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Circular Intuitionistic Fuzzy ELECTRE Approach: A Novel Multiple-Criteria Choice Model

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Abstract. This paper presents a multiple-criteria choice model, the circular intuitionistic fuzzy (C-IF) ELECTRE, designed to resolve C-IF ambiguities through built-in circular functions. Joint generalized scoring functions establish contrast relationships between C-IF evaluation values, facilitating concordance and discordance analyses for option ranking. The efficacy of C-IF ELECTRE I and II—leveraging tools such as the prioritization Boolean matrix, average outflows and inflows, and overall net flow—is validated through a multi-expert supplier evaluation, with outcomes benchmarked against alternative methods. A comparative analysis explores the impact of parameter variations, underscoring how integrating C-IF sets with ELECTRE enhances decision-making in complex, multifaceted environments.

Key words: multiple-criteria choice model, circular intuitionistic fuzzy (C-IF), ELECTRE, joint generalized scoring function, comparative analysis.

1. Introduction

Multiple-criteria analysis is crucial for addressing decision-making challenges in practical scenarios (Chen, 2024; Liu, 2024). In the ELECTRE (i.e. ÉLimination Et Choix Traduisant la REalité in French) framework, evaluative criteria serve as standards to assess and prioritize options based on their performance (Akram *et al.*, 2023a, 2023b). However, uncertainty can disrupt the ELECTRE process, leading to inaccurate assessments, inconsistent decisions, and suboptimal choices, undermining decision-makers' confidence (Ramya *et al.*, 2023; Wu *et al.*, 2023). Insufficient or unreliable data may introduce biases, distort rankings, and increase subjective judgments, compromising reliability (Liu *et al.*, 2023; Yüksel and Dinçer, 2023). This heightened risk complicates the appraisal of options and limits the efficacy of ELECTRE-based analysis (Zhang *et al.*, 2023; Zhou *et al.*, 2022).

To meet the demands of uncertain environments, researchers have expanded the ELECTRE methodology to various fuzzy scenarios. Beyond conventional fuzzy models, recent efforts have focused on higher-order fuzzy frameworks. Akram *et al.* (2023b) introduced the fuzzy ELECTRE IV method using triangular fuzzy numbers for imprecise data.

Wu *et al.* (2023) developed a triadic decision model combining ELECTRE III with attribute ratio evaluation in a spherical fuzzy setting for customer selection. Wang and Chen (2021) used T-spherical fuzzy score functions and Minkowski distance-dependent indices to design T-spherical fuzzy ELECTRE I and II methods. Yüksel and Dinçer (2023) evaluated sustainability in circular industrialization using quantum spherical fuzzy ELECTRE. Zhang *et al.* (2023) created an ELECTRE II method with cosine similarity measures to assess financial logistics firms' efficiency using double hierarchy hesitant fuzzy information. Ramya *et al.* (2023) proposed a disposal technique selection framework for e-waste using ELECTRE III with wiggly hesitant Pythagorean fuzzy sets. Zhou *et al.* (2022) developed a Fermatean fuzzy ELECTRE method using Jensen-Shannon divergence and cross-entropy to handle uncertain decision-maker and criteria weights. Akram *et al.* (2023a) investigated a 2-tuple linguistic Fermatean fuzzy ELECTRE II for group decision-making with linguistic variables. Pinar and Boran (2022) introduced a q-rung picture fuzzy ELECTRE model, integrating the technique for order preference by similarity to ideal solutions (TOPSIS) for group decision analysis. These advancements are a result of continuous work to improve the applicability and efficacy of ELECTRE-based strategies for tackling challenging and ambiguous decision-making situations.

Applying fuzzy extensions of the ELECTRE-based outranking methodology to multi-criteria analysis enhances decision-making, particularly in uncertain environments. These approaches incorporate fuzzy logic to handle uncertainty, offering deeper insights for decision-makers. However, advancements specifically tailored to the emerging circular intuitionistic fuzzy (C-IF) sets remain limited. Intuitionistic fuzzy (IF) sets, introduced by Atanassov (1986), include degrees of hesitancy alongside membership and non-membership, addressing vagueness more effectively than conventional fuzzy sets (Chen, 2024; Liu, 2024). Their integration with various tools enables more precise handling of uncertain information (Liu *et al.*, 2023). Atanassov later extended IF sets into C-IF sets, which represent circles with radii encompassing membership and non-membership components (Atanassov, 2020; Çakır and Taş, 2023). C-IF sets have demonstrated versatility across realistic applications (Alinejad *et al.*, 2024; Ci, 2024; Kong, 2024) and hold promise for future advancements in ELECTRE-based methods.

C-IF sets offer greater flexibility by integrating membership, hesitancy, and non-membership components (Ci, 2024; Jameel *et al.*, 2024). Represented by circular shapes, they capture multiple qualities or attributes, with the circular function enhancing their expressiveness in a triangular distribution space (Kong, 2024; Pratama *et al.*, 2024). When the circle's radius is set to zero, C-IF sets revert to standard IF sets. C-IF sets excel in handling uncertain, imprecise information, making them suitable for diverse decision-making tasks (Alinejad *et al.*, 2024; Chen, 2023a). Applications include decision assistance and support (Chen, 2023b, 2024), supplier evaluation and selection (Çakır and Taş, 2023; Chen, 2024), sustainable renewable energy management (Jameel *et al.*, 2024), biomass resource strategies (Alinejad *et al.*, 2024), food supply chain monitoring (Alsattar *et al.*, 2024), pattern recognition and medical diagnosis (Khan *et al.*, 2022), and public health risk assessments (Kong, 2024). These examples underscore the adaptability and utility of C-IF sets for addressing uncertainty in practical applications across various domains.

The versatility of C-IF sets makes them valuable for decision-making; however, effective analysis requires reliable methods to interpret and manipulate uncertain information (Pinar and Boran, 2022; Zhang *et al.*, 2023). Chen (2023b) formulated two generalized C-IF distance metrics – three-term and four-term Minkowski-like measures – designed to handle imperfect information. The four-term approach accounts for radius, membership, non-membership, and hesitancy, offering a complete representation of C-IF numbers. In contrast, the three-term model excludes hesitancy. This study adopts the four-term model for its comprehensive nature, providing a solid foundation for developing a C-IF ELECTRE approach to better address research objectives.

C-IF sets are highly effective for managing complex and ambiguous information. However, there remains a significant gap in developing ELECTRE-based methods tailored for C-IF decision environments. While existing ELECTRE approaches have shown success in handling uncertainty across various fuzzy frameworks – including spherical fuzzy (Wu *et al.*, 2023), T-spherical fuzzy (Wang and Chen, 2021), quantum spherical fuzzy (Yüksel and Dinçer, 2023), double hierarchy hesitant fuzzy (Zhang *et al.*, 2023), normal wiggly Pythagorean hesitant fuzzy (Ramya *et al.*, 2023), Fermatean fuzzy (Zhou *et al.*, 2022), 2-tuple linguistic Fermatean fuzzy (Akram *et al.*, 2023a), and q-rung picture fuzzy (Pinar and Boran, 2022) – their application within C-IF contexts remains limited. This study addresses this gap with the following motivations:

(1) Uncertainty Challenges: Current ELECTRE-based approaches struggle with complex uncertainties, highlighting the need for more robust and enhanced methods.
(2) Limited C-IF Applications: Despite progress in C-IF methods, there has been little integration of ELECTRE in C-IF multiple-criteria analysis, highlighting a key research gap.
(3) Distance Measurement Importance: Accurate measurement of C-IF distances is crucial for distinguishing complex information. This study adopts the four-term Minkowski-like distance model for its completeness in developing the C-IF ELECTRE approach.

This research aims to develop the C-IF ELECTRE, a multiple-criteria decision model for discrete decisions involving conflicting or incomparable criteria. It integrates circular intuitionistic fuzziness with a joint generalized scoring function, based on Hezam *et al.*'s (2023) natural exponential function, to address uncertainty. The concepts of aggressive and cautious IF estimates (inspired by Chen, 2023a) are employed to provide upper and lower estimations within the C-IF context, with several theorems outlining their properties and relationships. The C-IF ELECTRE approach involves establishing concordance and discordance sets to determine when one alternative is superior, equal, or inferior to another. C-IF ELECTRE I uses consistency and inconsistency indicators to build a dominance graph for partial-priority rankings. C-IF ELECTRE II introduces consistency-dependent outflow, inconsistency-dependent inflow, and net flow values for complete-priority rankings. Applied to supplier evaluations, the method aligns with comparative results and highlights the impact of parameter settings, such as distance and divergence measures, on ranking outcomes.

This study offers key advancements in ELECTRE-based decision-making:

(1) Joint Generalized Scoring Function: Introduces an inclination parameter to capture the decision-maker's aggressive, neutral, or cautious tendencies, integrating these attitudes into assessments.
(2) C-IF ELECTRE Framework: Develops a practical model tailored for C-IF contexts with step-by-step algorithms for defining problems, calculating scores, measuring consistency/inconsistency, and generating rankings.
(3) C-IF ELECTRE I and II Techniques: Combines C-IF sets with ELECTRE to manage complex multi-criteria decisions, validated through multi-expert supplier assessments, producing reliable, consistent rankings aligned with comparative methods.

This article is structured as follows: Section 1 highlights research motivations and the need for innovations in ELECTRE-based methodologies for C-IF contexts. Section 2 covers fundamental mathematical notations for IF and C-IF configurations. Section 3 develops a joint generalized scoring function to address C-IF uncertainty. Section 4 introduces the C-IF ELECTRE model for complex decision-making in uncertain contexts. Section 5 demonstrates the application of C-IF ELECTRE I and II techniques through a multi-expert supplier evaluation case. Section 6 analyses the impact of inclination parameter settings on results, highlighting the approach's advantages. Section 7 provides conclusions and suggests directions for future research.

2. Foundational Concepts Related to C-IF Sets

Definition 1 (Atanassov, 1986). Assume $\aleph$ is a nonempty set of elements. Let $u_I(\chi)$ and $v_I(\chi)$ be functions mapping $\aleph$ to [0, 1], representing the degree to which an element $\chi \in \aleph$ belongs to or does not belong to an IF set I, respectively, subject to $0 \leqslant u_I(\chi) + v_I(\chi) \leqslant 1$. The IF set I within $\aleph$ is defined as:

$$I = \{\langle \chi, u_I(\chi), v_I(\chi)\rangle \mid \chi \in \aleph\}. \tag{1}$$

Definition 2 (Hezam *et al.*, 2023). Let $i(\chi) = (u_I(\chi), v_I(\chi))$ represent an IF number within the IF set I. The degree of hesitancy is given by $h_I(\chi) = 1 - u_I(\chi) - v_I(\chi)$. The scoring mechanism for $i(\chi)$ uses a natural exponential function with Euler's number e as:

$$M\big(i(\chi)\big) = \frac{1}{2}\left[u_I(\chi) - v_I(\chi) + h_I(\chi) \cdot \left(\frac{e^{(u_I(\chi)-v_I(\chi))}}{e^{(u_I(\chi)-v_I(\chi))}+1} - \frac{1}{2}\right) + 1\right]. \tag{2}$$

Definition 3 (Atanassov, 2020). Let $\mathcal{L}^*$ represent an L-fuzzy set, defined as $\mathcal{L}^* = \{\langle \ell, \ell'\rangle \mid \ell, \ell' \in [0,1] \text{ and } \ell + \ell' \leqslant 1\}$. The membership degree $u_C(\chi) : \aleph \longrightarrow [0,1]$ and non-membership degree $v_C(\chi) : \aleph \longrightarrow [0,1]$ capture the extent to which $\chi \in \aleph$ belongs to or does not belong to a C-IF set C. These degrees satisfy $0 \leqslant u_C(\chi) + v_C(\chi) \leqslant 1$. The degree of hesitancy is calculated as: $h_C(\chi) = 1 - u_C(\chi) - v_C(\chi)$. The built-in circular function $\mathfrak{O}_r$ has a radius $r_C(\chi) : \aleph \longrightarrow [0, \sqrt{2}]$ with the centre at $(u_C(\chi), v_C(\chi))$. A C-IF number is represented as $c(\chi) = (u_C(\chi), v_C(\chi); r_C(\chi))$. The C-IF set C and its circular function $\mathfrak{O}_r$ within the domain $\aleph$ are structured as follows:

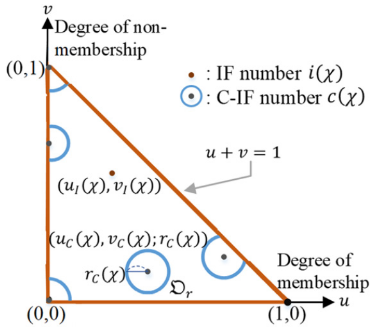


Fig. 1. The visualization of the relationship between standard IF and C-IF constructs.

$$C = \{\langle \chi, u_C(\chi), v_C(\chi); r_C(\chi)\rangle | \chi \in \aleph\} = \{\langle \chi, \mathfrak{O}_r(u_C(\chi), v_C(\chi))\rangle | \chi \in \aleph\}, \tag{3}$$

$$\mathfrak{O}_r(u_C(\chi), v_C(\chi)) = \left\{\langle \ell, \ell' \rangle | \ell, \ell' \in [0, 1] \text{ and } \sqrt{(u_C(\chi) - \ell)^2 + (v_C(\chi) - \ell')^2} \leqslant r_C(\chi)\right\} \cap \mathcal{L}^*. \tag{4}$$

A standard fuzzy set describes membership, while an IF set adds flexibility by allowing for hesitation between membership and non-membership. Building on this, the C-IF set introduces a circular structure, capturing more complex, ambiguous characteristics. Unlike IF sets, C-IF sets represent uncertainty more precisely using their circular form. Figure 1 compares IF and C-IF sets within a triangular space defined by the vertices (0, 0), (1, 1), and (0, 1), illustrating their overlap and differences. In this space, the C-IF number $c(\chi)$ is represented by the centre $(u_C(\chi), v_C(\chi))$ and radius $r_C(\chi)$ of the circular function $\mathfrak{O}_r$, while the IF number $i(\chi)$ corresponds to the point $(u_I(\chi), v_I(\chi))$. Figure 1 shows five ways to depict $\mathfrak{O}_r$, demonstrating how it constrains $\mathfrak{O}_r(u_C(\chi), v_C(\chi))$ within the L-fuzzy set $\mathcal{L}^*$. When the radius $r_C(\chi) = 0$ for all elements in the domain $\aleph$, the C-IF set C reduces to an IF set I, and the C-IF number $c(\chi)$ becomes equivalent to the IF number $i(\chi)$.

Chen (2023b) introduced two generalized C-IF distance metrics to overcome the limitations of traditional measures and improve adaptability. These metrics use triadic and quadripartite representations of C-IF Minkowski-like distances. The triadic model includes radius, membership, and non-membership, while the quadripartite version adds hesitancy, providing a more comprehensive assessment of C-IF dimensionality.

Definition 4 (Chen, 2023b). Consider two C-IF numbers, $c(\chi) = (u_C(\chi), v_C(\chi); r_C(\chi))$ and $c(\chi') = (u_C(\chi'), v_C(\chi'); r_C(\chi'))$. A positive integer $\xi \in Z^+$ serves as the metric parameter. The C-IF Minkowski-like distance between them is defined using three-term and four-term strategies as follows:

$$D^{\xi}_{\mathfrak{M}^0}(c(\chi), c(\chi')) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}|r_C(\chi) - r_C(\chi')| + \sqrt[\xi]{\frac{1}{2}\left(|u_C(\chi) - u_C(\chi')|^{\xi} + |v_C(\chi) - v_C(\chi')|^{\xi}\right)}\right), \tag{5}$$

$$D_{\mathfrak{M}}^{\xi}(c(\chi), c(\chi')) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}|r_C(\chi) - r_C(\chi')| + \sqrt[\xi]{\frac{1}{2}\left(|u_C(\chi) - u_C(\chi')|^{\xi} + |v_C(\chi) - v_C(\chi')|^{\xi} + |h_C(\chi) - h_C(\chi')|^{\xi}\right)}\right). \quad (6)$$

These distance measures quantify the dissimilarity between C-IF numbers, varying by approach (three-term or four-term) and the chosen metric parameter ξ. The triadic model omits hesitancy, while the quaternary model offers a more exhaustive representation. This research adopts the four-term Minkowski-like distance as the foundation for the C-IF ELECTRE methodology due to its comprehensive coverage of all relevant dimensions, ensuring alignment with the study's objectives.

