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Interval-Valued and Circular Intuitionistic Fuzzy Present Worth Analyses

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Abstract. Present worth (PW) analysis is an important technique in engineering economics for investment analysis. The values of PW analysis parameters such as interest rate, first cost, salvage value and annual cash flow are generally estimated including some degree of uncertainty. In order to capture the vagueness in these parameters, fuzzy sets are often used in the literature. In this study, we introduce interval-valued intuitionistic fuzzy PW analysis and circular intuitionistic fuzzy PW analysis in order to handle the impreciseness in the estimation of PW analysis parameters. Circular intuitionistic fuzzy sets are the latest extension of intuitionistic fuzzy sets defining the uncertainty of membership and non-membership degrees through a circle whose radius is r. Thus, we develop new fuzzy extensions of PW analysis including the uncertainty of membership functions. The methods are given step by step and an application for water treatment device purchasing at a local municipality is illustrated in order to show their applicability. In addition, a multi-parameter sensitivity analysis is given. Finally, discussions and suggestions for future research are given in conclusion section.

Key words: Present Worth analysis, Circular intuitionistic fuzzy sets, engineering economics, interval-valued intuitionistic fuzzy sets.

1. Introduction

Engineering economics is a collection of mathematical techniques which easify the comparison of investment alternatives. The main investment analysis techniques of engineering economics are benefit/cost ratio analysis (B/C), rate of return analysis (ROR), present worth analysis (PW), annual cash flow analysis (ACF) and payback period analysis (PPA). PW analysis is the major technique of engineering economics which finds the equivalent present worth of the future cash flows based on these parameters: first cost (FC), salvage value (SV), interest rate (i), annual benefits (AB), annual cost (AC), and life (n).

Fuzzy sets theory was developed by Zadeh (1965) in 1965 and the extensions of these ordinary fuzzy sets (OFSs) have been developed by numerous fuzzy set researchers. These fuzzy set extensions have been used in estimating, decision making, engineering economics, and controlling together with other intelligent systems. The extensions of ordinary fuzzy sets can be given as type-2 fuzzy sets in Zadeh (1975), intuitionistic fuzzy sets (IFSs)

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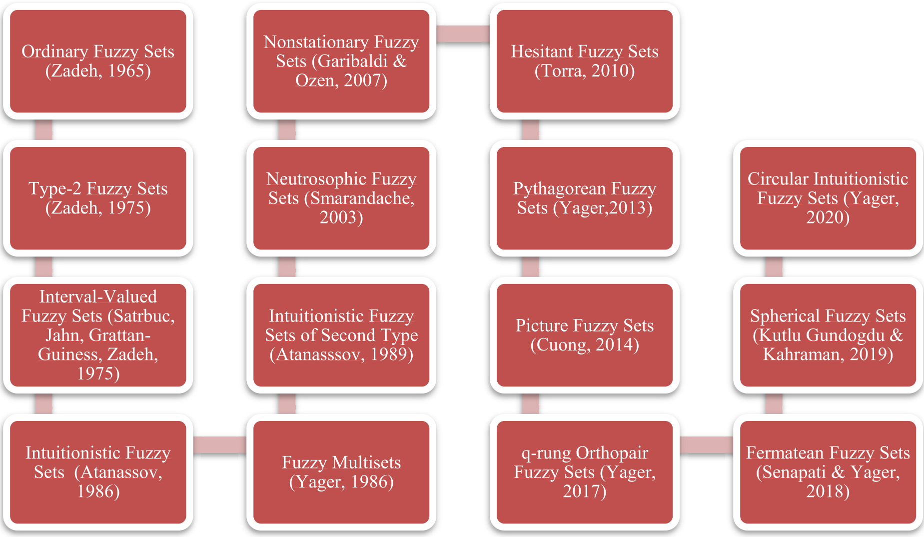

Fig. 1. Fuzzy set extensions.

in Atanassov (1986), fuzzy multisets in Yager (1986), intuitionistic fuzzy sets of second type in Atanassov (1989), neutrosophic sets (NSs) in Smarandache (1999), nonstationary fuzzy sets in Garibaldi and Ozen (2007), hesitant fuzzy sets (HFSs) in Torra (2010), Pythagorean fuzzy sets (PFSs) in Yager (2013), picture fuzzy sets in Cuong (2014), q-rung orthopair fuzzy sets (q-ROFs) in Yager (2017), fermatean fuzzy sets (FFSs) in Senapati and Yager (2020), spherical fuzzy sets (SFSs) in Kutlu Gündoğdu and Kahraman (2019) and circular intuitionistic fuzzy sets (C-IFSs) in Atanassov (2020). In Fig. 1, the historical progress of the fuzzy set theory is given.

Consideration of vagueness in the definition of membership functions is an old issue first time handled by Zadeh (1975). Type-2 fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets try to incorporate this vagueness into their models. Similarly, Atanassov (2020) developed C-IFSs as an extension of IFSs in order to handle this issue. A circle around the single valued intuitionistic fuzzy number is defined by a radius r. In this study, the main aim and contribution is to introduce a new extension of PW analysis with interval-valued intuitionistic fuzzy sets (IVIF) and C-IFSs.

The estimation of investment parameters generally involves uncertainty and vagueness. This uncertainty is best handled by the fuzzy set theory in the literature. Most of the publications on fuzzy engineering economics employ ordinary fuzzy sets. However, there are some papers employing intuitionistic fuzzy sets, Pythagorean fuzzy sets, neutrosophic sets, hesitant fuzzy sets, type-2 fuzzy sets, and fermatean fuzzy sets in the investment analysis techniques. In the following, we present the review results on fuzzy PW analysis. Kahraman *et al.* (1995) presented financial models based on some discounting techniques such as fuzzy equivalent uniform annual worth and fuzzy PW analyses. Iliev and Fustik (2003) used fuzzy net present analysis in evaluating hydroelectric projects based on fuzzy profitability index. This model was created by using triangular fuzzy numbers. Omitaomu *et al.* (2004) used present value model with triangular fuzzy numbers in the evaluation of information system projects. Kahraman *et al.* (2004) proposed fuzzy present worth based fuzzy models for quantifying manufacturing flexibility. The fuzzy model included uncertain cash flows and discount rates that were handled as triangular fuzzy numbers. Kahraman and Kaya (2008) studied equivalent fuzzy annual worth analysis in investment assessment. Matos and Dimitrovski (2008) introduced studies using equivalent uniform annual worth analysis with trapezoidal fuzzy numbers. Kuchta (2008) presented fuzzy PW analysis applications in optimization. Dimitrovski and Matos (2008) introduced uncorrelated and correlated cash flows in fuzzy PW analysis with arithmetic operations. Shahriari (2011) proposed a fuzzy net present value methodology that uses triangular fuzzy numbers in investment analysis. Kahraman *et al.* (2015) presented hesitant and intuitionistic fuzzy present and annual worth analyses. These developed methods use triangular hesitant fuzzy data, triangular intuitionistic fuzzy data, interval-valued hesitant data, and interval-valued intuitionistic fuzzy data in engineering economic problems for better forecasting. Kahraman *et al.* (2018a) introduced Pythagorean PW analysis for investment decision problems. Sarı and Kahraman (2017) applied net PW analysis with type-2 fuzzy sets. One of the PW analysis papers which uses neutrosophic sets belongs to Aydin *et al.* (2018). They introduced simplified neutrosophic PW analysis. The investment parameters' membership functions were defined by neutrosophic sets. The method was compared with classical and intuitionistic fuzzy PW analysis. Kahraman *et al.* (2018b) proposed ordinary fuzzy PW analysis, type-2 fuzzy PW analysis, intuitionistic fuzzy PW analysis, and hesitant fuzzy PW analysis in wind energy investment analysis. Aydin and Kabak (2020) developed future and present worth techniques in investment analysis with neutrosophic sets. Sergi and Sari (2021) extended PW analysis with fermatean fuzzy sets. The literature review is summarized in Table 1.

