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Quasi-Owen Value for Games on Augmenting Systems with a Coalition Structure

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Abstract. This paper focuses on games on augmenting systems with a coalition structure that can be seen as an extension of games with a coalition structure and games on augmenting systems. Considering the player payoffs, the quasi-Owen value is defined. To show the rationality of this payoff index, five representative axiomatic systems are established. The population monotonic allocation scheme (PMAS) and the core are introduced. Moreover, the relationships between the PMAS and quasi-Owen value as well as the core and quasi-Owen value are discussed. Finally, an illustrative example is given to show the concrete application of the new payoff indices.

Key words: cooperative game with a coalition structure, augmenting system, axiomatic system, quasi-Owen value.

1. Introduction

In some cooperative situations, the players join in coalitions that form a partition or coalitional structure of the set of players to get more payoffs or to gain the competitive advantage. Aumann and Dreze (1974) first established a model of games with a coalition structure, where the coalitions are independent with each other. Different from the cooperative model in reference (Aumann and Dreze, 1974; Owen, 1977) introduced games with a coalitional structure where the probability of cooperation among coalitions is considered and defined the Owen value for this type of games, which is an extension of the Shapley value (Peleg, 1986). Following the idea of the Banzhaf value, Owen (1978) further proposed the Banzhaf-Owen value for games with a coalitional structure. Later, Alonso-Meijde and Fiestras-Janeiro (2002) noted that the Banzhaf-Owen value dissatisfies the symmetry in quotient games and gave another solution concept for games with a coalition structure, which is known as the symmetric Banzhaf value. Meanwhile, the axiomatic systems of the Owen value are studied in references (Hart and Kurz, 1983; Peleg, 1989; Hamiache, 1999; Khmelnitskaya and Yanovskaya, 2007; Albizuri, 2008; Lorenzo-Freire,

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2019; Hu, 2021), and the axiomatic characterizations of the Banzhaf-Owen coalition value are discussed in the literature (Amer *et al.*, 2002; Alonso-Meijide *et al.*, 2007; Lorenzo-Freire, 2017).

Different from games with a coalitional structure (Aumann and Dreze, 1974; Owen, 1977), due to political, economic and/or other reasons, not all coalitions can be formed in some cooperation. People usually call such games as games under precedence constraints. Myerson (1977) first considered this situation and introduced games with communication structures using graph theory. Then, the Shapley value for this type of games is researched. Faigle and Kern (1992) discussed a special type of games under precedence constraints that satisfies the offered order relationship and closes under union and intersection and discussed the axiomatic system of the given Shapley value using hierarchical strength. Following the work of Edelman and Jamison (1985), Bilbao (1998) introduced games on convex geometries. Further, Bilbao (1998) and Bilbao and Edelman (2000) studied the characterizations of the Shapley value for games on convex geometries using chain axiom and hierarchical strength, respectively. Bilbao *et al.* (1998, 1999) researched the Banzhaf value and the core of games on convex geometries. Later, Bilbao *et al.* (2001, 2002) discussed another special kind of games under precedence constraints, which is named as games on matroids. Considering the player payoffs, the authors researched the Shapley value for two cases of games on matroids. Algaba *et al.* (2003) presented games on antimatroids and researched the Shapley value for this kind of games. Recently, Bilbao (2003) proposed the concept of games on augmenting systems and discussed the relationship between augmenting system, antimatroid and convex geometry. Then, the author introduced the Shapley value and the Banzhaf value for games on augmenting systems. Further, Bilbao and Ordonez (2009) researched two axiomatic systems of the Shapley value for games on augmenting systems using hierarchical strength and chain axiom. Algaba *et al.* (2010) proposed the α value for games on augmenting systems by generalizing the Myerson value for graph games and the Shapley value for games with permission structures. Wang *et al.* (2022) provided a new axiomatization of the α value for games on augmenting systems in view of marginality. In addition, Meng *et al.* (2023) studied the profit allocation on a four-echelon supply chain from the perspective of cooperative games on augmenting systems.

In general, games with a coalition structure are formed by the players' internal factor for obtaining more payoffs, while games under precedence constraints are due to the external factor as listed above. Considering these two aspects simultaneously, Meng and Zhang (2012) introduced games on convex geometries with a coalition structure, where all feasible coalitions in each union and in the coalition structure both form a convex geometry. After that, Meng and Zhang (2012) and Meng *et al.* (2015) studied three payoff indices for this type of cooperative games. However, as Meng *et al.* (2015) noted the application of convex geometries has limitations. Recently, Meng *et al.* (2016) introduced games on augmenting systems with a coalition structure, where all subsets of the coalition structure and those of each union both form an augmenting system. Then, the authors defined the augmenting symmetric Banzhaf coalitional value that is used as the payoff index of the players. However, this payoff index does not satisfy the efficiency, which is one of the most important properties of indices. To address this issue, this paper defines another payoff index for games on augmenting systems with a coalition structure: the quasi-Owen value, which can be seen as an extension of the Owen value (Owen, 1977). Then, we build five axiomatic systems to show its rationality. The first two axiomatic systems are based on *linearity*, the third one uses *strong monotonicity*, the fourth one employs the *potential function*, and the last one adopts the *balanced contributions*. Then, the concepts of the population monotonic allocation scheme (PMAS) and the core of games on augmenting systems with a coalition structure are introduced. Further, the relationship between the quasi-Owen value and the core is discussed, and the sufficient conditions for the quasi-Owen value to be a PMAS are provided.

The rest of this paper is organized as follows: In Section 2, some notations and basic definitions that will be used in the following sections are reviewed. In Section 3, the concept of games on augmenting systems with a coalition structure is introduced, and the quasi-Owen value is defined. Meanwhile, five axiomatic systems are built, each of which can be used to prove the existence and uniqueness of the quasi-Owen value. In Section 4, the core and the PMAS for games on augmenting systems with a coalition structure are introduced, and the relationships between them and the quasi-Owen value are studied. In Section 5, a numerical example is provided to concretely illustrate the application of the new indices. The conclusion is made in the last section.

2. Some Basic Concepts

Let $N = \{1, 2, \dots, n\}$ be the finite player set. The cardinality of any coalition $S \subseteq N$ is denoted by the corresponding lower case s. As we know, the coalitional values of a cooperative game can be seen as a fuzzy measure, and the unique proofs of some payoff functions are based on the Möbius transformation. Thus, let us first review the expression of fuzzy measures using the Möbius transformation.

Let $f : \{0, 1\}^n \to \Re$ be a pseudo-Boolean function. Grabisch (1997) noted that any fuzzy measure μ can be seen as a particular case of the pseudo-Boolean function and put under a multilinear polynomial with n variables:

$$\mu(A) = \sum_{T \subseteq N} \left[a_T \prod_{i \in T} y_i \right], \quad \forall A \subseteq N, \tag{1}$$

where $a_T \in \Re$, $y = (y_1, y_2, \dots, y_n) \in \{0, 1\}^n$, and $y_i = 1$ if and only if $i \in A$.

The set of coefficients a_T with $T \subseteq N$ corresponds to the Möbius transformation, denoted by $a_T = \sum_{S \subseteq T} (-1)^{|T \setminus S|} \mu(S)$. Because the transformation is inversible, μ can be recovered from a_T by $\mu(A) = \sum_{B \subseteq A} a_B$.

2.1. *Games with a Coalition Structure*

For the finite set $N = \{1, 2, \dots, n\}$, a coalition structure Γ on N is a partition of N, i.e. $\Gamma = \{B_1, B_2, \dots, B_m\}$ is a coalition structure if it satisfies $\bigcup_{1 \leqslant h \leqslant m} B_h = N$ and

$B_h \cap B_l = \varnothing$ for all $h, l \in M = \{1, 2, \dots, m\}$ with $h \neq l$, denoted by (N, Γ). We also assume $B_k \neq \varnothing$ for all $k \in M$. Each $B_k \in \Gamma$ is called a "union". There are two trivial coalition structures: $\Gamma = \{N\}$ and $\Gamma = \{\{1\}, \{2\}, \dots, \{n\}\}$, where each union is a singleton.

Let $L(N, \Gamma) = \{S \mid S = \bigcup_{l \in R \subseteq M \setminus \{k\}} B_l \cup T,\ \forall T \subseteq B_k,\ \forall k \in M\}$. A game with a coalition structure is a set function $v : L(N, \Gamma) \to \Re_+$ such that $v(\varnothing) = 0$. By $G(N, \Gamma)$, we denote the set of all games with a coalition structure. The restriction of Γ to S is $\Gamma|_S = \{T \in L(N, \Gamma) : T \subseteq S\}$ for any $S \in L(N, \Gamma)$. In order to denote simply, we will omit braces for singleton, e.g. writing $\varnothing$, i, k instead of $\{\varnothing\}$, $\{i\}$ and $\{k\}$ for any $\{i\} \subseteq N$ and any $\{k\} \subseteq M$.

Let $v \in G(N, \Gamma)$, Owen (1977) defined the following Owen value:

$$\psi_i(N, v, \Gamma) = \sum_{R \subseteq M \setminus k} \sum_{i \in S \subseteq B_k} \frac{r!(m-r-1)!}{m!} \frac{(s-1)!(b_k - s)!}{b_k!} \big(v(Q \cup S) - v\big((Q \cup S) \setminus i\big)\big), \quad \forall i \in N, \tag{2}$$

where $Q = \bigcup_{l \in R} B_l$, m and r denote the cardinalities of M and R, respectively.

Definition 1 (Alonso-Meijde and Fiestras-Janeiro, 2002). Let $v \in G(N, \Gamma)$, if $v^B(R) = v(\bigcup_{r \in R} B_r)$ for any $R \subseteq M$, then v^B is said to be a quotient game on (N, Γ), where Γ and M as given above, denoted by (M, v^B).

2.2. *Games on Augmenting Systems*

A set system on N is a pair of $(N, \mathscr{F})$, where $\mathscr{F} \subseteq 2^N$ is a family of subsets.

Definition 2 (Bilbao, 2003). An augmenting system is a set system $(N, \mathscr{F})$ with the following properties:

A1: $\varnothing \in \mathscr{F}$;
A2: If $S, T \in \mathscr{F}$ with $S \cap T \neq \varnothing$, then $S \cup T \in \mathscr{F}$;
A3: If $S, T \in \mathscr{F}$ with $S \subseteq T$ then there is $i \in T \setminus S$ such that $S \cup i \in \mathscr{F}$.

It is noteworthy that when A2 is defined as: if $S, T \in \mathscr{F}$, then $S \cap T \in \mathscr{F}$. $(N, \mathscr{F})$ is a convex geometry system (Bilbao, 2003). Further, when we delete the condition $S \cap T \neq \varnothing$ in A2, then $(N, \mathscr{F})$ is an antimatroid system (Bilbao, 2003).