3. Joint Generalized Scoring Function

This section aims to develop a joint generalized scoring function that addresses the C-IF uncertainty. Utilizing a natural exponential-based scoring mechanism (Hezam *et al.*, 2023), it effectively handles IF scenarios. Building on Chen's (2023a) concepts of aggressive and conservative estimates, this section introduces practical notions of aggressive and cautious IF estimates. It also explores their fundamental properties, highlighting their role in representing upper and lower bounds within C-IF information.

Definition 5. For a C-IF number $c(\chi) = (u_C(\chi), v_C(\chi); r_C(\chi))$, its aggressive IF estimate $i_c^{\alpha}(\chi)$ and cautious IF estimate $i_c^{\beta}(\chi)$ are defined as follows:

$$i_c^{\alpha}(\chi) = \left(u_I^{\alpha}(\chi), v_I^{\alpha}(\chi)\right) = \left(\min\left\{1, u_C(\chi) + \frac{r_C(\chi)}{\sqrt{2}}\right\}, \max\left\{0, v_C(\chi) - \frac{r_C(\chi)}{\sqrt{2}}\right\}\right), \quad (7)$$

$$i_c^{\beta}(\chi) = \left(u_I^{\beta}(\chi), v_I^{\beta}(\chi)\right) = \left(\max\left\{0, u_C(\chi) - \frac{r_C(\chi)}{\sqrt{2}}\right\}, \min\left\{1, v_C(\chi) + \frac{r_C(\chi)}{\sqrt{2}}\right\}\right). \quad (8)$$

Theorem 1. *For a C-IF number $c(\chi)$, the aggressive IF estimate $i_c^{\alpha}(\chi) = (u_I^{\alpha}(\chi), v_I^{\alpha}(\chi))$ and cautious IF estimate $i_c^{\beta}(\chi) = (u_I^{\beta}(\chi), v_I^{\beta}(\chi))$ follow a quasi-ordering relationship: $i_c^{\alpha}(\chi) \succcurlyeq_Q i_c^{\beta}(\chi)$. When defined as $i_c^{\alpha}(\chi) = \left(u_C(\chi) + r_C(\chi)/\sqrt{2}, v_C(\chi) - r_C(\chi)/\sqrt{2}\right)$ and $i_c^{\beta}(\chi) = \left(u_C(\chi) - r_C(\chi)/\sqrt{2}, v_C(\chi) + r_C(\chi)/\sqrt{2}\right)$, both estimates exhibit equal hesitancy: $h_I^{\alpha}(\chi) = h_I^{\beta}(\chi)$.*

Proof. The conditions for $i_c^{\alpha}(\chi) \succcurlyeq_Q i_c^{\beta}(\chi)$ are $u_I^{\alpha}(\chi) \geqslant u_I^{\beta}(\chi)$ and $v_I^{\alpha}(\chi) \leqslant v_I^{\beta}(\chi)$ for each $\chi \in \aleph$. From Eqs. (7) and (8), it follows that $\min\{1, u_C(\chi) + r_C(\chi)/\sqrt{2}\} \geqslant \max\{0, u_C(\chi) - r_C(\chi)/\sqrt{2}\}$, leading to $u_I^{\alpha}(\chi) \geqslant u_I^{\beta}(\chi)$. Since $\max\{0, v_C(\chi) - r_C(\chi)/\sqrt{2}\} \leqslant \min\{1, v_C(\chi) + r_C(\chi)/\sqrt{2}\}$, it follows that $v_I^{\alpha}(\chi) \leqslant v_I^{\beta}(\chi)$. It is established that $i_c^{\alpha}(\chi) \succcurlyeq_Q i_c^{\beta}(\chi)$ if and only if $u_I^{\alpha}(\chi) \geqslant u_I^{\beta}(\chi)$ and $v_I^{\alpha}(\chi) \leqslant v_I^{\beta}(\chi)$.

Given $i_c^\alpha(\chi) = (u_C(\chi) + r_C(\chi)/\sqrt{2}, v_C(\chi) - r_C(\chi)/\sqrt{2})$ and $i_c^\beta(\chi) = (u_C(\chi) - r_C(\chi)/\sqrt{2}, v_C(\chi) + r_C(\chi)/\sqrt{2})$, the degrees of hesitancy are: $h_I^\alpha(\chi) = 1 - u_C(\chi) - r_C(\chi)/\sqrt{2} - v_C(\chi) + r_C(\chi)/\sqrt{2} = 1 - u_C(\chi) - v_C(\chi)$ and $h_I^\beta(\chi) = 1 - u_C(\chi) - v_C(\chi)$. Thus, $i_c^\alpha(\chi)$ and $i_c^\beta(\chi)$ have the same degrees of hesitancy. □

Theorem 2. *The scoring mechanisms* $M(i_c^\alpha(\chi))$ *and* $M(i_c^\beta(\chi))$ *exhibit the fundamental properties*: (1) $0 \leqslant M(i_c^\alpha(\chi)) \leqslant 1$ *and* $0 \leqslant M(i_c^\beta(\chi)) \leqslant 1$; (2) $M(i_c^\alpha(\chi)) = 1$ *and* $M(i_c^\beta(\chi)) = 1$ *if* $i_c^\alpha(\chi) = (1, 0)$ *and* $i_c^\beta(\chi) = (1, 0)$, *respectively*; *and* (3) $M(i_c^\alpha(\chi)) = 0$ *and* $M(i_c^\beta(\chi)) = 0$ *if* $i_c^\alpha(\chi) = (0, 1)$ *and* $i_c^\beta(\chi) = (0, 1)$, *respectively.*

Proof. Applying Eq. (2), the scoring mechanisms $M(i_c^\alpha(\chi))$ and $M(i_c^\beta(\chi))$ are established:

$$M\big(i_c^\alpha(\chi)\big) = \frac{1}{2}\left[\big(u_I^\alpha(\chi) - v_I^\alpha(\chi)\big) + h_I^\alpha(\chi)\cdot\left(\frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2}\right) + 1\right],$$
$$M\big(i_c^\beta(\chi)\big) = \frac{1}{2}\left[\big(u_I^\beta(\chi) - v_I^\beta(\chi)\big) + h_I^\beta(\chi)\cdot\left(\frac{e^{(u_I^\beta(\chi)-v_I^\beta(\chi))}}{e^{(u_I^\beta(\chi)-v_I^\beta(\chi))}+1} - \frac{1}{2}\right) + 1\right].$$

Given the property in (1), the aggressive IF estimate $i_c^\alpha(\chi)$ satisfies $u_I^\alpha(\chi) + v_I^\alpha(\chi) + h_I^\alpha(\chi) = 1$, where $u_I^\alpha(\chi)$, $v_I^\alpha(\chi)$, and $h_I^\alpha(\chi)$ are constrained to the interval [0, 1]. The scoring mechanism uses the natural exponential function. Since $0 \leqslant u_I^\alpha(\chi) - v_I^\alpha(\chi) \leqslant 1$, we have $0 < e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))} \leqslant e^1 \approx 2.71828$. This leads to the following outcomes:

$$0 < \frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} \leqslant \frac{e}{e+1},$$
$$-\frac{1}{2} < \frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2} \leqslant \frac{e}{e+1} - \frac{1}{2} = \frac{e-1}{2e+2} = 0.2311.$$

Aa a result, it is known that $M(i_c^\alpha(\chi)) \leqslant 1$, because:

$$u_I^\alpha(\chi) - v_I^\alpha(\chi) + h_I^\alpha(\chi)\cdot\left(\frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2}\right)$$
$$\leqslant u_I^\alpha(\chi) + v_I^\alpha(\chi) + h_I^\alpha(\chi) \leqslant 1,$$
$$\frac{1}{2}\left[\big(u_I^\alpha(\chi) - v_I^\alpha(\chi)\big) + h_I^\alpha(\chi)\cdot\left(\frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2}\right) + 1\right] \leqslant \frac{1}{2}(1+1) = 1.$$

On the other hand, it is inferred that $M(i_c^\alpha(\chi)) \geqslant 0$, because:

$$u_I^\alpha(\chi) - v_I^\alpha(\chi) + h_I^\alpha(\chi)\cdot\left(\frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2}\right) + u_I^\alpha(\chi) + v_I^\alpha(\chi) + h_I^\alpha(\chi)$$
$$= 2u_I^\alpha(\chi) + h_I^\alpha(\chi)\cdot\left(\frac{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}}{e^{(u_I^\alpha(\chi)-v_I^\alpha(\chi))}+1} - \frac{1}{2} + 1\right)$$
$$> 2u_I^\alpha(\chi) + h_I^\alpha(\chi)\left(-\frac{1}{2} + 1\right) \geqslant 0.$$

It follows that $0 \leqslant M(i_c^{\alpha}(\chi)) \leqslant 1$ and $0 \leqslant M(i_c^{\beta}(\chi)) \leqslant 1$. According to property (2), $M(i_c^{\alpha}(\chi)) = 1$ and $M(i_c^{\beta}(\chi)) = 1$ if $i_c^{\alpha}(\chi) = (1,0)$ and $i_c^{\beta}(\chi) = (1,0)$, respectively. In these cases, both aggressive and cautious IF estimates yield a score of 1. For property (3), $M(i_c^{\alpha}(\chi)) = 0$ and $M(i_c^{\beta}(\chi)) = 0$ if $i_c^{\alpha}(\chi) = (0,1)$ and $i_c^{\beta}(\chi) = (0,1)$. Here, both IF numbers yielding (0, 1) lead to a score of 0. □

Theorem 3. *For the aggressive and cautious IF estimates $i_c^{\alpha}(\chi)$ and $i_c^{\beta}(\chi)$, the scoring mechanisms satisfy the inequality $M(i_c^{\alpha}(\chi)) \geqslant M(i_c^{\beta}(\chi))$ for each $\chi \in \aleph$.*

Proof. Consider the scoring mechanism $M(i(\chi))$ for an IF number $i(\chi) = (u_I(\chi), v_I(\chi))$ in Eq. (2). Expanding the formula of $M(i(\chi))$, the following representation is yielded:

$$M\big(i(\chi)\big) = \frac{1}{2} + \frac{1}{2}u_I(\chi) - \frac{1}{2}v_I(\chi) - \frac{1}{4}h_I(\chi) + \frac{1}{2}h_I(\chi) \cdot \left(\frac{e^{(u_I(\chi)-v_I(\chi))}}{e^{(u_I(\chi)-v_I(\chi))}+1}\right).$$

To find the first partial derivative of $M(i(\chi))$ in terms of $u_I(\chi)$, we treat $v_I(\chi)$ and $h_I(\chi)$ as constants. The derivatives of the first four terms with respect to $u_I(\chi)$ are 0, 0.5, 0, and 0, respectively. For the fifth term, which has $u_I(\chi)$ in the exponent, we apply the chain rule. To simplify this process, we introduce a new function:

$$\mathfrak{H}\big(i(\chi)\big) = \frac{e^{(u_I(\chi)-v_I(\chi))}}{e^{(u_I(\chi)-v_I(\chi))}+1}.$$

Applying the quotient rule, the following outcome can be generated:

$$\begin{aligned}\frac{d\mathfrak{H}(i(\chi))}{du_I(\chi)} &= \frac{(e^{(u_I(\chi)-v_I(\chi))}+1)\cdot e^{(u_I(\chi)-v_I(\chi))} - e^{(u_I(\chi)-v_I(\chi))}\cdot e^{(u_I(\chi)-v_I(\chi))}}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2}\\ &= \frac{e^{(u_I(\chi)-v_I(\chi))}}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2}.\end{aligned}$$

The first partial derivative of $M(i(\chi))$ regarding $u_I(\chi)$ can be obtained:

$$\frac{\partial M(i(\chi))}{\partial u_I(\chi)} = \frac{1}{2} + \frac{1}{2}h_I(\chi)\frac{e^{(u_I(\chi)-v_I(\chi))}}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2} \geqslant 0.$$

Since the partial derivative $\partial M(i(\chi))/\partial u_I(\chi)$ is non-negative, the scoring mechanism $M(i(\chi))$ is non-decreasing with respect to $u_I(\chi)$ when $v_I(\chi)$ and $h_I(\chi)$ are held constant. Similarly, to find the first partial derivative of $M(i(\chi))$ in terms of $v_I(\chi)$, we differentiate the expression while treating $u_I(\chi)$ and $h_I(\chi)$ as constants. The result of differentiating the function $\mathfrak{H}(i(\chi))$ with respect to $v_I(\chi)$ is as follows:

$$\begin{aligned}\frac{d\mathfrak{H}(i(\chi))}{dv_I(\chi)} &= \frac{(e^{(u_I(\chi)-v_I(\chi))}+1)\cdot(-e^{(u_I(\chi)-v_I(\chi))}) - e^{(u_I(\chi)-v_I(\chi))}\cdot(-e^{(u_I(\chi)-v_I(\chi))})}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2}\\ &= \frac{-e^{(u_I(\chi)-v_I(\chi))}}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2}.\end{aligned}$$

The first partial derivative of $M(i(\chi))$ concerning $v_I(\chi)$ is generated:

$$\frac{\partial M(i(\chi))}{\partial v_I(\chi)} = -\frac{1}{2} - \frac{1}{2}h_I(\chi)\frac{e^{(u_I(\chi)-v_I(\chi))}}{(e^{(u_I(\chi)-v_I(\chi))}+1)^2} \leqslant 0.$$

Thus, one can conclude that the scoring mechanism $M(i(\chi))$ is non-increasing in relation to $v_I(\chi)$ when $u(\chi)$ and $h_I(\chi)$ are fixed. As established in Theorem 1, the aggressive and cautious IF estimates have a quasi-ordering relationship: $i_c^\alpha(\chi) \succcurlyeq_Q i_c^\beta(\chi)$, leading to $u_I^\alpha(\chi) \geqslant u_I^\beta(\chi)$ and $v_I^\alpha(\chi) \leqslant v_I^\beta(\chi)$. Given that $M(i(\chi))$ is non-decreasing in $u_I(\chi)$ and non-increasing in $v_I(\chi)$, it can be deduced that $M(i_c^\alpha(\chi)) \geqslant M(i_c^\beta(\chi))$. □

Definition 6. Consider $c(\chi) = (u_C(\chi), v_C(\chi); r_C(\chi))$ as a C-IF number within the C-IF set C. Let φ be an inclination parameter ranging from 0 to 1. The joint generalized scoring function for $c(\chi)$, denoted as $S^\varphi(c(\chi))$, is formulated as follows:

$$S^\varphi\big(c(\chi)\big) = \varphi \cdot M\big(i_c^\alpha(\chi)\big) + (1-\varphi) \cdot M\big(i_c^\beta(\chi)\big). \tag{9}$$

Theorem 4. *The following characteristics are fulfilled by the joint generalized scoring function $S^\varphi(c(\chi))$ ($\varphi \in [0,1]$) of a C-IF number $c(\chi)$:* (1) $0 \leqslant S^\varphi(c(\chi)) \leqslant 1$; (2) $S^\varphi(c(\chi)) = M(i_c^\alpha(\chi))$ *when* $\varphi = 1$; *and* (3) $S^\varphi(c(\chi)) = M(i_c^\beta(\chi))$ *when* $\varphi = 0$.

Proof. From Definition 6, the inclination parameter φ ranges from 0 to 1. As per Theorem 2, the scoring mechanisms for $i_c^\alpha(\chi)$ and $i_c^\beta(\chi)$ are also bounded between 0 and 1. Given properties $0 \leqslant \varphi \leqslant 1$, $0 \leqslant M(i_c^\alpha(\chi)) \leqslant 1$, and $0 \leqslant M(i_c^\beta(\chi)) \leqslant 1$, we conclude that $0 \leqslant S^\varphi(c(\chi)) \leqslant 1$. For properties (2) and (3), the equations $S^\varphi(c(\chi)) = M(i_c^\alpha(\chi))$ and $S^\varphi(c(\chi)) = M(i_c^\beta(\chi))$ follow from the cases where $\varphi = 1$ and $\varphi = 0$, respectively. □

When the inclination parameter φ exceeds 0.5, it indicates that the decision-maker places greater importance on the aggressive IF estimate, reflecting an aggressive stance. Conversely, if φ is less than 0.5, the decision-maker favours the cautious IF estimate, demonstrating a cautious attitude. When φ equals 0.5, both aggressive and cautious perspectives are given equal weight, reflecting a neutral disposition.

