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ON CHANNEL ACCESS PROBABILITIES WHICH MAXIMIZE THROUGHPUT OF SLOTTED ALOHA

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Abstract. We consider finite population slotted ALOHA where each of n terminals has its own transmission probability p_i. Given the overall traffic load λ, the probabilities p_i are determined in such a way as to maximize throughput. This is achieved by solving a constrained optimization problem. The results of Abramson (1970) are obtained as a special case. Our recent results are improved (Mathar and Žilinskas, 1993).

Key words: throughput, networks, optimization.

The throughput d_{A} of the ALOHA–protocol is well known to be $d_{\mathrm{A}} = \lambda e^{-2\lambda}$ (Abramson, 1970; Tanenbaum, 1988), where λ denotes the total traffic load offered to the system. To obtain this result it is supposed that data packets arrive according to a homogeneous Poisson process with constant intensity $\lambda > 0$. Without loss of generality it may be assumed that data packets have a length of one time unit, i.e., time units are determined by the packet length. Throughput is defined as the average number of successfully received packets per time unit.

The maximum throughput of ALOHA is $1/2e \approx 0.184$, which is achieved at $\lambda = 0.5$ (one packet arriving each two time units on the average). This is a relatively small value, and Roberts (1972) published a protocol for doubling the capacity. His method is known as *slotted*

ALOHA and works as follows. Time is devided into slots of a packet length. A station is not allowed to send whenever it wishes, but instead has to wait for the beginning of the next slot. Slotted ALOHA has been analyzed in an approximate model to have throughput $d_{\mathrm{sA}} = \lambda e^{-\lambda}$ which is maximum for $\lambda = 1$ with value $1/e$. Analogously to the continuous model λ means the expected number of packets transmitted per slot. If two or more packets are transmitted in the same slot collisions occur and all packets involved are destroyed by superposition. The corresponding model assumes a large number of users K, each independently transmitting with equal probability λ/K. Throughput is obtained by considering the limit as $K \to \infty$ (Roberts, 1972; Tanenbaum, 1988).

In this note we consider finite population slotted ALOHA with different access probabilities. Abramson (1985) has investigated a particular model of this type with fixed traffic load $\lambda = 1$. He considered two groups of n_1 and n_2 users, respectively, each with different access probabilities g_1 and g_2. He concludes that the asymmetric case (g_1 large and g_2 small such that $n_1 g_1 + n_2 g_2 = \lambda = 1$) achieves large overall throughput given in (1).

We extend the results of Abramson (1985) by maximizing throughput over all access probabilities of n individual users such that the traffic load is fixed. As a special case we observe the claim of Abramson concerning asymmetricity, but only for small values of λ. For λ approximately larger than $e = 2.718...$ again a symmetric distribution of traffic load turns out to be most favorable. The method we use is to show that many local maxima are dominated by only two, and in spite of multimodality of the objective function, the global maximum is obtained without numerical comparison of a large set of potential solutions, as is suggested in (Mathar and Žilinskas, 1993).

Let us assume a finite community of n users. Each of them transmits in a slot independently of all others with probability p_i, $0 \leqslant p_i \leqslant 1$, $i = 1, \ldots, n$. The expected traffic load, i.e., the average number of packets ready for transmission in a slot is $\lambda = \sum_{i=1}^{n} p_i$. The probability that a packet will be successfully transmitted in a particular slot obviously coincides with the throughput, and is given by

$$d(p_1, \dots, p_n) = \sum_{i=1}^{n} p_i \prod_{j \neq i} (1 - p_j). \tag{1}$$

Maximizing throughput w.r.t. a fixed traffic load $0 < \lambda \leqslant n$ may be formulated as to

maximize $d(p_1, \dots, p_n)$ such that $0 \leqslant p_i \leqslant 1,\ \sum_{i=1}^{n} p_i = \lambda > 0.$

Let $p = (p_1, \dots, p_n)$. In case $\lambda \leqslant 1$ the solution $p^* = (p_1^*, \dots, p_n^*)$ is obvious since

$$d(p) \leqslant \sum_{i=1}^{n} p_i = \lambda,$$

and equality $d(p^*) = \lambda$ holds whenever $p_k^* = \lambda$ for some k, and $p_i^* = 0$ for $i \neq k$.

Hence, we can restrict our attention to the more complicated case $\lambda > 1$, and first state some preliminary results concerning relative boundary points of the constraining set

$$\mathcal{C} = \left\{ p = (p_1, \dots, p_n) \mid 0 \leqslant p_i \leqslant 1,\ \sum_{i=1}^{n} p_i = \lambda \right\}.$$

If $p_i = 1$ for at least two different components of p we have $d(p) = 0$, which excludes p as a maximum point. If exactly one component of p equals 1, without loss of generality we may assume $p_n = 1$, then

$$d(p_1, \dots, p_n) = \prod_{j=1}^{n-1} (1 - p_j).$$

From Marshall and Olkin (1979, p. 79) it is easily concluded that $\prod_{j=1}^{n-1}(1 - p_j)$ is a Schur–concave function such that the maximum of $d(p_1, \dots, p_{n-1}, 1)$ over $\mathcal{C}$ is attained at $p_i^* = \frac{\lambda - 1}{n-1}$, $i = 1, \dots, n-1$, and by symmetry of d, for any vector with permuted components.

