fuzzy. *Informatica*, 34(4), 679–711.

Barron, F., Barrett, B.E. (1996). Decision quality using ranked attribute weights. *Management Science*, 42(11), 1515–1523.

Biswas, P., Pramanik, S., Giri, B.C. (2019). NH-MADM strategy in Neutrosophic hesitant fuzzy set environment based on extended GRA. *Informatica*, 30(2), 213–242.

Brauers, W.K.M., Zavadskas, E.K. (2006). The MOORA method and its application to privatization in transition economy. *Control and Cybernetics*, 35(2), 445–469.

Brauers, W.K.M., Zavadskas, E.K. (2010). Project management by MULTIMOORA as an instrument for transition economies. *Technological and Economical Development of Economy*, 16(1), 5–24.

Chiclana, F., Herrera, F., Herrera-Viedma, E. (1998). Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations. *Fuzzy Sets and Systems*, 97, 33–48.

Churchman, C.W., Ackoff, R.L. (1954). An approximate measure of value. *Journal of the Operations Research Society*, 2(2), 172–187.

Crawford, G., Williams, C. (1985). A note on the analysis of subjective judgment matrices. *Journal of Mathematical Psychology*, 29(1), 387–405.

Das, M.C., Sarkar, B., Ray, S. (2012). Decision making under conflicting environment: a new MCDM method. *International Journal of Applied Decision Science*, 5(2), 142–162.

Deng, J. (1982). Control problems and Grey systems. *Systems and Control Letters*, 5(2), 288–294.

Fan, Z.P., Ma, J., Zhang, Q. (2002). An approach to multiple attribute decision making based on fuzzy preference information on alternatives. *Fuzzy Sets and Systems*, 131(1), 101–106.

Hatefi, M.A. (2019). Indifference threshold-based attribute ratio analysis: a method for assigning the weights to the attributes in multiple attribute decision making. *Applied Soft Computing*, 74, 643–651.

Hatefi, M.A. (2021). BRAW: block-wise rating the attribute weights in MADM. *Computers and Industrial Engineering*, 156 107274.

Hatefi, M.A. (2023). An Improved Rank Order Centroid method (IROC) for criteria weight estimation: an application in the engine/vehicle selection problem. *Informatica*, 34(2), 249–270.

Hovanov, N.V., Kolari, J.W., Sokolov, M.V. (2008). Deriving weights from general pairwise comparison matrices. *Mathematical Social Sciences*, 55, 205–220.

Hwang, C.L., Yoon, K. (1981). *Multiple Attribute Decision Making: Methods and Applications*. Springer, Berlin.

Jahan, A., Edwards, K.L. (2015). A state-of-the-art survey on the influence of normalization techniques in ranking: improving the materials selection process in engineering design. *Materials and Design*, 65, 335–342.

Katsikopoulos, K.V., Fasolo, B. (2006). New tools for decision analysts. *IEEE Transactions on Systems, Man, and Cybernetics – Part A: Systems and Humans*, 36(5), 960–967.

Kacprzyk, J. (1986). Group decision making with a fuzzy majority. *Fuzzy Sets and Systems*, 18(2), 105–118.

Kendall, M., Gibbons, J. (1990). *Rank Correlation Method*. Edward Arnold, London.

Keshavarz Ghorabaee, M., Amiri, M., Zavadskas, E.K., Turskis, Z., Antucheviciene, J. (2021). Determination of objective weights using a new Method based on the Removal Effects of Criteria (MEREC). *Symmetry*, 13, 525.

Lotfi, V., Steward, T.J., Zionts, S. (1992). An aspiration level interactive model for multiple criteria decision making. *Computer and Operations Research*, 19(7), 671–681.

McCrimmon, K.R. (1968). *Decision Making Among Multiple-Attribute Alternatives: Survey and Consolidated Approach. Santa Monica*. RAND Memorandum, RM-4823-ARPA.

Miller, D.W., Starr, M.K. (1969). *Executive Decisions and Operations Research*. Prentice-Hall, Inc., Englewood Cliffs.

Roy, B. (1978). ELECTRE III: algorithme de classement base surune presentation floue des preferences en presence de criteres multiples. *Cahiers du CERO*, 20(1), 3–24.

Saaty, T.L. (1980). *The Analytic Hierarchy Process*. McGraw-Hill, NY, USA.

Sarabando, P., Dias, L.C. (2010). Simple procedures of choice in multi-criteria problems without precise information about the alternatives' values. *Computers and Operations Research*, 37(12), 2239–2247.

Stevic, Z., Pamucar, D., Puska, A., Chatterjee, P. (2020). Sustainable supplier selection in healthcare industries using a new MCDM method: Measurement of Alternatives and Ranking according to Compromise Solution (MARCOS). *Computers and Industrial Engineering*, 140, 106231.

Triantaphyllou, E. (2000). *Multi-Criteria Decision Making: A Comparative Study*. Kluwer Academic Publishers, Dordrecht, Netherlands.

Tversky, A. (1969). Intransitivity of preferences. *Psychological Review*, 76(1), 31–48.

Winkler, R.L., Hays, W.L. (1985). In: *Statistics: Probability, Inference, and Decision*, New York, Holt, Rinehart and Winston.

Yazdani, M., Zarate, P., Zavadskas, E.K., Turskis, Z. (2018). A Combined Compromise Solution (COCOSO) method for multi-criteria decision-making problems. *Management Decision*, 57(9), 2501–2519.

Zadeh, L. (1965). Fuzzy sets. *Information and Control*, 8, 338–353.

Zavadskas, E.K., Kaklauskas, A. (1996). Determination of an efficient contractor by using the new method of multi-criteria assessment. In: Langford, D.A., Retik, A. (Eds.), *International Symposium for The Organization and Management of Construction, Shaping Theory and Practice, Vol. 2, Managing the Construction Project and Managing Risk. CIB W/65*. E. & F.N. Spon, London, pp. 94–104.

Zavadskas, E.K., Turskis, Z. (2008). A new logarithmic normalization method in games theory. *Informatica*, 19(2), 303–314.

Zavadskas, E.K., Turskis, Z. (2010a). A new Additive Ratio Assessment (ARAS) method in multi-criteria decision making. *Technological and Economic Development of Economy*, 16(2), 159–172.

Zavadskas, E.K., Turskis, Z. (2010b). A novel method for multiple criteria analysis: Grey Additive Ratio Assessment (ARAS-G) method. *Informatica*, 21(4), 597–610.

Zavadskas, E.K., Turskis, Z., Antucheviciene, J., Zakarevicius, A. (2012). Optimization of weighted aggregated sum product assessment. *Elektronika ir Elektrotechnika*, 122(6), 3–6.

Zhang, Q., Chen, J.C.H., Chong, P.P. (2004). Decision consolidation: criteria weight determination using multiple preferences formats. *Decision Support Systems*, 38(2), 247–258.

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