Because the power set of N is an augmenting system, traditional games are also games on augmenting systems. When $N = \bigcup_{S \in \mathscr{F}} S$, then the augmenting system $(N, \mathscr{F})$ is normal. Bilbao and Ordonez (2009) introduced the following concepts:

A compatible ordering of an augmenting system $(N, \mathscr{F})$, as the total ordering of N, is given by $i_1 < i_2 < \cdots < i_n$ such that $\{i_1, i_2, \dots, i_j\} \in \mathscr{F}$ for all $j = 1, 2, \dots, n$. A compatible ordering of $(N, \mathscr{F})$ corresponds to a maximal chain in $\mathscr{F}$. The set of all maximal chains in $\mathscr{F}$ is denoted by $\text{Ch}(\mathscr{F})$. Given an element $i \in N$ and a compatible ordering $C \in \text{Ch}(\mathscr{F})$, let $C(i)$ be the maximal chain C with i being the last element. For a set $S \in \mathscr{F}$, let $S^* = \{i \in N \backslash S : S \cup i \in \mathscr{F}\}$. The restriction of $\mathscr{F}$ to S is $\mathscr{F}|_S = \{T \in \mathscr{F} : T \subseteq S\}$ for any $S \in \mathscr{F}$.

Similar to Faigle and Kern (1992), Bilbao and Ordonez (2009) introduced the Shapley value for games on augmenting systems as:

$$\phi_i(N, v, \mathscr{F}) = \sum_{\{S \in \mathscr{F} : i \in S^*\}} \frac{c(S)c(S \cup i, N)}{c(N)} \big(v(S \cup i) - v(S)\big), \quad \forall i \in N, \tag{3}$$

where the set function $v : (N, \mathscr{F}) \to \Re_+$ is a game on augmenting systems such that $v(\varnothing) = 0$, $c(N) = |\text{Ch}(\mathscr{F})|$ is the total number of maximal chains in $\mathscr{F}$, $c(S) = c(\varnothing, S)$ and $c(N) = c(\varnothing, N)$ are the numbers of maximal chains from $\varnothing$ to S and from $\varnothing$ to N, respectively, and $c(S \cup i, N)$ is the number of maximal chains from $S \cup i$ to N.

3. Games on Augmenting Systems with a Coalition Structure

In this section, we discuss cooperative games on augmenting systems with a coalition structure, which can be seen as an extension of games with a coalition structure (Owen, 1977, 1978) and games on convex geometries with a coalition structure (Meng and Zhang, 2012; Meng *et al.*, 2015).

3.1. *The Concept of Games on Augmenting Systems with a Coalition Structure*

Similar to the concept of augmenting system on N, Meng *et al.* (2016) gave the concept of augmenting systems on $M = \{1, 2, \dots, m\}$ for $\Gamma = \{B_1, B_2, \dots, B_m\}$, namely, an augmenting system on M is a set system $(M, \mathscr{F}_M)$ with the following properties:

M1: $\varnothing \in \mathscr{F}_M$;
M2: If $K, H \in \mathscr{F}_M$ with $K \cap H \neq \varnothing$, then $K \cup H \in \mathscr{F}_M$;
M3: If $K, H \in \mathscr{F}_M$ with $K \subseteq H$, then there is $l \in H \backslash K$, such that $K \cup l \in \mathscr{F}_M$.

The number of maximal chains from R to K is denoted by $c(R, K)$, and $c(R)$ is the number of maximal chains from $\varnothing$ to R.

From Definition 2, one can check when the domain of N is restricted to B_k, we get an augmenting system $(B_k, \mathscr{F}_{B_k})$, where $\mathscr{F}_{B_k} \subseteq 2^{B_k}$ is a family of subsets that satisfies the conditions given in Definition 2. Augmenting systems with a coalition structure mean that the subsets of $M = \{1, 2, \dots, m\}$ and those of each $B_k \in \Gamma$ $(k \in M)$ form an augmenting system, respectively, denoted by $(N, \Gamma, \mathscr{F})$. Let

$$L(N, \Gamma, \mathscr{F}) = \left\{ S \,\middle|\, S = \bigcup_{l \in R \in \mathscr{F}_M, k \in R^*} B_l \cup T, \ \forall T \in \mathscr{F}_{B_k}, \ \forall k \in M \right\} \tag{4}$$

with $R^* = \{k \in M \backslash R : R \cup k \in \mathscr{F}_M\}$, which denotes the set of formed coalitions.

Definition 3. A game on augmenting system with a coalition structure is a set function $v : L(N, \Gamma, \mathscr{F}) \to \Re_+$ such that $v(\varnothing) = 0$.

Let $G(N, \Gamma, \mathscr{F})$ be the set of all games on augmenting systems with a coalition structure. Without special explanation, for any $(N, \Gamma, \mathscr{F})$, we always assume $B_k \in \mathscr{F}_{B_k}$ for any $k \in M$ and $M \in \mathscr{F}_M$, namely, the augment systems on each union and on the coalition structure are normal.

For $(N, \Gamma, \mathscr{F})$ with $N \in \mathscr{F}$, following the works of Faigle and Kern (1992) and Bilbao and Ordonez (2009), Meng *et al.* (2016) defined the *hierarchical strength* $h_S^{B_k}(i)$ of $i \in S$ for the coalition $S \in F_{B_k}$ as follows:

$$h_S^{B_k}(i) = \frac{|\{C \in \mathrm{Ch}(\mathscr{F}_{B_k}) : S \subseteq C(i)\}|}{c(B_k)}, \tag{5}$$

where $\mathrm{Ch}(\mathscr{F}_{B_k})$ is the set of all maximal chains in $\mathscr{F}_{B_k}$, $c(B_k) = |\mathrm{Ch}(\mathscr{F}_{B_k})|$ is the number of maximal chains in $\mathscr{F}_{B_k}$, and $h_S^{B_k}(i)$ is the average number of maximal chains in which the player $i \in S$ is the last member of S in the chain (Bilbao and Ordonez, 2009).

Similarly, we define the *hierarchical strength* $h_R^M(k)$ of $k \in R$ for the coalition $R \in \mathscr{F}_M$ as follows:

$$h_R^M(k) = \frac{|\{C \in \mathrm{Ch}(\mathscr{F}_M) : R \subseteq C(k)\}|}{c(M)}, \tag{6}$$

where $C(k) = \{k \text{ is the last element in } R \in \mathscr{F}_M\}$, $\mathrm{Ch}(\mathscr{F}_M)$ is the set of all maximal chains in $\mathscr{F}_M$, and $c(M) = |\mathrm{Ch}(\mathscr{F}_M)|$ is the total number of maximal chains in $\mathscr{F}_M$.

Example 1. Let $N = \{1, 2, 3, 4, 5\}$, and $\Gamma = \{B_1, B_2\}$ be a coalition structure on N, where $B_1 = \{1, 2, 3\}$ and $B_2 = \{4, 5\}$. If $\mathscr{F}_{B_1} = \{\varnothing, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, B_1\}$, $\mathscr{F}_{B_2} = \{\varnothing, \{4\}, \{5\}, B_2\}$ and $\mathscr{F}_M = \{\varnothing, \{2\}, M\}$, then it is an augmenting system with a coalition structure, where

$$L(N, \Gamma, \mathscr{F}) = \big\{\varnothing, \{4\}, \{5\}, \{4, 5\}, \{1, 4, 5\}, \{3, 4, 5\}, \{1, 2, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\big\}.$$

Then, we have $h_S^{B_1}(3) = 1/2$ for $S = \{3\}$, and we get $h_R^M(2) = 1$ for $R = \{2\}$.

Because $\{1, 3\} \notin \mathscr{F}_{B_1}$, one can check that $\mathscr{F}_{B_1}$ is not an antimatroid. Further, $\mathscr{F}_{B_1}$ is not a convex geometry for $\{1, 2\} \cap \{2, 3\} \notin \mathscr{F}_{B_1}$.

Definition 4. Let $v \in G(N, \Gamma, \mathscr{F})$. $T \in L(N, \Gamma, \mathscr{F})$ is said to be a carrier if $v(S \cap T) = v(S)$ for any $S \in L(N, \Gamma, \mathscr{F})$.

Because v is defined on $L(N, \Gamma, \mathscr{F})$, the value $v(S \cap T)$ can be any real number for $S \cap T \notin L(N, \Gamma, \mathscr{F})$. In this case, we assume that $v(S \cap T) = v(S)$ for defining the concept of carrier. For instance, in Example 1, if $T = \{1, 2\}$ and $S = \{2, 3\}$, then $S \cap T = \{2\} \notin \mathscr{F}_{B_1} \in L(N, \Gamma, \mathscr{F})$. In this case, we consider $v(2) = v(2, 3)$, where $v(2)$ is not real existence for $\{2\}$ being a virtual coalition. Note that when in the above case, we adopt this process.

3.2. *The Quasi-Owen Value*

Similar to Owen (1977) and Bilbao and Ordonez (2009), the quasi-Owen value for games on augmenting systems with a coalition structure is expressed as:

$$\varphi_i(N,\Gamma,v,\mathscr{F}) = \sum_{\{R\in F_M:k\in R*\}} \sum_{\{S\in F_{B_k}:i\in S*\}} \frac{c(R)c(R\cup k,M)}{c(M)} \frac{c(S)c(S\cup i,B_k)}{c(B_k)} \times \big(v(Q\cup S\cup i)-v(Q\cup S)\big), \quad \forall i\in N, \tag{7}$$

where $R^* = \{k\in M\backslash R : R\cup k\in\mathscr{F}_M\}$, $S^* = \{i\in B_k\backslash S : S\cup i\in\mathscr{F}_{B_k}\}$ and $Q = \bigcup_{l\in R} B_l$.

From $R\in\mathscr{F}_M$, we know that $Q=\bigcup_{l\in R} B_l$ can be formed. When there is only one union in M, then the quasi-Owen value degenerates to the Shapley value for games on augmenting systems. When we restrict the domain of $(N,\Gamma,\mathscr{F})$ in the setting of (N,Γ), then the quasi-Owen value degenerates to the Owen value. In summary, the quasi-Owen value is an extension of the Owen value, which is used for games on augmenting systems with a coalition structure.

For any $T\in L(N,\Gamma,\mathscr{F})\backslash\varnothing$, we define the unanimity game u_T as

$$u_T(S) = \begin{cases} 1, & T\subseteq S\in L(N,\Gamma,\mathscr{F}),\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 1 (Meng *et al.*, 2016). *Let $v\in G(N,\Gamma,\mathscr{F})$, then there is a unique set of coefficients $\{c_T : \varnothing\neq T\in L(N,\Gamma,\mathscr{F})\}$ such that $v=\sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} c_T u_T$. Moreover,*

$$c_T = \sum_{\{H\in\mathscr{F}_M:H\subseteq R\}} (-1)^{r-h}\Bigg(\sum_{\{D\in\mathscr{F}_{B_k}:D\subseteq A\}} (-1)^{a-d} v(Q\cup D)\Bigg), \tag{8}$$

where $T = A\bigcup_{l\in R\in\mathscr{F}_M,k\in R^*} B_l$, $A\in\mathscr{F}_{B_k}\backslash\varnothing$ and $Q=\bigcup_{l\in H} B_l$, r and h denote the cardinalities of R and H, respectively.

