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# STABILITY AND THE MATRIX LYAPUNOV EQUATION FOR DIFFERENTIAL SYSTEMS WITH POINT, DISTRIBUTED AND/OR MIXED POINT-DISTRIBUTED DELAYS

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**Abstract.** This paper establishes sufficient conditions for stability of linear and time-invariant delay differential systems including their various usual subclasses (i.e., point, distributed and mixed point-distributed delay systems). Sufficient conditions for stability are obtained in terms of the Schur's complement of operators and the frequency domain Lyapunov equation. The basic idea in the analysis consists in the use of modified Laplace operators which split the characteristic equation into two separate multiplicative factors whose roots characterize the system stability. The method allows a simple derivation of stabilizing control laws.

**Key words:** Lyapunov theory, stability, systems with delays.

**1. Introduction.** Delay differential systems are continuous systems with a time delay. The usual class of systems includes those involving point and/or distributed delays. Delays can be finite or infinite. A typical example of distributed infinite delays is the so called Volterra equation (Burton, 1985). The stability of the above kinds of systems is exhaustively investigated in Burton (1985) through the use of Lyapunov's stability theory. A major problem in the analysis of linear control systems with time delay is their stabilization using linear feedback with or without memory. Most of the stabilization schemes are complicated when compared with those for systems without delay; some of them require solving the matrix Riccati equation (Kwon and Pearson, 1977) or a trascendental equation (Osipov, 1965). Others involve transforming the system equation in a canonical form (Ikeda and Ohta, 1976). In Mori *et al.* (1983) a simple method is derived to stabilize linear systems with internal point delay by memoryless linear feedback. Stabilizability can be checked by inspecting the negativity of a symmetric matrix containing two free parameters so that, subsequently, a stabilizing feedback law can be composed from the solution of a matrix Lyapunov equation containing these parameters. In Agathoklis and Foda (1989), state-space models for 2-D (two-dimensional) and $n$-D ($n$-dimensional) systems were used to describe delay differential systems with commensurate and non-commensurate delays, respectively. Sufficient conditions for the asymptotic stability independent of delay are derived in terms of frequency-dependent Lyapunov's equations. The objetive of this paper is to investigate the stability of the various typical classes of delay systems by using frequency-dependent Lyapunov equations and a decomposition of the closed-loop characteristic equation in (at least) two parts related to modified Laplace (or mixed Laplace and discrete) operators. Both parts of the characteristic equation arise in a natural way from the appearance of both the Laplace operator "s" and the delay operator "exp(-sh)" in such an equation.

The paper is organozed as follows. Section 2 is devoted to obtain sufficient conditions for stabilizability for the case of point delay systems. Some particular stabilizing control laws are obtained. Section 3 presents a similar stability analysis technique for distributed delay systems and mixed point-distributed delay systems. Extensions are given in Section 4 for these systems in the context of differential difference representations. The case of commensurate delays is also included by using a 2-D state representation. Section 5 discusses the connections between the so-called continuous (strictly) bounded real and the (strictly) positive real lemmas and the stability conditions of the above sections. Section 6 is devoted to present abbreviately an iterative computational method related to some basic operators presented in Section 2 for alternative tests of stability. Finally, conlusions end the paper.

**Notation.**

- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of integer, real and complex numbers;
- $\mathbb{Z}_0$, $\mathbb{R}_0$, $\mathbb{C}_0$ are the above sets excluding zero;
- $T = \{z \in \mathbb{C} : |z| \geqslant 1\}$, i.e., the unit circle; $T_1 = T - \{1\}$;
- $U = \{z \in \mathbb{C} : |z| \geqslant 1\}$, i.e., the complex complement of the open unit disc; $U_1 = U - \{1\}$;
- $D = \{s \in \mathbb{C} : s \geqslant 0\}$, i.e., the closed complex right-half plane; $D_0 = D - \{0\}$.
- $\langle \cdot , \cdot \rangle$ denotes the inner product in a Hilbert space;
- p.d.h. (or $> 0$) stands for positive definite hermitian operators; p.d.s. for positive hermitian operators; "$\geqslant 0$" stands for semidefinite positive hermitian operators, and n.d.s. means negative semidefinite symmetric;
- superscript "-" stands for complementation of sets;
- Det$(.)$ is the determinant of the $(.)$-matrix and $\lambda_{\max}(.)$ is its maximum eigenvalue. $A^*$ and $A^T$ are the conjugate transpose and transpose of $A$;
- $I$ is the identity matrix; $I_n$ is the $n$-identity matrix;
- $L^1(a,b)$, $L^2(a,b)$ are the spaces of integrable and square-integrable functions on the real interval $(a,b)$, respectively; $l^2(.)$ stands for square summable sequences.

## 2. Point delay systems

**2.1. Stability.** Consider the linear and time-invariant system

$$\begin{aligned}(S_p) : \dot{x}(t) &= Ax(t) + A_0 x(t-h);\\ x(t) &= \varphi(t),\ t \in [-h, 0),\end{aligned} \tag{1}$$

where $h > 0$ is a point delay, $\varphi : [-h, 0] \to \mathbb{R}^n$ is an absolutely continuous function of initial conditions for the $n$-vector $x$, and $A$ and $A_0$ are $n \times n$-matrices of constant entries. A unique solution on $(0, \infty)$ exists for (1) for each function $\varphi$ (De la Sen, 1988). System (1) is asymptotically stable iff

$$E(s) = \text{Det}(sI - A - e^{-hs}A_0) \neq 0, \quad \forall s \in D \tag{2.a}$$

or, equivalently

$$E(0) = \mathrm{Det}(A + A_0) \neq 0; \quad E(s) \neq 0, \quad \forall s \in D_0. \quad (2.b)$$

The next results stands.

**Lemma 1.** *The following proposition are true.*

(i) *The next identity holds:*

$$sI - A - e^{-hs}A_0 = (I - z_1 A_1)\big(z_2 I - S(z_1)\big), \qquad (3.a)$$

*for all nonzero* $s \in \mathbb{C}$ *with* $z_1 = s^{-1}$ *(for* $s \neq 0$*),* $z_2 = s$ *and* $A_1$ *being an arbitrary constant* $n \times n$*-matrix, and*

$$S(z_1) = S\left(\frac{1}{s}\right) = s(A_1 - sI)^{-1}(A_1 - A - e^{-hs}A_0), \quad (3.b)$$
$$\forall s \in \mathbb{C}_0;$$

(ii) *The next identity holds:*

$$sI - A - zA_0 = \big(I - z\,Z(s)\big)(sI - A), \text{ all } s \in \mathbb{C},\ z = e^{-sh}, \quad (4.a)$$

*where*

$$Z(s) = A_0(sI - A)^{-1}. \qquad (4.b)$$

*Proof.* (i) Note that identity (3.a) becomes

$$-\big(A + e^{-hs}A_0\big) = -A_1 + \left(\frac{1}{s}A_1 - I\right)S\left(\frac{1}{s}\right) \qquad (5)$$

for $z_1 = 1/s$; $z_2 = s$. The solution $S(s^{-1})$ to (5) is (3.b) and the proof is complete.

(ii) identity (4.a) follows from direct substitution of (4.b) in its right-hand-side.