This paper is organized as follows: In Section 2, preliminaries of IVIF and C-IFSs are given. In Section 3, interval-valued intuitionistic fuzzy PW analysis extension is given. In Section 4, circular intuitionistic fuzzy PW analysis extension is given. In Section 5, a real life problem is solved with these proposed extensions in order to show the applicability of these proposed methods with a sensitivity analysis. Finally, conclusion is given in Section 6 with future suggestions.

2. Preliminaries

In this section, the preliminaries of interval-valued intuitionistic fuzzy sets and C-IFSs are given with definitions.

2.1. *Interval-Valued Intuitionistic Fuzzy Sets*

Intuitionistic fuzzy sets (IFSs) (Atanassov, 1986) were introduced by Atanassov in 1986. IVIFSs are an extension of IFSs that is developed by Atanassov and Gargov (1989) which have extensively been employed in the literature. IVIF numbers' preliminaries are summarized in the following:

Table 1
Fuzzy PW analysis publications.

Authors	Year	Type of fuzzy sets	Problem	Publication type
Kahraman *et al.* (1995)	1995	OFSs	Fuzzy flexibility evaluation	Conference paper
Iliev and Fustik (2003)	2003	OFSs	Hydroelectric project economical evaluation	Conference paper
Omitaomu *et al.* (2004)	2004	OFSs	Information system project for engineering economic analysis	Conference paper
Kahraman *et al.* (2004)	2004	OFSs	Fuzzy present worth models a for quantifying manufacturing flexibility	Article
Kahraman and Kaya (2008)	2008	OFSs	Fuzzy equivalent annual worth analysis in investment assessment	Book chapter
Matos and Dimitrovski (2008)	2008	OFSs	Fuzzy equivalent uniform annual worth analysis	Book chapter
Kuchta (2008)	2008	OFSs	Project selection optimization problem fuzzy net present value analysis	Book chapter
Dimitrovski and Matos (2008)	2008	OFSs	Uncorrelated and correlated cash flow in fuzzy PW analysis	Book chapter
Shahriari (2011)	2011	OFSs	Triangular fuzzy net present value for projects presentation	Book chapter
Kahraman *et al.* (2015)	2015	HFSs & Triangular hesitant data sets Interval-valued intuitionistic fuzzy sets & Triangular interval-valued intuitionistic fuzzy sets	Hesitant and intuitionistic fuzzy present worth and annual worth analyses	Article
Kahraman *et al.* (2018a)	2018	PFSs	Pythagorean fuzzy PW analysis in investments.	Conference paper
Sarı and Kahraman (2017)	2017	Type-2 fuzzy sets	Solid waste collection system for selection between roadside and underground waste bins.	Book chapter
Aydin *et al.* (2018)	2018	NSs	Investment evaluation problem with present value analysis.	Article
Kahraman *et al.* (2018b)	2018	OFSs Type-2 fuzzy sets HFSs	Ordinary fuzzy PW analysis, type-2 fuzzy PW analysis, intuitionistic fuzzy PW analysis, and hesitant fuzzy PW analysis	Book chapter
Aydin and Kabak (2020)	2020	NSs	Single valued neutrosophic present and future worth analysis	Article
Sergi and Sari (2021)	2021	FFSs	Fermatean fuzzy net PW analysis	Conference paper

Definition 1. Let X be a non-empty set. An IVIF set in X is an object $\tilde{A}$ given as in Eq. (1) (Atanassov and Gargov, 1989):

$$\tilde{X} = \{\langle x, [\mu_{\tilde{x}}^-, \mu_{\tilde{x}}^+], [\upsilon_{\tilde{x}}^-, \upsilon_{\tilde{x}}^+]\rangle; x \in X\}, \tag{1}$$

where $0 \leqslant \mu_{\tilde{x}}^+ + \upsilon_{\tilde{x}}^+ \leqslant 1$ for every $x \in X$.

Definition 2. Let $\tilde{A} = ([\mu_{\tilde{A}}^{-}, \mu_{\tilde{A}}^{+}], [v_{\tilde{A}}^{-}, v_{\tilde{A}}^{+}])$ and $\tilde{B} = ([\mu_{\tilde{B}}^{-}, \mu_{\tilde{B}}^{+}], [v_{\tilde{B}}^{-}, v_{\tilde{B}}^{+}])$ be two IVIF numbers (Xu, 2007). Then

$$\tilde{A} \oplus \tilde{B} = \left([\mu_{\tilde{A}}^{-} + \mu_{\tilde{B}}^{-} - \mu_{\tilde{A}}^{-}\mu_{\tilde{B}}^{-}, \mu_{\tilde{A}}^{+} + \mu_{\tilde{B}}^{+} - \mu_{\tilde{A}}^{+}\mu_{\tilde{B}}^{+}], [v_{\tilde{A}}^{-}v_{\tilde{B}}^{-}, v_{\tilde{A}}^{+}v_{\tilde{B}}^{+}]\right), \tag{2}$$

$$\tilde{A} \otimes \tilde{B} = \left([\mu_{\tilde{A}}^{-}\mu_{B}^{-}, \mu_{\tilde{A}}^{+}\mu_{B}^{+}], [v_{\tilde{A}}^{-} + v_{B}^{-} - v_{\tilde{A}}^{-}v_{B}^{-}, v_{\tilde{A}}^{+} + v_{B}^{+} - v_{\tilde{A}}^{+}v_{B}^{+}]\right). \tag{3}$$

Definition 3. Let $\tilde{r}_{ij}^{i} = ([\mu_{\tilde{r}}^{-}, \mu_{\tilde{r}}^{+}], [v_{\tilde{r}}^{-}, v_{\tilde{r}}^{+}])$ be the IVIF number where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Aggregated IVIF number $(\tilde{r}_{ij}^{Agg})$ by interval-valued intuitionistic fuzzy hybrid geometric operator is obtained as in Eq. (4) (Wei and Wang, 2007):

$$\tilde{r}_{ij}^{Agg} = \left\langle \left[\prod_{j=1}^{n} (\mu_j^{-})^{\omega_j}, \prod_{j=1}^{n} (\mu_j^{+})^{\omega_j}\right], \left[1 - \prod_{j=1}^{n} (1 - v_j^{-})^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - v_j^{+})^{\omega_j}\right]\right\rangle, \tag{4}$$

where $i = 1, 2, \ldots, m$ and $s = 1, 2, \ldots, k$. ω_j is the weights of expert i, where $\sum_{s=1}^{k} \omega_s = 1$.