Let f be a solution on $G(N,\Gamma,\mathscr{F})$. To show the axiomatic systems of the augmenting symmetric Banzhaf coalitional value, Meng *et al.* (2016) introduced the following two properties:

- *Linearity* (***L***). Let $v_1, v_2\in G(N,\Gamma,\mathscr{F})$ and $\alpha,\beta\in\Re$, then

$$f(N,\Gamma,\alpha v_1+\beta v_2,\mathscr{F}) = \alpha f(N,\Gamma,v_1,\mathscr{F}) + \beta f(N,\Gamma,v_2,\mathscr{F}). \tag{9}$$

- *Hierarchical strength in coalitions* (***HSC***). Let $v\in G(N,\Gamma,\mathscr{F})$. For any $T\in L(N,\Gamma,\mathscr{F})$, without loss of generality, suppose $T = S\bigcup_{l\in R\in\mathscr{F}_M,k\in R^*} B_l$ such that $S\in\mathscr{F}_{B_k}\backslash\varnothing$. For all $i,j\in S$,

$$h_S^{B_k}(j) f_i(N,\Gamma,u_T,\mathscr{F}) = h_S^{B_k}(i) f_j(N,\Gamma,u_T,\mathscr{F}). \tag{10}$$

Similar to the axiomatizations of the Owen value (Owen, 1977) and the Shapley value for games on augmenting systems (Bilbao and Ordonez, 2009), we introduce the following two properties:

- *Efficiency* (***EFF-1***). Let $v \in G(N, \Gamma, \mathscr{F})$. If T is a carrier, then $v(T) = \sum_{i \in T} f_i(N, \Gamma, v, \mathscr{F})$.
- *Hierarchical strength on unions* (***HSU***). Let $v \in G(N, \Gamma, \mathscr{F})$. For any $H \in \mathscr{F}_M$ and all $k, p \in H$,

$$h_H^M(k) \sum_{j \in B_p} f_j(N, \Gamma, u_T, \mathscr{F}) = h_H^M(p) \sum_{i \in B_k} f_i(N, \Gamma, u_T, \mathscr{F}), \tag{11}$$

where $T \in L(N, \Gamma, \mathscr{F})$ with $\bigcup_{l \in H} B_l \in T$.

Remark 1. If there is only one coalition in Γ, then *hierarchical strength in coalitions* degenerates to *hierarchical strength* for games on augmenting system. If all subsets of M and those of each $B_k \in \Gamma$ are both feasible, then *hierarchical strength in coalitions* and *hierarchical strength on unions* degenerate to *symmetry in the unions* and *symmetry in the quotient games* for games with a coalition structure, respectively. It is noteworthy that *hierarchical strength on unions* defined in Meng *et al.* (2016) is different from the above one.

Next, we apply the above listed axioms to show the existence and uniqueness of the quasi-Owen value. First, let us consider the following lemma:

Lemma 2. *Let $v \in G(N, \Gamma, \mathscr{F})$. Then, the quasi-Owen value defined on the unanimity game u_T can be expressed as*:

$$\varphi_i(N, \Gamma, u_T, \mathscr{F}) = \begin{cases} h_R^M(k) h_S^{B_k}(i), & \text{if } i \in S, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

where $T = S \bigcup_{l \in R \in \mathscr{F}_M, k \in R^*} B_l$ and $S \in \mathscr{F}_{B_k} \backslash \varnothing$.

Proof. From the expression of the quasi-Owen value, we have

$$\varphi_i(N, \Gamma, u_T, \mathscr{F}) = \frac{1}{c(M)} \sum_{H \in \mathrm{Ch}(\mathscr{F}_M)} \left(\frac{1}{c(B_k)} \sum_{C \in \mathrm{Ch}(\mathscr{F}_{B_k})} \big(u_T\big(Q \cup C(i)\big) - u_T\big(\big(Q \cup C(i)\big) \backslash i\big)\big)\right). \tag{13}$$

Case (1): If $T \not\subset Q \cup C(i)$, then $u_T(Q \cup C(i)) - u_T((Q \cup C(i)) \backslash i) = 0$.

Case (2): If $T \subseteq Q \cup C(i)$ and $i \notin S$, then $T \subseteq Q \cup C(i)$ implies $T \subseteq (Q \cup C(i)) \backslash i$, and $u_T(Q \cup C(i)) - u_T((Q \cup C(i)) \backslash i) = 0$.

Case (3): If $T \subseteq Q \cup C(i)$ and $i \in S$, we derive $u_T(Q \cup C(i)) - u_T((Q \cup C(i))\backslash i) = 1$. Thus, for every chain $C \in \mathrm{Ch}(\mathscr{F}_{B_k})$ and $H \in \mathrm{Ch}(\mathscr{F}_M)$, we obtain

$$(u_T)_{B_k}\big(C(i)\big) - (u_T)_{B_k}\big(C(i)\backslash i\big) = u_S\big(C(i)\big) - u_S\big(C(i)\backslash i\big) = 1 \tag{14}$$

and

$$u_T^B\big(H(k)\big) - u_T^B\big(H(k)\backslash k\big) = u_{R'}\big(H(k)\big) - u_{R'}\big(H(k)\backslash k\big) = 1, \tag{15}$$

where $S \subseteq C(i)$, $R' \subseteq H(k)$, $R' = R \cup k$ and $u_{R'}(H) = \begin{cases} 1, & R' \subseteq H, \\ 0, & \text{otherwise.} \end{cases}$

Thus,

$$\begin{aligned} &u_T\big(Q \cup C(i)\big) - u_T\big(\big(Q \cup C(i)\big)\backslash i\big) \\ &\quad = \big(u_S\big(C(i)\big) - u_S\big(C(i)\backslash i\big)\big)\big(u_{R'}\big(H(k)\big) - u_{R'}\big(H(k)\backslash k\big)\big). \end{aligned} \tag{16}$$

We get $\varphi_i(N, \Gamma, u_T, \mathscr{F}) = h_R^M(k) h_S^{B_k}(i)$. The result is obtained. □

Theorem 1. *There is a unique solution f defined on $G(N, \Gamma, F)$ that satisfies* ***L, EFF-1, HSC*** *and* ***HSU****.*

Proof. Existence. From Eq. (7), we know that ***L*** holds.

From Definition 4 and Eq. (7), we have $\varphi_i(N, v, \Gamma, \mathscr{F}) = 0$ for any $i \in N\backslash T$. When $i \in T$, let $v^Q(S) = v(Q \cup S) - v(Q)$ for any $S \in \mathscr{F}_{B_k}$, then

$$\begin{aligned} &\sum_{i \in T} \varphi_i(N, v, \Gamma, \mathscr{F}) \\ &\quad = \sum_{i \in N} \varphi_i(N, v, \Gamma, \mathscr{F}) \\ &\quad = \sum_{i \in N} \sum_{\{R \in \mathscr{F}_M : k \in R^*\}} \sum_{\{S \in \mathscr{F}_{B_k} : i \in S^* \wedge i \in B_k\}} \frac{c(R)c(R \cup k, M)}{c(M)} \frac{c(S)c(S \cup i, B_k)}{c(B_k)} \\ &\qquad \times \big(v(Q \cup S \cup i) - v(Q \cup S)\big) \\ &\quad = \sum_{i \in N} \sum_{\{R \in \mathscr{F}_M : k \in R^*\}} \sum_{\{S \in \mathscr{F}_{B_k} : i \in S^* \wedge i \in B_k\}} \frac{c(R)c(R \cup k, M)}{c(M)} \frac{c(S)c(S \cup i, B_k)}{c(B_k)} \\ &\qquad \times \big(v^Q(S \cup i) - v^Q(S)\big) \\ &\quad = \sum_{k \in M} \sum_{\{R \in \mathscr{F}_M : k \in R^*\}} \frac{c(R)c(R \cup k, M)}{c(M)} \sum_{i \in B_k} \sum_{\{S \in \mathscr{F}_{B_k} : i \in S^*\}} \frac{c(S)c(S \cup i, B_k)}{c(B_k)} \\ &\qquad \times \big(v^Q(S \cup i) - v^Q(S)\big) \\ &\quad = \sum_{k \in M} \sum_{\{R \in \mathscr{F}_M : k \in R^*\}} \frac{c(R)c(R \cup k, M)}{c(M)} v^Q(B_k) \end{aligned}$$

$$= \sum_{k \in M} \sum_{\{R \in \mathscr{F}_M : k \in R^*\}} \frac{c(R)c(R \cup k, M)}{c(M)} \big(v^B(R \cup k) - v^B(R)\big)$$
$$= v^B(M) = v(N) = v(T).$$

Thus, ***EFF-1*** holds.

From Lemma 2, we get ***HSC***.

Further, according to Lemma 2 we have

$$\varphi_i(N, \Gamma, u_T, \mathscr{F}) = h_H^M(k) h_S^{B_k}(i) \tag{17}$$

and

$$\varphi_j(N, \Gamma, u_T, \mathscr{F}) = h_H^M(p) h_E^{B_p}(j). \tag{18}$$

From ***EFF-1***, we obtain

$$\sum_{i \in B_k} \varphi_i(N, \Gamma, u_T, \mathscr{F}) = \sum_{i \in B_k} h_H^M(k) h_S^{B_k}(i) = h_H^M(k) \tag{19}$$

and

$$\sum_{j \in B_p} \varphi_j(N, \Gamma, u_T, \mathscr{F}) = \sum_{j \in B_p} h_H^M(p) h_E^{B_p}(j) = h_H^M(p). \tag{20}$$

Thus, ***HSU*** holds.

Uniqueness. From Lemma 1 and ***L***, we only need to prove the uniqueness of Eq. (7) on u_T for any $T \in L(N, \Gamma, \mathscr{F})$ with $T \neq \varnothing$. Let $M' = \{k \in M : B_k \cap T \neq \varnothing\}$ and $B'_k = B_k \cap T$ for any $k \in M$, define the unanimity quotient game u_T^B on Γ as follows:

$$u_T^B(R) = \begin{cases} 1, & M' \subseteq R, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

where $R \subseteq M$.

Let f be a solution on $(N, \Gamma, u_T, \mathscr{F})$ that satisfies the above axioms. From ***EFF-1*** and ***HSU***, we have

$$\sum_{i \in B_k} f_i(N, u_T, \Gamma, \mathscr{F}) = \begin{cases} 0, & k \notin M', \\ h_{M'}^M(k), & k \in M'. \end{cases} \tag{22}$$

For any $k \in M'$, from ***HSC*** we get

$$h_{B'_k}^{B_k}(j) f_i(N, \Gamma, u_T, \mathscr{F}) = h_{B'_k}^{B_k}(i) f_j(N, \Gamma, u_T, \mathscr{F}). \tag{23}$$

From $\sum_{i\in B'_k} h^{B_k}_{B'_k}(i) = 1$, we derive

$$f_i(N,\Gamma,u_T,\mathscr{F}) = \begin{cases} 0, & i \in B_k \backslash B'_k, \\ h^M_{M'}(k) h^{B_k}_{B'_k}(i), & i \in B'_k. \end{cases} \tag{24}$$

According to Lemma 2, we know that f and φ coincide on u_T. □

Similar to Bilbao and Ordonez (2009), we define the identify game for $G(N,\Gamma,\mathscr{F})$ to research another axiomatization of the quasi-Shapley value. For any $S \in L(N,\Gamma,\mathscr{F})\backslash\varnothing$, the identify game $\delta_S : L(N,\Gamma,\mathscr{F}) \to \Re$ is defined as:

$$\delta_S(T) = \begin{cases} 1, & S = T, \\ 0, & \text{otherwise}. \end{cases} \tag{25}$$

Using the identify game, we offer the following axiom which is an extension of *Chain axiom* for games on augmenting systems (Bilbao and Ordonez, 2009).