The inclination parameter φ in the joint generalized scoring function $S^\varphi(c(\chi))$ represents the decision-maker's psychological predisposition toward aggressive, neutral, or cautious attitudes. Assigning a specific value to φ reflects the decision-maker's personal tendencies or preferences. Within $S^\varphi(c(\chi))$, φ indicates a neutral tendency or preference for either aggressive or cautious outcomes.

4. Proposed C-IF ELECTRE Approach

This section introduces the C-IF ELECTRE decision-making model, designed to address complex discrete choice situations with multiple, incommensurable, and contradictory criteria. Proposed within the C-IF uncertain context, this model offers a novel approach for tackling these intricate decision-making challenges.

4.1. *Suggested Methodology*

From a mathematical perspective, a multiple-criteria choice problem involves a set of m choice options, signified as $O = \{O_1, O_2, \dots, O_m\}$, and a set of n evaluative criteria, denoted as $E = \{E_1, E_2, \dots, E_n\}$. The decision-maker can distinguish between two subsets of evaluative criteria: E_B, which includes beneficial criteria to be maximized, and E_N, which includes non-beneficial criteria to be minimized. These subsets are mutually exclusive ($E_B \cap E_N = \emptyset$) and together comprise the entire set of criteria ($E_B \cup E_N = E$).

The significance of a criterion $E_j \in E$ is represented by its C-IF weight $w_j = (\omega_j, \varpi_j; r_j)$, where $\omega_j \in [0, 1]$ and $\varpi_j \in [0, 1]$ indicate the degree of membership and non-membership of E_j in the fuzzy concept of "importance." The radius $r_j \in [0, \sqrt{2}]$ reflects the extent of uncertainty within the circular structure. The degree of hesitancy is given by $\hbar_j = 1 - \omega_j - \varpi_j$. A C-IF set W represents the weight characteristic, defined as follows:

$$W = \left\{\langle E_j, \omega_j, \varpi_j; r_j\rangle \,\middle|\, E_j \in E\right\} = \left\{\left\langle E_j, \mathfrak{O}_r(\omega_j, \varpi_j)\right\rangle \middle| E_j \in E\right\}, \tag{10}$$

$$\mathfrak{O}_r(\omega_j, \varpi_j) = \left\{\left\langle \ell, \ell'\right\rangle \middle| \ell, \ell' \in [0, 1], \sqrt{(\omega_j - \ell)^2 + \left(\varpi_j - \ell'\right)^2} \leqslant r_j, \text{ and } \ell + \ell' \leqslant 1\right\}. \tag{11}$$

The C-IF evaluation value of a choice option O_k (where $k = 1, 2, \dots, m$) assessed by an evaluative criterion E_j (where $j = 1, 2, \dots, n$) is represented as $c_{kj} = (u_{kj}, v_{kj}; r_{kj})$. The hesitation is calculated using the formula $h_{kj} = 1 - u_{kj} - v_{kj}$. The fuzzy characteristic associated with each choice option $O_k \in O$ is represented on this wise:

$$C_k = \left\{\langle E_j, u_{kj}, v_{kj}; r_{kj}\rangle \,\middle|\, E_j \in E\right\} = \left\{\left\langle E_j, \mathfrak{O}_r(u_{kj}, v_{kj})\right\rangle \middle| E_j \in E\right\}, \tag{12}$$

$$\mathfrak{O}_r(u_{kj}, v_{kj}) = \left\{\left\langle \ell, \ell'\right\rangle \middle| \ell, \ell' \in [0, 1], \sqrt{(u_{kj} - \ell)^2 + \left(v_{kj} - \ell'\right)^2} \leqslant r_{kj}, \text{ and } \ell + \ell' \leqslant 1\right\}. \tag{13}$$

Based on Definition 5, the aggressive IF estimate $i_{kj}^{\alpha} = (u_{kj}^{\alpha}, v_{kj}^{\alpha})$ and cautious IF estimate $i_{kj}^{\beta} = (u_{kj}^{\beta}, v_{kj}^{\beta})$ related to the C-IF evaluation value c_{kj} are calculated using the formulas: $i_{kj}^{\alpha} = (\min\{1, u_{kj} + r_{kj}/\sqrt{2}\}, \max\{0, v_{kj} - r_{kj}/\sqrt{2}\})$ and $i_{kj}^{\beta} = (\max\{0, u_{kj} - r_{kj}/\sqrt{2}\}, \min\{1, v_{kj} + r_{kj}/\sqrt{2}\})$. The degrees of hesitancy are given by: $h_{kj}^{\alpha} = 1 - u_{kj}^{\alpha} - v_{kj}^{\alpha}$ and $h_{kj}^{\beta} = 1 - u_{kj}^{\beta} - v_{kj}^{\beta}$. As per Definition 2, the natural exponential function-based scoring mechanisms for i_{kj}^{α} and i_{kj}^{β} are defined as: $M(i_{kj}^{\alpha}) = (1/2) \cdot \{(u_{kj}^{\alpha} - v_{kj}^{\alpha}) + h_{kj}^{\alpha} \cdot [e^{(u_{kj}^{\alpha} - v_{kj}^{\alpha})}/(e^{(u_{kj}^{\alpha} - v_{kj}^{\alpha})} + 1) - (1/2)] + 1\}$ and $M(i_{kj}^{\beta}) = (1/2) \cdot \{(u_{kj}^{\beta} - v_{kj}^{\beta}) + h_{kj}^{\alpha} \cdot [e^{(u_{kj}^{\beta} - v_{kj}^{\beta})}/(e^{(u_{kj}^{\beta} - v_{kj}^{\beta})} + 1) - (1/2)] + 1\}$. Utilizing these results and setting the inclination parameter φ, the joint generalized scoring function is generated as: $S^{\varphi}(c_{kj}) = \varphi \cdot M(i_{kj}^{\alpha}) + (1 - \varphi) \cdot M(i_{kj}^{\beta})$. This scoring function will be used to establish the sets of concordance and discordance for all pairs of choice options.

Definition 7. The contrast relationship between the C-IF evaluation values $c_{kj} = (u_{kj}, v_{kj}; r_{kj})$ and $c_{lj} = (u_{lj}, v_{lj}; r_{lj})$ for choice options O_k and O_l assessed by E_j can be identified using the joint generalized scoring function-based relations: "$\succ_S$" (more advantageous than), "$\sim_S$" (indifferent), and "$\prec_S$" (more disadvantaged than), as follows:

1. For $E_j \in E_B$: (a) If $S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj})$, then $c_{kj} \succ_S c_{lj}$; (b) If $S^{\varphi}(c_{kj}) = S^{\varphi}(c_{lj})$, then $c_{kj} \sim_S c_{lj}$; and (c) If $S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj})$, then $c_{kj} \prec_S c_{lj}$.
2. For $E_j \in E_N$: (a) If $S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj})$, then $c_{kj} \succ_S c_{lj}$; (b) If $S^{\varphi}(c_{kj}) = S^{\varphi}(c_{lj})$, then $c_{kj} \sim_S c_{lj}$; and (c) If $S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj})$, then $c_{kj} \prec_S c_{lj}$.

Definition 8. The concordance set $\mathbb{C}^{\varphi}(O_k/O_l)$ and the discordance set $\mathbb{D}^{\varphi}(O_k/O_l)$ for the case where choice option O_k outperforms O_l ($O_k, O_l \in O$ and $k \neq l$) can be defined using the joint generalized scoring function-based relations in Definition 7, as seen below:

$$\begin{aligned}\mathbb{C}^{\varphi}(O_k/O_l) &= \{E_j \big| c_{kj} \succcurlyeq_S c_{lj} \text{ for } E_j \in E\} \\ &= \{E_j \big| \big(S^{\varphi}(c_{kj}) \geqslant S^{\varphi}(c_{lj}) \text{ fo r} E_j \in E_B\big), \big(S^{\varphi}(c_{kj}) \leqslant S^{\varphi}(c_{lj}) \text{ for } E_j \in E_N\big)\big\},\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbb{D}^{\varphi}(O_k/O_l) &= \{E_j \big| c_{kj} \prec_S c_{lj} \text{ for } E_j \in E\} \\ &= \{E_j \big| \big(S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj}) \text{ for } E_j \in E_B\big), \big(S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj}) \text{ for } E_j \in E_N\big)\big\}.\end{aligned} \tag{15}$$

Theorem 5. *The concordance set $\mathbb{C}^{\varphi}(O_k/O_l)$ and the discordance set $\mathbb{D}^{\varphi}(O_k/O_l)$ satisfy the following characteristics in a fixed setting of the inclination parameter φ:* (1) $\mathbb{C}^{\varphi}(O_k/O_l) \cap \mathbb{D}^{\varphi}(O_k/O_l) = \emptyset$; (2) $\mathbb{C}^{\varphi}(O_k/O_l) \cup \mathbb{D}^{\varphi}(O_k/O_l) = E$; (3) $\mathbb{C}^{\varphi}(O_k/O_l) = E \backslash \mathbb{D}^{\varphi}(O_k/O_l)$; *and* (4) $\mathbb{C}^{\varphi}(O_k/O_l) = \mathbb{D}^{\varphi}(O_l/O_k)$ *when* $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ *for* $E_j \in E$.

Proof. Properties (1) to (3) are trivially valid. Property (1) states that $\mathbb{C}^{\varphi}(O_k/O_l)$ and $\mathbb{D}^{\varphi}(O_k/O_l)$ do not share any common criteria. Property (2) indicates that the union of $\mathbb{C}^{\varphi}(O_k/O_l)$ and $\mathbb{D}^{\varphi}(O_k/O_l)$ includes all criteria in the set E. Property (3) clarifies that $\mathbb{C}^{\varphi}(O_k/O_l)$ contains all criteria in E except those in $\mathbb{D}^{\varphi}(O_k/O_l)$. When $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$, property (4) determines that $\mathbb{C}^{\varphi}(O_k/O_l) = \{E_j|(S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj}) \text{ for } E_j \in E_B), (S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj}) \text{ for } E_j \in E_N)\}$. Conversely, it is known that $\mathbb{D}^{\varphi}(O_l/O_k) = \{E_j|(S^{\varphi}(c_{lj}) < S^{\varphi}(c_{kj}) \text{ for } E_j \in E_B), (S^{\varphi}(c_{lj}) > S^{\varphi}(c_{kj}) \text{ for } E_j \in E_N)\}$. Therefore, it is concluded that $\mathbb{C}^{\varphi}(O_k/O_l) = \mathbb{D}^{\varphi}(O_l/O_k)$. □

This study employs the joint generalized scoring function with W to define consistency indicators between choice pairs. Given the C-IF weight $w_j = (\omega_j, \varpi_j; r_j)$, the aggressive and cautious IF estimates are: $i^{\alpha}_{Wj} = (\omega^{\alpha}_j, \varpi^{\alpha}_j) = (\min\{1, \omega_j + r_j/\sqrt{2}\}, \max\{0, \varpi_j - r_j/\sqrt{2}\})$ and $i^{\beta}_{Wj} = (\omega^{\beta}_j, \varpi^{\beta}_j) = (\max\{0, \omega_j - r_j/\sqrt{2}\}, \min\{1, \varpi_j + r_j/\sqrt{2}\})$. The hesitancy degrees are: $\hbar^{\alpha}_j = 1 - \omega^{\alpha}_j - \varpi^{\alpha}_j$ and $\hbar^{\beta}_j = 1 - \omega^{\beta}_j - \varpi^{\beta}_j$. Using Definition 2, the scoring mechanisms for i^{α}_{Wj} and i^{β}_{Wj} are based on the natural exponential function:

$$M\big(i^{\alpha}_{Wj}\big) = \frac{1}{2}\bigg[\big(\omega^{\alpha}_{j} - \varpi^{\alpha}_{j}\big) + \hbar^{\alpha}_{j} \cdot \bigg(\frac{e^{(\omega^{\alpha}_{j}-\varpi^{\alpha}_{j})}}{e^{(\omega^{\alpha}_{j}-\varpi^{\alpha}_{j})}+1} - \frac{1}{2}\bigg) + 1\bigg],$$

$$M\big(i^{\beta}_{Wj}\big) = \frac{1}{2}\bigg[\big(\omega^{\beta}_{j} - \varpi^{\beta}_{j}\big) + \hbar^{\beta}_{j} \cdot \bigg(\frac{e^{(\omega^{\beta}_{j}-\varpi^{\beta}_{j})}}{e^{(\omega^{\beta}_{j}-\varpi^{\beta}_{j})}+1} - \frac{1}{2}\bigg) + 1\bigg].$$

Using the derived results and assuming the same inclination parameter φ, the joint generalized scoring function is defined as: $S^{\varphi}(w_j) = \varphi \cdot M(i^{\alpha}_{Wj}) + (1-\varphi) \cdot M(i^{\beta}_{Wj})$. This function combines the scoring mechanisms $M(i^{\alpha}_{Wj})$ and $M(i^{\beta}_{Wj})$ for aggressive and cautious IF estimates, weighted by φ and $(1-\varphi)$, to produce an overall score for w_j.

Definition 9. When the choice option O_k outperforms O_l ($O_k, O_l \in O$ and $k \neq l$), the consistency indicator $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ is defined via the joint generalized scoring functions as:

$$\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) = \frac{\sum_{E_j \in \mathbb{C}^{\varphi}(O_k/O_l)} S^{\varphi}(w_j) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|}{\sum_{j'=1}^{n} S^{\varphi}(w_{j'}) \cdot |S^{\varphi}(c_{kj'}) - S^{\varphi}(c_{lj'})|}. \tag{16}$$

Theorem 6. *Assuming that $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for at least one evaluative criterion $E_j \in E$, without loss of generality, the consistency indicator $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ within a fixed setting of the inclination parameter φ satisfies the boundary condition:* $0 \leqslant \mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) \leqslant 1$.

Proof. Applying Theorem 4, we know $0 \leqslant S^{\varphi}(c_{kj}) \leqslant 1$ and $0 \leqslant S^{\varphi}(c_{lj}) \leqslant 1$, implying $0 \leqslant |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| \leqslant 1$. Since the joint generalized scoring function is non-negative, $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) \geqslant 0$. Given that $\mathbb{C}^{\varphi}(O_k/O_l) \subseteq E$ by Theorem 5 (where $\mathbb{C}^{\varphi}(O_k/O_l) \cup \mathbb{D}^{\varphi}(O_k/O_l) = E$), it follows: $\sum_{E_j \in \mathbb{C}^{\varphi}(O_k/O_l)} S^{\varphi}(w_j) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| \leqslant \sum_{E_j \in E} S^{\varphi}(w_j) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| = \sum_{j'=1}^{n} S^{\varphi}(w_{j'}) \cdot |S^{\varphi}(c_{kj'}) - S^{\varphi}(c_{lj'})|$. Thus, $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) \leqslant 1$. In cases of zero denominator issues, at least one $|S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|$ must be non-zero, as $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for at least one criterion $E_j \in E$. Hence, $0 \leqslant \mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) \leqslant 1$. □

Theorem 7. *Assuming $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for $E_j \in E$, the consistency indicator $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ exhibits the ensuing properties for a fixed inclination parameter φ:* (1) $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) + \mathcal{I}^{\varphi}_{\mathbb{C}}(O_l/O_k) = 1$; (2) $\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} \mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l) = m(m-1)/2$; *and* (3) $\overline{\mathcal{I}}^{\varphi}_{\mathbb{C}} = 0.5$, *where $\overline{\mathcal{I}}^{\varphi}_{\mathbb{C}}$ is the average consistency indicator across all $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$, with $k \neq l$.*

Proof. As per Definition 8, the concordance sets are derived as: $\mathbb{C}^{\varphi}(O_k/O_l) = \{E_j \mid (S^{\varphi}(c_{kj}) \geqslant S^{\varphi}(c_{lj})$ for $E_j \in E_B), (S^{\varphi}(c_{kj}) \leqslant S^{\varphi}(c_{lj})$ for $E_j \in E_N)\}$ and $\mathbb{C}^{\varphi}(O_l/O_k) = \{E_j \mid (S^{\varphi}(c_{lj}) \geqslant S^{\varphi}(c_{kj})$ for $E_j \in E_B), (S^{\varphi}(c_{lj}) \leqslant S^{\varphi}(c_{kj})$ for $E_j \in E_N)\}$. Assuming $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for $E_j \in E$, the simplified expressions become: $\mathbb{C}^{\varphi}(O_k/O_l) = \{E_j \mid (S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj})$ for $E_j \in E_B), (S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj})$ for $E_j \in E_N)\}$ and $\mathbb{C}^{\varphi}(O_l/O_k) =$