Definition 4. Let $\tilde{r} = ([\mu_{\tilde{r}}^{-}, \mu_{\tilde{r}}^{+}], [v_{\tilde{r}}^{-}, v_{\tilde{r}}^{+}])$ be an IVIF number. Defuzzification formula $(\mathfrak{D}(\mathrm{x}))$ for $\tilde{r}$ is given as in Eq. (5) (Atanassov and Gargov, 1989):

$$\mathfrak{D}(\mathrm{x}) = \frac{\mu_{\tilde{r}}^{-} + \mu_{\tilde{r}}^{+} + (1 - v_{\tilde{r}}^{-}) + (1 - v_{\tilde{r}}^{+}) + \mu_{\tilde{r}}^{-} \times \mu_{\tilde{r}}^{+} - \sqrt{(1 - v_{\tilde{r}}^{-}) \times (1 - v_{r}^{+})}}{4}. \tag{5}$$

Definition 5. Let $\tilde{r} = ([\mu_{\tilde{r}}^{-}, \mu_{\tilde{r}}^{+}], [v_{\tilde{r}}^{-}, v_{\tilde{r}}^{+}])$ be an IVIF number. The score function of an IVIF number is given as in Eq. (6) (Xu, 2007, 2010):

$$S(\tilde{r}) = \left(\frac{1}{2}\right) \times \left(\mu_{\tilde{r}}^{-} - v_{\tilde{r}}^{-} + \mu_{\tilde{r}}^{+} - v_{\tilde{r}}^{+}\right), \tag{6}$$

where $S(\tilde{r}) \in [-1, 1]$.

Definition 6. Let $\tilde{r} = (\mu_{\tilde{r}}, v_{\tilde{r}})$ be an intuitionistic fuzzy number. The score function of this number is given as in Eq. (7).

$$S(\tilde{r}) = \mu_{\tilde{r}} - v_{\tilde{r}}, \tag{7}$$

where $S(\tilde{r}) \in [-1, 1]$.

Definition 7. Let $\tilde{r} = ([\mu_{r}^{-}, \mu_{\tilde{r}}^{+}], [v_{\tilde{r}}^{-}, v_{\tilde{r}}^{+}])$ be an IVIF number. The accuracy function of an IVIF number is given in Eq. (8) (Xu, 2007):

$$AF(\tilde{r}) = \left(\frac{1}{2}\right) \times \left(\mu_{\tilde{r}}^{-} + v_{\tilde{r}}^{-} + \mu_{\tilde{r}}^{+} + v_{\tilde{r}}^{+}\right), \tag{8}$$

where $S(\tilde{r}) \in [-1, 1]$.

2.2. *Circular Intuitionistic Fuzzy Sets*

C-IFSs are introduced by Atanassov (2020) as an extension of the IFSs which each element is represented by a circle with radius r. C-IFSs is defined in Definition 8.

Definition 8. Let E be a fixed universe. A C-IFS C_r in E is an object having the form as in Eq. (9).

$$C_r = \{\langle x, \mu_C(x), \nu_C(x); r\rangle \mid x \in E\}, \tag{9}$$

where

$$0 \leqslant \mu_C(x) + \nu_C(x) \leqslant 1 \tag{10}$$

and $r \in [0, 1]$ is a radius of the circle around each element $x \in E$, is called Circular-IFS and the functions $\mu_C : E \to [0, 1]$ and $\upsilon_C : E \to [0, 1]$ represent the degree of membership and the degree of non-membership of the element $x \in E$ to the set $C \subseteq E$, respectively.

The degree of indeterminacy is calculated as in Eq. (11):

$$\pi_C(x) = 1 - \mu_C(x) - \nu_C(x). \tag{11}$$

In contrast with the standard IFSs, where each element is represented by a point in the intuitionistic fuzzy interpretation triplet, each element in C-IFSs is represented by a circle with centre $\langle \mu_C(x), \nu_C(x)\rangle$ and radius r.

Definition 9. In an IFS C_j, let intuitionistic fuzzy pairs have the form $\{\langle m_{j,1}, n_{j,1}\rangle, \langle m_{j,2}, n_{j,2}\rangle, \ldots\}$. j is the number of IFSs C_j, each including k_j IF pairs. Then, C-IFSs is calculated as follows. The arithmetic average of the IF pairs is given as in Eq. (12):

$$\langle \mu(C_i), \nu(C_i)\rangle = \left\langle \frac{\sum_{s=1}^{k_j} m_{i,j}}{k_j}, \frac{\sum_{s=1}^{k_j} n_{i,j}}{k_j} \right\rangle, \tag{12}$$

where k_j is the number of intuitionistic fuzzy pairs in C_j.

Then, the radius of the $\langle \mu(C_j), \nu(C_j)\rangle$ is the maximum of the Euclidean distances given as in Eq. (13):

$$r_j = \max_{1 \leqslant j \leqslant k_j} \sqrt{\left(\mu(C_j) - m_{i,j}\right)^2 + \left(\nu(C_j) - n_{i,j}\right)^2}. \tag{13}$$

For universe $W = \{C_1, C_2, \ldots\}$, the C-IFS can be built as in Eq. (14):

$$A_r = \{\langle C_j, \mu(C_j), \nu(C_j); r\rangle \mid C_j \in W\} = \{\langle C_j, O_r(\mu(C_j), \nu(C_j))\rangle \mid C_j \in W\}. \tag{14}$$

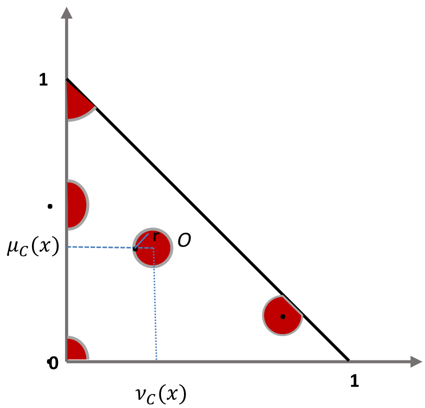

Fig. 2. C-IFS geometrical representation.

Definition 10. Let's have five circle forms as it is shown in Fig. 2, where the basic geometric interpretation of C-IFS is given.

$$L^* = \{\langle a, b\rangle \mid a, b \in [0, 1] \;\&\; a + b \leqslant 1\}. \tag{15}$$

Therefore, C_r can be rewritten in the form

$$C_r^* = \{\langle x, O(\mu_C(x), \nu_C(x)); r\rangle \mid x \in E\}, \tag{16}$$

where O is a function representing a circle, whose radius is r and whose centre is $(\mu_C(x), \nu_C(x))$.

$$\begin{aligned} & O(\mu_C(x), \nu_C(x)) \\ & \quad = \left\{\langle a, b\rangle \mid a, b \in [0, 1], \sqrt{(\mu_C(x) - a)^2 + (\nu_C(x) - b)^2} \leqslant r\right\} \cap L^* \\ & \quad = \left\{\langle a, b\rangle \mid a, b \in [0, 1], \sqrt{(\mu_C(x) - a)^2 + (\nu_C(x) - b)^2} \leqslant r, a + b \leqslant 1\right\}. \end{aligned}$$

C-IFSs is an extension of the standard IFSs and each standard IFS has the form $C = C_0 = \{\langle x, O(\mu_C(x), \nu_C(x)); 0\rangle \mid x \in E\}$, therefore, C-IFS with $r > 0$ can't be represented by a standard IFS.