Note that Lemma 1 also stands for any matrix $A_1(s)$ of complex entries. Eq. 2.b together with Lemma 1 yield:

**Lemma 2.** *The following propositions hold.*

(i) *System $S_p$, Eq. 1, is asymptotically stable (in the sence that $\|x(t)\|$ is bounded on $(0, \infty)$ and $\lim_{t\to\infty} x(t) = 0$) if the two following conditions hold:*
  (1) $E(0) \neq 0$,
  (2) $Det\big(z_2 I - S(z_1)\big) \neq 0,\ \forall (z_1, z_2) \in D_0 \times D_0$ (6)
  *for any arbitrary (strictly) Hurwitz matrix $A_1$.*

(ii) *Proposition* (i) *can be equivalently ennounced under the conditions*
  (1) $E(0) \neq 0$,
  (2) $\overline{S}(s) = s^{-1}(A_1 - sI)S(s^{-1}) = (A_1 - A - e^{-hs}A_0)$ *has no eigenvalues in $\overline{D}_0$, $s \neq 0$, for any strictly Hurwitz matrix $A_1$ (i.e., $A_1$ has all eigenvalues in $\overline{D}_0$).*

(iii) *System $S_p$ is asymptotically stable if $A$ is strictly Hurwitz; $E(0) \neq 0$ and $Z(s)$, Eq.* 4.b, *has all its eigenvalues in $\overline{U}_1$ for all $s \in \mathbb{R}_0$.*

*Proof.* Proposition (i) follows directly from (2)–(4) (in fact $E(0) \neq 0$ means that the characteristic equation has no zero eigenvalues). Proposition (ii) follows by noting that if $A_1$ is strictly Hurwitz then $\overline{S}(s)$ has eigenvalues in $\overline{D}_0$ iff $S(s)$ has eigenvalues in $\overline{D}_0$ and the fact that direct calculus using Schur's formula (Agathoklis and Foda, 1989) yields:

$$
\begin{aligned}
&\mathrm{Det}\begin{bmatrix} I - z_1 A_1 & -z_1(A_1 - A - e^{-h/z_1}A_0) \\ -z_1 I & z_2 I \end{bmatrix} \\
&= \mathrm{Det}(I - z_1 A_1)\mathrm{Det}\big(z_2 I - S(z_1)\big) \qquad (7) \\
&= \mathrm{Det}(I - z_1 A_1)\mathrm{Det}\big[z_2 I + (I - z_1 A_1)^{-1}(A_1 - A - e^{-h/z_1}A_0)\big]
\end{aligned}
$$

for $z_1 \neq 0$ which, according to (3.a), is equal to $E(s)$ for $s \neq 0$. The proof of (ii) is complete by introducing Condition 1. Proposition (iii) follows directly from equating the roots of both sides of (5).

REMARK 1. Note from Lemma 2 that if $A_1 = A$ (strictly Hurwitz), then system $S_p$, Eq. 1, is asymptotically stable if $A_0$ is strictly Hurwitz. By extending Lemma 2 to the use of a complex matrix $A_1(s)$, note also that Conditions 1–2 of (ii) contain the general asymptotic stability condition for $S_p$, namely $A_1(s) = A + e^{-hs}A_0$ has eigenvalues in $\overline{D}_0$ and Det$(A + A_0) \neq 0$. Note also that the condition $E(0) = -$Det$(A + A_0) \neq 0$ in Lemma 2 means that there are no zero roots in the characteristic equation of $S_p$. The independent test for existence of zero roots or unstable roots in the open right-half complex plane given by Conditions 1–2 in Lemma 2 (i)–(ii) is motivated by the facts that $S(z_1)$, Eq. 3.b, cannot be defined for $s = 0$, and, furthermore, such a matrix is a key definition in the computation of Schur's complement.

### 2.2. Stabilizing control laws

**2.2.1. Free external delay system.** Assume that $S_p$, Eq. 1, is now forced as follows:

$$(S_p'): \ \dot{x}(t) = Ax(t) + A_0x(t-h) + Bu(t), \qquad (8)$$

where $u \in \mathbb{R}^m$ is the control vector and $B \in \mathbb{R}^{n\times m}$ is the control matrix. Lemma 2 and Remark 1 directly imply the next result.

**Lemma 3.** *System $S_p'$, Eq. 8, is asymptotically Lyapunov stable for the control law*

$$u(t) = Kx(t) + K_0x(t-h), \qquad (9)$$

*where $K$ and $K_0$ are $m \times n$ constant matrices if Lemma 2 stands with the changes $A \rightarrow (A + BK)$ and $A_0 \rightarrow (A_0 + BK_0)$. In particular, the closed-loop system$S_p'$, Eqs. 8–9, is asymptotically stable if $(A+BK)$ and $(A_0+BK_0)$ are strictly Hurwitz matrices. Stabilizing matrices $K$ and $K_0$ exist provided that $(A, B)$ and $(A_0, B)$ are stabilizable pairs.*

Note that a necessary condition for closed-loop asymptotic stability of $S_p'$ is its open-loop stabilizability, namely, rank $[sI - A - e^{-hs}A_0 : B] = n$, all $s \in D$. Note also that the stabilizability of the $(A, B)$ and $(A_0, B)$ pairs reffered to in Lemma 3 means that the linear undelayed systems $\dot{z}_1 = Az_1 + Bu_1$; $\dot{z}_2 = A_0z_2 + Bu_2$ are both stabilizable under linear feedback controls $u_1 = Kz_1,\ u_2 = K_0z_2$ for some constant matrices $K$ and $K_0$.

REMARK 2. Note from Lemma 3 that the closed-loop system $S_p'$, Eq. 8, is asymptotically stable for some linear control law $u = K_0x(t-h)$ if $A$ is strictly Hurwitz and $(A_0, B)$ is stabilizable. It is also asymptotically stable for some $K$ and the control law $u = Kx(t)$ provided that $A_0$ is strictly Hurwitz and the pair $(A, B)$ is stabilizably under such a control law. It suffices that $(A_0 + BK_0)$ and $(A + BK)$ be strictly Hurwitz matrices, respectively.

**2.2.2. External delay system.** Now, system $S_p'$, Eq. 8, is modified as follows

$$(S_p''):\ \dot{x}(t) = Ax(t) + A_0x(t-h) + Bu(t) + B_0u(t-h'), \quad (10)$$

where $h$ and $h'$ are internal and external positive delays. This system subjected to the control law (9) becomes

$$\begin{aligned}\dot{x}(t) = (A + BK)x(t) + (A_0 + BK_0)x(t-h)\\ + B_0Kx(t-h') + B_0K_0x(t-h-h'). \quad (11)\end{aligned}$$

Thus Lemmas 1–2 can be extended to the use of a complex $A_1(s)$. In Lemma 2 (ii), Condition (1) becomes $E(0) = \text{Det}[A + A_0 + (B + B_0)(K + K_0)] \neq 0$ while Condition 2 becomes $\overline{S}(s) = A_1(s) - (A+BK) - e^{-hs}(A_0 + BK_0) + B_0[e^{-h's}K + e^{-(h+h')s}K_0]$ has no eigenvalues in $D_0$ for any $A_1(s)$ of eigenvalues in $\overline{D}_0$. These conditions include the closed-loop stability condition, namely:

$\text{Det}(A_1(s)) = \text{Det}[A + BK + e^{-hs}(A_0 + BK_0) + e^{-h's}B_0[K + e^{-hs}K_0]] \neq 0$, all $s \in D_0$, which includes several particular cases. For instance,

a) If the system with internal (i.e., in the state) delay is asymptotically stable, namely: $\text{Det}[A + A_0 + B(K + K_0)] \neq 0$ and $[A + BK + e^{-hs}(A_0 + BK_0)]$ has all its eigenvalues in $\overline{D}$ (make $h' = 0,\ B_0 = 0$ in (10)), then choose $A_1(s) = A + BK + e^{-hs}(A_0 + BK_0)$ (Lemma 2 (ii)) which has all its eigenvalues in $\overline{D}_0$. Thus, the current system is asymptotically stable if $\overline{S}(s) = -B_0[K + e^{-hs}K_0]$ has all its eigenvalues in $\overline{D}_0$ or, equivalently, if all the eigenvalues of $B_0[K + e^{-hs}K_0]$ are in $D_0$. In summary, if the pairs $(A, B)$ and $(A_0, B)$ are stabilizable and $K$ and $K_0$ are chosen such that $(A_0 + BK_0)$ are strictly Hurwitz, then system (11) (obtained from $S_p''$ with the control law (9)) is asymptotically stable provided that the following holds:

1) $\mathrm{Det}[A + (B + B_0)(K + K_0)] \neq 0$.