This discussion shows that $p^* = (p_1^*, \ldots, p_n^*)$ with $p_k^* = 1$ for one component and $p_i^* = \frac{\lambda-1}{n-1}$ for $i \neq k$ is a candidate for a maximum point over the boundary of $\mathcal{C}$ with value $d(p^*) = (\frac{n-\lambda}{n-1})^{n-1}$.

The points $(0, \ldots, 0, \frac{\lambda-1}{k-1}, \ldots, \frac{\lambda-1}{k-1}, 1)$ (and those with permuted components) with the throughput value $f(k) = (\frac{k-\lambda}{k-1})^{k-1}$ should also be considered as a candidate for a local maximum in corresponding $(k-1)$-dimensional hyperplanes on the boundary of the feasible set. But the function $f(x)$ of the continuous variable x is monotonically increasing because $f'(x) > 0$, as it is shown below. We have

$$f'(x) = f(x) \cdot \Big(\ln(1-z) + \frac{z}{1-z}\Big),$$

where $0 < z = \frac{\lambda-1}{x-1} < 1, \; 1 < \lambda < n, \; f(x) > 0$, and

$$\ln(1-z) + \frac{z}{1-z} = -\sum_{k=1}^{\infty} \frac{z^k}{k} + \sum_{k=1}^{\infty} z^k > 0. \qquad (2)$$

This shows that the local maximum $d(p^*)$ dominates the local maxima attained at extreme points of lower dimensional faces of the feasible set.

We now investigate interior points of $\mathcal{C}$ by a Langrangian setup. We search for stationary points by solving the system of equations

$$\nabla d(p) \; + \gamma \, \nabla g(p) = 0,$$

where $g(p) = \sum_{i=1}^{n} p_i - \lambda = 0$ describes the restrictions. Carrying out differentiation yields the following system

$$\begin{aligned} &\prod_{i \neq j}(1-p_i) - \sum_{k \neq j} p_k \prod_{i \neq j, i \neq k} (1-p_i) + \gamma = 0, \\ &\qquad j = 1, \ldots, n, \\ &g(p) = 0. \end{aligned} \qquad (3)$$

After multiplying the j-th equation by $(1 - p_j)$ we get

$$\prod_{i=1}^{n}(1-p_i) - \sum_{k\neq j} p_k \prod_{i\neq k}(1-p_i) + \gamma(1-p_j) = 0.$$

This leads to

$$\prod_{i\neq j}(1-p_i) + \gamma(1-p_j) = d(p) \quad \text{for all } j = 1,\dots,n, \tag{4}$$

and the differences of the j-th and k-th equation give

$$\Big(\prod_{i\neq j, i\neq k}(1-p_i) - \gamma\Big)(p_j - p_k) = 0 \quad \text{for all } j,k = 1,\dots,n.$$

If $p_j \neq p_k$ for some $j \neq k$ it follows that $\prod_{i\neq j,i\neq k}(1-p_i) = \gamma$, and from $p_k \neq p_\ell$ for some $\ell \neq k$ it follows that $\prod_{i\neq k,i\neq \ell}(1-p_i) = \gamma$, which yields $p_j = p_\ell$. This shows that stationary points p have at most two different components. Thus, each stationary point p may be represented as $p = (a,\dots,a,b,\dots,b)$ with k entries a and $n-k$ of them equal to b, where $ka + (n-k)b = \lambda$, $k \in \{0,\dots,n\}$.

$k = 0$ and $k = n$ means $a = b = \frac{\lambda}{n}$. By (3), with $\gamma = (\lambda-1)(1-\frac{\lambda}{n})^{n-2}$, it is easily verified that the corresponding point $p = (\frac{\lambda}{n},\dots,\frac{\lambda}{n})$ is stationary.

Now let $1 \leqslant k \leqslant n-1$, $0 \leqslant a,b < 1$, $a \neq b$, and consider the system (4) for corresponding stationary points p:

$$\begin{aligned}(1-a)^{k-1}(1-b)^{n-k} + \gamma(1-a) &= d(p),\\ (1-a)^{k}(1-b)^{n-k-1} + \gamma(1-b) &= d(p).\end{aligned}$$

By elementary transformations it follows that

$$\gamma = (1-a)^{k-1}(1-b)^{n-k-1}.$$

Substituting γ in (4) and observing that

$$d(p) = (1-a)^{k-1}(1-b)^{n-k-1}(\lambda - nab) = \gamma(\lambda - nab),$$

we obtain the following system for stationary points with two different components:

$$\begin{aligned} nab - a - b &= \lambda - 2, \\ ka + (n-k)b &= \lambda, \\ 0 \leqslant a, b &< 1, \ a \neq b. \end{aligned} \tag{5}$$