$\{E_j \mid (S^{\varphi}(c_{lj}) > S^{\varphi}(c_{kj})$ for $E_j \in E_B), (S^{\varphi}(c_{lj}) < S^{\varphi}(c_{kj})$ for $E_j \in E_N)\}$. Thus, $\mathbb{C}^{\varphi}(O_k/O_l) \cup \mathbb{C}^{\varphi}(O_l/O_k) = E$. For property (1), the calculation follows:

$$\begin{aligned}
&\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) + \mathcal{I}_{\mathbb{C}}^{\varphi}(O_l/O_k) \\
&\quad = \frac{\sum_{E_j \in \mathbb{C}^{\varphi}(O_k/O_l)} S^{\varphi}(w_j)|S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| + \sum_{E_j \in \mathbb{C}^{\varphi}(O_l/O_k)} S^{\varphi}(w_j)|S^{\varphi}(c_{lj}) - S^{\varphi}(c_{kj})|}{\sum_{j'=1}^{n} S^{\varphi}(w_{j'}) \cdot |S^{\varphi}(c_{kj'}) - S^{\varphi}(c_{lj'})|} \\
&\quad = \frac{\sum_{E_j \in E} S^{\varphi}(w_j) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|}{\sum_{E_j \in E} S^{\varphi}(w_j) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|} = 1.
\end{aligned}$$

The sum of consistency indicators for O_k outperforming O_l and vice versa equals 1, satisfying property (1). Property (2) asserts that the total sum of consistency indicators for all option pairs (excluding self-pairs) is $m(m-1)/2$. This follows from the relation $\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) + \mathcal{I}_{\mathbb{C}}^{\varphi}(O_l/O_k) = 1$, leading to:

$$\begin{aligned}
&\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} \mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) = \sum_{k=1,k\neq l}^{m} \big(\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) + \mathcal{I}_{\mathbb{C}}^{\varphi}(O_l/O_k)\big) \\
&\quad = \sum_{k=1,k\neq l}^{m} 1 = \frac{m(m-1)}{2}.
\end{aligned}$$

This confirms property (2), representing the total pairwise comparisons. The average consistency indicator $\overline{\mathcal{I}}_{\mathbb{C}}^{\varphi}$ is computed over all $\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l)$ values, as:

$$\overline{\mathcal{I}}_{\mathbb{C}}^{\varphi} = \frac{\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} \mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l)}{m(m-1)} = \frac{m(m-1)}{2} \Big/ m(m-1) = \frac{1}{2}.$$

Accordingly, property (3) is confirmed. □

This study introduces an inconsistency indicator for each choice pair using the joint generalized scoring function S^{φ} and the C-IF Minkowski-like distance $D_{\mathfrak{M}^0}^{\xi}$ or $D_{\mathfrak{M}}^{\xi}$. In Definition 4, $D_{\mathfrak{M}^0}^{\xi}(c_{kj}, c_{lj})$ employs a three-term model (radius, membership, non-membership), excluding hesitancy. Conversely, the four-term distance $D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj})$ incorporates all key elements: radius, membership, non-membership, and hesitancy. This study adopts the four-term Minkowski-like distance model to fully capture the uncertainty in C-IF information and leverage its comprehensive representation. By setting $\xi = 1$ (Manhattan) or $\xi = 2$ (Euclidean), the four-term distance is computed as:

$$D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) = \frac{1}{2}\left\{\frac{1}{\sqrt{2}}|r_{kj} - r_{lj}| + \sqrt[\xi]{\frac{1}{2}\big(|u_{kj} - u_{lj}|^{\xi} + |v_{kj} - v_{lj}|^{\xi} + |h_{kj} - h_{lj}|^{\xi}\big)}\right\}.$$

Definition 10. When choice option O_k outperforms O_l (where $O_k, O_l \in O$ and $k \neq l$), the inconsistency indicator $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ is formulated using the (four-term strategy-based)

C-IF Minkowski-like distance and the joint generalized scoring function as follows:

$$\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) = \frac{\sum_{E_j \in \mathbb{D}^{\varphi}(O_k/O_l)} D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|}{\sum_{j'=1}^{n} D_{\mathfrak{M}}^{\xi}(c_{kj'}, c_{lj'}) \cdot |S^{\varphi}(c_{kj'}) - S^{\varphi}(c_{lj'})|}. \tag{17}$$

Theorem 8. *Assuming that $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for at least one evaluative criterion $E_j \in E$, without loss of generality, the inconsistency indicator $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ within a fixed setting of the inclination parameter φ satisfies the boundary condition*: $0 \leqslant \mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) \leqslant 1$.

Proof. Both the C-IF Minkowski-like distance and the joint generalized scoring function possess the non-negativity property. Next, this theorem's proving procedure follows a resemble approach to that of Theorem 6. □

Theorem 9. *Assuming $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for $E_j \in E$, the inconsistency indicator $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ satisfies the following beneficial characteristics for a fixed inclination parameter φ*: (1) $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) + \mathcal{I}_{\mathbb{D}}^{\varphi}(O_l/O_k) = 1$; (2) $\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} \mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) = m(m-1)/2$; *and* (3) $\overline{\mathcal{I}}_{\mathbb{D}}^{\varphi} = 0.5$, *where $\overline{\mathcal{I}}_{\mathbb{D}}^{\varphi}$ represents the average inconsistency indicator calculated by considering all $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ values for $O_k, O_l \in O$, and $k \neq l$.*

Proof. By Definition 8, the discordance sets are defined as follows: $\mathbb{D}^{\varphi}(O_k/O_l) = \{E_j \mid (S^{\varphi}(c_{kj}) < S^{\varphi}(c_{lj})$ for $E_j \in E_B), (S^{\varphi}(c_{kj}) > S^{\varphi}(c_{lj})$ for $E_j \in E_N)\}$ and $\mathbb{D}^{\varphi}(O_l/O_k) = \{E_j \mid (S^{\varphi}(c_{lj}) < S^{\varphi}(c_{kj})$ for $E_j \in E_B), (S^{\varphi}(c_{lj}) > S^{\varphi}(c_{kj})$ for $E_j \in E_N)\}$. With the precondition $S^{\varphi}(c_{kj}) \neq S^{\varphi}(c_{lj})$ for $E_j \in E$, it implies that $\mathbb{D}^{\varphi}(O_k/O_l) \cup \mathbb{D}^{\varphi}(O_l/O_k) = E$. The C-IF Minkowski-like distance exhibits symmetry: $D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) = D_{\mathfrak{M}}^{\xi}(c_{lj}, c_{kj})$. Property (1) can be confirmed as follows: $\Sigma_{E_j \in \mathbb{D}^{\varphi}(O_k/O_l)} D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| + \Sigma_{E_j \in \mathbb{D}^{\varphi}(O_l/O_k)} D_{\mathfrak{M}}^{\xi}(c_{lj}, c_{kj}) \cdot |S^{\varphi}(c_{lj}) - S^{\varphi}(c_{kj})| = \Sigma_{E_j \in E} D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})|$. Consequently, $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) + \mathcal{I}_{\mathbb{D}}^{\varphi}(O_l/O_k) = \Sigma_{E_j \in E} D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| / \Sigma_{E_j \in E} D_{\mathfrak{M}}^{\xi}(c_{kj}, c_{lj}) \cdot |S^{\varphi}(c_{kj}) - S^{\varphi}(c_{lj})| = 1$. Thus, property (1) is satisfied. The proofs for properties (2) and (3) follow a similar approach as in Theorem 7. In summary, property (1) states that the sum of the inconsistency indicators for O_k and O_l equals one. Property (2) indicates the total number of possible pairwise discordance comparisons is $m(m-1)/2$. Property (3) asserts that, on average, the inconsistency indicators reflect a balanced level of inconsistency between the choice options. □

To construct the C-IF ELECTRE I prioritization procedure, this study first compares the consistency indicator $\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l)$ with the average consistency indicator $\overline{\mathcal{I}}_{\mathbb{C}}^{\varphi}$ for $O_k, O_l \in O$ and $k \neq l$. This comparison allows for the establishment of the consistency entry $\mathcal{B}_{\mathbb{C}}^{\varphi}(O_k/O_l)$ and the creation of the consistency Boolean matrix $\mathfrak{B}_{\mathbb{C}}^{\varphi}$. The process investigates and articulates consistencies between pairs of choice options as follows:

$$\mathcal{B}_{\mathbb{C}}^{\varphi}(O_k/O_l) = \begin{cases} 1 & \text{if } \mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) \geqslant \overline{\mathcal{I}}_{\mathbb{C}}^{\varphi}, \\ 0 & \text{if } \mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l) < \overline{\mathcal{I}}_{\mathbb{C}}^{\varphi}, \end{cases} \tag{18}$$

$$\mathfrak{B}_{\mathbb{C}}^{\varphi}=\begin{bmatrix} - & \mathcal{B}_{\mathbb{C}}^{\varphi}(O_1/O_2) & \cdots & \mathcal{B}_{\mathbb{C}}^{\varphi}(O_1/O_m) \\ \mathcal{B}_{\mathbb{C}}^{\varphi}(O_2/O_1) & - & \cdots & \mathcal{B}_{\mathbb{C}}^{\varphi}(O_2/O_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{\mathbb{C}}^{\varphi}(O_m/O_1) & \mathcal{B}_{\mathbb{C}}^{\varphi}(O_m/O_2) & \cdots & - \end{bmatrix}. \tag{19}$$

This study compares the inconsistency indicator $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ with the average inconsistency indicator $\overline{\mathcal{I}}_{\mathbb{D}}^{\varphi}$ to generate the inconsistency entry $\mathcal{B}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ for $O_k, O_l \in O$ and $k \neq l$. The inconsistency Boolean matrix $\mathfrak{B}_{\mathbb{D}}^{\varphi}$ is constructed. The following procedure assesses and depicts the inconsistencies between pairs of choice options:

$$\mathcal{B}_{\mathbb{D}}^{\varphi}(O_k/O_l)=\begin{cases} 1 & \text{if } \mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) \leqslant \overline{\mathcal{I}}_{\mathbb{D}}^{\varphi}, \\ 0 & \text{if } \mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l) > \overline{\mathcal{I}}_{\mathbb{D}}^{\varphi}, \end{cases} \tag{20}$$

$$\mathfrak{B}_{\mathbb{D}}^{\varphi}=\begin{bmatrix} - & \mathcal{B}_{\mathbb{D}}^{\varphi}(O_1/O_2) & \cdots & \mathcal{B}_{\mathbb{D}}^{\varphi}(O_1/O_m) \\ \mathcal{B}_{\mathbb{D}}^{\varphi}(O_2/O_1) & \cdots & \mathcal{B}_{\mathbb{D}}^{\varphi}(O_2/O_m) & \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{\mathbb{D}}^{\varphi}(O_m/O_1) & \mathcal{B}_{\mathbb{D}}^{\varphi}(O_m/O_2) & \cdots & - \end{bmatrix}. \tag{21}$$

Using the specified φ, this paper formulates the overall prioritization entry $\mathcal{B}_{\mathbb{O}}^{\varphi}(O_k/O_l)$ for $O_k, O_l \in O$ $(k \neq l)$ and constructs the overall prioritization Boolean matrix $\mathfrak{B}_{\mathbb{O}}^{\varphi}$ as:

$$\mathcal{B}_{\mathbb{O}}^{\varphi}(O_k/O_l)=\mathcal{B}_{\mathbb{C}}^{\varphi}(O_k/O_l)\cdot\mathcal{B}_{\mathbb{D}}^{\varphi}(O_k/O_l), \tag{22}$$

$$\mathfrak{B}_{\mathbb{O}}^{\varphi}=\begin{bmatrix} - & \mathcal{B}_{\mathbb{O}}^{\varphi}(O_1/O_2) & \cdots & \mathcal{B}_{\mathbb{O}}^{\varphi}(O_1/O_m) \\ \mathcal{B}_{\mathbb{O}}^{\varphi}(O_2/O_1) & - & \cdots & \mathcal{B}_{\mathbb{O}}^{\varphi}(O_2/O_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_{\mathbb{O}}^{\varphi}(O_m/O_1) & \mathcal{B}_{\mathbb{O}}^{\varphi}(O_m/O_2) & \cdots & - \end{bmatrix}. \tag{23}$$

If $\mathcal{B}_{\mathbb{O}}^{\varphi}(O_k/O_l)=1$, it indicates that O_k is preferred to O_l based on both consistency and inconsistency indicators. If $\mathcal{B}_{\mathbb{O}}^{\varphi}(O_k/O_l)=0$, it means that either O_k is less preferred than O_l or they are considered unrelated. A dominance graph can be constructed to illustrate a partial priority ranking of the m options based on the results from the overall prioritization Boolean matrix $\mathfrak{B}_{\mathbb{O}}^{\varphi}$, forming the basis for the C-IF ELECTRE I prioritization process.

This study presents the concepts of consistency-dependent average outflow, inconsistency-dependent average inflow, and overall net flow for ranking choice options in the C-IF ELECTRE II prioritization process. These concepts are inspired by the leaving and entering flows developed by Wang and Chen (2021). Specifically, the consistency-dependent average outflow $\mathcal{A}_{\mathbb{C}}^{\varphi}(O_k)$, the inconsistency-dependent average inflow $\mathcal{A}_{\mathbb{D}}^{\varphi}(O_k)$, and the overall net flow $\mathcal{N}_{\mathbb{O}}^{\varphi}(O_k)$ for choice option O_k are defined as follows:

$$\mathcal{A}_{\mathbb{C}}^{\varphi}(O_k)=\frac{\sum_{l=1,l\neq k}^{m}\mathcal{I}_{\mathbb{C}}^{\varphi}(O_k/O_l)}{m-1}, \tag{24}$$

$$\mathcal{A}_{\mathbb{D}}^{\varphi}(O_k) = \frac{\sum_{l=1, l\neq k}^{m} \mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)}{m-1}, \tag{25}$$

$$\mathcal{N}_{\mathbb{O}}^{\varphi}(O_k) = \mathcal{A}_{\mathbb{C}}^{\varphi}(O_k) - \mathcal{A}_{\mathbb{D}}^{\varphi}(O_k). \tag{26}$$

The average outflow, average inflow, and overall net flow satisfy these properties: (1) $0 \leqslant \mathcal{A}_{\mathbb{C}}^{\varphi}(O_k) \leqslant 1$; (2) $0 \leqslant \mathcal{A}_{\mathbb{D}}^{\varphi}(O_k) \leqslant 1$; and (3) $-1 \leqslant \mathcal{N}_{\mathbb{O}}^{\varphi}(O_k) \leqslant 1$. A higher value of $\mathcal{N}_{\mathbb{O}}^{\varphi}(O_k)$ indicates a better option. The C-IF ELECTRE II process ranks options in O by descending net flows, offering a thorough assessment to guide decision-making effectively.

4.2. *Suggested Algorithm*

Figure 2 presents a systematic framework for applying the C-IF ELECTRE I and II approaches to multiple-criteria choice problems under C-IF uncertainty. The C-IF ELECTRE I method involves several stages: (1) Problem Definition: Identifies choice options and evaluative criteria; (2) Data Creation: Establishes C-IF weights for criteria and generates C-IF evaluation values for options; (3) C-IF Scoring: Computes aggressive and cautious IF estimates, and determines scoring mechanisms based on hesitancy degrees; (4) Consistency/Inconsistency Indexing: Identifies concordance and discordance sets, calculates Minkowski-like distances, and determines consistency/inconsistency indicators; and (5) C-IF ELECTRE I Prioritization: Produces overall prioritization entries and Boolean matrices, and creates a dominance graph for partial-priority ranking. The C-IF ELECTRE II approach follows similar steps but incorporates average outflows and inflows alongside net flows to establish complete-priority rankings via overall net flows.

The implementation steps of the C-IF ELECTRE I approach are as follows:

Problem definition stage: See Steps I.1 and I.2

Step I.1. Define a multiple-criteria decision problem with evaluative criteria $E = \{E_1, E_2, \ldots, E_n\}$ and choice options $O = \{O_1, O_2, \ldots, O_m\}$.

Step I.2. Split criteria into beneficial E_B and non-beneficial E_N subsets.

Data creation stage: See Steps I.3 and I.4

Step I.3. Establish C-IF weights $w_j = (\omega_j, \varpi_j; r_j)$ for each criterion E_j and construct the weight characteristic W using Eqs. (10) and (11).

Step I.4. Generate C-IF evaluation values $c_{kj} = (u_{kj}, v_{kj}; r_{kj})$ for each option O_k under criterion E_j and build the fuzzy characteristic C_k using Eqs. (12) and (13).