- *Chain axiom in coalitions* (***CAC***). Let $v \in G(N,\Gamma,\mathscr{F})$. For any $T \in L(N,\Gamma,\mathscr{F})$, without loss of generality, suppose that $T = S \bigcup_{l\in R\in\mathscr{F}_M, k\in R^*} B_l$, where $S \in \mathscr{F}_{B_k}\backslash\varnothing$. Then, for all $i, j \in exS$, we have

$$c(S\backslash i) f_j(N,\Gamma,\delta_S,\mathscr{F}) = c(S\backslash j) f_i(N,\Gamma,\delta_S,\mathscr{F}). \tag{26}$$

From Eq. (7), one can easily check that the quasi-Owen value can be equivalently expressed as:

$$\begin{aligned} &\varphi_i(N,\Gamma,v,\mathscr{F}) \\ &\quad = \sum_{\{R\in\mathscr{F}_M : k\in R^*\}} \sum_{\{S\in\mathscr{F}_{B_k} : i\in exS \wedge i\in B_k\}} \frac{c(R)c(R\cup k,M)}{c(M)} \frac{c(S\backslash i)c(S,B_k)}{c(B_k)} \\ &\qquad \times \big(v(Q\cup S) - v\big((Q\cup S)\backslash i\big)\big). \quad \forall i \in N, \end{aligned} \tag{27}$$

where $exS = \{i \in S : S\backslash i \in \mathscr{F}_{B_k}\}$.

Theorem 2. *There is a unique solution f defined on $G(N,\Gamma,F)$ that satisfies **L**, **EFF-1**, **CAC** and **HSU**.*

Proof. From $u_S = \sum_{S\subseteq T} \delta_T$ for any $S \in L(N,\Gamma,\mathscr{F})\backslash\varnothing$ and Theorem 1, one can easily derive the conclusion. □

Next, let us consider another axiomatization of the quasi-Owen value. Young (1985) proposed a characterization of the Shapley value using *strong monotonicity*. According to Young (1985), we propose *strong monotonicity* in the framework of games on augmenting systems with a coalition structure.

- *Strong monotonicity* (***SM***). Let $v, w \in G(N,\Gamma,\mathscr{F})$. If $v(S\cup i) - v(S) \geqslant w(S\cup i) - w(S)$ for any $S \in L(N,\Gamma,\mathscr{F})\backslash\varnothing$ with $i \in S^*$, then $f_i(N,\Gamma,v,\mathscr{F}) \geqslant f_i(N,\Gamma,w,\mathscr{F})$.

Theorem 3. *There is a unique solution f defined on $G(N, \Gamma, F)$ that satisfies **EFF-1**, **HSC**, **HSU** and **SM**.*

Proof. From Theorem 1 and Eq. (7), it is easy to conclude that φ satisfies these properties. Next, let us consider the uniqueness. From Lemma 1, for any $v \in G(N, \Gamma, \mathscr{F})$ it can be uniquely expressed as:

$$v = \sum_{\varnothing \neq T \in L(N,\Gamma,\mathscr{F})} c_T u_T. \tag{28}$$

Let I be the minimum number of non-zero terms in some expression for v in (5). As in Young (1985), the theorem is proved using induction on I.

(I) If $c_T = 0$ for all $T \in L(N, \Gamma, \mathscr{F})\backslash\varnothing$, it is easy to derive $f_i(N, \Gamma, v, \mathscr{F}) = 0$ for any $i \in N$.

(II) If there is one $T \in L(N, \Gamma, \mathscr{F})\backslash\varnothing$ such that $c_T \neq 0$, we get $v = c_T u_T$. From ***EFF-1***, ***HSC*** and ***HSU***, we derive

$$f_i(N, \Gamma, c_T u_T, \mathscr{F}) = \begin{cases} 0, & i \in B_k\backslash B'_k, \\ c_T h^M_{M'}(k) h^{B_k}_{B'_k}(i), & i \in B'_k, \end{cases} \tag{29}$$

where M' and B' as shown in Theorem 1.

(III) Assume that f is unique whenever the index of v is at most I. Let v have the index $I + 1$ with the following expression:

$$v = \sum_{r=1}^{I+1} c_{T_r} u_{T_r}, \tag{30}$$

where $T_r \in L(N, \Gamma, \mathscr{F})\backslash\varnothing$ such that $c_{T_r} \neq 0$.

Let $T = \bigcap_{r=1}^{I+1} T_r$, for any $i \in N\backslash T$ we construct the game

$$w = \sum_{r:(T_r)_i \neq 0} c_{T_r} u_{T_r}. \tag{31}$$

Then, the index of w is at most I, Because $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for all $S \in L(N, \Gamma, \mathscr{F})$ with $i \in S^*$ and $i \in N\backslash T$. According to induction and ***SM***, we have

$$f_i(N, \Gamma, w, \mathscr{F}) = f_i(N, \Gamma, v, \mathscr{F}) = \begin{cases} 0, & \text{otherwise}, \\ \sum_{r:(T_r)_i \neq 0} c_{T_r} h^M_{M_{T_r}}(k) h^{B_k}_{B^k_{T_r}}(i), & i \in B^{k*}_{T_r}, \end{cases} \tag{32}$$

where $M_{T_r} = \{k \in M : B_k \cap T_{T_r} \neq \varnothing\}$ and $B^k_{T_r} = B_k \cap T_{T_r}$ for any $k \in M$.

On the other hand, for any $i \in T$, by ***EFF-1***, ***HSC*** and ***HSU*** we obtain

$$f_i(N, \Gamma, v, \mathscr{F}) = \sum_{r=1}^{I+1} c_{T_r} h^M_{M_{T_r}}(k) h^{B_k}_{B^k_{T_r}}(i). \tag{33}$$

The conclusion is obtained. □

Next, we will give another two axiomatic systems to characterize the quasi-Owen value from the perspective of the *potential function* and *balanced contributions*, respectively.

Hart and Mas-Colell (1989) first introduced the concept of the *potential function*. Later, Winter (1992) extended the potential function to games with a coalition structure, by which an axiomatic system of the Owen value (Owen, 1977) was characterized. Now, we define the potential function for games on augmenting systems with a coalition structure to characterize the quasi-Owen value.

Definition 5. Let $v \in G(N, \Gamma, \mathscr{F})$. Given a function $P : G(N, \Gamma, \mathscr{F}) \to \Re^m$, where $P^k(\varnothing, \Gamma, v, \mathscr{F}) = 0$. Let $\Gamma = \{B_1, B_2, \ldots, B_m\}$. The marginal contribution of player i to $G(N, \Gamma, v, \mathscr{F})$ is

$$D^i P(N, \Gamma, v, \mathscr{F}) = P(N, \Gamma, v, \mathscr{F}) - P\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big), \tag{34}$$

P is said to be a potential function for the game $G(N, \Gamma, v, \mathscr{F})$ if it satisfies

$$\sum_{i \in B_k} D^i P(N, \Gamma, v, \mathscr{F}) = D^k P\big(M, v^B, \mathscr{F}_M\big) \tag{35}$$

for all $k \in M$, and

$$\sum_{i \in N} D^i P(N, \Gamma, v, \mathscr{F}) = v(N). \tag{36}$$

Using the potential function for games on augmenting systems with a coalition structure, we offer the following theorem for the quasi-Owen value.

Theorem 4. *There is a unique potential function P for the game $G(N, \Gamma, v, \mathscr{F})$. Moreover, for any $v \in G(N, \Gamma, \mathscr{F})$ and any $i \in N$, $D^i P(N, \Gamma, v, \mathscr{F}) = \varphi_i(N, \Gamma, v, \mathscr{F})$, where φ is the quasi-Owen value.*

Proof. Existence. For any $B_k \in \Gamma$ and any $T \in L(N, \Gamma, \mathscr{F})$, let

$$d_T(B_k) = \begin{cases} \frac{c_T}{m' b'_k}, & B_k \cap T \neq \varnothing, \\ 0, & B_k \cap T = \varnothing, \end{cases} \tag{37}$$

where m' and b'_k are the cardinalities of $M' = \{k \in M : B_k \cap T \neq \varnothing\}$ and $B'_k = B_k \cap T$, $k \in M$, respectively, and c_T as shown in Eq. (28).

Let

$$P(N,\Gamma,v,\mathscr{F}) = \Bigg(\sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_1), \sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_2), \dots, \sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_m)\Bigg).$$

Without loss of generality, suppose that $i \in B_k$ for any $i \in T$, we have

$$D^i P(N,\Gamma,v,\mathscr{F}) = \sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_k) - \sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_k\backslash i) = \sum_{\varnothing\neq T\in L(N,\Gamma,\mathscr{F})} d_T(B_k) = \varphi_i(N,\Gamma,v,\mathscr{F}). \tag{38}$$

From $\varphi_k(M, v^B, \mathscr{F}_M) = \sum_{i\in B_k} \varphi_i(N,\Gamma,v,\mathscr{F})$, we have

$$D^k P\big(M, v^B, \mathscr{F}_M\big) = \varphi_k\big(M, v^B, \mathscr{F}_M\big) = \sum_{i\in B_k} \varphi_i(N,\Gamma,v,\mathscr{F}) = \sum_{i\in B_k} D^i P(N,\Gamma,v,\mathscr{F}). \tag{39}$$

From Eq. (38) and ***EFF-2***, P satisfies Eq. (36). Therefore, P is a potential function.

Uniqueness. Note that Eq. (35) can be written as

$$P(N,\Gamma,v,\mathscr{F}) = \frac{1}{b_k}\Bigg(D^k P\big(M, v^B, \mathscr{F}_M\big) + \sum_{i\in B_k} P\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big)\Bigg), \tag{40}$$

where b_k is the cardinality of B_k.

When there is only one union in the coalition structure, we can conclude that there is the Hart-Mas-Colell potential function (Hart and Mas-Colell, 1989) which is known to be unique. By Eq. (35) and Eq. (36), we further derive that P defined on $(M, v^B, \mathscr{F}_M)$ is unique. Then, we can obtain the uniqueness of $P(N,\Gamma,v,\mathscr{F})$ recursively according to Eq. (40) with the initial condition $P(\varnothing,\Gamma,v,\mathscr{F}) = 0$. □

Different to ***EFF-1,*** which is defined in view of carrier, we define the following ***EFF-2*** which will be used in the following two axiomatic systems.

- *Efficiency* (***EFF-2***). Let $v \in G(N,\Gamma,\mathscr{F})$, then $v(N) = \sum_{i\in N} f_i(N,\Gamma,v,\mathscr{F})$.

