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PARAMETERS ESTIMATION OF QUASIHOMOGENEOUS AUTOREGRESSIVE RANDOM FIELDS

Aloyzas KAPUSTINSKAS

Lithuanian Energy Institute
3035 Kaunas, Breslaujos St. 3, Lithuania

Abstract. The problem of the parameters estimation of quasihomogeneous autoregressive random field is considered. An algorithm is proposed for the parameter estimation of certain classes of such fields.

Key words: random field, autoregressive field, quasihomogeneous field, identification, parameters estimation.

1. Introduction. It is known, that random fields are divided into homogeneous and nonhomogeneous (Rytov, 1978). Statistical characteristics of a homogeneous field are constant – they are independent of the space and the time coordinates. Nonhomogeneous fields has inconstant characteristics – they are any functions of the space and the time. Quasihomogeneous (QH) fields belong to the class of nonhomogeneous fields. Statistical characteristics of such fields are jump functions of the space and the time coordinates.

QH autoregressive (AR) fields has inconstant variance – it is any jump function either of the space, or of the time, or of both of them (Kapustinskas, 1993). They are divided into five classes:

– quasihomogeneous-quasistationary (QHQS),
– quasihomogeneous-stationary (QHS),
– homogeneous–quasistationary (HQS),
– quasihomogeneous-nonstationary (QHNS),
– nonhomogeneous–quasistationary (NHQS),

QH AR fields.

The problem of the identification of homogeneous AR fields were considered in the papers (Kapustinskas, 1987, 1988).

In this paper we shall consider the problem of the parameters estimation of a QHQS, QHS and HQS AR fields, existing in one-dimensional space R^1 and the time.

2. The problem. Let QH AR field exists in one-dimensional space R^1 and the time, i.e., the values of the field are located at the points of the space (x,t), where x, t are discrete values of the space and the time coordinates $\big(x,\ t \in (-\infty,\infty)\big)$, and is described by following difference equation (Kapustinskas, 1993)

$$\xi_t^x = \sum_{k=1}^{n_t} \sum_{i=-n'_x}^{n''_x} a_k^i\, \xi_{t-k}^{x+i} + b\, g_t^x, \tag{1}$$

where ξ_t^x is the value of the field at point x and moment t $\big(\xi_t^x \in (-\infty,\infty)\big)$, n_t is the order of the field with regard to coordinate t, $\{n'_x,\ n''_x\}$ is the order of the field with regard to space coordinate x, $\{g_t^x\}$ is the sequence of independent normal random values with zero average and finite variance $\sigma_g^2 = 1$ (a white noise field), $\{a_k^i\}$, b are the parameters of the field – any jump functions either of the space, or of the time, or of both of them.

Let the parameters $c = \{a_k^i,\ b\}$ be unknown. Let we know, that considered field is either a QHQS, or a QHS, or a HQS AR field, i.e., at least one of the parameters a_k^i, b changes by a jumps either in the space, or in the time, or in both of them. Let the field be observed inside a rectangle field consideration (FC) area Ω, i.e., we have observations $\{\xi_t^x\}$ $(x = \overline{1, W^x},\ t = \overline{1, W^t})$ at everyone point $(x,\ t)$ of this area (Fig. 1).

The dimensions of this area are $W^x \times W^t$. Let QH areas Ω_{rq} and the parameters changing by not a jumps (PCNJ) areas $\triangle_{rq}^p$ be rectangle and has the variable step net type structure (Fig. 1) (Kapustinskas, 1993). Let the dimensions either of QH areas (Fig. 2a)

$$w_r^x \times w_q^t \quad (r = \overline{1, N_x},\ q = \overline{1, N_t})$$

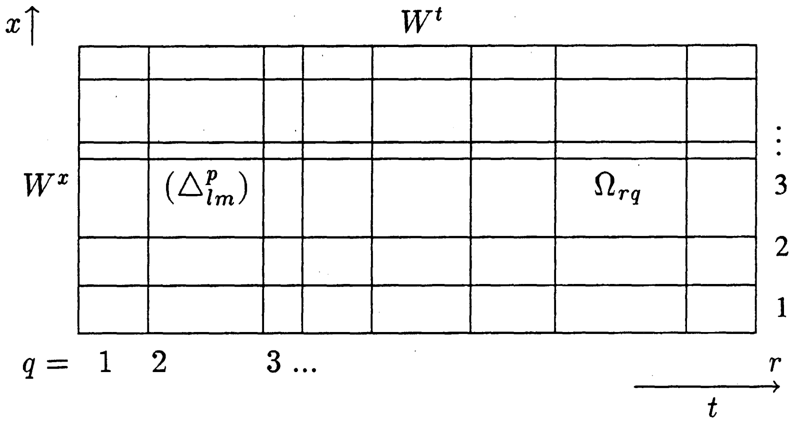

Fig. 1. A rectangle FC area Ω and variable step net type location of rectangle QH areas Ω_{rq} or PCNJ areas $\triangle^{p}_{lm}$ inside of them. $W^x \times W^t$ are the dimensions of the FC area.