C-IF scoring stage: See Steps I.5–I.7

Step I.5. Compute the aggressive IF estimates $i_{kj}^{\alpha} = (u_{kj}^{\alpha}, v_{kj}^{\alpha})$ and $i_{Wj}^{\alpha} = (\omega_j^{\alpha}, \varpi_j^{\alpha})$ using Eq. (7), as well as the cautious IF estimates $i_{kj}^{\beta} = (u_{kj}^{\beta}, v_{kj}^{\beta})$ and $i_{Wj}^{\beta} = (\omega_j^{\beta}, \varpi_j^{\beta})$ by Eq. (8).

Step I.6. Derive hesitancy degrees h_{kj}^{α}, h_{kj}^{β}, $\hbar_j^{\alpha}$, and $\hbar_j^{\beta}$, as well as scoring mechanisms $M(i_{kj}^{\alpha})$, $M(i_{kj}^{\beta})$, $M(i_{Wj}^{\alpha})$, and $M(i_{Wj}^{\beta})$ using the natural exponential function in Eq. (2).

Step I.7. Designate a value to the inclination parameter $\varphi \in [0, 1]$, and generate joint scoring functions $S^{\varphi}(c_{kj})$ and $S^{\varphi}(w_j)$ using Eq. (9).

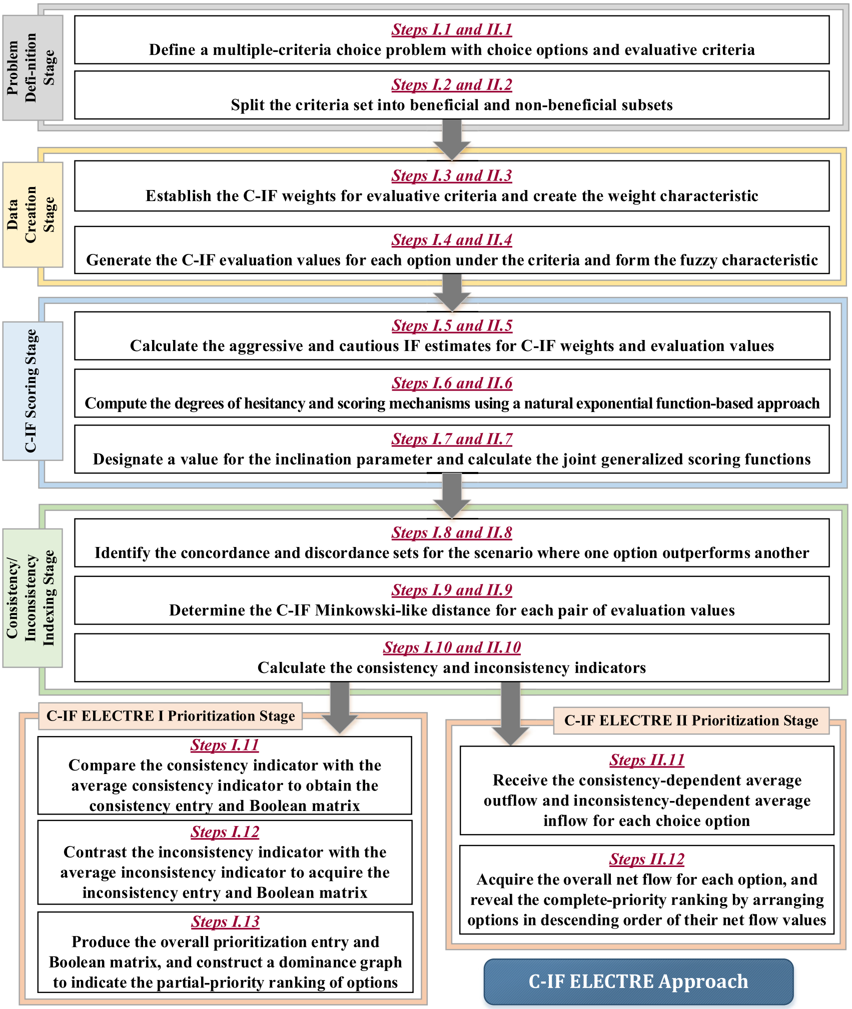


Fig. 2. The systematic procedure of the C-IF ELECTRE I and II approaches.

Consistency/inconsistency indexing stage: See Steps I.8–I.10

Step I.8. Employ Eqs. (14) and (15), respectively, to establish the concordance set $\mathbb{C}^{\varphi}(O_k/O_l)$ and the discordance set $\mathbb{D}^{\varphi}(O_k/O_l)$ ($O_k, O_l \in O$ and $k \neq l$).

Step I.9. Allocate a value for the metric parameter $\xi \in Z^{+}$, and derive the C-IF Minkowski-like distance $D^{\xi}_{\mathfrak{M}}(c_{kj}, c_{lj})$ using the four-term strategy described in Eq. (6).

Step I.10. Determine the consistency indicator $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ and the inconsistency indicator $\mathcal{I}^{\varphi}_{\mathbb{D}}(O_k/O_l)$ using Eqs. (16) and (17), respectively.

C-IF ELECTRE I prioritization stage: See Steps I.11–I.13

Step I.11. Compare the consistency indicator $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ with the average consistency indicator $\overline{\mathcal{I}}^{\varphi}_{\mathbb{C}} = 0.5$ (from Theorem 7) to obtain the consistency entry $\mathcal{B}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ and the consistency Boolean matrix $\mathfrak{B}^{\varphi}_{\mathbb{C}}$, as provided in Eqs. (18) and (19), respectively.

Step I.12. Contrast the inconsistency indicator $\mathcal{I}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ with the average inconsistency indicator $\overline{\mathcal{I}}_{\mathbb{D}}^{\varphi} = 0.5$ (from Theorem 9) to acquire the inconsistency entry $\mathcal{B}_{\mathbb{D}}^{\varphi}(O_k/O_l)$ and the inconsistency Boolean matrix $\mathfrak{B}_{\mathbb{D}}^{\varphi}$, as presented in Eqs. (20) and (21), respectively.
Step I.13. Produce the overall prioritization entry $\mathcal{B}_{\mathbb{O}}^{\varphi}(O_k/O_l)$ and the overall prioritization Boolean matrix $\mathfrak{B}_{\mathbb{O}}^{\varphi}$ using Eqs. (22) and (23), respectively. Construct a dominance graph to indicate the partial-priority ranking of options in the set O.

The implementation steps of the C-IF ELECTRE II approach include:
Steps II.1 to II.10. Same as Steps I.1 to I.10.

C-IF ELECTRE II prioritization stage: See Steps II.11 and II.12
Step II.11. Calculate the consistency-dependent average outflow $\mathcal{A}_{\mathbb{C}}^{\varphi}(O_k)$ the inconsistency-dependent average inflow $\mathcal{A}_{\mathbb{D}}^{\varphi}(O_k)$ using Eqs. (24) and (25), respectively.
Step II.12. Obtain the overall net flow $\mathcal{N}_{\mathbb{O}}^{\varphi}(O_k)$ using Eq. (26), and then rank the options in descending order based on their overall net flow values.

5. Application to a Problem with Supplier Evaluation

This section delves into the practical quandary of multi-expert supplier appraisal, as posed by Otay and Kahraman (2022), through the application of the proposed C-IF ELECTRE framework. Moreover, the operationalization of the advocated C-IF ELECTRE I and II methodologies is elucidated, highlighting their prowess in both efficacy and efficiency.

The C-IF ELECTRE approach is applied to evaluate and select suppliers in an engineering company, focusing on a specific component from various supplied options. Initially, potential suppliers are listed as choice options. Environmental considerations, particularly pollution control and ISO standards, are prioritized in the assessment. Suppliers failing to meet these criteria are eliminated from further analysis. The remaining options—Supplier Options #1 to #3—are evaluated across three key dimensions: cost, service, and technology/quality. These dimensions are assessed through nine evaluative criteria, including payment terms, on-time delivery, and quality management systems. Figure 3 illustrates the hierarchical structure of the multi-expert supplier evaluation, detailing the relationships among dimensions, evaluative criteria, and supplier options.

Step I.1 was conducted based on the problem setting from Otay and Kahraman (2022) and the hierarchical structure shown in Fig. 3. Three choice options were evaluated: Supplier Options #1, #2, and #3, represented as $O = \{O_1, O_2, O_3\}$. These suppliers were assessed across three dimensions: (1) the cost dimension, consisting of price (E_1), terms of payments (E_2), and handling and transportation (E_3); and (2) the service dimension, encompassing flexibility (E_4), on-time delivery (E_5), and past performance (E_6); and (3) the technology and quality dimension, including quality management systems (E_7), technological capability (E_8), and R&D studies (E_9). The set of evaluative criteria was defined as $E = \{E_1, E_2, \dots, E_9\}$.

Step I.2 divides the set E into two subsets: E_B (beneficial criteria) and E_N (non-beneficial criteria). Beneficial criteria reflect attributes where higher values are preferred, while non-beneficial criteria favour lower values. For instance, price (E_1) and terms of

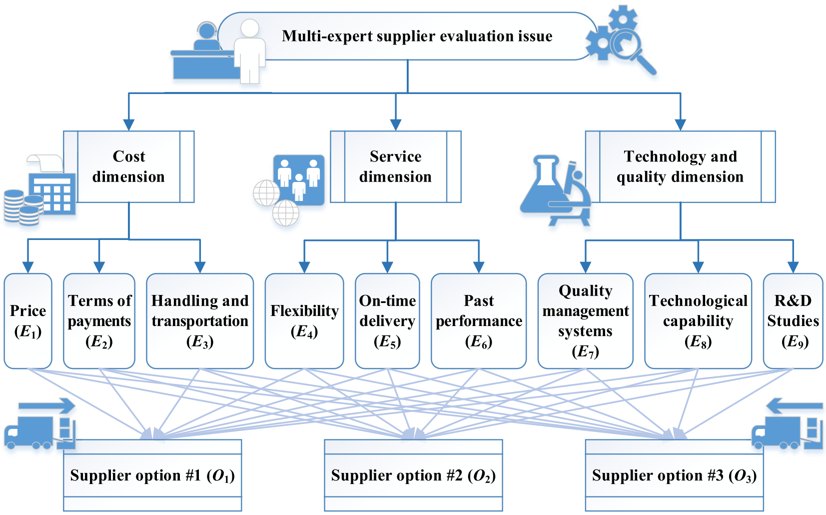


Fig. 3. The hierarchical structure of the multi-expert supplier evaluation issue.

payments (E_2) are non-beneficial, as lower values are preferred. In contrast, quality management systems (E_7) and technological capability (E_8) are beneficial, with higher values indicating desirable traits. Although the nine criteria $E_1, E_2, \dots, E_9$ include both types, they are standardized as beneficial following Otay and Kahraman (2022). The sets of beneficial and non-beneficial criteria were defined as $E_B = \{E_1, E_2, \dots, E_9\}$ and $E_N = \emptyset$.

The data creation stage focuses on establishing the C-IF weight w_j and C-IF evaluation value c_{kj} as outlined in Steps I.3 and I.4. This study follows a structured nine-point linguistic rating scale, as proposed by Chen (2024), involving the following steps: (1) Select a Linguistic Scale: Use a nine-point scale to streamline evaluations; (2) Collect Evaluations: Gather semantic evaluations from decision-makers regarding choice options and criterion importance; (3) Convert to IF Numbers: Transform the semantic evaluations into corresponding IF numbers; (4) Calculate C-IF Evaluation Values: For each decision-maker's assessment, define an IF number. Then, average these IF numbers to find the centre and determine the radius based on maximum deviation; and (5) Establish C-IF Weights: Similarly, calculate C-IF weights by averaging the importance weights to find the center and radius. These steps synthesize insights from multiple decision-makers into a cohesive C-IF framework for evaluation values and weights.

In the context of multi-expert supplier evaluation, the data from Otay and Kahraman (2022) were used, consolidating evaluations from three procurement specialists to derive collective C-IF weights. In Step I.3, the C-IF weight w_j is presented as a C-IF number $(\omega_j, \varpi_j; r_j)$ in the second column of Table 1. Using Eqs. (10) and (11), the weight characteristic was established as: $W = \{\langle E_j, \omega_j, \varpi_j; r_j\rangle | E_j \in \{E_1, E_2, \dots, E_9\}\} = \{\langle E_j, \mathfrak{O}_r(\omega_j, \varpi_j)\rangle | E_j \in \{E_1, E_2, \dots, E_9\}\}$, where $\mathfrak{O}_r(\omega_j, \varpi_j) = \{\langle \ell, \ell'\rangle \mid \ell, \ell' \in [0,1], [(\omega_j - \ell)^2 + (\varpi_j - \ell')^2]^{0.5} \leqslant r_j,$ and $\ell + \ell' \leqslant 1\}$ for $j \in \{1, 2, \dots, 9\}$.

In Step I.4, the C-IF evaluation value c_{kj}, expressed as a C-IF number $(u_{kj}, v_{kj}; r_{kj})$, was derived from the aggregated evaluation data in Otay and Kahraman's (2022) study

Table 1
Information regarding the C-IF weight w_j and the C-IF evaluation value c_{kj}.

E_j	$(\omega_j, \varpi_j; r_j)$	$(u_{1j}, v_{1j}; r_{1j})$	$(u_{2j}, v_{2j}; r_{2j})$	$(u_{3j}, v_{3j}; r_{3j})$
E_1	(0.573, 0.360; 0.158)	(0.346, 0.528; 0.271)	(0.624, 0.254; 0.163)	(0.618, 0.277; 0.100)
E_2	(0.327, 0.589; 0.112)	(0.439, 0.439; 0.142)	(0.650, 0.250; 0.000)	(0.675, 0.207; 0.190)
E_3	(0.456, 0.469; 0.158)	(0.346, 0.528; 0.271)	(0.569, 0.327; 0.112)	(0.629, 0.267; 0.114)
E_4	(0.454, 0.480; 0.174)	(0.401, 0.480; 0.198)	(0.698, 0.194; 0.074)	(0.615, 0.267; 0.178)
E_5	(0.571, 0.366; 0.174)	(0.578, 0.316; 0.098)	(0.718, 0.175; 0.102)	(0.618, 0.277; 0.100)
E_6	(0.342, 0.574; 0.161)	(0.618, 0.277; 0.100)	(0.737, 0.140; 0.145)	(0.776, 0.096; 0.199)
E_7	(0.324, 0.592; 0.074)	(0.578, 0.316; 0.098)	(0.615, 0.267; 0.178)	(0.729, 0.166; 0.115)
E_8	(0.577, 0.360; 0.074)	(0.574, 0.293; 0.341)	(0.598, 0.296; 0.072)	(0.569, 0.327; 0.112)
E_9	(0.458, 0.473; 0.000)	(0.528, 0.346; 0.271)	(0.569, 0.327; 0.112)	(0.598, 0.296; 0.072)

*: Refer to Otay and Kahraman (2022).

Table 2
Specifics on the aggressive IF estimates i^{α}_{Wj} and i^{α}_{kj}, alongside the cautious IF estimates i^{β}_{Wj} and i^{β}_{kj}.