2) $B_0[K + e^{-hs}K_0]$ has all its eigenvalues in $D_0$. In fact, notice that if $A+BK)$ and $(A_0+BK_0)$ are strictly Hurwitz, then $S_p'$, subjected to (9), i.e., the free external dalay system $(h' = 0,\ B_0 = 0$ in (10)) is asympptotically stable according to Lemma 3. In other words, the asymptotic stability of $(S_p'')$ subjected to (9) is guaranteed if $(S_p')$, subjected to (9), is asymptotically stable under the sufficient condition $(A+BK)$ and $(A_0+BK_0)$ are strictly Hurwitz and $\dot{z}(t) = B_0[Kz(t)+K_0z(t-h)]$ is asymptotically stable.

b) In the case (a), a sufficient condition for asymptotic stability of $\dot{z}(t) = B_0[Kz(t) + K_0z(t - h)]$ is that $B_0K$ and $B_0K_0$ are both strictly Hurwitz. This follows from Lemma 2 (ii). Note that a necessary condition for the absence of zero eigenvalues in these two matrices is that the input dimension and the state dimension are coincident, i.e., $m = n$.

**2.3. Frequency-dependent Lyapunov equations.** The stability conditions of Lemma 2 (subsequent results are centered in the implications Lemma 2 (ii)) are now interpreted in terms of frequency-dependent Lyapunov's equations as follows.

**Theorem 1.** (Stability from frequency-dependent continuous and discrete Lyapunov equations). *The next two propositions hold:*

(i) $S_p$, *Eq.* 1, *is asymptotically stable if the following holds:*
   (i1) $E(0) \neq 0$;
   (i2) $\overline{S}(s)$, *defined in Lemma* 2 (ii), *is a stability (or Hurwitz) matrix with respect to* $\overline{D}_0$, *that is, for any positive definite hermitian (p.d.h.) matrix* $Q(s)$, *there exists a unique*

*$P(s)$ p.d.h. such that*

$$\overline{S}^*(s)\widetilde{P}(s) + \widetilde{P}(s)\overline{S}(s) = -Q(s), \text{ all } s \in \mathbb{R}_0, \quad (12.a)$$

*where $\overline{S}^*(jw) = \overline{S}^T(-jw)$, or, equivalently, $Z(s)$ defined in* (4.b) *(and referred to in Lemma* 2 (iii)*) is a stability matrix with respect to $\overline{U}_1$, that is, for any p.d.h. $Q(s)$, there is a unique p.d.h. $\widetilde{P}(s)$ such that*

$$P(s) - Z^*(s)P(s)Z(s) = Q(s), \text{ all } s \in R_0, \quad (12.b)$$

*where $Z^*(j\omega) = Z^T(-j\omega)$.*

(ii) *Assume that $E(0) \neq 0$ and $\sup_{s\in\mathbb{R}_0}(\lambda_{\max}|Z^*(s)Z(s)|) < 1$. Let $C(s) \geqslant 0$, $s \in \mathbb{R}_0$ (a semidefinite positive matrix in $\mathbb{R} - \{0\}$) be arbitrary and of $n$-order. Then, there exists a positive matrix $X(.) \in \mathbb{R}^{n\times m}$, all $s \in \mathbb{R}_0$, such that*

$$\begin{aligned} F(C) &= \begin{bmatrix} C & Z^* \\ Z & X \end{bmatrix} \geqslant 0; \\ X(s) &= GS\left(\begin{bmatrix} C & Z^* \\ Z & X \end{bmatrix}\right) \\ &= C(s) - Z^*(s)X^{-1}(s)Z(s) \end{aligned} \quad (13)$$

*if and only if $\overline{Z}_\infty(s) > 0$, which case $X(s) = GS(\overline{Z}_\infty(s))$, all $s \in \mathbb{R}_0$, is a solution where $GS(.)$ denotes the generalized Schur complement and $\overline{Z}_n(s)$ is given by*

$$\overline{Z}_n(s) = \begin{bmatrix} C & Z^* & & & \\ Z & C & Z^* & & 0 \\ & Z & & & \\ 0 & & Z & C & Z^* \\ & & & Z & C \end{bmatrix}; \quad (14)$$

*$n \in \mathbb{Z}$ is greater than unity,*

*where $\overline{Z}_{\infty}(.)$ is the block tridiagonal matrix of infinite blocks whose diagonal entries are all $A$, whose subdiagonal and superdiagonal entries are $Z$ and $Z^*$, respectively ($\overline{Z}_n(.)$ in (14) is defined by the first $n \times n$ blocks of the upper right portion of $\overline{Z}_{\infty}(.)$).*

*Also, there exists a solution $P(s) > 0$ to (12.b) for*

$$\begin{aligned} Q(s) &= X(s) + X^{-1}(s) - C(s) \\ &= \overline{Z}_{\infty}(s) - \overline{Z}_{\infty}^{-1}(s) - C(s), \quad \forall s \in \mathbb{R}_0, \end{aligned} \tag{15}$$

*and $S_p$, Eq. 1, is asymptotically stable.*

*Proof.* Proposition (i) follows from Lemma 2 (ii), which requires Condition (i1), and the equivalence between the continuous frequency-dependent Lyapunov equation (12.a) in (i2) and Condition 2 in Lemma 2 (ii) (Agathoklis and Foda, 1989). Proposition (ii) is proved as follows. Let $GS[F(C)] = C - Z^*X^{-1}Z$ be the generalized Schur complement (or shorted operator) of the partitioned matrix $F(C)$, Eq. 13. The (bounded) matrix $F(C)$ on the Hilbert space $\mathcal{H}$ of the square integrable functions $x \in L^2(-\infty, \infty)$ of the solutions of (1) endowed with the norm $\langle x, x\rangle = \int_{-\infty}^{\infty} \|x(j\omega)\|^2 d\omega$. Such functions are square integrable as a direct consequence of proposition (i). Consider $\overline{Z}_{\infty} = \lim_{n\to\infty} \overline{Z}_n$ with $\overline{Z}_n$ defined in (14), as an operator on $l^2(\mathcal{P}_j, \mathcal{H})$, the Hilbert space of all sequences $\overline{x} = \{\overline{x}_i\}_{i=1}^{\infty}$, with $\overline{x}_1 = x_1$ in $\mathcal{H}$, and $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$ and the $x_i$ generated from (14) defined for any arbitrary partition $\mathcal{P}_j$ of $Z$ any $j \in \mathbb{Z}_0^+$, for the solution $x(t)$. First note that a positive definite solution $X(s)$ fulfilling (13), all $s \in \mathbb{R}$ verifies

$$\begin{bmatrix} C & X^* \\ Z & X \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Z^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X^{-1/2} & 0 \\ 0 & X^{-1/2} \end{bmatrix} + \begin{bmatrix} I & I \\ I & I \end{bmatrix} \begin{bmatrix} X^{-1/2} & 0 \\ 0 & X^{1/2} \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix}, \tag{16}$$