Definition 11. Let $C_1 = \langle \mu_{C_1}(x), \nu_{C_1}(x);\ r_1\rangle$ and $C_2 = \langle \mu_{C_2}(x), \nu_{C_2}(x); r_2\rangle$ be two circular intuitionistic fuzzy numbers. The operations given here are based on the minimum and maximum of the radiuses separately since they give the results with minimum and maximum level of uncertainty, respectively. Smaller radius represents smaller vagueness whereas larger radius represents larger vagueness for IF pairs. Their operations can be described in Eqs. (17)–(24):

$$\begin{aligned}
& C_1 \cap_{\min} C_2 \\
& \quad = \{\langle x, \min(\mu_{C_1}(x), \mu_{C_2}(x)), \max(\nu_{C_1}(x), \nu_{C_2}(x)); \min(r_1, r_2)\rangle \mid x \in E\}, \quad (17) \\
& C_1 \cap_{\max} C_2 \\
& \quad = \{\langle x, \min(\mu_{C_1}(x), \mu_{C_2}(x)), \max(\nu_{C_1}(x), \nu_{C_2}(x)); \max(r_1, r_2)\rangle \mid x \in E\}, \quad (18) \\
& C_1 \cup_{\min} C_2 \\
& \quad = \{\langle x, \max(\mu_{C_1}(x), \mu_{C_2}(x)), \min(\nu_{C_1}(x), \nu_{C_2}(x)); \min(r_1, r_2)\rangle \mid x \in E\}, \quad (19) \\
& C_1 \cup_{\max} C_2 \\
& \quad = \{\langle x, \max(\mu_{C_1}(x), \mu_{C_2}(x)), \min(\nu_{C_1}(x), \nu_{C_2}(x)); \max(r_1, r_2)\rangle \mid x \in E\}, \quad (20) \\
& C_1 \oplus_{\min} C_2 \\
& \quad = \{\langle x, \mu_{C_1}(x) + \mu_{C_2}(x) - \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) * \nu_{C_2}(x); \\
& \qquad \min(r_1, r_2)\rangle \mid x \in E\}, \quad (21) \\
& C_1 \oplus_{\max} C_2 \\
& \quad = \{\langle x, \mu_{C_1}(x) + \mu_{C_2}(x) - \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) * \nu_{C_2}(x); \\
& \qquad \max(r_1, r_2)\rangle \mid x \in E\}, \quad (22) \\
& C_1 \otimes_{\min} C_2 \\
& \quad = \{\langle x, \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) + \nu_{C_2}(x) - \nu_{C_1}(x) * \nu_{C_2}(x); \\
& \qquad \min(r_1, r_2)\rangle \mid x \in E\}, \quad (23) \\
& C_1 \otimes_{\max} C_2 \\
& \quad = \{\langle x, \mu_{C_1}(x) * \mu_{C_2}(x), \nu_{C_1}(x) + \nu_{C_2}(x) - \nu_{C_1}(x) * \nu_{C_2}(x); \\
& \qquad \max(r_1, r_2)\rangle \mid x \in E\}. \quad (24)
\end{aligned}$$

Aggregation of the intuitionistic fuzzy numbers is realized by using Definition 12.

Definition 12. Let $\tilde{A}_i = (\mu_{\tilde{A}_i}, \nu_{\tilde{A}_i})$ $(i = 1, 2, \dots, n)$ be a set of IFNs and $w = (w_1, w_2, \dots, w_n)^T$ be weight vector of $\tilde{A}_i$ with $\sum_{i=1}^{n} w_i = 1$, then an intuitionistic fuzzy weighted geometric (*IFWG*) operator is given in Eq. (25) (Xu and Yager, 2006):

$$IFWG(\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_n) = \left(\prod_{i=1}^{n} \mu_{\tilde{A}_i}^{w_i}, \left(1 - \prod_{i=1}^{n} (1 - \upsilon_{\tilde{A}_i})^{w_i}\right)\right). \quad (25)$$

3. Interval-Valued Intuitionistic Fuzzy PW Analysis

The parameters of *FC*, *SV*, *AB*, *AC*, i, and n are given with interval-valued intuitionistic fuzzy values which are expressed by circular intuitionistic fuzzy numbers (IVFN) in

Eqs. (26)–(31):

$$\widetilde{FC}_I = \left\{ \begin{array}{c} \langle fc_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle fc_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle fc_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}, \tag{26}$$

$$\widetilde{AC}_I = \left\{ \begin{array}{c} \langle ac_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle ac_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle ac_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}, \tag{27}$$

$$\widetilde{AB}_I = \left\{ \begin{array}{c} \langle ab_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle ab_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle ab_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}, \tag{28}$$

$$\widetilde{SV}_I = \left\{ \begin{array}{c} \langle sv_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle sv_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle sv_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}, \tag{29}$$

$$\tilde{\imath}_I = \left\{ \begin{array}{c} \langle i_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle i_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle i_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}, \tag{30}$$

$$\tilde{n}_I = \left\{ \begin{array}{c} \langle n_1, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \langle n_2, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle, \dots, \\ \langle n_k, \mathrm{IVFN}_1, \dots, \mathrm{IVFN}_m \rangle \end{array} \right\}. \tag{31}$$

Aggregation of IVIF numbers is performed by Eq. (25). Later, parameter values can be computed by multiplying the defuzzified values of membership functions with parameter values. The score function and defuzzification function of memberships are used for obtaining crisp values for each parameter. After defuzzification process the present worth is obtained by Eq. (32):

$$\widetilde{PW}_I = -\widetilde{FC}_I - \widetilde{AC}_I \left[\frac{(1+\tilde{\imath}_I)^{\tilde{n}_I} - 1}{\tilde{\imath}_I (1+\tilde{\imath}_I)^{\tilde{n}_I}} \right] + \widetilde{AB}_I \left[\frac{(1+\tilde{\imath}_I)^{\tilde{n}_I} - 1}{\tilde{\imath}_I (1+\tilde{\imath}_I)^{\tilde{n}_I}} \right] + \widetilde{SV}_I (1+\tilde{\imath}_I)^{-I}, \tag{32}$$

where $\widetilde{PW}_I$: Intuitionistic fuzzy PW, $\widetilde{FC}_I$: Intuitionistic fuzzy FC, $\widetilde{AC}_I$: Intuitionistic fuzzy AC, $\widetilde{AB}_I$: Intuitionistic fuzzy AB, $\widetilde{SV}_I$: Intuitionistic fuzzy SV, $\tilde{\imath}_I$: Intuitionistic fuzzy interest rate (i), $\tilde{n}_I$: Intuitionistic fuzzy life (n).

4. Circular Intuitionistic Fuzzy PW Analysis

The parameters of FC, SV, AB, AC, i, and n are given by circular intuitionistic fuzzy numbers (CIFN) as in Eqs. (33)–(38):

$$\widetilde{FC}_{CIF} = \left\{ \begin{array}{c} \langle fc_1, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \langle fc_2, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \dots, \\ \langle fc_k, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{33}$$

$$\widetilde{AC}_{CIF} = \left\{ \begin{array}{c} \langle ac_1, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \langle ac_2, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \dots, \\ \langle ac_k, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{34}$$

$$\widetilde{AB}_{CIF} = \left\{ \begin{array}{c} \langle ab_1, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \langle ab_2, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle, \dots, \\ \langle ab_k, \mathrm{CIFN}_1, \dots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{35}$$

$$\widetilde{SV}_{CIF} = \left\{ \begin{array}{c} \langle sv_1, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \langle sv_2, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \ldots, \\ \langle sv_k, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{36}$$

$$\tilde{\imath}_{CIF} = \left\{ \begin{array}{c} \langle i_1, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \langle i_2, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \ldots, \\ \langle i_k, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{37}$$

$$\tilde{n}_{CIF} = \left\{ \begin{array}{c} \langle n_1, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \langle n_2, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle, \ldots, \\ \langle n_k, \mathrm{CIFN}_1, \ldots, \mathrm{CIFN}_m \rangle \end{array} \right\}, \tag{38}$$

where $\mathrm{CIFN} = (\langle x, O(\mu_C(x), \nu_C(x)); r \rangle)$.