Next, we consider the last axiomatic system of the quasi-Owen value. Myerson (1980) proposed a characterization of the Myerson value using *Balanced contributions*. Later, Zou *et al.* (2020) presented the *Intracoalitional quasi-balanced contributions with respect to* α and *Coalitional quasi-balanced contributions with respect to* α to characterize the α-Egalitarian Owen value for cooperative games with a coalition structure. Now, we propose *Intra-coalitional balanced contributions* and *Coalitional balanced contributions* for games on augmenting system with a coalition structure as:

- *Intra-coalitional balanced contributions* (***IBC***). Let $v \in G(N, \Gamma, \mathscr{F})$. For any $T \in L(N, \Gamma, \mathscr{F})$, without loss of generality, suppose that $T = S \bigcup_{l \in R \in \mathscr{F}_M, k \in R^*} B_l$, where $S \in \mathscr{F}_{B_k} \backslash \varnothing$. Then

$$f_i(N, \Gamma, v, \mathscr{F}) - f_i\big(N\backslash j, \Gamma\big|_{N\backslash j}, v, \mathscr{F}\big|_{N\backslash j}\big)$$
$$= f_j(N, \Gamma, v, \mathscr{F}) - f_j\big(N\backslash i, \Gamma\big|_{N\backslash i}, v, \mathscr{F}\big|_{N\backslash i}\big) \quad (41)$$

for all $i, j \in S \in \mathscr{F}_{B_k}$ such that $i \neq j$.
- *Coalitional balanced contributions* (***CBC***). Let $v \in G(N, \Gamma, \mathscr{F})$, then

$$\sum_{i \in B_k} f_i(N, \Gamma, v, \mathscr{F}) - \sum_{i \in B_k} f_i\big(N\backslash B_l, \Gamma\big|_{N\backslash B_l}, v, \mathscr{F}\big|_{N\backslash B_l}\big)$$
$$= \sum_{i \in B_l} f_i(N, \Gamma, v, \mathscr{F}) - \sum_{i \in B_l} f_i\big(N\backslash B_k, \Gamma\big|_{N\backslash B_k}, v, \mathscr{F}\big|_{N\backslash B_k}\big) \quad (42)$$

for all $k, l \in R \in \mathscr{F}_M$ such that $k \neq l$.

Theorem 5. *There is a unique solution f defined on $G(N, \Gamma, \mathscr{F})$ that satisfies **EFF-2**, **IBC**, and **CBC**, which equals to φ.*

Proof. Existence. Obviously, φ satisfies ***EFF-2***. From Theorem 4, we know that $\varphi_i(N, \Gamma, v, \mathscr{F}) = D^i P(N, \Gamma, v, \mathscr{F})$, where P is the unique potential function for the game $v \in G(N, \Gamma, \mathscr{F})$. Therefore, for any $T \in L(N, \Gamma, \mathscr{F})$, without loss of generality, suppose that $T = S \bigcup_{l \in R \in \mathscr{F}_M, k \in R^*} B_l$, where $S \in \mathscr{F}_{B_k} \backslash \varnothing$. Then,

$$\varphi_i(N, \Gamma, v, \mathscr{F}) - \varphi_j(N, \Gamma, v, \mathscr{F}) = D^i P(N, \Gamma, v, \mathscr{F}) - D^j P(N, \Gamma, v, \mathscr{F})$$
$$= P^j(N\backslash j, \Gamma, v, \mathscr{F}) - P^j\big(N\backslash i, \Gamma\big|_{N\backslash i}, v, \mathscr{F}\big|_{N\backslash i}\big)$$
$$= P^j\big(N\backslash j, \Gamma\big|_{N\backslash j}, v, \mathscr{F}\big|_{N\backslash j}\big) - P^j\big(N\backslash ij, \Gamma\big|_{N\backslash ij}, v, \mathscr{F}\big|_{N\backslash ij}\big)$$
$$+ P^j\big(N\backslash ij, \Gamma\big|_{N\backslash ij}, v, \mathscr{F}\big|_{N\backslash ij}\big) - P^j\big(N\backslash i, \Gamma\big|_{N\backslash i}, v, \mathscr{F}\big|_{N\backslash i}\big)$$
$$= \varphi_i\big(N\backslash j, \Gamma\big|_{N\backslash j}, v, \mathscr{F}\big|_{N\backslash j}\big) - \varphi_j\big(N\backslash i, \Gamma\big|_{N\backslash i}, v, \mathscr{F}\big|_{N\backslash i}\big)$$

for all $i, j \in S \in \mathscr{F}_{B_k}$ such that $i \neq j$.

Thus, ***IBC*** holds.

Similarly, one can show that ***CBC*** is true.

To prove uniqueness, we just need to show that f admits a potential function. Let $v \in G(N, \Gamma, \mathscr{F})$. For any $T \in L(N, \Gamma, \mathscr{F})$, without loss of generality, suppose that $T = S \bigcup_{l \in R} B_l$, where $S \in \mathscr{F}_{B_k}$ and $R \in \mathscr{F}_M \backslash k$. When $t = 1$ with t being the cardinality of T, by ***EFF-2*** we have $f_i(T, \Gamma|_T, v, \mathscr{F}|_T) = v(i) = D^i P(T, \Gamma|_T, v, \mathscr{F}|_T)$, where P is the unique potential function for the game $G(N, \Gamma, v, \mathscr{F})$. By induction, when $t \leqslant n-1$ $(n \geqslant 2)$, for any $i \in S$, we have

$$f_i\big(T, \Gamma\big|_T, v, \mathscr{F}\big|_T\big) = D^i P\big(T, \Gamma\big|_T, v, \mathscr{F}\big|_T\big)$$
$$= P\big(T, \Gamma\big|_T, v, \mathscr{F}_T\big) - P\big(T\backslash i, \Gamma\big|_{T\backslash i}, v, \mathscr{F}\big|_{T\backslash i}\big). \quad (43)$$

By Eq. (43), we derive

$$\sum_{i\in S} f_i\big(T, \Gamma\big|_T, v, \mathscr{F}_T\big) = \sum_{i\in B_k} f_i\big(T, \Gamma\big|_T, v, \mathscr{F}_T\big) = \sum_{i\in B_k} D^i P\big(T, \Gamma\big|_T, v, \mathscr{F}\big|_T\big). \tag{44}$$

From the definition of the potential function and Eq. (44), we get

$$\sum_{i\in B_k} f_i\big(T, \Gamma\big|_T, v, \mathscr{F}_T\big) = D^k P\big(R\cup k, v^B, \mathscr{F}_M\big|_{R\cup k}\big). \tag{45}$$

Next, we prove the conclusion is true when $t = n$. By ***CBC***, we obtain

$$\sum_{i\in B_k} f_i(N, \Gamma, v, \mathscr{F}) - \sum_{i\in B_k} f_i\big(N\backslash B_l, \Gamma\big|_{N\backslash B_l}, v, \mathscr{F}\big|_{N\backslash B_l}\big)$$
$$= \sum_{i\in B_l} f_i(N, \Gamma, v, \mathscr{F}) - \sum_{i\in B_l} f_i\big(N\backslash B_k, \Gamma\big|_{N\backslash B_k}, v, \mathscr{F}\big|_{N\backslash B_k}\big).$$

By Eq. (45), we derive

$$\sum_{i\in B_k} f_i(N, \Gamma, v, \mathscr{F}) - D^k P\big(M\backslash l, v^B, \mathscr{F}_M\big|_{M\backslash l}\big)$$
$$= \sum_{i\in B_l} f_i(N, \Gamma, v, \mathscr{F}) - D^l P\big(M\backslash k, v^B, \mathscr{F}_M\big|_{M\backslash k}\big)$$

by which we get

$$\sum_{i\in B_k} f_i(N, \Gamma, v, \mathscr{F}) + P\big(M\backslash k, v^B, \mathscr{F}_M\big|_{M\backslash k}\big)$$
$$= \frac{1}{m}\Bigg(\sum_{l\in M}\sum_{i\in B_l} f_i(N, \Gamma, v, \mathscr{F}) + \sum_{l\in M} P\big(M\backslash l, v^B, \mathscr{F}_M\big|_{M\backslash l}\big)\Bigg).$$

According to ***EFF-2***, we derive

$$\sum_{i\in B_k} f_i(N, \Gamma, v, \mathscr{F}) + P\big(M\backslash k, v^B, \mathscr{F}_M\big|_{M\backslash k}\big)$$
$$= \frac{1}{m}\Bigg(v(N) + \sum_{l\in M} P\big(M\backslash l, v^B, \mathscr{F}_M\big|_{M\backslash l}\big)\Bigg).$$

From the concept of the potential function, we obtain

$$\sum_{i\in B_k} f_i(N, \Gamma, v, \mathscr{F}) = P\big(M, v^B, \mathscr{F}_M\big) - P\big(M\backslash k, v^B, \mathscr{F}_M\big|_{M\backslash k}\big)$$
$$= D^k P\big(M, v^B, \mathscr{F}_M\big). \tag{46}$$

By ***IBC***, we get

$$\begin{aligned} & f_i(N, \Gamma, v, \mathscr{F}) - f_j(N, \Gamma, v, \mathscr{F}) \\ & \quad = f_i\big(N\backslash j, \Gamma|_{N\backslash j}, v, \mathscr{F}|_{N\backslash j}\big) - f_j\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big) \\ & \quad = P\big(N\backslash j, \Gamma|_{N\backslash j}, v, \mathscr{F}|_{N\backslash j}\big) - P\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big) \end{aligned}$$

for all $i, j \in S \in \mathscr{F}_{B_k}$.

Then,

$$\begin{aligned} & f_i(N, \Gamma, v, \mathscr{F}) - \frac{1}{b_k} \sum_{j \in B_k} f_j(N, \Gamma, v, \mathscr{F}) \\ & \quad = \frac{1}{b_k} \sum_{j \in B_k} P\big(N\backslash j, \Gamma|_{N\backslash j}, v, \mathscr{F}|_{N\backslash j}\big) - P\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big), \end{aligned} \tag{47}$$

where b_k is the cardinality of B_k.

From Eq. (46), we have

$$\begin{aligned} & f_i(N, \Gamma, v, \mathscr{F}) + P\big(N\backslash i, \Gamma|_{N\backslash i}, v, \mathscr{F}|_{N\backslash i}\big) \\ & \quad = \frac{1}{b_k}\Big(D^k P\big(M, v^B, \mathscr{F}_M\big) + \sum_{j \in B_k} P\big(N\backslash j, \Gamma|_{N\backslash j}, v, \mathscr{F}|_{N\backslash j}\big)\Big). \end{aligned} \tag{48}$$

By Eq. (40), we know that the right hand side of Eq. (48) equals to $P(N, \Gamma, v, \mathscr{F})$. Thus, Eq. (43) is true for $t = n$. Theorem 4 shows that the conclusion is obtained. □

In this subsection, we focus on the axioms of the quasi-Owen value and give five axiomatic systems. These axiomatic systems can be divided into two categories in view of the axiom of linearity. The first two are based on *linearity*, while other three suggest alternative foundations of the quasi-Owen value without *linearity*. It is noteworthy that we can similarly build other axiomatic systems.

4. The Core and the PMASs

In this section, we introduce the core and the PMAS for games on augmenting systems with a coalition structure. Further, the relationship between the quasi-Owen value and the core is discussed, and the conditions for the quasi-Owen value to be a PMAS are given.