$\forall s \in \mathbb{R}_0$ and, thus, $F(C) \geqslant 0$, any matrix $C \in \mathbb{R}^{n\times n}$ all $s \in \mathbb{R}_0$ and the right-hand-side of the second equation in (13) is indeed a shorted operator. In the case of singular $X$, the positive condition will be required as an additional hypothesis. Now, $X = GS[F(C)]$ is a positive operator iff $\overline{Z}_\infty$ is positive in which case $X = GS(\overline{Z}_\infty)$ is a solution to the second equation in (13) (Anderson *et al.*, 1990). The remainder of the proof is done by assuming that $P(s) > 0$ exists for (12.b), all $s \in R_0$. Define $\widetilde{X}(s) = P(s) - X^{-1}(s)$, all $s \in R_0$. Substitution of this identity into (12.b) yields

$$Q(s) = \left(X^{-1}(s) + \widetilde{X}(s)\right) - \overline{S}^*(s)\left(X^{-1}(s) + \widetilde{X}(s)\right)Z(s). \quad (17)$$

By adding $(C + X)$ to both sides of the (17) and by taking into account that there exists a solution $X = GS(\overline{Z}_\infty)$ to the second equation in (13), it follows that

$$\begin{aligned} &Q(s) + C(s) + X(s) = X^{-1}(s) + \widetilde{X}(s) + X(s) \\ &\quad - Z^*(s)\widetilde{X}(s)Z(s) + C(s) - Z(s)X^{-1}(s)Z(s) \Rightarrow \\ &Q(s) = P(s) + X(s) - Z^*(s)\widetilde{X}(s)Z(s) - C(s) \\ &\quad = X(s) + X^{-1}(s) + \widetilde{X}(s) - Z^*(s)\widetilde{X}(s)Z(s) - C(s), \quad (18) \end{aligned}$$

since $P(s) = X^{-1}(s) + \widetilde{X}(s)$. Substitution of the second equation of (13) into (18) yields

$$\begin{aligned} Q(s) =& X^{-1}(s) - Z^*(s)X^{-1}(s)Z(s) \\ &+ \widetilde{X}(s) - Z^*(s)\widetilde{X}(s)Z(s), \end{aligned} \quad (19)$$

which is positive definite for any $\widetilde{X} \in R^{n\times n}$, all $s \in R_0^+$ provided that $\lambda_{\max}(Z^*(s)Z(s)) < 1$, all $s \in R_0^+$. The equivalence between (12.b) and (19) yields a unique solution $P(s) > 0$ to (12.b) from Proposition (i) since $Q(.) > 0$. The proof is complete.

Note that Eq. 12.a is the continuous Lyapunov equation while Eq. 12.b is the discrete Lyapunov equation whose existence of a solution is dealt with again in Theorem 1 (ii). Extensions of proposition (ii) to the continuous Lyapunov equation (12.a) can be done althouhg they are more involved. Tests for the existence of a solution to $X = C - Z^*X^{-1}Z$ leading to the existence of a solution for (12.a) can be obtained by combining several results in Anderson *et al.* (1990) and Theorem 2 as follows.

**Theorem 2.** *The next propositions hold:*

(i) *Let* $F(C) \geqslant 0$. *Then,* $range(Z) \subset range(X^{1/2})$ *and there is a unique matrix* $M$ *satisfying the following three conditions:*
(i.1) $Z = X^{1/2}M$,
(i.2) $Ker(Z) = Ker(M)$,
(i.3) $range(M) \subseteq range(X)$,
*and this matrix satisfies*

$$X = GS[F(C)] = C - M^*M.$$

(ii) *For any* $n$-*vector* $c$,

$$\left\langle GS[F(C)c], c\right\rangle = \inf_y \left\langle F(C)\begin{bmatrix} c \\ y \end{bmatrix}, \begin{bmatrix} c \\ y \end{bmatrix}\right\rangle,$$

*and, futhermore,*

$$GS[F(X)] = \sup\{X : \text{Diag }(X : 0) \leqslant F(X),\ 0 \leqslant X\}.$$

(iii) *Assume that* $E$ *is a positive matrix, partitioned as*

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} = \begin{bmatrix} E & & E_{13} \\ & & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}.$$

Let $GS_2(E)$ denote the shorted operator of $E$ to the upper $2 \times 2$ block, for example in the invertible case, $GS_2(E)$ may be written

$$GS_2(E) = \widetilde{E} - [E_{13}^T : E_{23}^T]^T E_{33}^{-1}[E_{31} : E_{32}].$$

Then $GS(E) = GS(GS_2(E))$.

(iv) *Given* $C \geqslant 0$ *and*

$$Y = \begin{bmatrix} C/2 & Z^* \\ Z & C/2 \end{bmatrix} \geqslant 0,$$

*then* $\overline{Z}_\infty = \lim_{n\to\infty} \overline{Z}_n \geqslant 0$ *in* (14) *and a solution* $X \geqslant 0$ *exists for the second equation in* (13). *If* $E(0) \neq 0$ *and* $Q(.)$ *is defined by* (15), *the* $S_p$ *is asymptotically stable provided* $\sup_{s\in \mathbb{R}_0}(\lambda_{\max}|Z^*(s)Z(s)| < 1)$.

(v) *If in proposition* (iv) $C$ *and* $Z$ *are both positive (and hence* $Z = Z^*$*), then* $\overline{Z}_\infty \geqslant 0$ *if* $Y \geqslant 0$. *If, in addition,* $E(0) \neq 0$ *and* $Q(.)$ *is defined by* (15) *for all* $s \in \mathbb{R}_0$, *then* $S_p$ *is asymptotically stable.*

Note that Theorems 1–2 can be easily extended to deal with stability of the closed-loop systems (8)–(9) and (11) obtained by linear feedback and eventually subject to external delays. Such an extension can be performed by modifying the matrices $A$ and $A_0$ of (1) by including the controller matrices $K$ and $K_0$. Note that the usefulness of Theorem 2 (i)–(iii) lies in checking the positiveness of $GS(F(C))$ for some matrix $C \geqslant 0$ so that the existence of a solution $X$ to (13) is guaranteed. This implies the existence of a solution to a particular discrete Lyapunov equation (see (12.b) and (15) and see Theorem 1) so that $S_p$ is asymptotically Lyapunov stable.

### 3. Distributed delay systems

**3.1. Stability.** Consider the different system

$$(S_d): \ \dot{x}(t) = Ax(t) + \int_0^h \widetilde{A}_0(\theta)x(t-\theta)d\theta, \qquad (20)$$

where $x(.)$ is an $n$-vector initilized on $[-h, 0]$ by an absolutely continuous function $\varphi[-h, 0] \to \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $A_0([-h, 0], \mathbb{R}^{n\times n}) \in L^1(-h, 0)$.

From (20), the characteristic equation of $S_d$ is

$$\mathrm{Det}\left[sI - A - \overline{A}_0(s,h)\right] = 0, \tag{21}$$

where $\overline{A}_0(s,h) = \int_0^h \widetilde{A}_0(\theta)e^{-\theta s}d\theta$. A particular case of interest is when $\widetilde{A}_0(\theta) = e^{A_0'\theta}$ with $A_0' \in \mathbf{R}^{n\times n}$. In this case, the particular system $S_d'$ of $S_d$ is given by (20) with

$$\begin{aligned}\widetilde{A}_0(s,h) &= \int_0^h e^{A_0'\theta}e^{-\theta s}d\theta \\ &= \int_0^h e^{(A_0'-sI)\theta}d\theta = \int_0^h e^{\overline{A}(s)\theta}\,d\theta,\end{aligned} \tag{22}$$

since the series $\sum_{K=0}^{\infty} \overline{A}^K t^K/K!$ converges uniformly to the function $e^{\overline{A}t}$ for all real $t$, Eq. 22 can be rewritten for all $s$ not being an eigenvalue of $A_0'$ as

$$\begin{aligned}\overline{A}_0(s,h) &= \int_0^h \sum_{k=0}^{\infty} \frac{\overline{A}^k\theta^k}{k!}d\theta = \sum_{k=0}^{\infty}\int_0^h \frac{\overline{A}^k\theta^k}{k!}d\theta \\ &= \sum_{k=0}^{\infty}\frac{\overline{A}^k h^{k+1}}{(k+1)!} = \overline{A}^{-1}\sum_{k=0}^{\infty}\frac{A^{k+1}h^{k+1}}{(k+1)!} \\ &= \overline{A}^{-1}\sum_{k=0}^{\infty}\frac{\overline{A}^k h^k}{k!} = \overline{A}^{-1}\left[\sum_{k=0}^{\infty}\frac{\overline{A}^k h^k}{k!} - I\right] \\ &= (A_0' - sI)^{-1}\left(e^{(A_0'-sI)h} - I\right).\end{aligned} \tag{23}$$