In this method, aggregation of IFSs is performed by Eq. (25).

Later, parameter values can be computed by multiplying the defuzzified values of membership functions with parameter values. The score function and defuzzification function of memberships are used for obtaining crisp values for each parameter. After defuzzification process the present worth is obtained by Eq. (39):

$$\begin{aligned} \widetilde{PW}_{CIF} = {} & -\widetilde{FC}_{CIF} - \widetilde{AC}_{CIF}\left[\frac{(1+\tilde{\imath}_{CIF})^{\tilde{n}_{CIF}} - 1}{\tilde{\imath}_{CIF}(1+\tilde{\imath}_{ICIF})^{\tilde{n}_I}}\right] \\ & + \tilde{AB}_{CIF}\left[\frac{(1+\tilde{\imath}_{CIF})^{\tilde{n}_{CIF}} - 1}{\tilde{\imath}_{CIF}(1+\tilde{\imath}_{CIF})^{\tilde{n}_{CIF}}}\right] + \widetilde{SV}_{CIF}(1+\tilde{\imath}_{CIF})^{-I}, \end{aligned} \tag{39}$$

where $\widetilde{PW}_{CIF}$: Circular intuitionistic fuzzy PW, $\widetilde{FC}_{CIF}$: Circular intuitionistic fuzzy FC, $\widetilde{AC}_{CIF}$: Circular intuitionistic fuzzy AC, $\widetilde{AB}_{CIF}$: Circular intuitionistic fuzzy AB, $\widetilde{SV}_{CIF}$: Circular intuitionistic fuzzy SV, $\tilde{\imath}_{CIF}$: Circular intuitionistic fuzzy interest rate (i), $\tilde{n}_{CIF}$: Circular intuitionistic fuzzy life (n).

5. Application

A water treatment device will be purchased by a local municipality. The interval-valued intuitionistic fuzzy data of this device are given in Table 2. Three experts having different experiences assign three different intervals for each of investment parameters and determine their interval-valued intuitionistic fuzzy membership and non-membership degrees. w values in Table 2 represent the weights of experts based on their experience levels.

We apply two levels of aggregation: Aggregation within each parameter and aggregation between parameters. In Table 3, the aggregated membership functions and expected weighted interval-values within each parameter are given. Then, aggregation between parameters is applied. For both of aggregation levels, Eq. (4) is used.

Expected value calculations are given in Table 4. For instance, the value [218, 650; 259, 325] is calculated as follows:

$$\begin{aligned} FC1_{Expected\ Value1} &= 0.5\times 200{,}000 + 0.2\times 220{,}000 + 0.3\times 240{,}000 = 216{,}000, \\ FC2_{Expected\ Value1} &= 0.25\times 200{,}000 + 0.45\times 220{,}000 + 0.3\times 240{,}000 = 221{,}000, \\ FC3_{Expected\ Value1} &= 0.35\times 200{,}000 + 0.3\times 220{,}000 + 0.35\times 240{,}000 = 216{,}000, \end{aligned}$$

Table 2
Interval-valued intuitionistic fuzzy parameters.

Parameter	Expert	w	Alternative	$\langle[\mu^-, \mu^+], [\vartheta^-, \vartheta^+]\rangle$
First cost	E1	0.5	[200000, 250000]	⟨[0.7, 0.8], [0.1, 0.2]⟩
		0.2	[220000, 260000]	⟨[0.6, 0.7], [0.05, 0.25]⟩
		0.3	[240000, 270000]	⟨[0.5, 0.9], [0.05, 0.1]⟩
	E2	0.25	[200000, 250000]	⟨[0.7, 0.8], [0.1, 0.2]⟩
		0.45	[220000, 260000]	⟨[0.6, 0.7], [0.05, 0.25]⟩
		0.3	[240000, 270000]	⟨[0.5, 0.9], [0.05, 0.1]⟩
	E3	0.35	[200000, 250000]	⟨[0.7, 0.8], [0.1, 0.2]⟩
		0.3	[220000, 260000]	⟨[0.6, 0.7], [0.05, 0.25]⟩
		0.35	[240000, 270000]	⟨[0.5, 0.9], [0.05, 0.1]⟩
Annual benefit	E1	0.5	[40000, 50000]	⟨[0.8, 0.85], [0.1, 0.15]⟩
		0.2	[45000, 55000]	⟨[0.9, 0.95], [0, 0.05]⟩
		0.3	[50000, 60000]	⟨[0.8, 0.9], [0.05, 0.1]⟩
	E2	0.25	[40000, 50000]	⟨[0.8, 0.85], [0.1, 0.15]⟩
		0.45	[45000, 55000]	⟨[0.9, 0.95], [0, 0.05]⟩
		0.3	[50000, 60000]	⟨[0.8, 0.85], [0.1, 0.15]⟩
	E3	0.35	[40000, 50000]	⟨[0.8, 0.85], [0.1, 0.15]⟩
		0.3	[45000, 55000]	⟨[0.9, 0.95], [0, 0.05]⟩
		0.35	[50000, 60000]	⟨[0.8, 0.85], [0.1, 0.15]⟩
Annual cost	E1	0.5	[10000, 20000]	⟨[0.7, 0.8], [0.1, 0.2]⟩
		0.2	[15000, 25000]	⟨[0.9, 0.95], [0, 0.05]⟩
		0.3	[20000, 30000]	⟨[0.7, 0.85], [0.1, 0.15]⟩
	E2	0.25	[10000, 20000]	⟨[0.7, 0.9], [0.0, 0.1]⟩
		0.45	[15000, 25000]	⟨[0.9, 0.95], [0, 0.05]⟩
		0.3	[20000, 30000]	⟨[0.7, 0.85], [0.1, 0.15]⟩
	E3	0.35	[10000, 20000]	⟨[0.6, 0.7], [0.1, 0.3]⟩
		0.3	[15000, 25000]	⟨[0.7, 0.8], [0.05, 0.2]⟩
		0.35	[20000, 30000]	⟨[0.85, 0.9], [0.05, 0.1]⟩
Salvage value	E1	0.5	[80000, 100000]	⟨[0.65, 0.7], [0.15, 0.3]⟩
		0.2	[90000, 110000]	⟨[0.75, 0.80], [0.15, 0.2]⟩
		0.3	[100000, 120000]	⟨[0.7, 0.75], [0.1, 0.25]⟩
	E2	0.25	[80000, 100000]	⟨[0.60, 0.7], [0.1, 0.15]⟩
		0.45	[90000, 110000]	⟨[0.8, 0.90], [0.05, 0.10]⟩
		0.3	[100000, 120000]	⟨[0.7, 0.85], [0.1, 0.15]⟩
	E3	0.35	[80000, 100000]	⟨[0.55, 0.60], [0.2, 0.40]⟩
		0.3	[90000, 110000]	⟨[0.65, 0.75], [0.15, 0.25]⟩
		0.35	[100000, 120000]	⟨[0.75, 0.85], [0.1, 0.15]⟩
Interest rate	E1	0.5	[0.08, 0.10]	⟨[0.6, 0.65], [0.2, 0.3]⟩
		0.2	[0.09, 0.11]	⟨[0.7, 0.75], [0.15, 0.20]⟩
		0.3	[0.10, 0.12]	⟨[0.6, 0.65], [0.1, 0.25]⟩
	E2	0.25	[0.08, 0.10]	⟨[0.6, 0.7], [0.2, 0.3]⟩
		0.45	[0.09, 0.11]	⟨[0.7, 0.8], [0.15, 0.2]⟩
		0.3	[0.10, 0.12]	⟨[0.6, 0.9], [0.05, 0.10]⟩
	E3	0.35	[0.08, 0.10]	⟨[0.6, 0.7], [0.2, 0.3]⟩
		0.3	[0.09, 0.11]	⟨[0.7, 0.8], [0.15, 0.2]⟩
		0.35	[0.10, 0.12]	⟨[0.6, 0.8], [0.1, 0.2]⟩
Life	E1	0.5	[20, 23]	⟨[0.6, 0.75], [0.2, 0.25]⟩
		0.2	[21, 22]	⟨[0.7, 0.85], [0.1, 0.15]⟩
		0.3	[22, 23]	⟨[0.6, 0.7], [0.15, 0.3]⟩
	E2	0.25	[20, 24]	⟨[0.6, 0.8], [0.1, 0.2]⟩