4.1. *The Concept of the Core*

In a similar way to the core of games with a coalition structure (Pulido and Sánchez-Soriano, 2009), the definition of the core of games on augmenting systems with a coalition structure is defined as:

Definition 6. Let $v \in G(N, \Gamma, \mathscr{F})$. The core $C(N, \Gamma, v, \mathscr{F})$ of v is defined as:

$$C(N, \Gamma, v, \mathscr{F}) = \left\{ x \in \Re^n \middle| \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geqslant v(S), \forall S \in L(N, \Gamma, \mathscr{F}) \right\}. \tag{49}$$

Now, we investigate some properties of the core, which are extended from reference (Pulido and Sánchez-Soriano, 2009). First, we introduce the concept of reduced games for games on augmenting systems with a coalition structure.

Definition 7. Let $v \in G(N, \Gamma, \mathscr{F})$, and let x be a corresponding solution. For any $S \in L(N, \Gamma, \mathscr{F})$, the reduced game $G(S, \Gamma|_S, v_S^x, \mathscr{F}|_S)$ is defined as:

$$v_S^x = \begin{cases} 0, \quad T = \varnothing \\ v(N) - x(N\backslash T), \quad T = S \\ \max\left\{v(T \cup R) - x(R) : R \subseteq N\backslash S, T \cup R \in L(N, \Gamma, \mathscr{F})\right\}, \\ \quad T \in \mathscr{F}|_S\backslash\{\varnothing, S\}, \end{cases} \tag{50}$$

where $x(R) = \sum_{i\in R} x_i$, $v(N) - x(N\backslash T) = x(T)$ and $v_N^x = v$.

From reduced games, we further offer the following concepts of the coalitional reduced game property (***C-RGP***) and the coalitional converse reduced game property (***C-CRGP***).

Definition 8. Let $v \in G(N, \Gamma, \mathscr{F})$, $S \in L(N, \Gamma, \mathscr{F})\backslash\varnothing$ and σ be a solution. If $x \in \sigma(N, \Gamma, v, \mathscr{F})$ implies $x|_S \in \sigma(S, \Gamma|_S, v_S^x, \mathscr{F}|_S)$, then the solution σ satisfies **C-RGP**. Further, if $x|_S \in \sigma(S, \Gamma|_S, v_S^x, \mathscr{F}|_S)$ means $x \in \sigma(N, \Gamma, v, \mathscr{F})$, then the solution σ satisfies **C-CRGP**, where $x|_S$ denotes the restriction of x to the coalition S.

Lemma 3. *Let $v \in G(N, \Gamma, \mathscr{F})$. The core $C(N, \Gamma, v, \mathscr{F})$ satisfies **C-RGP**.*

Proof. Let $S \in L(N, \Gamma, \mathscr{F})\backslash\varnothing$. For any $x \in C(N, \Gamma, v, \mathscr{F})$ and any $T \in \mathscr{F}|_S$, if $T = S$, then

$$v_S^x(T) = v(N) - x(N\backslash T) = x(T) = x(S).$$

Otherwise,

$$\begin{aligned} v_S^x(T) - x(T) &= \max\left\{v(T \cup R) - x(R) : R \subseteq N\backslash S, T \cup R \in L(N, \Gamma, \mathscr{F})\right\} - x(T) \\ &= \max\left\{v(T \cup R) - x(T \cup R) : R \subseteq N\backslash S, T \cup R \in L(N, \Gamma, \mathscr{F})\right\} \\ &\leqslant 0. \end{aligned}$$

Hence, $v_S^x(T) \leqslant x(T)$. Therefore, the conclusion is true. □

Next, we show that the core satisfies ***C-CRGP***. To do this, let us consider the following lemma.

Definition 9. If for any $k \in M$ and any $i \in B_k$, we have $k \in \mathscr{F}_M$ and $i \in \mathscr{F}_{B_k}$, then we call $(N, \Gamma, \mathscr{F})$ an atomic augmenting system with a coalition structure.

Lemma 4. *Let* $(N, \Gamma, \mathscr{F})$ *be an atomic augmenting system with a coalition structure. For any* $S \in L(N, \Gamma, \mathscr{F})\backslash\{N, \varnothing\}$ *and any* $j \in N\backslash S$ *such that* $S \cup j \in L(N, \Gamma, \mathscr{F})$*, there is a player* $i \in S$ *such that* $\{i, j\} \in L(N, \Gamma, \mathscr{F})$.

Proof. From the assumption, we have $S \cup j \in L(N, \Gamma, \mathscr{F})$. Since $(N, \Gamma, \mathscr{F})$ is atomic, we have $j \in L(N, \Gamma, \mathscr{F})$. Then, there is a chain from j to $S \cup j$. In this chain, there is a set $\{i, j\} \in L(N, \Gamma, \mathscr{F})$ such that $\{j\} \subseteq \{i, j\} \subseteq S \cup j$. Therefore, $i \in S$, which concludes the proof. □

According to Lemma 4, we offer the following proof of ***C-CRGP***.

Lemma 5. *Let* $v \in G(N, \Gamma, \mathscr{F})$*,* $(N, \Gamma, \mathscr{F})$ *be an atomic augmenting system with a coalition structure, and* x *be a solution. If* $x|_S \in \sigma(S, \Gamma|_S, v_S^x, \mathscr{F}|_S)$ *for all* $S \in L(N, \Gamma, \mathscr{F})$ *with* $s = 2$*, then* $x \in \sigma(N, \Gamma, v, \mathscr{F})$*, where* s *is the cardinality of* S.

Proof. If $n \leqslant 2$, the statement obviously holds. Assume that $n \geqslant 3$. We show that the statement still holds for all $S \in L(N, \Gamma, \mathscr{F})$ with $s = 2$. Let $\sum_{i \in N} x_i = x(N) = v(N)$ such that $x|_S \in C(S, \Gamma|_S, v_S^x, \mathscr{F}|_S)$ for all $S \in L(N, \Gamma, \mathscr{F})$ with $s = 2$. From $S \in L(N, \Gamma, \mathscr{F})\backslash\{N, \varnothing\}$ and Lemma 4, we know that there are two players $i \in S$ and $j \in N\backslash S$ such that $\{i, j\} \in L(N, \Gamma, \mathscr{F})$. Since $(N, \Gamma, \mathscr{F})$ is atomic and $x|_{\{i,j\}} \in C(\{i, j\}, \Gamma|_{\{i,j\}}, v^x_{\{i,j\}}, \mathscr{F}|_{\{i,j\}})$, we have $v^x_{\{i,j\}}(i) - x_i \leqslant 0$. Then,

$$\begin{aligned} v^x_{\{i,j\}}(i) - x_i &= \max\big\{v(i \cup R) - x(R) : R \subseteq N\backslash\{i, j\}, R \cup i \in L(N, \Gamma, \mathscr{F})\big\} - x_i \\ &= \max\big\{v(R \cup i) - x(R \cup i) : R \subseteq N\backslash\{i, j\}, R \cup i \in L(N, \Gamma, \mathscr{F})\big\} \\ &\geqslant v(S) - x(S). \end{aligned}$$

Therefore, $v(S) \leqslant x(S)$. □

The following corollary is immediate from Lemma 5, which shows that the core of $v \in G(N, \Gamma, \mathscr{F})$ satisfies the ***C-CRGP*** when $(N, \Gamma, \mathscr{F})$ is atomic.

Corollary 1. *Let* $v \in G(N, \Gamma, \mathscr{F})$*, and* $(N, \Gamma, \mathscr{F})$ *be an atomic augmenting system with a coalition structure. Then, the core* $C(N, \Gamma, v, \mathscr{F})$ *satisfies* ***C-CRGP***.

To build the axiomatic system of the core $C(N, \Gamma, v, \mathscr{F})$, we further review the property of the individual rationality (***IR***): Let $v \in G(N, \Gamma, \mathscr{F})$ and x be a solution. If $x_i \geqslant v(i)$ for any $i \in L(N, \Gamma, \mathscr{F})$, then the solution x owns ***IR***.

From the above analysis, one can check that the core of games on atomic augmenting systems with a coalition structure satisfies: ***EFF-2***, ***IR***, ***C-RGP*** and ***C-CRGP***. In fact, these four properties can characterize the core of games on atomic augmenting systems with a coalition structure.

Theorem 6. *Let $v \in G(N, \Gamma, \mathscr{F})$. If $(N, \Gamma, \mathscr{F})$ is an atomic augmenting system with a coalition structure, then the core $C(N, \Gamma, v, \mathscr{F})$ is the unique solution on v that satisfies* ***EFF-2***, ***IR***, ***C-RGP*** *and* ***C-CRGP***.

Proof. The proof of Theorem 6 is similar to that of Theorem 5.14 in Peleg (1986), hence it is omitted. □

REMARK 2. If there is only one coalition in Γ, then ***C-RGP*** and ***C-CRGP*** degenerate to the reduced game property (***RGP***) and the converse reduced game property (***CRGP***) for games on augmenting system, respectively. If all subsets of M and those of each $B_k \in \Gamma$ are both feasible, then ***C-RGP*** and ***C-CRGP*** degenerate to the corresponding properties for traditional games with a coalition structure, respectively.

Similar to the Owen value for games with a coalition structure, we can prove that the quasi-Owen value for games on augmenting systems with a coalition structure belongs to the core. Based on the work of Pulido and Sánchez-Soriano (2009), we first give the following definition of quasi coalitional strong-convex games.

DEFINITION 10. Let $v \in G(N, \Gamma, \mathscr{F})$. It is said to be quasi coalitional strong-convex if $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$ for any $S, T \in L(N, \Gamma, \mathscr{F})$ such that $S \cup T, S \cap T \in L(N, \Gamma, \mathscr{F})$.

Following the work of Pulido and Sánchez-Soriano (2009), one can conclude that the quasi Owen value belongs to the core of quasi coalitional strong-convex games. However, as the next example shows, even if the game $v \in G(N, v, \Gamma, \mathscr{F})$ is not quasi coalitional strong-convex, the quasi Owen value may still belong to the core.

EXAMPLE 2. Let $N = \{1, 2, 3, 4, 5\}$ be the player set, and $\Gamma = \{B_1, B_2\}$ be a coalition structure on N, where $B_1 = \{1, 2, 3\}$ and $B_2 = \{4, 5\}$. If $\mathscr{F}_{B_1} = \{\varnothing, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}, B_1\}$, $\mathscr{F}_{B_2} = \{\varnothing, \{4\}, \{5\}, B_2\}$ and $\mathscr{F}_M = \{\varnothing, \{1\}, \{2\}, M\}$, then it is an augmenting system with a coalition structure, where

$$\begin{aligned} L(N, \Gamma, \mathscr{F}) = \big\{&\varnothing, \{1\}, \{3\}, \{4\}, \{5\}, \{4, 5\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 4, 5\}, \\ &\{3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \\ &\{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\big\}. \end{aligned}$$

Further, the values of the coalitions are $v(1, 2) = v(1, 2, 3) = v(2, 3, 4, 5) = 1$, $v(1, 4, 5) = v(1, 2, 3, 4) = v(1, 2, 3, 5) = v(1, 2, 4, 5) = 1$, $v(N) = 3$, and $v(S) = 0$ for other coalitions in $S \in L(N, v, \mathscr{F})$.