Thus, the characteristic equation (21) reduces to

$$\mathrm{Det}\left[sI - A - (sI - A_0')^{-1}(I - e^{(A_0'-sI)h})\right] = 0. \tag{24}$$

Define the matrix function $A_0(s,h) = e^{hs}\overline{A}_0(s,h)$ which, in the particular case (22), becomes

$$A_0 = A_0(s,h) = e^{hs}\overline{A}_0(s,h) = (sI - A_0')^{-1}(e^{hs}I - e^{A_0'h}). \quad (25)$$

The characteristic equations (21) and (24) can be rewritten, respectively, as

$$\text{Det}\big[sI - A - e^{hs}A_0(s,h)\big] = 0, \quad (26)$$

and

$$\text{Det}\big[sI - A - e^{-hs}(sI - A_0')^{-1}(e^{hs}I - e^{A_0'h})\big] = 0. \quad (27)$$

Thus, Lemma 1 remains valid with the change $A_0 \to A_0(s,h)$ so that Eq. 3.a becomes

$$sI - A - e^{-hs}A_0(s,h) = (I - z_1A_1)\big(z_2I - S(z_1)\big) = 0 \quad (28)$$

for $z_1 = s^{-1}\ (s \neq 0),\ z_2 = s$ and $A_1$ being an arbitrary $n \times n$-matrix with

$$S(z_1) = S(1/s) = s(A_1 - sI)^{-1}(A_1 - A - e^{-hs}A_0(s,h)) \quad (29)$$

with $A_0(s,h) = e^{hs}\overline{A}_0(s,h)$, $\overline{A}_0(s,h) = \int_0^h \widetilde{A}_0(\theta)e^{-\theta s}\,d\theta$. In the particular case of (22) and (25), one gets

$$\begin{aligned} S(z_1) = S(1/s) =& s(A_1 - sI)^{-1}\big[A_1 - A - e^{-hs}(sI - A_0')^{-1} \\ & \times (e^{hs}I - e^{A_0'h})\big], \end{aligned} \quad (30)$$

and $\overline{S}(s)$ in Lemma 2 (ii) becomes

$$\overline{S}(s) = \big(A_1 - A - e^{-hs}A_0(s,h)\big) \quad (31)$$

with $A_0(s,h) = \int_0^h \widetilde{A}_0(\theta)e^{-\theta s}d\theta$ and

$$\overline{S}(s) = \big[A_1 - A - e^{-hs}(sI - A_0')^{-1}(e^{hs}I - e^{A_0'h})\big]. \quad (32)$$

In both cases, a strictly Hurwitz matrix $A_1(s)$ (i.e., Det $A_1(s) \neq 0$, all $s \in D$) can be considered. The next results stands.

**Lemma 4.** *Lemmas 1–2 and thus Theorems 1–2 apply mutatis-mutandis for system $S_d$, Eq. 20, (including the particular case of system $S'_d$ Eq. 20 and 22) provided that $A'_0$ is nonsingular and $E(0) = -Det(A + A_0(s,h)) \neq 0$ (Or, in particular, $E(0) = -Det(A + A_0'^{-1}(I - e^{A'_0 h})) \neq 0$) subjected to the definitions Eqs. 29–32.*

**3.2. Free external delay system.** Assume that $S_d$ is forced with free external delay as follows

$$(S''_d): \ \dot{x}(t) = Ax(t) + \int_0^h \widetilde{A}_0(\theta)x(t-\theta)\,d\theta + Bu(t), \qquad (33)$$

where $u \in IR^m$ and $B \in IR^{n\times m}$ are, respectively, the control vector and control matrix.

**3.2.1. Stabilizing control law.** Assume the control law

$$u(t) = Kx(t) + \int_0^h \widetilde{K}_0(\theta)x(t-\theta)\,d\theta,$$

where $K$ and $\widetilde{K}_0(.) \in \mathbb{R}^{n\times m}$. Define $\overline{K}_0(s,h)$ and $K_0(s,h)$ as follows

$$\overline{K}_0(s,h) = e^{-hs}K_0(s,h) = \int_0^h \widetilde{K}_0(\theta)e^{-\theta s}\,d\theta. \qquad (35)$$

As in the control law (9) for $S_p$, particular control laws may be obtained from (34) (see Remark 2.).

The closed-loop characteristic equation associated with (33)–(34) becomes:

$$\text{Det}\big[sI - (A + BK) - e^{-hs}(A_0(s,h) + BK_0(s,h))\big] = 0, \quad (36)$$

which cannot have roots in $D$ in order the system to be asymptotically stable. Lemma 3 applies to this characteristic equation with the change $(A_0 + BK_0) \to A_0(s,h) + BK_0(s,h)$. In particular, the closed-loop system is asymptotically stable if $(A + BK)$ is strictly Hurwitz (which requires that the pair $(A, B)$ is stabilizable) and Det $[A_0(s,h) + BK_0(s,h)] \neq 0$, all $s \in D$.

In the particular case $S'_d$, Eq. 22, and provided that

$$\overline{K}_0(s,h) = \left[\int_0^h e^{(K'_0 - sI)\theta} d\theta\right] K_0, \quad (37)$$

and $A'_0$ and $K'_0$ are $n \times n$ and $m \times m$ nonsingular matrices, Eq. 36 becomes

$$\text{Det}\Big\{sI - (A + BK) - e^{-hs}\Big[(sI - A'_0)^{-1}\big(e^{hs}I - e^{A'_0 h}\big) + B(sI - K'_0)^{-1}\big(e^{hs}I - e^{K'_0 h}\big)\Big]K_0\Big\} \neq 0, \ \forall s \in D \quad (38)$$

and the results of Sections 2–3 can be again applied.

**3.3. External delay system.** Assume that the closed-loop system possess an external delay $h' > 0$, namely:

$$(S'''_d) : \dot{x}(t) = Ax(t) + \int_0^h \widetilde{A}_0(\theta)x(t-\theta)\, d\theta + Bu(t) + \int_0^{h'} \widetilde{B}_0(\theta)u(t-\theta)\, d\theta, \quad (39)$$

where $\widetilde{B}_0(.) \in \mathbb{R}^{n \times m}$, and is subjected to the control law (34). The closed-loop system becomes

$$\dot{x}(t) = [A + BK]x(t) + \int_0^h [\widetilde{A}_0(\theta) + B\widetilde{K}_0(\theta)]\, x(t-\theta)\, d\theta$$
$$+ \int_0^{h'} \widetilde{B}_0(\theta) \left[ Kx(t-\theta) + \int_0^h \widetilde{K}_0(\tau)x(t-\theta-\tau)\, d\tau \right] d\theta. \quad (40)$$

and the condition for asymptotic stability is

$$\mathrm{Det}\Big\{ sI + BK) - \int_0^h [\widetilde{A}_0(\theta) + B\widetilde{K}_0)]\, e^{-\theta s}\, d\theta - \int_0^{h'} \widetilde{B}_0(\theta) K e^{-\theta s}\, d\theta$$
$$- \int_0^{h'}\int_0^h \widetilde{B}_0(\theta)\widetilde{K}_0(\tau) e^{-(\theta+\tau)s}\, d\tau d\theta \Big\} \neq 0, \quad \forall s \in D. \quad (41)$$