(*continued on next page*)

Table 2
(*continued*)

Parameter	Expert	w	Alternative	$\langle[\mu^-, \mu^+], [\vartheta^-, \vartheta^+]\rangle$
		0.45	[22, 25]	⟨[0.7, 0.8], [0.1, 0.2]⟩
		0.3	[24, 26]	⟨[0.8, 0.85], [0.1, 0.15]⟩
	E3	0.35	[19, 23]	⟨[0.6, 0.7], [0.1, 0.3]⟩
		0.3	[21, 25]	⟨[0.7, 0.8], [0.1, 0.15]⟩
		0.35	[23, 27]	⟨[0.6, 0.8], [0.15, 0.2]⟩

Table 3
Aggregation within each parameter.

Parameter	Experts	Experts' weights	Weighted interval-values		Aggregated membership functions $\langle[\mu^-, \mu^+], [\vartheta^-, \vartheta^+]\rangle$
FC	E1	0.4	216000	258000	⟨[0.614, 0.807], [0.075, 0.182]⟩
	E2	0.25	221000	260500	⟨[0.590, 0.780], [0.063, 0.195]⟩
	E3	0.35	220000	260000	⟨[0.594, 0.801], [0.068, 0.182]⟩
AB	E1	0.4	44000	54000	⟨[0.819, 0.884], [0.066, 0.116]⟩
	E2	0.25	45250	55250	⟨[0.844, 0.894], [0.056, 0.106]⟩
	E3	0.35	45000	55000	⟨[0.829, 0.879], [0.071, 0.121]⟩
AC	E1	0.4	14000	24000	⟨[0.736, 0.843], [0.081, 0.157]⟩
	E2	0.25	15250	25250	⟨[0.784, 0.906], [0.031, 0.094]⟩
	E3	0.35	15000	25000	⟨[0.710, 0.796], [0.068, 0.204]⟩
SV	E1	0.4	88000	108000	⟨[0.684, 0.734], [0.135, 0.266]⟩
	E2	0.25	90500	110500	⟨[0.715, 0.831], [0.078, 0.128]⟩
	E3	0.35	90000	78500	⟨[0.645, 0.725], [0.151, 0.275]⟩
IR	E1	0.4	0.088	0.108	⟨[0.619, 0.669], [0.161, 0.266]⟩
	E2	0.25	0.091	0.111	⟨[0.643, 0.802], [0.134, 0.198]⟩
	E3	0.35	0.09	0.11	⟨[0.628, 0.763], [0.151, 0.237]⟩
n	E1	0.4	20.8	22.8	⟨[0.619, 0.753], [0.166, 0.247]⟩
	E2	0.25	22.1	25.05	⟨[0.701, 0.815], [0.113, 0.185]⟩
	E3	0.35	21	25	⟨[0.628, 0.763], [0.118, 0.223]⟩

$$\begin{aligned}
FC_{Expected\ Value1} &= 0.4\times FC1_{Expected\ Value} + 0.25\times FC2_{Expected\ Value}\\
&\quad + 0.35\times FC3_{Expected\ Value} = 218{,}650,\\
FC1_{Expected\ Value2} &= 0.5\times 250{,}000 + 0.2\times 260{,}000 + 0.3\times 270{,}000 = 258{,}000,\\
FC2_{Expected\ Value2} &= 0.25\times 250{,}000 + 0.45\times 260{,}000 + 0.3\times 270{,}000 = 260{,}500,\\
FC3_{Expected\ Value2} &= 0.35\times 250{,}000 + 0.3\times 260{,}000 + 0.35\times 270{,}000 = 260{,}000,\\
FC_{Expected\ Value2} &= 0.4\times FC1_{Expected\ Value} + 0.25\times FC2_{Expected\ Value}\\
&\quad + 0.35\times FC3_{Expected\ Value} = 259{,}325.
\end{aligned}$$

The aggregation between parameters is the next step. Lower and upper *PW* values are obtained with Eq. (31). For the lower *PW*, we select the upper values for *FC*, *AC*, *IR* and lower values for *AB*, *SV* and life. Similarly, for the lower *PW*, we select the upper values for *AB*, *SV* life and lower values for *FC*, *AC*, and *IR*. PW_{lower} and PW_{upper}

Table 4
Values after aggregation within each parameter.

Parameter	Expected interval-values	Aggregated membership functions $\langle[\mu^-,\mu^+],[\vartheta^-,\vartheta^+]\rangle$
FC	[218650; 259325]	⟨[0.601, 0.798], [0.070, 0.185]⟩
AB	[44662.5; 54,662.5]	⟨[0.829, 0.885], [0.065, 0.115]⟩
AC	[11862.5; 24,662.5]	⟨[0.738, 0.841], [0.064, 0.159]⟩
SV	[89325; 98300]	⟨[0.677, 0.754], [0.127, 0.237]⟩
IR	[0.089; 0.109]	⟨[0.628, 0.733], [0.151, 0.239]⟩
Life	[21195, 24.133]	⟨[0.642, 0.772], [0.136, 0.223]⟩

are obtained as −86,767.344$ and 212,187.693$, respectively. The aggregated membership function in order to calculate final PW, the membership functions are obtained by selecting the minimum values for membership degrees and maximum values for non-membership degrees. In this way, we finally obtained the interval-valued membership function as: ⟨[0.601, 0.733], [0.151, 0.239]⟩. In order to defuzzify membership function, Eq. (5) is used and we obtained it as 0.529. Final crisp PW using interval-valued intuitionistic fuzzy sets is calculated as follows:

$$PW = ((-86,767.344 + 212,187.693) \div 2) \times 0.529 = 33{,}162.894\$.$$

Now, circular intuitionistic fuzzy *PW* analysis will be applied. Table 5 presents the circular intuitionistic fuzzy investment parameters. First, *within aggregation operation*, then *between aggregation operation* is applied. The results of *within aggregation operation* are also given in Table 5.

A radius (r) is calculated for each parameter as in Eq. (13) and $r_{\max}$ value is obtained by selecting the maximum value among them. The weights vector of experts for aggregating membership functions is $w = [w_{E1}, w_{E2}, w_{E3}] = [0.4, 0.25, 0.35]$ where $\sum_{s=1}^{3} w_s = 1$. Eq. (25) is used for aggregation operations. Table 6 includes the expected values of parameters together with the aggregated membership functions and their $r_{\max}$ values.