or of PCNJ areas

$$\alpha_l^{px} \times \alpha_m^{pt} \quad (l = \overline{1, N_{\Delta x}},\ m = \overline{1, N_{\Delta t}}\ p = \overline{1, n_c})$$

be known. Here

$$n_c = \{n' + n'' + 1\}n_t \tag{2}$$

is total number of the parameters.

It is necessary to calculate an estimates $\widehat{a}^i_k$, $\widehat{b}$ of the parameters a^i_k, b on the ground of the field observations $\{\xi^x_t\}$ $(x = \overline{1, W^x},\ t = \overline{1, W^t})$ inside FC area.

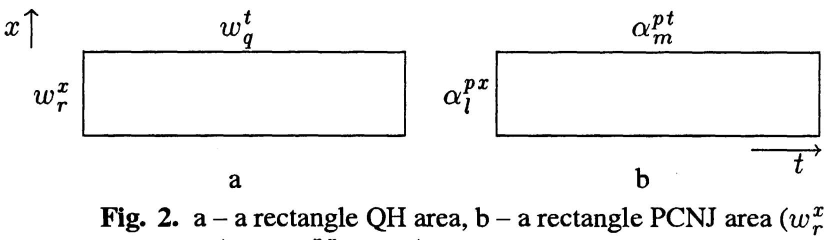

Fig. 2. a – a rectangle QH area, b – a rectangle PCNJ area ($w^x_r \times w^t_q$ and $\alpha^{px}_l \times \alpha^{pt}_m$ are the dimensions of them.)

3. The parameters estimation. QH AR fields takes intermediate place between nonhomogeneous and homogeneous AR fields. On the one hand such fields may be treated as nonhomogeneous, the parameters of which are any functions of the space and the time. On the other hand any kinds of such fields (QHQS, QHS and HQS AR fields) may be treated as homogeneous AR fields. It were shown, that although the parameters of QHQS, QHS or HQS AR field changes by jumps inside FC area, they are constant inside any QH area (Kapustinskas, 1993), i.e., such fields behaves inside any QH area as homogeneous AR fields with constant parameters. Therefore the two approaches to the parameters estimation are possible: one, based on the identification methods of nonhomogeneous AR fields, another – on the identification methods of homogeneous AR fields. The first approach, however, is complicative: at first the identification methods of nonhomogeneous AR fields are not created yet, and, at second, they should be comparatively complicative (see similar methods in the case of nonstationary AR random processes (Kapustinskas and Nemura, 1993)). Another approach is less complicative, because the identification methods of homogeneous AR fields are well known, they are comparatively simple (Kapustinskas, 1987). The properties of the parameters estimates are well known also. Therefore it is expedient to propose following method. At first, the observations $\{\xi_t^x\}$ of FC area are divided into the sets $\{\xi_t^x\}_{r,q}$ $(r = \overline{1, N_x},\ q = \overline{1, N_t})$ of the observations of QH areas (if QH areas are unknown, they must be determined of PCNJ areas). Then the estimates $\tilde{\gamma}_\tau^l(r, q)$ of the autocovariances γ_τ^l of the field are calculated for every QH area. Further the estimates $\widehat{a}_k^i(r, q)$, $\widehat{b}(r, q)$ of the parameters $a_k^i(r, q)$, $b(r, q)$ are calculated for every (r, q)–th QH area by the use homogeneous AR field parameters estimation methods. In detail this method is described below.