	Outcomes associated with the aggressive IF estimates $i^{\alpha}_{Wj} = (\omega^{\alpha}_j, \varpi^{\alpha}_j)$ and $i^{\alpha}_{kj} = (u^{\alpha}_{kj}, v^{\alpha}_{kj})$			
E_j	$(\omega^{\alpha}_j, \varpi^{\alpha}_j)$	$(u^{\alpha}_{1j}, v^{\alpha}_{1j})$	$(u^{\alpha}_{2j}, v^{\alpha}_{2j})$	$(u^{\alpha}_{3j}, v^{\alpha}_{3j})$
E_1	(0.685, 0.248)	(0.538, 0.336)	(0.739, 0.139)	(0.689, 0.206)
E_2	(0.406, 0.510)	(0.539, 0.339)	(0.650, 0.250)	(0.809, 0.073)
E_3	(0.568, 0.357)	(0.538, 0.336)	(0.648, 0.248)	(0.710, 0.186)
E_4	(0.577, 0.357)	(0.541, 0.340)	(0.750, 0.142)	(0.741, 0.141)
E_5	(0.694, 0.243)	(0.647, 0.247)	(0.790, 0.103)	(0.689, 0.206)
E_6	(0.456, 0.460)	(0.689, 0.206)	(0.840, 0.037)	(0.917, 0.000)
E_7	(0.376, 0.540)	(0.647, 0.247)	(0.741, 0.141)	(0.810, 0.085)
E_8	(0.629, 0.308)	(0.815, 0.052)	(0.649, 0.245)	(0.648, 0.248)
E_9	(0.458, 0.473)	(0.720, 0.154)	(0.648, 0.248)	(0.649, 0.245)
	Outcomes associated with the cautious IF estimates $i^{\beta}_{Wj} = (\omega^{\beta}_j, \varpi^{\beta}_j)$ and $i^{\beta}_{kj} = (u^{\beta}_{kj}, v^{\beta}_{kj})$			
	$(\omega^{\beta}_j, \varpi^{\beta}_j)$	$(u^{\beta}_{1j}, v^{\beta}_{1j})$	$(u^{\beta}_{2j}, v^{\beta}_{2j})$	$(u^{\beta}_{3j}, v^{\beta}_{3j})$
E_1	(0.461, 0.472)	(0.154, 0.720)	(0.509, 0.369)	(0.547, 0.348)
E_2	(0.248, 0.668)	(0.339, 0.539)	(0.650, 0.250)	(0.541, 0.341)
E_3	(0.344, 0.581)	(0.154, 0.720)	(0.490, 0.406)	(0.548, 0.348)
E_4	(0.331, 0.603)	(0.261, 0.620)	(0.646, 0.246)	(0.489, 0.393)
E_5	(0.448, 0.489)	(0.509, 0.385)	(0.646, 0.247)	(0.547, 0.348)
E_6	(0.228, 0.688)	(0.547, 0.348)	(0.634, 0.243)	(0.635, 0.237)
E_7	(0.272, 0.644)	(0.509, 0.385)	(0.489, 0.393)	(0.648, 0.247)
E_8	(0.525, 0.412)	(0.333, 0.534)	(0.547, 0.347)	(0.490, 0.406)
E_9	(0.458, 0.473)	(0.336, 0.538)	(0.490, 0.406)	(0.547, 0.347)

and is shown in the third to fifth columns of Table 1. The C-IF characteristic for each O_k was established using Eqs. (12) and (13) as follows: $C_k = \{\langle E_j, u_{kj}, v_{kj}; r_{kj}\rangle | E_j \in \{E_1, E_2, \dots, E_9\}\} = \{\langle E_j, \mathfrak{O}_r(u_{kj}, v_{kj})\rangle | E_j \in \{E_1, E_2, \dots, E_9\}\}$, where $\mathfrak{O}_r(u_{kj}, v_{kj}) = \{\langle \ell, \ell'\rangle \mid \ell, \ell' \in [0, 1], [(u_{kj} - \ell)^2 + (v_{kj} - \ell')^2]^{0.5} \leqslant r_{kj},\ \text{and}\ \ell + \ell' \leqslant 1\}$ for $k \in \{1, 2, 3\}$ and $j \in \{1, 2, \dots, 9\}$.

Table 2 presents the aggressive and cautious IF estimates for each criterion and supplier option. In Step I.5, the aggressive IF estimates $i^{\alpha}_{Wj} = (\omega^{\alpha}_j, \varpi^{\alpha}_j)$ and $i^{\alpha}_{kj} = (u^{\alpha}_{kj}, v^{\alpha}_{kj})$ are generated using Eq. (7). For example, with $w_1 = (0.573, 0.360; 0.158)$ and $c_{21} = (0.624, 0.254; 0.163)$, Eq. (7) yields: $i^{\alpha}_{W1} = (\min\{1, 0.573 + 0.158/\sqrt{2}\}, \max\{0, 0.360 -$

Table 3
Specifics on the scoring mechanisms $M(i^{\alpha}_{Wj})$, $M(i^{\alpha}_{kj})$, $M(i^{\beta}_{Wj})$, and $M(i^{\beta}_{kj})$.

	Outcomes related to aggressive IF estimates				Outcomes related to cautious IF estimates			
E_j	$M(i^{\alpha}_{Wj})$	$M(i^{\alpha}_{1j})$	$M(i^{\alpha}_{2j})$	$M(i^{\alpha}_{3j})$	$M(i^{\beta}_{Wj})$	$M(i^{\beta}_{1j})$	$M(i^{\beta}_{2j})$	$M(i^{\beta}_{3j})$
E_1	0.722	0.604	0.809	0.748	0.494	0.208	0.572	0.602
E_2	0.447	0.603	0.705	0.878	0.286	0.397	0.705	0.603
E_3	0.607	0.604	0.705	0.769	0.379	0.208	0.543	0.603
E_4	0.612	0.603	0.812	0.809	0.362	0.315	0.705	0.549
E_5	0.729	0.705	0.852	0.748	0.479	0.564	0.705	0.602
E_6	0.498	0.748	0.913	0.967	0.265	0.602	0.701	0.705
E_7	0.416	0.705	0.809	0.872	0.310	0.564	0.549	0.706
E_8	0.663	0.894	0.707	0.705	0.557	0.396	0.603	0.543
E_9	0.492	0.792	0.705	0.707	0.492	0.396	0.543	0.603

$0.158/\sqrt{2}\}) = (0.685, 0.248)$ and $i^{\alpha}_{21} = (\min\{1, 0.624 + 0.163/\sqrt{2}\}, \max\{0, 0.254 - 0.163/\sqrt{2}\}) = (0.739, 0.139)$. These results are shown in the upper portion of Table 2. Similarly, the cautious IF estimates $i^{\beta}_{Wj} = (\omega^{\beta}_j, \varpi^{\beta}_j)$ and $i^{\beta}_{kj} = (u^{\beta}_{kj}, v^{\beta}_{kj})$ were generated using Eq. (8) and presented in the bottom half of Table 2.

In Step I.6, the degrees of hesitancy $\hbar^{\alpha}_j$, h^{α}_{kj}, $\hbar^{\beta}_j$, and h^{β}_{kj} for the estimates i^{α}_{Wj}, i^{α}_{kj}, i^{β}_{Wj}, and i^{β}_{kj} were computed. For instance, for i^{α}_{W1}, hesitancy was calculated as: $\hbar^{\alpha}_1 = 1 - \omega^{\alpha}_1 - \varpi^{\alpha}_1 = 1 - 0.685 - 0.248 = 0.067$. The scoring mechanisms $M(i^{\alpha}_{Wj})$, $M(i^{\alpha}_{kj})$, $M(i^{\beta}_{Wj})$, and $M(i^{\beta}_{kj})$ were then derived using the natural exponential function approach from Eq. (2). For example, for $i^{\alpha}_{21} = (0.739, 0.139)$ with $h^{\alpha}_{21} = 0.122$:

$$M\left(i^{\alpha}_{21}\right) = \frac{1}{2}\left[(0.739 - 0.139) + 0.122 \cdot \left(\frac{e^{(0.739-0.139)}}{e^{(0.739-0.139)} + 1} - \frac{1}{2}\right) + 1\right] = 0.809.$$

The results, including the scoring mechanisms $M(i^{\alpha}_{Wj})$ and $M(i^{\alpha}_{kj})$ for aggressive IF estimates (left part), and $M(i^{\beta}_{Wj})$ and $M(i^{\beta}_{kj})$ for cautious IF estimates (right part), are illustrated in Table 3.

In Step I.7, the inclination parameter φ is assigned a value within [0, 1] to reflect the decision-maker's preference between aggressive and cautious IF estimates. Here, setting $\varphi = 0.5$ indicates equal importance to both. Next, the joint generalized scoring functions $S^{0.5}(w_j)$ and $S^{0.5}(c_{kj})$ were computed for each w_j and c_{kj} using Eq. (9). For instance, the calculation for $S^{0.5}(c_{21})$ is: $S^{0.5}(c_{21}) = 0.5 \times 0.809 + (1 - 0.5) \times 0.572 = 0.691$.

In the example with $\varphi = 0.5$, the expressions of Eqs. (14) and (15) were simplified since all criteria were beneficial. The concordance and discordance sets were defined as:

$$\mathbb{C}^{0.5}(O_k/O_l) = \left\{E_j \,\middle|\, S^{0.5}(c_{kj}) \geqslant S^{0.5}(c_{lj}) \text{ for } E_j \in E\right\},$$
$$\mathbb{D}^{0.5}(O_k/O_l) = \left\{E_j \,\middle|\, S^{0.5}(c_{kj}) < S^{0.5}(c_{lj}) \text{ for } E_j \in E\right\}.$$

As outlined in Step I.8, Table 4 presents the results for $\mathbb{C}^{0.5}(O_k/O_l)$ and $\mathbb{D}^{0.5}(O_k/O_l)$ when option O_k outperforms O_l $(k \neq l)$, along with relevant explanations.

Table 4
Specifics on the concordance and discordance sets for $\varphi = 0.5$ ($\mathbb{C}^{0.5}(O_k/O_l)$ and $\mathbb{D}^{0.5}(O_k/O_l)$).

O_k	O_l	$\mathbb{C}^{0.5}(O_k/O_l)$	$\mathbb{D}^{0.5}(O_k/O_l)$	Explanation
O_1	O_2	$\emptyset$	$\{E_1, E_2, \dots, E_9\}$	$S^{0.5}(c_{1j}) < S^{0.5}(c_{2j})$ for $j \in \{1, 2, \dots, 9\}$
	O_3	$\{E_8\}$	$\{E_1, E_2, \dots, E_7, E_9\}$	$S^{0.5}(c_{18}) \geqslant S^{0.5}(c_{38})$ and $S^{0.5}(c_{1j}) < S^{0.5}(c_{3j})$ for $j \in \{1, 2, \dots, 7, 9\}$
O_2	O_1	$\{E_1, E_2, \dots, E_9\}$	$\emptyset$	$S^{0.5}(c_{2j}) \geqslant S^{0.5}(c_{1j})$ for $j \in \{1, 2, \dots, 9\}$
	O_3	$\{E_1, E_4, E_5, E_8\}$	$\{E_2, E_3, E_6, E_7, E_9\}$	$S^{0.5}(c_{2j}) \geqslant S^{0.5}(c_{3j})$ for $j \in \{1, 4, 5, 8\}$ and $S^{0.5}(c_{2j}) < S^{0.5}(c_{3j})$ for $j \in \{2, 3, 6, 7, 9\}$
O_3	O_1	$\{E_1, E_2, \dots, E_7, E_9\}$	$\{E_8\}$	$S^{0.5}(c_{3j}) \geqslant S^{0.5}(c_{1j})$ for $j \in \{1, 2, \dots, 7, 9\}$ and $S^{0.5}(c_{38}) < S^{0.5}(c_{18})$
	O_2	$\{E_2, E_3, E_6, E_7, E_9\}$	$\{E_1, E_4, E_5, E_8\}$	$S^{0.5}(c_{3j}) \geqslant S^{0.5}(c_{2j})$ for $j \in \{2, 3, 6, 7, 9\}$ and $S^{0.5}(c_{3j}) < S^{0.5}(c_{2j})$ for $j \in \{1, 4, 5, 8\}$

In Step I.9, the metric parameter ξ was set to 1 and 2, reflecting the common use of the Manhattan ($\xi = 1$) and Euclidean ($\xi = 2$) metrics in practice. The C-IF distances $D^1_{\mathfrak{M}}(c_{kj}, c_{lj})$ and $D^2_{\mathfrak{M}}(c_{kj}, c_{lj})$ were computed using the four-term technique from Eq. (6):

$$D^1_{\mathfrak{M}}(c_{kj}, c_{lj}) = \frac{1}{2}\left[\frac{1}{\sqrt{2}}|r_{kj} - r_{lj}| + \frac{1}{2}\big(|u_{kj} - u_{lj}| + |v_{kj} - v_{lj}| + |h_{kj} - h_{lj}|\big)\right],$$

$$D^2_{\mathfrak{M}}(c_{kj}, c_{lj}) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}|r_{kj} - r_{lj}| + \sqrt{\frac{1}{2}\big(|u_{kj} - u_{lj}|^2 + |v_{kj} - v_{lj}|^2 + |h_{kj} - h_{lj}|^2\big)}\right).$$

These formulas account for differences in parameters r, u, v, and h between two C-IF evaluation values (c_{kj} and c_{lj}) to calculate the distance measures $D^1_{\mathfrak{M}}$ and $D^2_{\mathfrak{M}}$. Given $c_{13} = (0.346, 0.528; 0.271)$ and $c_{23} = (0.569, 0.327; 0.112)$ with hesitancy degrees of 0.126 and 0.104, the distances were: $D^1_{\mathfrak{M}}(c_{13}, c_{23}) = D^1_{\mathfrak{M}}(c_{23}, c_{13}) = (1/2) \cdot [(1/\sqrt{2}) \cdot |0.271-0.112|+(1/2)\cdot(|0.346-0.569|+|0.528-0.327|+|0.126-0.104|)] = 0.168$, and $D^2_{\mathfrak{M}}(c_{13}, c_{23}) = D^2_{\mathfrak{M}}(c_{23}, c_{13}) = (1/2) \cdot \{(1/\sqrt{2}) \cdot |0.271 - 0.112| + [(1/2) \cdot (|0.346 - 0.569|^2 + |0.528 - 0.327|^2 + |0.126 - 0.104|^2)]^{0.5}\} = 0.163$. These Manhattan- and Euclidean-like distances, detailed in Table 5, quantify the dissimilarity between the C-IF evaluation values, reflecting differences across their parameters.

Following Step I.10, the consistency indicator $\mathcal{I}^{0.5}_{\mathbb{C}}(O_k/O_l)$ and inconsistency indicator $\mathcal{I}^{0.5}_{\mathbb{D}}(O_k/O_l)$ (for $\varphi = 0.5$) were derived using Eqs. (16) and (17). In the case where O_3 outperforms O_2, Table 4 shows the concordance set $\mathbb{C}^{0.5}(O_3/O_2) = \{E_2, E_3, E_6, E_7, E_9\}$ and the discordance set $\mathbb{D}^{0.5}(O_3/O_2) = \{E_1, E_4, E_5, E_8\}$. The consistency indicator $\mathcal{I}^{0.5}_{\mathbb{C}}(O_3/O_2)$ was then computed as:

$$\mathcal{I}^{0.5}_{\mathbb{C}}(O_3/O_2) = \frac{\sum_{E_j \in \{E_2, E_3, E_6, E_7, E_9\}} S^{0.5}(w_j) \cdot |S^{0.5}(c_{3j}) - S^{0.5}(c_{2j})|}{\sum_{j'=1}^{9} S^{0.5}(w_{j'}) \cdot |S^{0.5}(c_{3j'}) - S^{0.5}(c_{2j'})|}$$

$$= \big(0.367 \cdot |0.741 - 0.705| + 0.493 \cdot |0.686 - 0.624| + 0.382 \cdot |0.836 - 0.807|$$

Table 5
Specifics on the C-IF Manhattan- and Euclidean-like distances $D^1_{\mathfrak{M}}(c_{kj}, c_{lj})$ and $D^2_{\mathfrak{M}}(c_{kj}, c_{lj})$.