For instance, the value of [218, 650; 259, 325] is calculated as follows:

$$\begin{aligned}
FC1_{\textit{Expected Value1}} &= 0.5\times200{,}000 + 0.2\times220{,}000 + 0.3\times240{,}000 = 216{,}000,\\
FC2_{\textit{Expected Value1}} &= 0.25\times200{,}000 + 0.45\times220{,}000 + 0.3\times240{,}000 = 221{,}000,\\
FC3_{\textit{Expected Value1}} &= 0.35\times200{,}000 + 0.3\times220{,}000 + 0.35\times240{,}000 = 220{,}000,\\
FC_{\textit{Expected Value1}} &= 0.4\times FC1_{\textit{Expected Value1}} + 0.25\times FC2_{\textit{Expected Value1}}\\
&\quad + 0.35\times FC3_{\textit{Expected Value1}} = 218{,}650,\\
FC1_{\textit{Expected Value2}} &= 0.5\times250{,}000 + 0.2\times260{,}000 + 0.3\times270{,}000 = 258{,}000,\\
FC2_{\textit{Expected Value2}} &= 0.25\times250{,}000 + 0.45\times260{,}000 + 0.3\times270{,}000 = 260{,}500,\\
FC3_{\textit{Expected Value2}} &= 0.35\times250{,}000 + 0.3\times260{,}000 + 0.35\times270{,}000 = 260{,}000,\\
FC_{\textit{Expected Value2}} &= 0.4\times FC1_{\textit{Expected Value2}} + 0.25\times FC2_{\textit{Expected Value2}}\\
&\quad + 0.35\times FC3_{\textit{Expected Value2}} = 259{,}325.
\end{aligned}$$

Table 5
Circular intuitionistic fuzzy parameters.

Parameters	Experts	W	Alternative	Membership function	Within aggregated membership functions (μ, ϑ)	r	$r_{\max}$
First Cost	E1	0.5	[200000, 250000]	(0.6, 0.3)	(0.648, 0.261)	0.486	0.627
		0.2	[220000, 260000]	(0.7, 0.25)		0.592	
		0.3	[240000, 270000]	(0.7, 0.2)		0.627	
	E2	0.25	[200000, 250000]	(0.8, 0.2)	(0.780, 0.195)	0.020	0.153
		0.45	[220000, 260000]	(0.7, 0.25)		0.097	
		0.3	[240000, 270000]	(0.9, 0.1)		0.153	
	E3	0.35	[200000, 250000]	(0.8, 0.2)	(0.801, 0.182)	0.018	0.129
		0.3	[220000, 260000]	(0.7, 0.25)		0.122	
		0.35	[240000, 270000]	(0.9, 0.1)		0.129	
Annual Benefit	E1	0.5	[40000, 50000]	(0.85, 0.15)	(0.884, 0.116)	0.048	0.093
		0.2	[45000, 55000]	(0.95, 0.05)		0.093	
		0.3	[50000, 60000]	(0.9, 0.1)		0.022	
	E2	0.25	[40000, 50000]	(0.85, 0.15)	(0.894, 0.106)	0.062	0.080
		0.45	[45000, 55000]	(0.95, 0.05)		0.080	
		0.3	[50000, 60000]	(0.85, 0.15)		0.062	
	E3	0.35	[40000, 50000]	(0.85, 0.15)	(0.879, 0.121)	0.041	0.080
		0.3	[45000, 55000]	(0.95, 0.05)		0.101	
		0.35	[50000, 60000]	(0.85, 0.15)		0.041	
Annual Cost	E1	0.5	[10000, 20000]	(0.8, 0.2)	(0.843, 0.157)	0.061	0.151
		0.2	[15000, 25000]	(0.95, 0.05)		0.151	
		0.3	[20000, 30000]	(0.85, 0.15)		0.010	
	E2	0.25	[10000, 20000]	(0.9, 0.1)	(0.906, 0.094)	0.100	0.160
		0.45	[15000, 25000]	(0.95, 0.05)		0.066	
		0.3	[20000, 30000]	(0.85, 0.15)		0.160	
	E3	0.35	[10000, 20000]	(0.7, 0.3)	(0.796, 0.204)	0.315	0.315
		0.3	[15000, 25000]	(0.8, 0.2)		0.200	
		0.35	[20000, 30000]	(0.9, 0.1)		0.145	
Salvage Value	E1	0.5	[80000, 100000]	(0.7, 0.3)	(0.734, 0.266)	0.048	0.093
		0.2	[90000, 110000]	(0.8, 0.2)		0.093	
		0.3	[100000, 120000]	(0.75, 0.25)		0.023	
	E2	0.25	[80000, 100000]	(0.7, 0.15)	(0.831, 0.128)	0.133	0.133
		0.45	[90000, 110000]	(0.9, 0.1)		0.075	
		0.3	[100000, 120000]	(0.85, 0.15)		0.029	
	E3	0.35	[80000, 100000]	(0.6, 0.4)	(0.725, 0.275)	0.176	0.177
		0.3	[90000, 110000]	(0.75, 0.25)		0.036	
		0.35	[100000, 120000]	(0.85, 0.15)		0.177	
Interest Rate	E1	0.5	[0.08, 0.10]	(0.65, 0.3)	(0.669, 0.266)	0.039	0.105
		0.2	[0.09, 0.11]	(0.75, 0.2)		0.105	
		0.3	[0.10, 0.12]	(0.65, 0.25)		0.025	
	E2	0.25	[0.08, 0.10]	(0.7, 0.3)	(0.802, 0.198)	0.144	0.144
		0.45	[0.09, 0.11]	(0.8, 0.2)		0.002	
		0.3	[0.10, 0.12]	(0.9, 0.1)		0.139	
	E3	0.35	[0.08, 0.10]	(0.7, 0.3)	(0.763, 0.237)	0.090	0.090
		0.3	[0.09, 0.11]	(0.8, 0.2)		0.052	
		0.35	[0.10, 0.12]	(0.8, 0.2)		0.052	
Life	E1	0.5	[20, 23]	(0.7, 0.3)	(0.675, 0.281)	0.032	0.085
		0.2	[21, 22]	(0.65, 0.2)		0.085	
		0.3	[22, 23]	(0.65, 0.3)		0.031	

(*continued on next page*)

Table 5
(*continued*)

Parameters	Experts	W	Alternative	Membership function	Within aggregated membership functions (μ, ϑ)	r	$r_{\max}$
	E2	0.25	[20, 24]	(0.7, 0.3)	(0.685, 0.257)	0.046	0.059
		0.45	[22, 25]	(0.7, 0.2)		0.059	
		0.3	[24, 26]	(0.65, 0.3)		0.055	
	E3	0.35	[19, 23]	(0.7, 0.3)	(0.682, 0.271)	0.034	0.074
		0.3	[21, 25]	(0.7, 0.2)		0,074	
		0.35	[23, 27]	(0.65, 0.3)		0,043	

Table 6
Expected parameters, aggregated membership functions and $r_{\max}$ values.

Parameters	Expected parameters	Within aggregated membership functions	$r_{\max}$
FC	[218,650; 259,325]	(0.731, 0.218)	0.627
AB	[44,662.5; 54,662.5]	(0.885, 0.115)	0.101
AC	[11,862.5; 24,662.5]	(0.841, 0.159)	0.315
SV	[89.325; 98.300]	(0.754, 0.237)	0.177
IR	[0.089; 0.109]	(0.733, 0.239)	0.144
Life	[21.195; 24.133]	(0.680, 0.272)	0.085

Table 7
Optimistic and pessimistic membership functions.