Now, let $p = (a, \ldots, a, b, \ldots, b)$ be a stationary point whose coordinates satisfy the system of equations (5). Then $b = b(a) = \frac{a+\lambda-2}{n \cdot a - 1}$. We consider the monotone transformation

$$\phi(k, a) = \ln d(p) = (k-1)\ln(1-a) + (n-k-1)\ln(1-b) \\ + \ln(\lambda - nab),$$

which does not alter the maximum points of $d(p)$. It holds that $\phi_a(k, a) = 0$ for all stationary points a, where ϕ_a denotes the partial derivative of $\phi(k, a)$ w.r.t. a. By the second equation of (5), $a = a(k)$ is a function of k. Substituting k by the continuous variable x, and taking account of $\phi_a(x, a) = 0$, we get

$$\phi_x(x, a(x)) = \ln(1-a) - \ln(1-b) + \phi_a(x, a) \cdot a'(x) = 0,$$

equivalently $\ln \frac{1-a}{1-b} = 0$, i.e., $1 - a = 1 - \frac{\lambda - k \cdot a}{n-k}$, thus $a = \frac{\lambda}{n}$.

Hence, stationary points of $\phi(x, a(x))$ satisfy $a = \frac{\lambda}{n}$, together with (5) implying that the maximum is attained at $x = n$. Therefore, the best local maximum over the interior of C is attained at the point $p^o = (\frac{\lambda}{n}, \ldots, \frac{\lambda}{n})$ with the value $\lambda \cdot (1 - \frac{\lambda}{n})^{n-1}$.

Comparing $d(p^*)$ and $d(p^o)$ shows that

$$d(p^o) > d(p^*), \quad \text{if } \lambda > \Big(\frac{n}{n-1}\Big)^{n-1},$$
$$d(p^o) \leqslant d(p^*), \quad \text{if } \lambda \leqslant \Big(\frac{n}{n-1}\Big)^{n-1}. \qquad (6)$$

The local maxima over the relative interior of the $(k-1)$-dimensional faces of the boundary of C have a similar structure as p^o, namely $\varphi(k) = d(p^o_k) = d(0,\ldots,0,\frac{\lambda}{k},\ldots,\frac{\lambda}{k}) = \lambda\cdot(1-\frac{\lambda}{k})^{k-1}$. In general φ is not a monotone increasing function of k. Assuming again a continuous variable x substituting $k \in \mathbb{N}_0$, and differentiating the function $F(x) = \ln\,\varphi(x)$ yields

$$F'(x) = \ln(1-z) + \frac{z}{1-z} - \frac{z}{\lambda}\cdot\frac{z}{1-z}, \; z = \frac{\lambda}{x}.$$

Similar to the argument in (2), a power series expansion shows that

$$\begin{aligned} F'(x) &= -\sum_{k=1}^{\infty}\frac{z^k}{k} + \sum_{k=1}^{\infty} z^k - \frac{1}{\lambda}\sum_{k=1}^{\infty} z^{k+1} \\ &= \sum_{k=2}^{\infty}\Big(\frac{k-1}{k} - \frac{1}{\lambda}\Big)z^k > 0, \end{aligned}$$

whenever $\lambda > 2$. The function $\varphi(k)$ is increasing in k for $\lambda > 2$, and in this case p^o dominates the maximum points (of similar structure) over all relative interior points of the faces of lower dimension. The case $\lambda \leqslant 2$ is irrelevant since according to (6) the point p^o is dominated by p^*.

Summarizing our results we conclude with the following

Theorem. *The throughput $d(p_1,\ldots,p_n)$ of slotted ALOHA with n users applying access probabilities $0 \leqslant p_1,\ldots,p_n \leqslant 1$, $\sum_{i=1}^n p_i = \lambda$, is maximized at*

$$p = (p_1,\ldots,p_n) = \begin{cases} (0,\ldots,0,\lambda), & \text{if } \lambda \leqslant 1, \\ (\frac{\lambda-1}{n-1},\ldots,\frac{\lambda-1}{n-1},1), & \text{if } 1 < \lambda \leqslant (\frac{n}{n-1})^{n-1}, \\ (\frac{\lambda}{n},\ldots,\frac{\lambda}{n}), & \text{if } \lambda > (\frac{n}{n-1})^{n-1}. \end{cases}$$

REFERENCES

Abramson, N. (1970). The ALOHA system – another alternative for computer communication AFIPS. *Spring Joint Computer Conference,* Vol.37. pp. 281–285.

Abramson, N. (1985). Development of the ALOHANET. *IEEE Transactionson Information Theory,* Vol.IT–31. pp. 119–123.

Marshall, A.W., and I.Olkin (1979). *Inequalities, Theory of Majorization and its Applications.* Academic Press, New York.

Mathar, R., and A.Žilinskas (1993). Maximizing throughput of finite population slotted ALOHA. *Informatica,* **4**(1–2), 163–171.

Roberts, L.G. (1972). Extensions of packet communication technology to a hand held personal terminal. *AFIPS, Spring Joint Computer Conference,* Vol.4.0. pp. 295–298.

Tanenbaum, A.S. (1988). *Computer Networks.* Prentice Hall, Englewood Cliffs, NJ.

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