Notice that this game is not quasi coalitional strong-convex as $v(1,2)+v(1,4,5) > v(1,2,4,5) + v(1)$. However, one can easily check that the quasi Owen value $\varphi(N, v, \Gamma, \mathscr{F}) = \left(\frac{5}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$ is an element in the core according to Definition 6. To fill this gap, we relax the condition and consider the convexity of the coalitions in the same chain.

Definition 11. Let $v \in G(N, \Gamma, \mathscr{F})$. It is said to be a quasi-chain coalitional convex game if for each $k \in R^*$ such that $R \in \mathscr{F}_M$,

$$v\left((S \cup T) \cup \bigcup_{l \in R} B_l\right) + v\left((S \cap T) \cup \bigcup_{l \in R} B_l\right) \geqslant v\left(S \cup \bigcup_{l \in R} B_l\right) + v\left(T \cup \bigcup_{l \in R} B_l\right)$$

for any $S, T \in \mathscr{F}_{B_k}$ such that $S \subseteq T$.

According to Definition 11, one can easily check that the game offered in Example 2 is a quasi-chain coalitional convex. By Definitions 10 and 11, one can conclude that quasi coalitional strong-convex game is quasi-chain coalitional convex game. Similar to classical case, we derive the following theorem.

Theorem 7. *Let $v \in G(N, \Gamma, \mathscr{F})$. If v is quasi-chain coalitional convex, then $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i \in N} \in C(N, \Gamma, v, \mathscr{F})$.*

Proof. For any $S \in L(N, \Gamma, \mathscr{F})$, without loss of generality, suppose that $S = T \cup \bigcup_{k_l \in H} B_{k_l}$, where $T \in \mathscr{F}_{B_{k_p}}$, $H \in \mathscr{F}_M$, and $k_p \in H^*$. Then, there is a compatible ordering from $\varnothing$ to $H \cup k_p$ and a compatible ordering from $\varnothing$ to T. Assume that $\{\varnothing, \{k_1\}, \{k_1, k_2\}, \ldots, H, H \cup k_p, H \cup k_p \cup k_q, \ldots, M\} \in \mathrm{Ch}(M)$ and $\big\{\varnothing, \{i_1^{k_p}\}, \{i_1^{k_p}, i_2^{k_p}\}, \ldots, T, \ T \cup \{i_j^{k_p}\}, \ldots, \{i_1^{k_p}, i_2^{k_p}, \ldots, i_{b_{k_p}}^{k_p}\}\big\} \in \mathrm{Ch}(B_{k_p})$, where b_{k_p} is the cardinality of coalition B_{k_p}. $\big\{\varnothing, \{i_1^{k_l}\}, \{i_1^{k_l}, i_2^{k_l}\}, \ldots, \{i_1^{k_l}, i_2^{k_l}, \ldots, i_{b_{k_l}}^{k_l}\}\big\} \in \mathrm{Ch}(B_{k_l})$ for any $k_l \in M \backslash k_p$. Let

$$\begin{aligned}
x_{i_1^{k_1}} &= v\big(i_1^{k_1}\big),\\
x_{i_2^{k_1}} &= v\big(i_1^{k_1}, i_2^{k_1}\big) - v\big(i_1^{k_1}\big),\\
&\ldots,\\
x_{i_{b_{k_1}}^{k_1}} &= v(B_{k_1}) - v\big(B_{k_1} \backslash i_{b_{k_1}}^{k_1}\big),\\
x_{i_{b_{k_1}+1}^{k_1}} &= v\big(B_{k_1} \cup i_1^{k_2}\big) - v(B_{k_1}),\\
&\ldots,\\
x_n &= v(N) - v\big(N \backslash i_{b_{k_m}}^{k_m}\big).
\end{aligned}$$

It is obvious that $\sum_{i \in N} x_i = v(N)$.

If $T \neq B_{k_p}$, then $N\backslash S = (B_{k_p}\backslash T) \cup \bigcup_{k_l \in M\backslash H} B_{k_l} = \{i_j^{k_p}, i_{j+1}^{k_p}, \ldots, i_{b_{k_p}}^{k_p}\} \cup \bigcup_{k_l \in M\backslash H} B_{k_l}$, where $B_{k_p}\backslash T = \{i_j^{k_p}, i_{j+1}^{k_p}, \ldots, i_{b_{k_p}}^{k_p}\}$. Let $Q = (T \cup i_j^{k_p}) \cup \bigcup_{k_l \in H} B_{k_l}$, then $S \cup Q = S \cup i_j^{k_p}$ and $S \cap Q = Q\backslash i_j^{k_p}$. From the quasi-chain coalitional chain convexity of v, we get $\sum_{i\in S} x_i - v(S) \geqslant \sum_{i \in S\cup i_j^{k_p}} x_i - v(S \cup i_j^{k_p})$.

If $T = B_{k_p}$, then $N\backslash S = \bigcup_{k_l \in M\backslash H} B_{k_l}$. Let $Q = \bigcup_{k_l \in H\cup k_p} B_{k_l} \cup i_1^{k_q}$, where $i_1^{k_q} \in \mathscr{F}_{B_{k_q}}$. Since $k_q \in (H \cup k_p)^*$, by the quasi-chain coalitional chain convexity of v, we get $\sum_{i\in S} x_i - v(S) \geqslant \sum_{i\in S\cup i_1^{k_q}} x_i - v(S \cup i_1^{k_q})$.

By recursive relation, we get $\sum_{i\in S} x_i - v(S) \geqslant \sum_{i\in N} x_i - v(N) = 0$. Thus, $(x_i)_{i\in N} \in C(N, v, \Gamma, \mathscr{F})$. From Eq. (7), we know that $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i\in N}$ is a convex combination of $c(M)\sum_{k\in M} c(B_k)!$ elements in $C(N, v, \Gamma, \mathscr{F})$. Since $C(N, v, \Gamma, \mathscr{F})$ is a convex set, we get $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i\in N} \in C(N, \Gamma, v, \mathscr{F})$. □

The above theorem shows that when games on augmenting systems with a coalition structure are convex, there is no player who can make his own payoff larger than the quasi-Owen value without reducing other players' payoff. Hence, there are no incentive to deviate from this allocation scheme.

4.2. *The Concept of PMASs*

Inspired by Sprumont (1990) who first introduced and studied the concept of PMASs for traditional cooperative games, we here introduce the notion of PMASs for games on augmenting systems with a coalition structure.

Definition 12. Let $v \in G(N, \Gamma, \mathscr{F})$. If the vector $x = (x_i(S))_{i\in S}$ satisfies

(i) $\sum_{i\in S} x_i(S) = v(S)$ for any $S \in L(N, \Gamma, \mathscr{F})$;
(ii) $x_i(S) \leqslant x_i(T)$ for all $i \in S$ and all $S, T \in L(N, \Gamma, \mathscr{F})$ such that $S \subseteq T$; then $x = (x_i(S))_{i\in S}$ is called a PMAS.

Next, we study the conditions under which the quasi-Owen value is a PMAS.

Theorem 8. *Let $v \in G(N, \Gamma, \mathscr{F})$, and $S, T \in L(N, \Gamma, \mathscr{F})$ with $S \subseteq T$. Without loss of generality, suppose that $S = D\bigcup_{l\in R\in\mathscr{F}_M, k\in R^*} B_l$ and $T = Y\bigcup_{l\in Q\in\mathscr{F}_M, k\in Q^*} B_l$, where $R \subseteq Q$ and $D \subseteq Y \in \mathscr{F}_{B_k}$.*

(i) *If $\mathscr{F}_M|_{Q\cup k} = \{P \mid P = U \cup O \in \mathscr{F}_M,$ where $O \subseteq Q\backslash R$ and $U \in \mathscr{F}_M|_{R\cup k}\}$ and $Q = R \cup h$ such that $h \in M\backslash R$, we have*

$$\frac{c(U)c(U\cup l, R\cup k)}{c(R\cup k)} \leqslant \frac{c(U)c(U\cup l, R\cup k\cup h)}{c(R\cup k\cup h)} + \frac{c(U\cup h)c(U\cup h\cup l, R\cup k\cup h)}{c(R\cup k\cup h)}, \tag{51}$$

where $l \in R^*$;

(ii) *If* $\mathscr{F}_{B_k}\big|_Y = \big\{E|\, E = A \cup C \in \mathscr{F}_{B_k},$ *where* $C \subseteq Y \backslash D$ *and* $A \in \mathscr{F}_{B_k}\big|_D\big\}$ *and* $Y = D \cup j$ *such that* $j \in Y \backslash D$, *we obtain*

$$\frac{c(A)c(A \cup i, D)}{c(D)} \leqslant \frac{c(A)c(A \cup i, D \cup j)}{c(D \cup j)} + \frac{c(A \cup j)c(A \cup j \cup i, D \cup j)}{c(D \cup j)}, \tag{52}$$

where $j \in A^*$;

(iii) *If* v *is quasi-chain coalitional convex, then the quasi-Owen value* $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i \in N}$ *is a PMAS.*

Proof. From Eq. (7), one can easily derive the condition (i) in Definition 12. As for the condition (ii), it is proved recursively. Suppose that $S \cup j = T$, where $j \notin S$. From Eq. (7), we have

$$\begin{aligned}
&\varphi_i(S, \Gamma|_S, v|_S, \mathscr{F}|_S) \\
&\quad= \sum_{\{U \in \mathscr{F}_M|_{R\cup k}: k \in U^*\}} \sum_{\{A \in \mathscr{F}_{B_k}|_D : i \in A^*\}} \frac{c(U)c(U \cup k, R \cup k)}{c(R \cup k)} \frac{c(A)c(A \cup i, D)}{c(D)} \\
&\qquad\times \big(v(W \cup A \cup i) - v(W \cup A)\big)
\end{aligned} \tag{53}$$

for any $i \in S$, where $W = \bigcup_{l \in U \in \mathscr{F}_M|_{R\cup k}} B_l$.

Case 1. $T = Y \bigcup_{l \in R \in \mathscr{F}_M, k \in R^*} B_l$, where $Y \in \mathscr{F}_{B_k}$ and $Y = D \cup j$. Then,

$$\begin{aligned}
&\varphi_i(T, \Gamma|_T, v|_T, \mathscr{F}|_T) \\
&\quad= \sum_{\{U \in \mathscr{F}_M|_{R\cup k}: k \in U^*\}} \sum_{\{E \in \mathscr{F}_{B_k}|_{D\cup j} : i \in B^*\}} \frac{c(U)c(U \cup k, R \cup k)}{c(R \cup k)} \frac{c(E)c(E \cup i, D \cup j)}{c(D \cup j)} \\
&\qquad\times \big(v(W \cup E \cup i) - v(W \cup E)\big)
\end{aligned} \tag{54}$$

for any $i \in S$.