The autocovariances estimates may be calculated by the use of following equations

$$\tilde{\gamma}_\tau^l(r, q) = \frac{1}{w_q^t} \sum_{t'=1}^{w_q^t - \tau} \xi_{t'}^{x'}\, \xi_{t'+\tau}^{x'+l}, \qquad (3)$$

$$\tilde{\gamma}_{\tau}^{l}(r,\, q) = \frac{1}{w_r^x} \sum_{x'=1}^{w_r^x - l} \xi_{t'}^{x'}\, \xi_{t'+\tau}^{x'+l}, \tag{4}$$

$$\tilde{\gamma}_{\tau}^{l}(r,\, q) = \frac{1}{w_q^t\, w_r^x} \sum_{t'=1}^{w_q^t - \tau} \sum_{x'=1}^{w_r^x - l} \xi_{t'}^{x'}\, \xi_{t'+\tau}^{x'+l}, \tag{5}$$

where

$$r = \overline{1,\, N_x},\; q = \overline{1,\, N_t},\; l = \overline{0,\, \max(n_x',\, n_x'')},\; \tau = \overline{0,\, n_t}, \tag{6}$$

$x',\, t'$ are the coordinates of the points of any QH area $(x' = \overline{1,\, w_r^x},\; t' = \overline{1,\, w_q^t})$.

Let the number of the observations inside any QH area, of which is expedient to calculate the parameters estimates, be Q. Although every equation (3)–(5) may be used for the autocovariances estimation, it is expedient to use the equation (3), if

$$(w_r^x \geqslant Q) \wedge (w_q^t < Q), \tag{7}$$

the equation (4) – if

$$(w_r^x < Q) \wedge (w_q^t \geqslant Q), \tag{8}$$

and the equation (5) – if

$$(w_r^x < Q) \wedge (w_q^t < Q) \wedge (w_r^x \times w_q^t \geqslant Q). \tag{9}$$

In the case, when

$$w_r^x \times w_q^t < Q, \tag{10}$$

the calculations are impossible – it is necessary to choose a smaller number Q of the observations.

The estimates of the parameters $a_k^i(r,\, q)$ are calculated after the estimation of the autocovariances by such equation (Kapustinskas, 1987):

$$\widehat{A}(r,\, q) = \widetilde{\Gamma}^{-1}(r,\, q)\widetilde{F}(r,\, q) \qquad (r = \overline{1,\, N_x},\; q = \overline{1,\, N_t}), \tag{11}$$

where

$$\widehat{A}^{\mathrm{T}}(r,q) = [\widehat{A}_1^{\mathrm{T}}(r,q) \vdots \widehat{A}_2^{\mathrm{T}}(r,q) \vdots \dots \vdots \widehat{A}_{n_t}^{\mathrm{T}}(r,q)], \tag{12}$$

$$\widehat{A}_k^{\mathrm{T}}(r,q) = [\widehat{a}_k^{-n'_x}(r,q)\;\; \widehat{a}_k^{-n'_x+1}(r,q) \dots \widehat{a}_k^{n''_x}(r,q)] \qquad (k = \overline{1,\, n_t}), \tag{13}$$

$$\widetilde{F}^{\mathrm{T}}(r,q) = [\widetilde{F}_1^{\mathrm{T}}(r,q) \vdots \widetilde{F}_2^{\mathrm{T}}(r,q) \vdots \dots \vdots \widetilde{F}_{n_t}^{\mathrm{T}}(r,q)], \tag{14}$$

$$\widetilde{F}_k^{\mathrm{T}}(r,q) = [\widetilde{\gamma}_k^{-n'_x}(r,q)\;\; \widetilde{\gamma}_k^{-n'_x+1}(r,q) \dots \widetilde{\gamma}_k^{n''_x}(r,q)] \qquad (k = \overline{1,\, n_t}), \tag{15}$$

$$\widetilde{\Gamma}(r,\, q) = \begin{bmatrix} \widetilde{\Gamma}_{11} & \vdots & \widetilde{\Gamma}_{12} & \vdots & \cdots & \vdots & \widetilde{\Gamma}_{1,n_t} \\ \widetilde{\Gamma}_{21} & \vdots & \widetilde{\Gamma}_{22} & \vdots & \cdots & \vdots & \widetilde{\Gamma}_{2,n_t} \\ \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ \widetilde{\Gamma}_{n_t,1} & \vdots & \widetilde{\Gamma}_{n_t,2} & \vdots & \cdots & \vdots & \widetilde{\Gamma}_{n_t,n_t} \end{bmatrix} \tag{16}$$

$$\widetilde{\Gamma}_{ik} = \begin{bmatrix} \widetilde{\gamma}_{i-k}^0(r,q) & \widetilde{\gamma}_{i-k}^{-1}(r,q) & \cdots & \widetilde{\gamma}_{i-k}^{-n'_x-n''_x}(r,q) \\ \widetilde{\gamma}_{i-k}^1(r,q) & \widetilde{\gamma}_{i-k}^0(r,q) & \cdots & \widetilde{\gamma}_{i-k}^{-n'_x-n''_x+1}(r,q) \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{\gamma}_{i-k}^{n'_x+n''_x}(r,q) & \widetilde{\gamma}_{i-k}^{n'_x+n''_x-1}(r,q) & \cdots & \widetilde{\gamma}_{i-k}^0(r,q) \end{bmatrix} \qquad (i,\, k = \overline{1,\, n_t}). \tag{17}$$