In the particular case of (22), one has

$$\begin{aligned} \overline{K}_0(s,h) &= \left[ \int_0^h e^{(K_0' - sI)\theta} d\theta \right] K_0; \\ B_0(s,h) &= \left[ \int_0^{h'} e^{(B_0' - sI)\theta} d\theta \right] B_0, \end{aligned} \quad (42)$$

where $K_0$ and $B_0$ are $m \times n$ and $n \times m$ matrices, respectively while $K_0'$ and $B_0'$ are $m \times n$ matrices. In this case, Eq. 41 becomes modified as follows

$$\mathrm{Det}\Big\{ sI - (A + BK) - e^{-hs}\big[(sI - A_0')^{-1}(e^{hs}I - e^{A_0' h})$$

$$+ B(sI - K_0')^{-1}(e^{hs}I - e^{K_0'h})K_0 + e^{hs}(sI - B_0')^{-1}(I - e^{(B_0'-sI)h}) \times B_0(sI - K_0')^{-1}(I - e^{K_0'-sI)h'})K_0 + e^{hs}(sI - B_0')^{-1}(I - e^{(B_0'-sI)h'})B_0K]\Big\} \neq 0 \quad (43)$$

for all $s \in D$. Again, results of Sections 2 and 3 apply to the characteristic equations (41) and (43) with the changes $A \to A + BK$, $A_0 \to A_0(s,h)$ and $E(0)$ obtained from $\mathrm{Det}[sI - (A + BK) - e^{-hs}A_0(s,h)]|_{s=0}$. In particular, the necessary condition for stability $E(0) \neq 0$ is

$$E(0) = -\mathrm{Det}\big[A + BK - {A'}_0^{-1}(I - e^{A_0'h}) - B{K'}_0^{-1}(I - e^{K_0'h})K_0 - {B'}_0^{-1}(I - e^{B_0'h'})K + {B'}_0^{-1}(I - e^{B_0'h})B_0{K'}_0^{-1}(I - e^{K_0'h'})K_0\big] \neq 0, \quad (44)$$

provided that $n \times n$-matrices $A_0'$ and $B_0'$ and the $m \times m$ matrix $K_0'$ are nonsingular. Thus, a set of particular conditions (which imply together that Eq. 43 has all its roots in $\overline{D}$, namely, for $(S_d''')$, Eq. 39, subjected to distributed delays fulfilling (22), is asymptotically stable is (Lemma 2 (ii)).

(a) $E(0) \neq 0$ (Eq. 44); $A_0', B_0'$ and $K_0'$ are nonsingular;

(b) $(A+BK)$ is strictly Hurwitz (Note that it always exists a stabilizing $K$ provided that $(A, B)$ is a stabilizable pair) and $\mathrm{Det}[A_0(s, h, h')] \neq 0$, $\forall s \in D$, where

$$A_0(s,h) = (sI - A_0')^{-1}(e^{hs}I - e^{A_0'h}) + B(sI - K_0')^{-1}(e^{hs}I - e^{K_0'h})K_0 + e^{hs}(sI - B_0')^{-1}(I - e^{(B_0'-sI)h}) \times B_0(sI - K_0')^{-1}(I - e^{(K_0'-sI)h'})K_0 + e^{hs}(sI - B_0')^{-1}(I - e^{(B_0'-sI)h'})B_0K. \quad (45)$$

REMARK 3. Note that Lemmas 1–3 and Theorems 1–2, related to sufficient conditions for asymptotic stability apply to the various classes of distributed delay systems $(S_d, S'_d, S''_d, S'''_d)$ and its closed-loop versions for the given control laws with the only appropriate modifications of $E(0), \overline{S}(s), Z(s), A$ and $A_0(.)$

**3.4. Mixed point-distributed delay systems.** Consider the autonomous system

$$(S_m): \; \dot{x}(t) = Ax(t) + A_0 x(t-h_0) + \int_0^h e^{\widetilde{A}_0 \theta} x(t-\theta)\, d\theta \quad (46)$$

under the initial conditions given for systems $S_p$, Eq. 1, and $S'_d$, Eqs. 20 and 22. $h_0$ and $h$ are positive point and distributed delays, respectively. Eq. 46 can be extended to use of a matrix function $\widetilde{A}_0(\theta)$ of entries of bounded variation as in Section 3.1 (see $S_d$ Eq. 20). System $S_m$ is asymptotically Lyapunov stable iff

$$\begin{aligned} \mathrm{Det}\big[sI - A - e^{-h_0 s} A_0 \\ - (sI - A'_0 0^{-1}(I - e^{(A'_0 - sI)h})\big] \neq 0, \quad \forall s \in D. \quad (47) \end{aligned}$$

This condition can be splited in two, one being related to the static conditions through $E(0) = -\mathrm{Det}[A + A_0 - A_0'^{-1}(I - e_0^{A'h})] \neq 0$, provided that $A'_0$ is nonsingular, and another one related to the fact that the roots of the corresponding $\widetilde{S}(s)$-matrix (obtained as in Lemma 2) must belong to $\overline{D}_0$ In this way, the generalizations of Lemmas 1–3 and Theorems 1–2 to $S_m$ is inmediate. Their extensions for results on closed-loop stability are also direct.

## 4. Mixed point-distributed delay systems with extended differential-difference representations. Systems with commensurate delays

**4.1. Mixed point-distributed delay systems.** First, system $S_m$, Eq. 46, having mixed point-distributed delay is interpreted as a mixed differential-difference delay system. Taking Laplace transforms in (46) with zero initial conditions, one gets

$$\left[sI - A - e^{-h_0 s}A_0 - (I - e_0^{A_0' - sI)h})(sI - A_0')^{-1}\right]x(s) = 0. \quad (48)$$

Define the auxiliary variable $x_1(s) = (sI - A_0')^{-1}x(s)$. Thus, we have in the time-domain the next differential $2n$-system which has the same characteristic equation (48) as $S_m$, Eq. 46:

$$\begin{aligned}(S_m') : \ \dot{x}(t) =& Ax(t) + A_0 x(t - h_0) \\ &+ (I - e^{A_0' h})x_1(t) - x_1(t - h), \quad (49.a) \\ x_1(t) =& A_0' x_1(t) + x(t) \quad (49.b)\end{aligned}$$

with appropriate initial conditions. On the other hand, it is possible to rewrite (48)–(49) in operational form as follows

$$\begin{bmatrix} \widehat{A}(s) - z_1 A_0 & (e^{A_0' h} - I) + z_2 I \\ -I & sI - A_0' \end{bmatrix} \times \\ \times \begin{bmatrix} x(s) \\ x_1(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (50)$$

where $z_1 = e^{-h_0 s}$; $z_2 = e^{-hs}$ and $\widehat{A}(s) = sI - A$. Note that

$$E(s, z_1, z_2) = \\ = \text{Det}\begin{bmatrix} \widehat{A}(s) - z_1 A_0 & (e^{A_0' h} - I) + z_2 I \\ -I & sI - A_0' \end{bmatrix} \neq 0 \quad (51)$$

for all $(s, z_1, z_2) \in D \times U \times U$, for asymptotic stability. Direct calculus with Schur's complement in (51) gives

$$\begin{aligned}E(s, z_1, z_2) =& \text{Det}(\widehat{A}(s) - z_1 A_0)\text{Det}(sI - \widehat{Z}(z_1, z_2)) \ (52.a) \\ =& \text{Det}(\widehat{A}(s))\text{Det}(I - z_1, \widehat{A}^{-1}(s)A_0) \\ & \times \text{Det}(sI - \widehat{Z}(z_1, z_2)), \quad (52.b)\end{aligned}$$

where $\widehat{Z}(z_1,z_2) = A_0' + (z_1^{-1}\widehat{A}(s) - A_0)^{-1}[(z_2^{-1}I - e^{A_0'h}) - I]$. Note that (52) can be rewritten, by eliminating the complex argument $s$ in the second factor of (52.b), as

$$E(s,z_1,z_2) = \text{Det}(sI - A)\text{Det}(I - z_1\widehat{A}^{-1}(-1/h_0 \ln z_1)A_0) \times \text{Det}(sI - \widehat{Z}(z_1,z_2)). \quad (53)$$

The following result follows from (52.b) and (53).