Parameters	Optimistic membership functions	Pessimistic membership functions
FC	(1, 0)	(0.104, 0.845)
AB	(0.985, 0.015)	(0.784, 0.216)
AC	(1, 0)	(0.526, 0.474)
SV	(0.931, 0.060)	(0.577, 0.414)
IR	(0.877, 0.096)	(0.589, 0.383)
Life	(0.764, 0.187)	(0.595, 0.356)
Membership functions (Max μ, Min ϑ for optimistic), (Max ϑ, Min μ for pessimistic)	(1, 0)	(0.104, 0.845)

Table 7 gives us the values of the investment parameters for both optimistic and pessimistic cases, separately.

The score values for optimistic and pessimistic membership values are calculated as 1 and -0.741, respectively using Eq. (7).

Table 8 shows the average values of expected interval-values for the parameters.

Based on the parameter values in Table 8, PW is calculated as 62,710.2\$. The *PWs* for optimistic and pessimistic cases are obtained as in Eq. (40):

Table 8
Expected interval-values and average values of parameters.

Parameters	Expected interval-values	Midpoints
FC	[218,650; 259,325]	238,987.500
AB	[44,662.5; 54,662.5]	49,662.500
AC	[11,862.5; 24,662.5]	18,262.500
SV	[89,325; 98,300]	93,812.500
IR	[0.089; 0.109]	0.099
Life	[21.195; 24.133]	22.664

$$\mathrm{PW}_c = \mathrm{PW} \times (1 + \mathrm{SF}),\ c\text{: optimistic, pessimistic}, \tag{40}$$

$$\mathrm{PW}_{Optimistic} = 62{,}710.2 \times (1 + 1) = 102{,}300{,}45\$,$$

$$\mathrm{PW}_{Pessimistic} = 62{,}710.2 \times (1 - 0.741) = 13{,}248{,}41\$.$$

5.1. *Sensitivity Analysis*

Sensitivity analysis is based on the midpoints of the parameters which are given in Table 8. The experts indicate that the most critical parameters are AB and AC. In Eq. (41), annual worth (AW) formula is given:

$$AW = -FC(A/P, i\%, n) + AB(1 + x) - AC(1 + y) + SV(A/F, i\%, n), \tag{41}$$

$$\begin{aligned} AW = {} & -238{,}987.5(A/P, 9.9\%, 22.664) + 49,662.500(1 + x) \\ & - 18{,}262.500(1 + y) + 93{,}812.5(A/F, 9.9\%, 22.664), \end{aligned}$$

$$\begin{aligned} AW = {} & -238{,}987.5 \times \left(\frac{(1 + 0.099)^{22.664} - 1)}{0.099 \times (1 + 0.099)^{22.664}}\right)^{-1} + 49{,}662.5(1 + x) \\ & - 18{,}262.5(1 + y) + 93812.5 \times \left(\frac{(1 + 0.099)^{22.664} - 1}{0.099}\right)^{-1} \\ = {} & 6958.8 + 49{,}662.5 + 49{,}662.5x - 18{,}262.5 - 18{,}262.5y + 1{,}239.12 \\ = {} & 5822{,}7 + 49{,}662.5x - 18{,}262.5y, \end{aligned}$$

$$(x, y) = (-0.12; 0),$$

$$(x, y) = (0; 0.32). \tag{42}$$

Fig. 3 shows that possible AB and AC values over the square ±8.57% have no risk for the investor. In other words, the investor will remain insensitive to changes in AB and AC up to 8.57% changes in any direction.

6. Conclusion

C-IFSs (Atanassov, 2020) are the latest extension of IFSs based on the similar idea of type-2 fuzzy sets, which try to incorporate the vagueness and impreciseness of member-

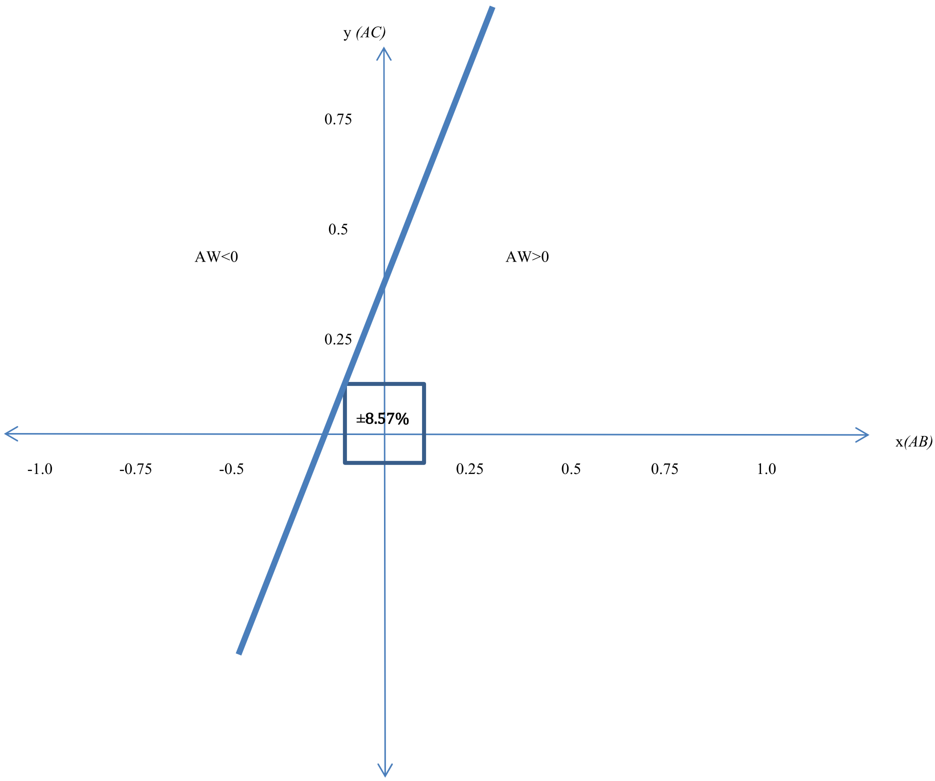

Fig. 3. Two parameters sensitivity analysis.

ship functions into the problem modelling. In this study, a comprehensive literature review has been presented and no investment analysis using circular intuitionistic fuzzy sets has been observed in this literature review. We have proposed new PW analysis methods based on C-IFSs and IVIF sets. All investment parameters have been handled as fuzzy variables. However, score and defuzzification functions have been used whenever it is required. C-IFSs could successfully model the uncertainty in the assignment of membership functions by incorporating a circle whose radius is r. Interval-valued intuitionistic fuzzy sets have been used in engineering economic analyses as an alternative to single valued circular intuitionistic fuzzy analyses. The expected value of the interval-valued intuitionistic fuzzy present worth has been calculated as \$33,162.894 whereas it is the interval \$ [16, 806.6; 122, 550.45] in single valued circular intuitionistic fuzzy *PW* analysis. The midpoint of this interval is \$69,678.525. The difference between these results comes from the different points of view to the uncertainty of membership and non-membership degrees. For further research, other extension types of fuzzy sets such as q-rung orthopair fuzzy sets and picture fuzzy sets can be employed for the calculation of fuzzy *PW* and the results can be compared with the methods presented in this paper.

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