As the autocovariances are symmetric (Kapustinskas, 1985), therefore

$$\widetilde{\gamma}_{i-k}^s(r,\, q) = \begin{cases} \widetilde{\gamma}_{i-k}^s(r,\, q) & (i > k), \\ \widetilde{\gamma}_{i-k}^{-s}(r,\, q) & (i < k), \\ \widetilde{\gamma}_0^{-s}(r,\, q) & (i = k,\; s < 0). \end{cases} \tag{18}$$

The estimates $\widehat{b}(r, q)$ of the parameters $b(r, q)$ are calculated by such equation

$$\widehat{b}(r, q) = \{\widetilde{\gamma}_0^0(r, q) - \sum_{k=1}^{n_t} \sum_{i=1}^{n_x''} \widehat{a}_k^i \widetilde{\gamma}_k^i\}^{1/2} \tag{19}$$

$$(r = \overline{1, N_x}, \; q = \overline{1, N_t}).$$

The QH areas can be various – smaller or larger. If at least one of the QH area is too small, i.e., if

$$w_r^x < \max(n_x', n_x'') \tag{20}$$

or

$$w_q^t < n_t, \tag{21}$$

the field cannot be identified. Therefore it is expedient to verify the identifiability conditions (20)–(21) before the beginning the parameters estimation.

Above we admitted the QH areas be known. If the QH areas are unknown, but are known the PCNJ areas, at first the QH areas must be determined by the help of the algorithm, described in the paper (Kapustinskas, 1993), and then the parameters estimates calculated.

In such way the parameters estimates inside everyone QH area are calculated. If the PCNJ areas are unknown, we must admit, that they are the same, as the QH areas, and then the calculated above parameters estimates of the QH areas are the parameters estimates and of the PCNJ areas too. Other way – to use special methods for determination of the PCNJ areas.

If the PCNJ areas are known, the parameters estimates of the (l, m)-th PCNJ area are determined, as an average of the parameters estimates of those (r^*, q^*)-th QH areas, which crosses this (l, m)-th PCNJ area, i.e.,

$$\widehat{a}_k^i(l, m) = \frac{1}{N^*} \sum_{r^*, q^*} \widehat{a}_k^i(r^*, q^*), \tag{22}$$

$$\widehat{b}(l, m) = \frac{1}{N^*} \sum_{r^*, q^*} \widehat{b}(r^*, q^*), \tag{23}$$

where N^* is total number of those QH areas, which crosses $(l,\ m)$-th PCNJ area.

The problem is to determine these QH areas. The three cases are possible:

a) when the PCNJ area is inside the QH area (Fig. 3);

b) when a PCNJ area is crossed by any QH area (Fig. 4);

c) when a PCNJ area is not crossed by any QH area.

If

$$\beta'_x(r,\ q) \leqslant \lambda'_x(l,\ m), \qquad \beta''_x(r,\ q) \geqslant \lambda''_x(l,\ m), \tag{24}$$

$$\beta'_t(r,\ q) \leqslant \lambda'_t(l,\ m), \qquad \beta''_t(r,\ q) \geqslant \lambda''_t(l,\ m), \tag{25}$$

then there is the case a) and if

$$\beta''_x(r,\ q) > \lambda'_x(l,\ m), \qquad \beta'_x(r,\ q) < \lambda''_x(l,\ m), \tag{26}$$

$$\beta''_t(r,\ q) > \lambda'_t(l,\ m), \qquad \beta'_t(r,\ q) < \lambda''_t(m,\ l), \tag{27}$$

then there is the case b), where $\beta,\ \lambda$ are the coordinates of the QH and PCNJ areas tops:

$$\begin{aligned} &\lambda'_x(l,\ m) = \begin{cases} 0, & (l=1), \\ \sum\limits_{j=1}^{l} \alpha_j^{px}, & (l>1); \end{cases} \\ &\lambda''_x(l,\ m) = \lambda'_x(l,\ m) + \alpha_l^{px}, \end{aligned} \tag{28}$$

$$\begin{aligned} &\lambda'_t(l,\ m) = \begin{cases} 0, & (m=1), \\ \sum\limits_{j=1}^{m} \alpha_j^{pt}, & (m>1); \end{cases} \\ &\lambda''_t(l,\ m) = \lambda'_t(l,\ m) + \alpha_m^{pt}, \end{aligned} \tag{29}$$

$$\begin{aligned} &\beta'_x(r,\ q) = \begin{cases} 0, & (r=1), \\ \sum\limits_{j=1}^{r} w_j^{x}, & (r>1); \end{cases} \\ &\beta''_x(r,\ q) = \beta'_x(r,\ q) + w_r^{x}, \end{aligned} \tag{30}$$