	Outcome of the C-IF Manhattan-like distances			Outcome of the C-IF Euclidean-like distances		
E_j	$D^1_{\mathfrak{M}}(c_{1j}, c_{2j})$, $D^1_{\mathfrak{M}}(c_{2j}, c_{1j})$	$D^1_{\mathfrak{M}}(c_{1j}, c_{3j})$, $D^1_{\mathfrak{M}}(c_{3j}, c_{1j})$	$D^1_{\mathfrak{M}}(c_{2j}, c_{3j})$, $D^1_{\mathfrak{M}}(c_{3j}, c_{2j})$	$D^2_{\mathfrak{M}}(c_{1j}, c_{2j})$, $D^2_{\mathfrak{M}}(c_{2j}, c_{1j})$	$D^2_{\mathfrak{M}}(c_{1j}, c_{3j})$, $D^2_{\mathfrak{M}}(c_{3j}, c_{1j})$	$D^2_{\mathfrak{M}}(c_{2j}, c_{3j})$, $D^2_{\mathfrak{M}}(c_{3j}, c_{2j})$
E_1	0.177	0.196	0.034	0.176	0.192	0.033
E_2	0.156	0.135	0.089	0.151	0.134	0.086
E_3	0.168	0.197	0.031	0.163	0.192	0.031
E_4	0.192	0.114	0.078	0.190	0.114	0.076
E_5	0.072	0.021	0.052	0.072	0.020	0.051
E_6	0.084	0.126	0.041	0.080	0.120	0.040
E_7	0.053	0.082	0.079	0.050	0.081	0.076
E_8	0.109	0.098	0.030	0.108	0.097	0.029
E_9	0.077	0.105	0.030	0.074	0.102	0.029

$$
\begin{aligned}
&+ 0.363 \cdot |0.789 - 0.679| + 0.492 \cdot |0.655 - 0.624|\big)/\big(0.608 \cdot |0.675 - 0.691| \\
&+ 0.367 \cdot |0.741 - 0.705| + 0.493 \cdot |0.686 - 0.624| + 0.487 \cdot |0.679 - 0.759| \\
&+ 0.604 \cdot |0.675 - 0.779| + 0.382 \cdot |0.836 - 0.807| + 0.363 \cdot |0.789 - 0.679| \\
&+ 0.610 \cdot |0.624 - 0.655| + 0.492 \cdot |0.655 - 0.624|\big) = 0.458.
\end{aligned}
$$

By employing Eq. (17) and utilizing the C-IF Manhattan-like distance $D^1_{\mathfrak{M}}(c_{kj}, c_{lj})$, the inconsistency indicator $\mathcal{I}^{0.5}_{\mathbb{D}}(O_3/O_2)$ can be produced in this fashion:

$$
\begin{aligned}
\mathcal{I}^{0.5}_{\mathbb{D}}(O_3/O_2) &= \frac{\sum_{E_j \in \{E_1, E_4, E_5, E_8\}} D^1_{\mathfrak{M}}(c_{3j}, c_{2j}) \cdot |S^{0.5}(c_{3j}) - S^{0.5}(c_{2j})|}{\sum_{j'=1}^{9} D^1_{\mathfrak{M}}(c_{3j'}, c_{2j'}) \cdot |S^{0.5}(c_{3j'}) - S^{0.5}(c_{2j'})|} \\
&= \big(0.034 \cdot |0.675 - 0.691| + 0.078 \cdot |0.679 - 0.759| + 0.052 \cdot |0.675 - 0.779| \\
&\quad + 0.030 \cdot |0.624 - 0.655|\big)/\big(0.034 \cdot |0.675 - 0.691| + 0.089 \cdot |0.741 - 0.705| \\
&\quad + 0.031 \cdot |0.686 - 0.624| + 0.078 \cdot |0.679 - 0.759| + 0.052 \cdot |0.675 - 0.779| \\
&\quad + 0.041 \cdot |0.836 - 0.807| + 0.079 \cdot |0.789 - 0.679| + 0.030 \cdot |0.624 - 0.655| \\
&\quad + 0.030 \cdot |0.655 - 0.624|\big) = 0.451.
\end{aligned}
$$

Using the C-IF Euclidean-like distance $D^2_{\mathfrak{M}}(c_{kj}, c_{lj})$, the inconsistency indicator $\mathcal{I}^{0.5}_{\mathbb{D}}(O_3/O_2)$ equals 0.454. Additional results are provided in Table 6's third, fifth, and seventh columns.

Following Step I.11, the average consistency indicator $\overline{\mathcal{I}}^{0.5}_{\mathbb{C}}$ is 0.5 (Theorem 7). Using Eq. (18), the consistency entry $\mathcal{B}^{0.5}_{\mathbb{C}}(O_k/O_l)$ was generated by comparing $\mathcal{I}^{0.5}_{\mathbb{C}}(O_k/O_l)$ with $\overline{\mathcal{I}}^{0.5}_{\mathbb{C}}$. The fourth column of Table 6 presents these comparisons. The consistency Boolean matrix $\mathfrak{B}^{0.5}_{\mathbb{C}}$ was then constructed using Eq. (19):

$$
\mathfrak{B}^{0.5}_{\mathbb{C}} = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 1 \\ 1 & 0 & - \end{bmatrix}.
$$

Table 6
Specifics on the consistency/inconsistency indicators and entries for $\varphi = 0.5$.

				Use of the $D^1_{\mathfrak{M}}$ measure		Use of the $D^2_{\mathfrak{M}}$ measure	
O_k	O_l	$\mathcal{I}^{0.5}_{\mathbb{C}}(O_k/O_l)$	$\mathcal{B}^{0.5}_{\mathbb{C}}(O_k/O_l)$	$\mathcal{I}^{0.5}_{\mathbb{D}}(O_k/O_l)$	$\mathcal{B}^{0.5}_{\mathbb{D}}(O_k/O_l)$	$\mathcal{I}^{0.5}_{\mathbb{D}}(O_k/O_l)$	$\mathcal{B}^{0.5}_{\mathbb{D}}(O_k/O_l)$
O_1	O_2	0.000	$0\ (< \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	1.000	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	1.000	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$
	O_3	0.019	$0\ (< \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.990	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.990	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$
O_2	O_1	1.000	$1\ (\geqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.000	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.000	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$
	O_3	0.542	$1\ (\geqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.549	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.546	$0\ (> \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$
O_3	O_1	0.981	$1\ (\geqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.010	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.010	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$
	O_2	0.458	$0\ (< \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.451	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$	0.454	$1\ (\leqslant \overline{\mathcal{I}}^{0.5}_{\mathbb{C}})$

In Step I.12, the average inconsistency indicator $\overline{\mathcal{I}}^{0.5}_{\mathbb{D}}$ was yielded to be 0.5 (Theorem 9). By comparing $\mathcal{I}^{0.5}_{\mathbb{D}}(O_k/O_l)$ with $\overline{\mathcal{I}}^{0.5}_{\mathbb{D}}$, the inconsistency entry $\mathcal{B}^{0.5}_{\mathbb{D}}(O_k/O_l)$ was derived using Eq. (20). Results based on the C-IF Manhattan-like distance $D^1_{\mathfrak{M}}$ and Euclidean-like distance $D^2_{\mathfrak{M}}$ are shown in the sixth and eighth columns of Table 6, respectively. Table 6 summarizes the consistency and inconsistency indicators $\mathcal{I}^{0.5}_{\mathbb{C}}(O_k/O_l)$ and $\mathcal{I}^{0.5}_{\mathbb{D}}(O_k/O_l)$, along with their entries $\mathcal{B}^{0.5}_{\mathbb{C}}(O_k/O_l)$ and $\mathcal{B}^{0.5}_{\mathbb{D}}(O_k/O_l)$ for $\varphi = 0.5$. As noted from Eq. (21), both distance measures yield the same inconsistency Boolean matrix $\mathfrak{B}^{0.5}_{\mathbb{D}}$, as shown:

$$\mathfrak{B}^{0.5}_{\mathbb{D}} = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 0 & - \end{bmatrix}.$$

Following Step I.13, the overall prioritization entry $\mathcal{B}^{0.5}_{\mathbb{O}}(O_k/O_l)$ was calculated using $\mathcal{B}^{0.5}_{\mathbb{O}}(O_k/O_l) = \mathcal{B}^{0.5}_{\mathbb{C}}(O_k/O_l) \cdot \mathcal{B}^{0.5}_{\mathbb{D}}(O_k/O_l)$ from Eq. (22). These results were then used to build the overall prioritization Boolean matrix $\mathfrak{B}^{0.5}_{\mathbb{O}}$ via Eq. (23), as shown:

$$\mathfrak{B}^{0.5}_{\mathbb{O}} = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 0 & - \end{bmatrix}.$$

Continuing the procedure in Step I.13, a dominance graph was created to illustrate the partial-prioritization ranking of the three supplier options, as shown in Fig. 4. The overall outranking relationships using the C-IF ELECTRE I techniques were depicted utilizing orange arrows: $O_2 \succcurlyeq^{0.5}_{\mathbb{O}} O_1$ and $O_3 \succcurlyeq^{0.5}_{\mathbb{O}} O_1$. Consistency-based relationships are shown with green arrows: $O_2 \succcurlyeq^{0.5}_{\mathbb{C}} O_1$, $O_2 \succcurlyeq^{0.5}_{\mathbb{C}} O_3$, and $O_3 \succcurlyeq^{0.5}_{\mathbb{C}} O_1$. Inconsistency-based relationships are indicated by yellow arrows: $O_2 \succcurlyeq^{0.5}_{\mathbb{D}} O_1$ and $O_3 \succcurlyeq^{0.5}_{\mathbb{D}} O_1$. The ranking from the C-IF analytic hierarchy process (AHP) and VIKOR (i.e. VlseKriterijumska Optimizacija I Kompromisno Resenje in Serbian) methodology in Otay and Kahraman (2022) was $O_2 \succcurlyeq O_3 \succcurlyeq O_1$. The outranking relationships from the C-IF ELECTRE I approach with $\varphi = 0.5$ and $\xi = 1, 2$ align closely with the findings of Otay and Kahraman.

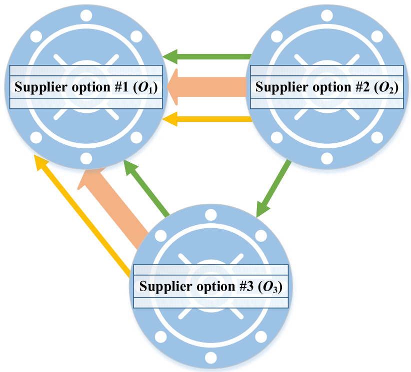


Fig. 4. The dominance graph for the multi-expert supplier evaluation issue.

In the C-IF ELECTRE II approach, Steps II.1–II.10 mirror Steps I.1–I.10. In Step II.11, Eq. (24) calculates the consistency-dependent average outflow $\mathcal{A}_{\mathbb{C}}^{0.5}(O_k)$. For example, for O_1:$\mathcal{A}_{\mathbb{C}}^{0.5}(O_1) = \sum_{l=1,l\neq 1}^{3} \mathcal{I}_{\mathbb{C}}^{0.5}(O_1/O_l)/(3-1) = [\mathcal{I}_{\mathbb{C}}^{0.5}(O_1/O_2) + \mathcal{I}_{\mathbb{C}}^{0.5}(O_1/O_3)]/2 = (0.000+0.019)/2 = 0.010$. Moreover, $\mathcal{A}_{\mathbb{C}}^{0.5}(O_2) = 0.771$ and $\mathcal{A}_{\mathbb{C}}^{0.5}(O_3) = 0.720$. In Step II.11, Eq. (25) computes the inconsistency-dependent average inflow $\mathcal{A}_{\mathbb{D}}^{0.5}(O_k)$. Using the C-IF Manhattan-like distance measure $D_{\mathfrak{M}}^{1}$: $\mathcal{A}_{\mathbb{D}}^{0.5}(O_1) = \sum_{l=1,l\neq 1}^{3} \mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_l)/(3-1) = [\mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_2) + \mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_3)]/2 = (1.000 + 0.990)/2 = 0.995$. Additionally, $\mathcal{A}_{\mathbb{D}}^{0.5}(O_2) = 0.275$ and $\mathcal{A}_{\mathbb{D}}^{0.5}(O_3) = 0.231$. Using the C-IF Euclidean-like distance measure $D_{\mathfrak{M}}^{2}$:$\mathcal{A}_{\mathbb{D}}^{0.5}(O_1) = \sum_{l=1,l\neq 1}^{3} \mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_l)/(3-1) = [\mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_2) + \mathcal{I}_{\mathbb{D}}^{0.5}(O_1/O_3)]/2 = (1.000 + 0.990)/2 = 0.995$, $\mathcal{A}_{\mathbb{D}}^{0.5}(O_2) = 0.273$, and $\mathcal{A}_{\mathbb{D}}^{0.5}(O_3) = 0.232$.

Finally, Eq. (26) calculates the overall net flow $\mathcal{N}_{\mathbb{O}}^{0.5}(O_k)$. Using the C-IF Manhattan-like distance measure $D_{\mathfrak{M}}^{1}$, the calculations yield: $\mathcal{N}_{\mathbb{O}}^{0.5}(O_1) = \mathcal{A}_{\mathbb{C}}^{0.5}(O_1) - \mathcal{A}_{\mathbb{D}}^{0.5}(O_1) = 0.010 - 0.995 = -0.985$, $\mathcal{N}_{\mathbb{O}}^{0.5}(O_2) = 0.496$, and $\mathcal{N}_{\mathbb{O}}^{0.5}(O_3) = 0.489$. Using the C-IF Euclidean-like distance $D_{\mathfrak{M}}^{2}$:$\mathcal{N}_{\mathbb{O}}^{0.5}(O_1) = \mathcal{A}_{\mathbb{C}}^{0.5}(O_1) - \mathcal{A}_{\mathbb{D}}^{0.5}(O_1) = 0.010 - 0.995 = -0.985$, $\mathcal{N}_{\mathbb{O}}^{0.5}(O_2) = 0.498$, and $\mathcal{N}_{\mathbb{O}}^{0.5}(O_3) = 0.488$. Arranging these in descending order, the complete prioritization ranking is $O_2 \succ_{\mathcal{N}}^{0.5} O_3 \succ_{\mathcal{N}}^{0.5} O_1$ for both measures $D_{\mathfrak{M}}^{1}$ and $D_{\mathfrak{M}}^{2}$. This aligns with the preference order from the C-IF AHP and VIKOR methodology in Otay and Kahraman (2022), confirming that the outranking relationships from the current C-IF ELECTRE II approach with $\varphi = 0.5$ and $\xi = 1, 2$ are consistent with their findings.

The proposed C-IF ELECTRE I and II techniques have demonstrated feasibility and efficiency in addressing multiple-criteria supplier assessments. By integrating C-IF theory with the established ELECTRE framework, these methodologies provide a comprehensive and reliable decision-making toolset. They effectively consider both beneficial and non-beneficial criteria while accounting for the decision-maker's attitudes toward aggression and caution. The rankings of supplier options generated from these techniques are coherent and dependable, aligning with the findings from the C-IF AHP and VIKOR approach by Otay and Kahraman (2022). This indicates that the C-IF ELECTRE I and II methods significantly enhance decision-making in complex supplier evaluation scenarios.

The C-IF ELECTRE approach provides valuable insights into how decision-makers evaluate suppliers across multiple criteria. Key dimensions—such as cost, service, technology, and quality—greatly influence supplier rankings, particularly factors like price, on-time delivery, and technological capability. By incorporating both aggressive and cautious estimates, this method addresses uncertainty and adapts to varying market conditions and expert judgments, improving the reliability of recommendations.

Key advantages of the C-IF ELECTRE methodology include: (1) Refined Uncertainty Representation: Circular structures capture uncertainty, offering a balanced view of fluctuating factors like supplier reliability; (2) Flexibility in Risk Management: The inclination parameter allows customization based on the decision-maker's risk tolerance, accommodating aggressive or cautious evaluations; and (3) Improved Sensitivity to Complex Criteria: This approach effectively manages complex, interdependent criteria, making it ideal for multidimensional supplier evaluations where factors are interconnected.

Despite its strengths, the C-IF ELECTRE approach has limitations: (1) Computational Complexity: The integration of C-IF sets and the need for precise calibration of parameters (φ and ξ) increases computational intricacy, potentially burdening decision-makers without advanced tools or expertise; and (2) Limited Sensitivity to Extremes: The approach may not adequately respond to suppliers excelling or underperforming in specific criteria, risking the omission of those with exceptional performance in niche areas but lacking broader competencies.

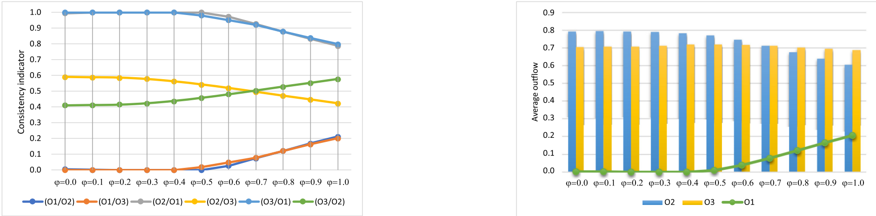
6. Comprehensive Comparative Analysis

This section examines the effects of different inclination parameter settings on the C-IF ELECTRE approach's results. It also incorporates the Chebyshev distance metric ($\xi \to \infty$) within the C-IF Minkowski-like distance measure, alongside the previously used Manhattan and Euclidean metrics. Moreover, it evaluates divergence functions proposed by Khan *et al.* (2022) to compare the results generated by the C-IF ELECTRE techniques.