By condition (ii), we have $\mathscr{F}_{B_k}\big|_Y = \mathscr{F}_{B_k}\big|_{D\cup j} = \{E|\, E = A \vee A \cup j \in \mathscr{F}_{B_k}$, where $A \in \mathscr{F}_{B_k}\big|_D\}$. By the property A2, we have $i \in (A \cup j)^*$ for any $i \in A^*$. Then, Eq. (55) can be written as:

$$\begin{aligned}
&\varphi_i(T, \Gamma|_T, v|_T, \mathscr{F}|_T) \\
&= \sum_{\{U \in \mathscr{F}_M|_{R\cup k}: k \in U*\}} \sum_{\{A \in \mathscr{F}_{B_k}|_D : i \in A*\}} \Bigg[\frac{c(U)c(U \cup k, R \cup k)}{c(R \cup k)} \frac{c(A)c(A \cup i, D \cup j)}{c(D \cup j)} \\
&\quad\times \big(v(W \cup A \cup i) - v(W \cup A)\big)
\end{aligned}$$

$$+\frac{c(U)c(U\cup k,R\cup k)}{c(R\cup k)}\frac{c(A\cup j)c(A\cup j\cup i,D\cup j)}{c(D\cup j)}$$
$$\times\big(v(W\cup A\cup j\cup i)-v(W\cup A\cup j)\big)\Big]. \tag{55}$$

According to the condition (iii) we obtain

$$v(W\cup A\cup i)-v(W\cup A)\leqslant v(W\cup A\cup j\cup i)-v(W\cup A\cup j). \tag{56}$$

By condition (ii), we have

$$\frac{c(U)c(U\cup k,R\cup k)}{c(R\cup k)}\frac{c(A)c(A\cup i,D)}{c(D)}$$
$$\leqslant\frac{c(U)c(U\cup k,R\cup k)}{c(R\cup k)}$$
$$\times\left(\frac{c(A)c(A\cup i,D\cup j)}{c(D\cup j)}+\frac{c(A\cup j)c(A\cup j\cup i,D\cup j)}{c(D\cup j)}\right). \tag{57}$$

According to Eqs. (54), (56) to (58), we derive $\varphi_i(S,\Gamma|_S,v|_S,\mathscr{F}_S)\leqslant\varphi_i(T,\Gamma|_T,v|_T,\mathscr{F}|_T)$ for all $i\in S$.

Case 2. $T=D\bigcup_{l\in Q\in\mathscr{F}_M,k\in Q^*}B_l$, where $Q=R\cup h$ and $B_h=\{j\}$. From Eq. (7), we have

$$\varphi_i(T,\Gamma|_T,v|_T,\mathscr{F}|_T)$$
$$=\sum_{\{U\in\mathscr{F}_M|_{R\cup k\cup h}:k\in P^*\}}\sum_{\{A\in\mathscr{F}_{B_k}|_D:i\in A*\}}\frac{c(U)c(U\cup k,R\cup k\cup h)}{c(U\cup k\cup h)}\frac{c(A)c(A\cup i,D)}{c(D)}$$
$$\times\big(v(W\cup A\cup i)-v(W\cup A)\big) \tag{58}$$

for any $i\in S$, where $W=\bigcup_{l\in U\in\mathscr{F}_M|_{R\cup h}}B_l$.

By condition (i), we obtain $\mathscr{F}_M|_{Q\cup k}=\{P|\,P=U\vee U\cup h\in\mathscr{F}_M$, where $U\in\mathscr{F}_M|_{R\cup k}\}$. By the property A2, we have $k\in(U\cup h)^*$ for any $k\in U^*$. Then, Eq. (59) can be written as:

$$\varphi_i(T,\Gamma|_T,v|_T,\mathscr{F}|_T)$$
$$=\sum_{\{U\in\mathscr{F}_M|_{R\cup k}:k\in U^*\}}\sum_{\{A\in\mathscr{F}_{B_k}|_D:i\in A^*\wedge i\in B_k\}}\Bigg[\frac{c(U)c(U\cup k,R\cup k\cup h)}{c(R\cup k\cup h)}\frac{c(A)c(A\cup i,D)}{c(D)}(v(W\cup A\cup i)$$
$$-v(W\cup A))+\frac{c(U\cup h)c(U\cup k\cup h,R\cup k\cup h)}{c(R\cup k\cup h)}\frac{c(A)c(A\cup i,D)}{c(D)}$$
$$\times\big(v(W\cup j\cup A\cup i)-v(W\cup j\cup A)\big)\Bigg]. \tag{59}$$

By conditions (i) and (iii), we have $\varphi_i(S,\Gamma|_S,v|_S,\mathscr{F}_S)\leqslant\varphi_i(T,\Gamma|_T,v|_T,\mathscr{F}|_T)$ for all $i\in S$.

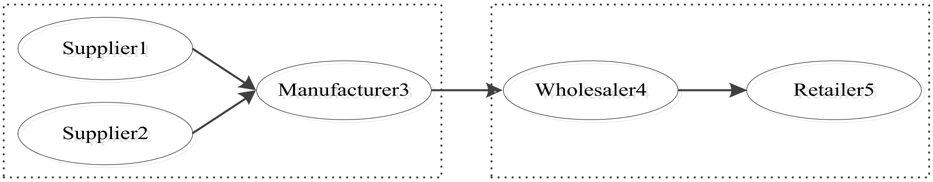

Fig. 1. The model of the food supply chain.

Hence, $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i\in N}$ is a PMAS. □

Remark 3. If there is only one coalition in Γ, we get the conditions for the Shapley value for games on augmenting systems to be a PMAS. If all subsets of M and those of each $B_k \in \Gamma$ are feasible, the three conditions in Theorem 8 reduce to the condition for the Owen value for games with a coalition structure to be a PMAS.

5. An Illustrative Example

In this section, we provide an application of games on augmenting system with a coalition structure in the food supply chain. Set up a supply chain consisting of food raw material supplier, food packaging supplier, food processing manufacturer, wholesaler and retailer. For the convenience of expression, the above members are set as 1, 2, 3, 4 and 5, respectively. The model of the food supply chain is shown in Fig. 1.

In this food supply chain, to gain more profits with lower cost, companies 1, 2 and 3 decide to cooperate and form the production union $\{1, 2, 3\}$ denoted as B_1, and companies 4 and 5 decide to cooperate and form the sales union $\{4, 5\}$ denoted as B_2. Because their skill levels and working procedures are different, they cannot cooperate freely. For example, the wholesaler 4 and retailer 5 carry on their work only after suppliers 1, 2 and manufacturer 3 have finished their production. Thus, when B_1 and B_2 cooperate, the coalitions that can be formed are $\varnothing$, $\{B_1\}$ and $\{B_1, B_2\}$. For the same reason, the coalitions formed by the companies 4 and 5 are $\varnothing$, $\{4\}$ and B_2. In the union $\{1, 2, 3\}$, the coalitions $\{1, 3\}$ and $\{2, 3\}$ can be formed for production. In addition, 1 and 2 are the suppliers of 3, which have a competitive relationship with each other, so the supply coalition $\{1, 2\}$ can't be formed. However, the coalition $\{1, 2, 3\}$ can be formed since the participation of 3. In conclusion, the formed coalitions are $\varnothing$, $\{1\}$, $\{2\}$, $\{1, 3\}$, $\{2, 3\}$ and B_1. The coalition values (million dollars/week) are offered as shown in Table 1. Thus, v is a game on augmenting system with a coalition structure denoted by $G(N, \Gamma, v, \mathscr{F})$, where $\Gamma = \{B_1, B_2\}$, $\mathscr{F}_{B_1} = \{\varnothing, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, B_1\}$, $\mathscr{F}_{B_2} = \{\varnothing, \{4\}, B_2\}$ and $\mathscr{F}_M = \{\varnothing, \{B_1\}, \{B_1, B_2\}\}$.

From Eq. (7), the quasi-Owen values of enterprises are

$$\varphi_1(N, \Gamma, v, \mathscr{F}) = 3.5, \quad \varphi_2(N, \Gamma, v, \mathscr{F}) = 4.5, \quad \varphi_3(N, \Gamma, v, \mathscr{F}) = 5,$$
$$\varphi_4(N, \Gamma, v, \mathscr{F}) = 4, \quad \varphi_5(N, \Gamma, v, \mathscr{F}) = 3.$$

Table 1
The coalition values (million dollars/week).

S	$v(S)$	(S)	
$\varnothing$	0	$\{2, 3\}$	9
$\{1\}$	3	$\{1, 2, 3\}$	13
$\{2\}$	4	$\{1, 2, 3, 4\}$	17
$\{1, 3\}$	8	$\{1, 2, 3, 4, 5\}$	20

It is apparent that v is quasi-chain coalitional convex. From Eq. (49), the core is defined as:

$$C(N, \Gamma, v, \mathscr{F}) = \left\{(x_1, x_2, x_3, x_4, x_5) \middle| \sum_{i=1}^{5} x_i = 20, x_1 \geqslant 3, x_2 \geqslant 4, x_1 + x_3 \geqslant 8, x_2 + x_3 \geqslant 9 \right.$$
$$\left. x_1 + x_2 + x_3 \geqslant 13, x_1 + x_2 + x_3 + x_4 \geqslant 17 \right\}.$$

One can easily check that $(\varphi_i(N, \Gamma, v, \mathscr{F}))_{i \in N} \in C(N, \Gamma, v, \mathscr{F})$. It shows that no enterprise can make its own payoff greater than the quasi-Owen value without reducing other players' payoffs. Namely, the quasi-Owen value is one of the best rule to distribute the coalition payoffs of the companies in the food supply chain.

This example shows that $L(N, \Gamma, \mathscr{F}) = \{\varnothing, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}\}$. For all $S, T \in L(N, \Gamma, \mathscr{F})$, such that $S \subseteq T$, one can check that the conditions in Theorem 8 are satisfied. Further, one can show that the quasi-Owen value is a PMAS. For example, when $S = \{1, 3\}$ and $T = \{1, 2, 3\}$, we have

$$\varphi_1(S, \Gamma|_S, v|_S, F|_S) = 3 \leqslant \varphi_1(T, \Gamma|_T, v|_T, F|_T) = 3 \quad \text{and}$$
$$\varphi_3(S, \Gamma|_S, v|_S, F|_S) = 3 \leqslant \varphi_3(T, \Gamma|_T, v|_T, F|_T) = 5.$$

From the above results, we know that each enterprise can get more payoffs from the larger coalitions than from the smaller coalitions.

6. Conclusion

From the relationships among augmenting system, antimatroid and convex geometry (Bilbao and Ordonez, 2009), one can easily check that when augmenting systems on a coalition structure and on each union are closed under intersection, they turn to games on convex geometries with a coalition structure. Further, when augmenting systems on a coalition structure and on each union are closed under union, they become games on antimatroids with a coalition structure. It is noteworthy that the power set is also an augmenting system. Thus, game on augmenting systems with a coalition structure is an extension of game with a coalition structure.

The above relationships about different types of games show that the quasi-Owen value can be seen as a payoff index for them under the corresponding special conditions. Further, all listed axiomatic systems still hold for the quasi-Owen value in the setting of the above mentioned cooperative games, where axiomatic systems are defined under the associated conditions. This paper only studies a special kind of games under precedence constraints with a coalition structure, and it will be interesting to take into account other types of games under precedence constraints. Moreover, similar to the offered numerical example, we can apply the quasi-Owen value into other practical cooperative cases.

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