**Lemma 5.** *$S_p$ is asymptotically stable if the following conditions hold*

(i.1) *$A$ has all its eigenvalues in $\overline{D}$.*

(i.2) *$\widehat{A}^{-1}(-1/h_0 \ln z_1)A_0$ has all its eigenvalues in $\overline{U}$ (or $\widehat{A}^{-1}(s)A_0$ has all its eigenvalues in $\widehat{U}$ for all $s \in \mathbb{R}$).*

(i.3) *$\widehat{Z}(z_1,z_2)$ has no eigenvalues in $D$ for all $(z_1,z_2) \in T \times T$ (or, $\widehat{Z}(z_1) = \widehat{Z}(z_1,\ z_2 = z_1 e^{\Delta h/h_0 \ln z_1})$, for $\Delta h = (h - h_0)$, has no eigenvalues in $D$ for all $z_1 \in T$).*

Thus, Lemma 5 gives sufficient stability conditions in the sense that $E(s,z_1,z_2) \neq 0$ in $D \times U \times U$ (i.e, the characteristic polynomial is void of zeros in the non-compact hyperplane composed of the closed right half-plane and the two dimensional closed unit disc. This zero criterion is stronger than the asymptotic stability condition (47) of Section 3 since $D \times U \times U$ has more points than $D \times \exp(-D) \times \exp(-D)$. Therefore, the stability results based only on this zero criterion are more conservative than those given in Section 3 Agathoklis and Foda (1989).

Frequency-dependent Lyapunov equations and the associated stability conditions follow inmediately from (52)–(53) in the same way, as in Section 2–3 (see Theorems 1–2 and Remark 3).

Now, assume that $S_m'$, Eq. 49, is substituted by:

$$(S_c):\ x(t) = A_0 x(t - h_0) + (I - e^{A_0'h})x_1(t) - x_1(t-h), \quad (54.a)$$

$$\dot{x}_1(t) = A_0' x_1(t) + x(t) \quad (54.b)$$

This system is called a point-commensurate delay system because it involves mixed differential-difference equations. The particular case dealt with in Agathoklis and Foda (1989) can be equivalently described by a differential equation involving point delays.

The characteristic equation implies that

$$E(s, z_1, z_2) = \mathrm{Det}\begin{bmatrix} I - z_1 A_0 & (e^{A'_0 h} - I) + z_2 I \\ -I & sI - A'_0 \end{bmatrix} \neq 0 \qquad (55)$$

all $(s, z_1, z_2) \in D \times U \times U$ for $z_1 = e^{-h_0 s}$, $z_2 = e^{-hs}$ in order $S_c$ to be asymptotically stable.

Note that

$$E(s, z_1, z_2) = \mathrm{Det}(I - z_1 A_0)\mathrm{Det}(sI - \widehat{Z}(z_1, z_2)),$$

where

$$\widehat{Z}(z_1, z_2) = A'_0(z_1^{-1} I - A_0)^{-1}\big[z_2^{-1}(I - e^{A'_0 h}) - I\big].$$

Thus, the next result stands.

**Lemma 6.** *System $S_c$ Eq. 54 is asymptotically stable if the following conditions hold*

(i.1) *$A$ has all its eigenvalues in $\overline{U}$.*

(i.2) *$\widehat{Z}(z_1, z_2)$ has no eigenvalues in $D$ for all $(z_1, z_2) \in T \times T$ (or $\widehat{Z}(z_1) = \widehat{Z}(z_1,\ z_2 = z_1 e^{\Delta h/h_0 \ln z_1})$, $\Delta h = (h - h_0)$, has no eigenvalues in $D$ for all $z_1 \in T$).*

*Assume that $h_0 = h$ so that $z = z_1 = z_2 = e^{-hs}$ so that $E(s, z) = Det(I - zA_0)Det(sI - \widehat{Z}(z))$ and $\widehat{Z}(z) = A'_0 + (z^{-1} I - A_0)^{-1}[z^{-1}(I - e^{A'_0 h}) - I]$. Thus, asymptotic stability is guaranteed by modifying (i.2) so that $\widehat{S}(z)$ has no eigenvalues in $D$ for all $z \in T$. If $A'_0 = 0$ in (54) (namely, there is a point delay only), then the result holds with $\widehat{Z}(z) = (z^{-1} I - A_0)^{-1}$.*

REMARK 4. Note that the use of Schur's formula in (55) for $S_c$ leads to

$$E(s, z_1, z_2) = \mathrm{Det}(I - z_1 Z(s))\mathrm{Det}(sI - A'_0), \qquad (56)$$

or the modified expression corresponding to (52), where $Z(s) = A_0 + [z_2^{-1}(I - e^{A_0' h}) - I](sI - A_0')^{-1}$. Thus, an equivalent criterion for asymptotic stability to that in Lemma 6 is: $A_0'$ is strictly Hurwitz and $Z(s)$ has all its eigenvalues in $\overline{U}$, all $s \in \mathbb{R}$.

As in section before, Theorems 1–2 can be applied to derive conditions for the determinant Eq. 56 to have all its zeros in $\overline{D} \times \overline{U}$ leading to frequency-dependent Lyapunov equations. Note that Lemma 1 (ii) applies "mutatis-mutandis" and so Theorem 1 (i) (as much as Eq. 12.b is concerned) and Theorem 1 (ii), and Theorem 2.

**4.2. Commensurate delays.** Note that if $A_0' = 0$ and $h = h_0 \Rightarrow z = z_1 = z_2 = e^{-hs}$, then the characteristic Eq. 55

$$(S_c'): \begin{bmatrix} x_1(t+h) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_0 & -I \\ I & 0 \end{bmatrix}; \quad y = [E_1 : E_2]x \tag{57}$$

with $x = [x_1^T : x_2^T]^T$ and appropriate matrices of parameters $E_{1,2}$. Eq. 57 describes a delay differential system with commensurate delay which is given by the functional differential equation

$$\frac{d^n}{dt^n}y(t) + \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\frac{d^i}{dt^i}y(t - jh) = 0, \tag{58}$$

whose autonomous 2-D state-space model is, in general, an extension of (57) to

$$\begin{bmatrix} x_1(t+h) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_0 & A_1 \\ A_2 & A_3 \end{bmatrix}; \quad \text{some } A_i \in \mathbb{R}^{n\times n} \quad (1 = 1,2,3,4). \tag{59}$$

**5. Stability and the CBR and DPR lemmas.** In this section, the stability interpreted in terms of real positiveness of a square matrix. First, the next definitions are borrowed and extended from Agathoklis and Foda (1989).

DEFINITIONS. (1) Let $Z(s)$ be a square matrix over $\mathbb{R}(s)$. It is called continuous strictly bounded real (CSBR) if the following holds:

(i) $Z(s)$ is analytic in $D$;

(ii) $I - Z^*(s)\ Z(s) > 0$, all $s \in \mathbb{R}$. (60)

If (i)–(ii) stand only on $D_0$ and $\mathbb{R}_0$, respectively, then $Z(s)$ will be called continuous bounded real (CBR).