The C-IF Chebyshev-like distance between c_{kj} and c_{lj} is derived by setting the metric parameter ξ in Definition 4 to infinity. This is calculated using the four-term strategy as:

$$D_{\mathfrak{M}}^{\infty}(c_{kj}, c_{lj}) = \frac{1}{2}\left\{\frac{1}{\sqrt{2}}|r_{kj} - r_{lj}| + \max\{|u_{kj} - u_{lj}|, |v_{kj} - v_{lj}|, |h_{kj} - h_{lj}|\}\right\}. \tag{27}$$

Divergence measures in fuzzy contexts help identify differences between fuzzy sets. For C-IF sets, Khan *et al.* (2022) developed various divergence functions to address higher-order uncertainties and evaluated their performance. They established five divergence functions based on chi-square and Canberra distances, as well as the exponential function. This study will adopt these five divergence measures to calculate inconsistency indicators. Let $D_{\mathfrak{E}}^{\epsilon}(c_{kj}, c_{lj})$ represent the divergence measure between c_{kj} and c_{lj}, where ϵ



(a) Contrast of $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ (b) Contrast of $\mathcal{A}^{\varphi}_{\mathbb{C}}(O_k)$

Fig. 5. Comparison outcomes of consistency indicators and consistency-dependent average outflows.

is an identifier parameter from the set $\{1, 2, \dots, 5\}$. The formulas for the five divergence measures based on chi-square distances ($D^1_{\mathfrak{C}}$ and $D^2_{\mathfrak{C}}$), Canberra distances ($D^3_{\mathfrak{C}}$ and $D^4_{\mathfrak{C}}$), and the exponential function ($D^5_{\mathfrak{C}}$) are presented below:

$$D^1_{\mathfrak{C}}(c_{kj}, c_{lj}) = \frac{(u_{kj} - u_{lj})^2}{1 + u_{kj} + u_{lj}} + \frac{(v_{kj} - v_{lj})^2}{1 + v_{kj} + v_{lj}} + \frac{(r_{kj} - r_{lj})^2}{1 + r_{kj} + r_{lj}}, \tag{28}$$

$$D^2_{\mathfrak{C}}(c_{kj}, c_{lj}) = \left[\frac{(u_{kj} - u_{lj})^2}{1 + u_{kj} + u_{lj}} + \frac{(v_{kj} - v_{lj})^2}{1 + v_{kj} + v_{lj}}\right] \cdot e^{\frac{(r_{kj} - r_{lj})^2}{1 + r_{kj} + r_{lj}}}, \tag{29}$$

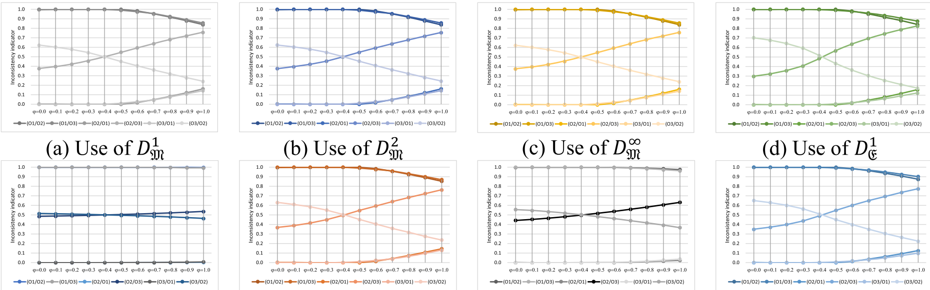
$$D^3_{\mathfrak{C}}(c_{kj}, c_{lj}) = \frac{|u_{kj} - u_{lj}|}{1 + u_{kj} + u_{lj}} + \frac{|v_{kj} - v_{lj}|}{1 + v_{kj} + v_{lj}} + \frac{|r_{kj} - r_{lj}|}{1 + r_{kj} + r_{lj}}, \tag{30}$$

$$D^4_{\mathfrak{C}}(c_{kj}, c_{lj}) = \left[\frac{|u_{kj} - u_{lj}|}{1 + u_{kj} + u_{lj}} + \frac{|v_{kj} - v_{lj}|}{1 + v_{kj} + v_{lj}}\right] \cdot e^{\frac{|r_{kj} - r_{lj}|}{1 + r_{kj} + r_{lj}}}, \tag{31}$$

$$\begin{aligned} D^5_{\mathfrak{C}}(c_{kj}, c_{lj}) = \frac{1}{2 \cdot (e^2 - 1)} \Big[&(u_{kj} - u_{lj}) \cdot \left(e^{\frac{4 \cdot u_{kj}}{1 + u_{kj} + u_{lj}}} - e^{\frac{4 \cdot u_{lj}}{1 + u_{kj} + u_{lj}}}\right) + (v_{kj} - v_{lj}) \cdot \\ &\left(e^{\frac{4 \cdot v_{kj}}{1 + v_{kj} + v_{lj}}} - e^{\frac{4 \cdot v_{lj}}{1 + v_{kj} + v_{lj}}}\right) + (r_{kj} - r_{lj}) \cdot \left(e^{\frac{4 \cdot r_{kj}}{1 + r_{kj} + r_{lj}}} - e^{\frac{4 \cdot r_{lj}}{1 + r_{kj} + r_{lj}}}\right) \Big]. \end{aligned} \tag{32}$$

In the initial comparative analysis, the inclination parameter φ was systematically varied from 0 to 1 in increments of 0.1. Using these eleven configurations, the study applied the C-IF ELECTRE to the same supplier evaluation issue. Figure 5 illustrates the results in two parts: Fig. 5(a) shows the distribution of consistency indicators $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_k/O_l)$ among option pairs. The distribution remains stable for φ values between 0 and 0.4, but shifts significantly when φ exceeds 0.5. As φ increases, $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_2/O_1)$, $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_2/O_3)$, and $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_3/O_1)$ decrease, while $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_1/O_2)$, $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_3/O_2)$, and $\mathcal{I}^{\varphi}_{\mathbb{C}}(O_1/O_3)$ increase. Figure 5(b) displays the consistency-dependent average outflows $\mathcal{A}^{\varphi}_{\mathbb{C}}(O_k)$ for each option. It reveals that the average outflow of O_1 is consistently the lowest across all φ values. As φ increases from 0 to 0.6, O_2 has the highest average outflow, followed by O_3. From $\varphi = 0.7$ to 1, O_3 surpasses O_2 in average outflow, indicating a shift in advantage. These findings suggest that O_2 is favoured at lower φ values, while O_3 gains an advantage at higher φ values.

In the second comparative analysis, the study examined the combined effects of different inclination parameter values ($\varphi = 0, 0.1, \dots, 1$) and various C-IF distance and



(a) Use of $D^1_{\mathfrak{M}}$ (b) Use of $D^2_{\mathfrak{M}}$ (c) Use of $D^\infty_{\mathfrak{M}}$ (d) Use of $D^1_{\mathfrak{C}}$

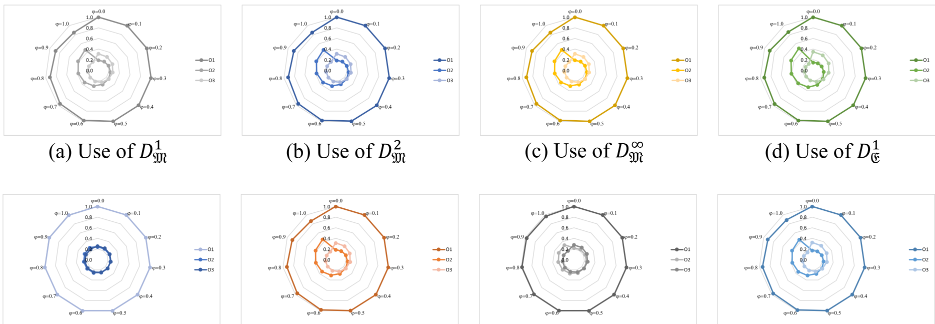
(e) Use of $D^2_{\mathfrak{C}}$ (f) Use of $D^3_{\mathfrak{C}}$ (g) Use of $D^4_{\mathfrak{C}}$ (h) Use of $D^5_{\mathfrak{C}}$

Fig. 6. Comparison outcomes of inconsistency indicators for distinct C-IF distance/divergence measures.

divergence measures on inconsistency indicators. It evaluated the C-IF Minkowski-like distance $D^\xi_{\mathfrak{M}}$ with metric parameters $\xi = 1, 2, \infty$, alongside five divergence measures $D^\epsilon_{\mathfrak{C}}$ based on chi-square, Canberra, and exponential functions. The study tested these eight measures: Manhattan-like $D^1_{\mathfrak{M}}$, Euclidean-like $D^2_{\mathfrak{M}}$, Chebyshev-like $D^\infty_{\mathfrak{M}}$, chi-square $D^1_{\mathfrak{C}}$ and $D^2_{\mathfrak{C}}$, Canberra $D^3_{\mathfrak{C}}$ and $D^4_{\mathfrak{C}}$, and exponential $D^5_{\mathfrak{C}}$. It compared outcomes for various parameter combinations, as shown in Fig. 6: (1) Similar trends: Inconsistency indicators calculated with $D^1_{\mathfrak{M}}$, $D^2_{\mathfrak{M}}$, and $D^\infty_{\mathfrak{M}}$ showed high consistency. Likewise, results from divergence measures $D^1_{\mathfrak{C}}$, $D^3_{\mathfrak{C}}$, and $D^5_{\mathfrak{C}}$ aligned closely with those from the Minkowski-like distances; (2) Distinct patterns: Line charts in Fig. 6(e) and (g) revealed unique trends, suggesting these cases differ from others; and (3) Stable results: Indicators based on $D^2_{\mathfrak{C}}$ and $D^4_{\mathfrak{C}}$ remained relatively consistent across all φ values, contrasting with other measures that captured parameter variations more effectively. These findings highlight the C-IF ELECTRE framework's robustness in accommodating diverse distance and divergence measures.

The third comparative analysis considered key parameters: inclination ($\varphi = 0, 0.1, \ldots, 1$), metric ($\xi = 1, 2, \infty$), and identifier ($\varepsilon = 1, 2, \ldots, 5$). Figure 7 contrasts outcomes of inconsistency-dependent average inflows across various C-IF distance/divergence measures, including Manhattan, Euclidean, and Chebyshev distances ($D^1_{\mathfrak{M}}$, $D^2_{\mathfrak{M}}$, and $D^\infty_{\mathfrak{M}}$) from Chen (2023b) and five divergence measures ($D^1_{\mathfrak{C}}$–$D^5_{\mathfrak{C}}$) from Khan *et al.* (2022). Key findings include: (1) Consistent trends: Radar charts in Fig. 7(a)–(d), (f), and (h) show similar patterns across distance measures $D^1_{\mathfrak{M}}$, $D^2_{\mathfrak{M}}$, and $D^\infty_{\mathfrak{M}}$ and divergence measures $D^1_{\mathfrak{C}}$, $D^3_{\mathfrak{C}}$, and $D^5_{\mathfrak{C}}$; (2) Distinct cases: Fig. 7(e) ($D^2_{\mathfrak{C}}$) and Fig. 7(g) ($D^4_{\mathfrak{C}}$) reveal unique patterns, indicating these scenarios differ from others; and (3) Disadvantage patterns: Across all φ values, option O_1 shows the highest average inflow (disadvantage). For φ from 0 to 0.4, O_2 has the lowest inflow, followed by O_3. When φ exceeds 0.5, O_3 becomes the least disadvantaged, followed by O_2. These results confirm that the C-IF ELECTRE techniques produce consistent and reliable findings, regardless of the measurement method used.

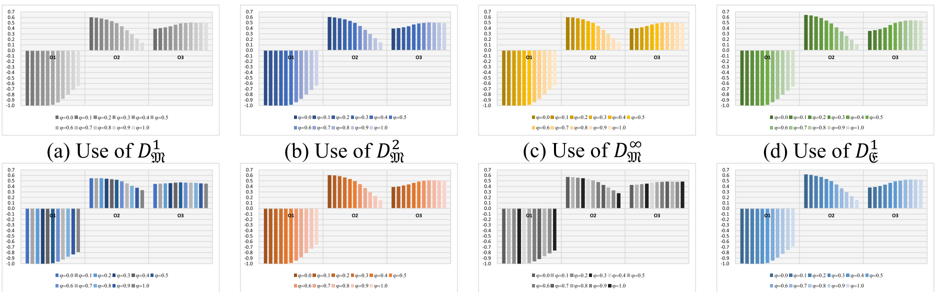
The fourth comparative study considered the parameters $\varphi = 0, 0.1, \ldots, 1$, $\xi = 1, 2, \infty$, and $\varepsilon = 1, 2, \ldots, 5$. Figure 8 illustrates the overall net flow $\mathcal{N}^\varphi_{\mathbb{O}}(O_k)$ for each option, reflecting both the advantages from consistency indicators and disadvantages



(a) Use of $D^1_{\mathfrak{M}}$ (b) Use of $D^2_{\mathfrak{M}}$ (c) Use of $D^\infty_{\mathfrak{M}}$ (d) Use of $D^1_{\mathfrak{E}}$

(e) Use of $D^2_{\mathfrak{E}}$ (f) Use of $D^3_{\mathfrak{E}}$ (g) Use of $D^4_{\mathfrak{E}}$ (h) Use of $D^5_{\mathfrak{E}}$

Fig. 7. Comparison outcomes of inconsistency-dependent average inflows.



(a) Use of $D^1_{\mathfrak{M}}$ (b) Use of $D^2_{\mathfrak{M}}$ (c) Use of $D^\infty_{\mathfrak{M}}$ (d) Use of $D^1_{\mathfrak{E}}$

(e) Use of $D^2_{\mathfrak{E}}$ (f) Use of $D^3_{\mathfrak{E}}$ (g) Use of $D^4_{\mathfrak{E}}$ (h) Use of $D^5_{\mathfrak{E}}$

Fig. 8. Comparison outcomes of overall net flows.

from inconsistency indicators. When using C-IF Manhattan, Euclidean, and Chebyshev distances ($D^1_{\mathfrak{M}}$, $D^2_{\mathfrak{M}}$, and $D^\infty_{\mathfrak{M}}$), or divergence measures based on Canberra distances ($D^3_{\mathfrak{E}}$, $D^4_{\mathfrak{E}}$) and the exponential function ($D^5_{\mathfrak{E}}$), the rankings were $O_2 \succ^{\varphi}_{\mathcal{N}} O_3 \succ^{\varphi}_{\mathcal{N}} O_1$ for φ values from 0 to 0.5, and shifted to $O_3 \succ^{\varphi}_{\mathcal{N}} O_2 \succ^{\varphi}_{\mathcal{N}} O_1$ for φ values from 0.6 to 1. With the chi-square-based divergence measure $D^1_{\mathfrak{E}}$, the rankings remained $O_2 \succ^{\varphi}_{\mathcal{N}} O_3 \succ^{\varphi}_{\mathcal{N}} O_1$ for φ between 0 and 0.4, but shifted to $O_3 \succ^{\varphi}_{\mathcal{N}} O_2 \succ^{\varphi}_{\mathcal{N}} O_1$ for φ between 0.5 and 1. Using $D^2_{\mathfrak{E}}$, the shift occurred later, with rankings changing from $O_2 \succ^{\varphi}_{\mathcal{N}} O_3 \succ^{\varphi}_{\mathcal{N}} O_1$ ($\varphi = 0$ to 0.6) to $O_3 \succ^{\varphi}_{\mathcal{N}} O_2 \succ^{\varphi}_{\mathcal{N}} O_1$ ($\varphi = 0.7$ to 1). These findings reveal how rankings vary with φ values under different distance and divergence measures, offering insights into the options' performance across varying conditions.

7. Conclusions

The proposed C-IF ELECTRE I and II approaches effectively handle uncertainties, hesitancies, and imprecise data through C-IF theory. They integrate a scoring mechanism, consistency and inconsistency indexing, and prioritization steps. Applied to supplier evaluation tasks, these methods demonstrate practical utility and reliability. The key contributions include: introducing the joint generalized scoring function, capturing decision-maker preferences, developing the C-IF ELECTRE framework, and validating its benefits through multi-expert supplier assessments.

The C-IF ELECTRE methodology, while effective for handling uncertainty in supplier evaluations or other decision-making issues, requires careful parameter tuning, may be complex for non-experts, and could face limitations in other fuzzy contexts or decision-making scenarios, requiring further validation and adaptation for broader applicability. Future research could broaden the application of the C-IF ELECTRE approach and strengthen its validity through comparisons with other methods and model quality metrics. While this study focused on supplier evaluation, future work should apply the approach to diverse fields—such as healthcare, project management, and environmental sustainability—to assess its versatility. Comparative analyses with other decision-making models, including traditional and fuzzy ELECTRE methods, would highlight the unique strengths of the C-IF ELECTRE. Additionally, evaluating its stability and reliability through metrics like accuracy and consistency would provide valuable quantitative insights, further validating its effectiveness and practical value.

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