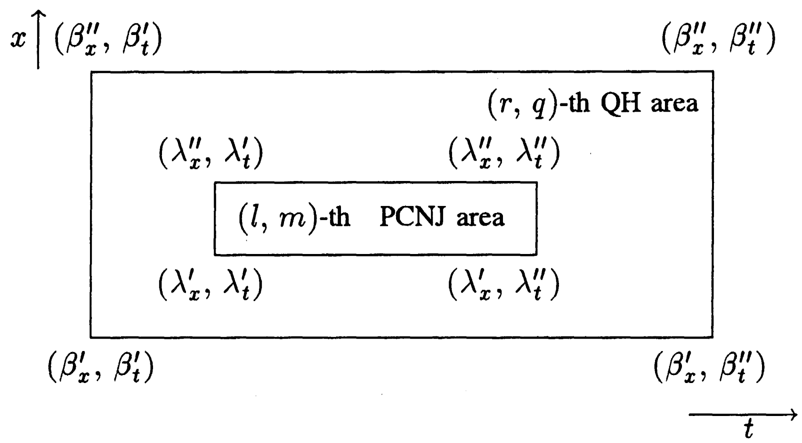

Fig. 3. The case, when (l, m)-th PCNJ area is inside (r, q)-th QH area. β, λ are the coordinates of the areas tops.

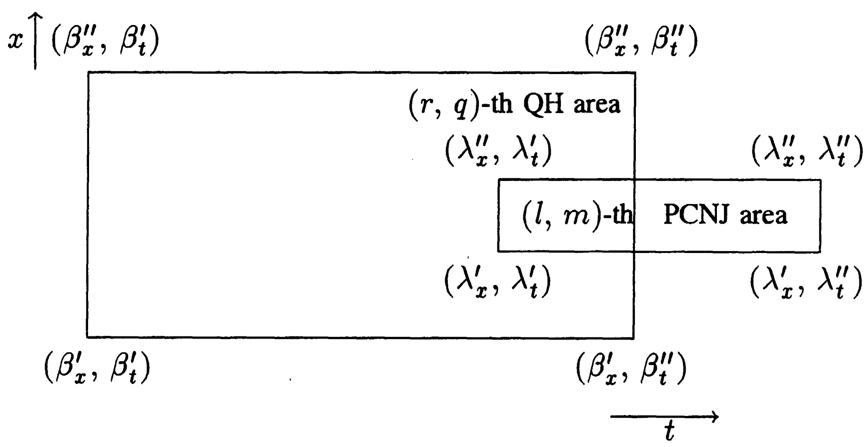

Fig. 4. The case, when (l, m)-th PCNJ area is crossed by (r, q)-th QH area. β, λ are the coordinates of the areas tops.

$$\beta_t'(r,\,q) = \begin{cases} 0, & (q=1), \\ \sum\limits_{j=1}^{q} w_j^t, & (q>1); \end{cases} \tag{31}$$
$$\beta_t''(r,\,q) = \beta_t'(r,\,q) + w_q^t,$$

Then we must search all QH areas for everyone PCNJ area. If the case a) is met, then searching is interrupted and the parameters estimates of $(l,\,m)$-th PCNJ area are equal to the parameters estimates of this QH area, i.e.,

$$\widehat{a}_k^i(l,\,m) = \widehat{a}_k^i(r,\,q), \qquad \widehat{b}(l,\,m) = \widehat{b}(r,\,q). \tag{32}$$

If the case b) is met, the searching is continued, while all the QH areas, which crosses $(l,\,m)$-th PCNJ area, will be not found. Then the parameters estimates of this PCNJ area are calculated by the equations (22)–(23).