(2) Let $S(z)$ be a square matrix over $\mathbb{R}(z)$. It is called discrete strictly positive real (DSPR) if the following holds:

(i) $S(z)$ is analytic in $U$.

(ii) $S^*(z) + S(z) > 0$, all $s \in T$. (61)

If (i)–(ii) stand only on $U_1$ and $T_1$, respectively, then $Z(s)$ will be called discrete positive real (CPR).

REMARK 5. As pointed out in the above reference, the CBR matrix definition compared to Theorem 1 (i) leads to the implication from (60) to (12.b) provided that this one is extended to all $s$ in $\mathbb{R}$ and admits the constant solution $P = \widehat{T}^*\widehat{T}$ over $\mathbb{R}$. However, the converse is not true.

REMARK 6. Note that condition 2 in Lemma 2 (ii) can be rewritten as $\overline{S}(z) = (A_1 - A - zA_0)$ with $z = e^{-hs}$. Thus, the fact that $\overline{S}(z)$ has no eigenvalues in $D_0$ for all $z \in T$ can be described in terms of a discrete frequency-dependent Lyapunov equation as follows; $\overline{S}(z)$ is a stability matrix with respect to $\overline{D}_0$ if for any p.d.h. $\widetilde{Q}(z)$, there exists a unique $\widetilde{P}(z)$ being p.d.h. such that

$$\overline{S}^*(z)\widetilde{P}(z) + \widetilde{P}(z)\overline{S}(z) = -\widetilde{Q}(z), \quad \text{all} \quad z \in T_1, \tag{62}$$

which is equivalently to (12.a).

REMARK 7. As a direct consequence of Theorem 1 (i) and Remarks 4–6, it follows that Condition 2 of Lemma 2 (ii) for $S_p$ can be tested through Eq. 12.b (Theorem 1 (i)) an similarly, guaranteed if (60) holds. Alternatively, it can be checked under (62) or (12.a) and guaranteed if (61) holds. Similar arguments can be used for the various open and closed-loop delay systems which have been dealt with in this paper.

**Lemma 7.** *The following propositions hold:*

(i) (CBR lemma). *Suppose that the quadruple* $\{F, G, H^T, J\}$ *is a minimal realization of* $Z(s)$, *Eq.* 60, *i.e.,*

$$Z(s) = H^T(sI - F)^{-1}G + J. \tag{63}$$

*The,* $Z(s)$ *is CSBR iff there exists* $P$ *p.d.s. such that*

$$Q_1 = \begin{bmatrix} J^TJ - I & (PG + HJ)^T \\ PG + HJ & F^TP + PF + HH^T \end{bmatrix} < 0, \tag{64}$$

*(i.e.,* $Q_1$ *is n.d.s.). Eq.* 64 *holds with* $Q_1 \leqslant 0$ *iff* $Z(s)$ *is CSBR.*

(ii) (DPR lemma). *Suppose that the quadruple* $\{F, G, H^T, J\}$ *is a minimal realization of* $Z(s)$, *Eq.* 61, *i.e.,*

$$S(z) = H^T(z^{-1}I - F)^{-1}G + J. \tag{65}$$

*Thus,* $S(z)$ *is DSPR iff there exists* $P$ *p.d.s. such that*

$$Q_2 = \begin{bmatrix} F^TPF - P & F^TPG - H \\ (F^TPG - H)^T & G^TPG - J - J^T \end{bmatrix} < 0. \tag{66}$$

Eq. 66 with "$\leqslant$" (i.e., $Q_2$ is semidefinite) iff $S(z)$ is DPR.

The next result stands as an alternative to Theorem 2 (iv)–(v).

**Lemma 8.** *Condition* (i2) *of Theorem* 1 (i) *is ensured if* $S(s)$ *is CBR (Definition* 1*) or alternatively and equivalently by* $S(z = e^{-hs})$ *being DBR (Definition* 2*) (see Remarks* 5–7*). Thus,* $S_p$, *Eq.* 1 *is asymptotically stable if Condition* (i1) *of Theorem* 1 (i) *holds.*

REMARK 8. Note that, according to Lemmas 3–5, and Remarks 4–5, Lemma 8 applies "mutatis-mutandis" for systems $S_d, S'_d, S''_d, S'''_d$, $S_m, S'_m S'_m, S_c$ and $S'_c$ as well as for their closed-loop versions through the appropriate changes in the various matrices of parameters.

**6. Iterative computational procedure.** Note that $\overline{Z}_n \oplus 0 \to \overline{Z}_\infty$ strongly and that $\overline{Z}_\infty > 0 \iff \overline{Z}_n > 0$ for all $n$. Formally define $\overline{Z}_\infty : l^2((\overline{\mathcal{H}}) \to l^2(\overline{\mathcal{H}})$ by

$$(\overline{Z}_\infty(C)\widetilde{x})_i = \begin{cases} C\widetilde{x}_1 + Z^*\widetilde{x}_2, & i = 1, \\ \overline{Z}\widetilde{x}_{i-1} + C\widetilde{x}_i + Z^*\widetilde{x}_{i+1}, & i > 1, \end{cases} \tag{67}$$

where $\overline{\mathcal{H}}$ is discrete Hilbert space obtained from initial conditions in $\mathcal{H}$ (i.e., the initial state vector – see Eq. 12.b) and the application of (67). Let $Z_n = [Z; 0; \ldots; 0]$ a block $1 \times n$ partioned matrix and let $\overline{Z}_\infty = [\overline{Z}; 0; \ldots; 0]$. Thus,

$$\begin{aligned} \overline{Z}_n(C) &= \begin{bmatrix} C & Z^*_{n-1} \\ Z_{n-1} & Z_{n-1} \end{bmatrix}, \\ \overline{Z}_\infty(C) &= \begin{bmatrix} C & Z^*_\infty \\ Z_\infty & \overline{Z}_\infty \end{bmatrix}. \end{aligned} \tag{68}$$

If $Z_\infty \geqslant 0$ then let $M_\infty$ (namely, the operator of Theorem 2) be such that $\overline{Z}^*_\infty = \overline{Z}^{1/2}_\infty M_\infty$ and $GS(\overline{Z}_\infty(C)) = C - M^*_\infty M_\infty$. Similarly, for $Z_n(C)$, define $M_{n-1}, Z_{n-1}, Z^*_{n-1}$ and $GS\left(\overline{Z}_n(C)\right) = C - M^*_{n-1}M_{n-1}$.

Define a sequence of positive operators

$$\begin{aligned} X_0 = C, \quad X_1 &= GS\left(\begin{bmatrix} C & Z \\ Z & X_0 \end{bmatrix}\right); \\ X_{n+1} &= GS\left(\begin{bmatrix} C & Z^* \\ Z & X_n \end{bmatrix}\right); \end{aligned} \tag{69}$$

$X_n \to X_\infty$ strongly in the strong operator topology and the shorts of the upper-left hand $n \times n$ blocks of an operator converge, in the strong operator topology, to the short of that operator (Anderson *et al.*, 1988), namely, $X_n \to GS(\overline{Z}_n C))$ as $n \to \infty$ strongly so that $GS(\overline{Z}_n(C)) \to X$ strongly in that topology.

**Conclusions.** This paper has presented a method to derive sufficient conditions for asymptotic stability of linear and time-invariant systems involving delays. Several cases have been considered by including point, distributed, mixed point-distributed and commensurate delays. The stability criteria are based upon the use of mixed Laplace and delay operators in 2-D and 3-D state representations which lead to (sufficient) conditions for asymptotic stability stated over biplanes or hyperplanes through the use of Schur's complements of operators. An interpretation of those stability conditions is given in terms of frequency-dependent Lyapunov equations. Such conditions can be tested by using an iterative procedure involving the use of (generalized) Schur's complement for operators.

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