As described above method is based on the homogeneous AR field identification method, the properties of the parameters estimates are the same as in the case of homogeneous AR field. The estimates, for example, are asymptotically unbiased. Therefore the accuracy of the estimates, calculated by this method in any QH area on the ground of certain number observations of this area, is the same as in the case of the parameters estimates of homogeneous AR field, when they are calculated on the ground of the same number of observations. For this reason total number of required observations for the parameters estimation of QH AR field is larger, than of homogeneous fields, because FC area consists of $(N^x \times N^t)$-number of QH areas.

3. The estimation algorithm. The following algorithm for the parameters estimation of a QH AR field can be proposed on the base of above considerations.

Step 1. If the QH areas are known, we go to *Step* 3, otherwise – to *Step* 2.

Step 2. The QH areas are determined by the help of the algorithm, described in the paper (Kapustinskas, 1993).

Step 3. Variable r is assigned a zero value.

Step 4. Variable r is incremented by one.

Step 5. Variable q is assigned a zero value.

Step 6. Variable q is incremented by one.

Step 7. If the identifiability conditions are dissatisfied, i.e., if $\left(w_r^x < \max(n'_x,\ n''_x)\right) \vee (w_q^t < n_t)$, the calculations are interrupted, otherwise we go to *Step* 8.

Step 8. If $w_r^x \times w_q^t < Q$, the calculations are interrupted, otherwise – we go to *Step* 9.

Step 9. If $q < N_t$, we return to *Step* 6, otherwise – to *Step* 0.

Step 10. If $r < N_x$, we return to *Step* 6, otherwise – to *Step* 10.

Step 11. Variable r is assigned a zero value.

Step 12. Variable r is incremented by one.

Step 13. Variable q is assigned a zero value.

Step 14. Variable q is incremented by one.

Step 15. Variable l is assigned a -1 value.

Step 16. Variable l is incremented by one.

Step 17. Variable τ is assigned a -1 value.

Step 18. Variable τ is incremented by one.

Step 19. If $(w_r^x \geqslant Q) \wedge (w_q^t < Q)$, we go to *Step* 23, otherwise – to *Step* 20.

Step 20. If $(w_r^x < Q) \wedge (w_q^t \geqslant Q)$, we go to *Step* 22, otherwise – to *Step* 21.

Step 21. The autocovariances estimates are calculated by the equation (5). Go to *Step* 24.

Step 22. The autocovariances estimates are calculated by the equation (4). Go to *Step* 24.

Step 23. The autocovariances estimates are calculated by the equation (3).

Step 24. If $\tau < n_t$, we go to *Step* 18, otherwise – to *Step* 25.

Step 25. If $l < \max(n'_x,\ n''_x)$, we go to *Step* 16, otherwise – to *Step* 26.

Step 26. The estimates $\widehat{A}_{rq}$ are calculated by the equation (11).

Step 27. The estimates $\widehat{b}(r,\ q)$ are calculated by the equation (19).

Step 28. If $q < N_t$, we return to *Step* 14, otherwise – to *Step* 29.

Step 29. If $r < N_x$, we return to *Step* 12, otherwise – to *Step* 30.

Step 30. If PCNJ areas are known, we go to *Step* 31, otherwise the calculations are ended.

Step 31. Variable p is assigned a zero value.

Step 32. Variable p is incremented by one.

Step 33. Variables l and λ_x are assigned a zero value.

Step 34. Variable l is incremented by one.

Step 35. Variable λ''_x is calculated by the equation (28).

Step 36. Variables m and λ'_t are assigned a zero value.

Step 37. Variable m is incremented by one.

Step 38. Variable λ''_t is calculated by the equation (29).

Step 39. Variables r and β'_x are assigned a zero value.

Step 40. Variable r is incremented by one.

Step 41. Variable β''_x is calculated by the equation (30).

Step 42. Variables q, N^* and β'_t are assigned a zero value.

Step 43. Variable q is incremented by one.

Step 44. Variable β''_t is calculated by the equation (31).

Step 45. If inequations (24)–(25) are satisfied, we calculate the parameters estimates by the equations (32) and go to *Step* 47, otherwise – to *Step* 46.

Step 46. If inequations (26)–(27) are satisfied, variable N^* is incremented by one and we go to *Step* 47. Otherwise – to *Step* 47.

Step 47. Variable β'_t is incremented by value w_q^{pt}.

Step 48. If $q < N_t$, we return to *Step* 43, otherwise – go to *Step* 49.

Step 49. Variable β'_x is incremented by value w_r^{px}.

Step 50. If $r < N_x$, we return to *Step* 40, otherwise – go to *Step* 51.

Step 51. Variable λ'_t is incremented by value α_m^t.

Step 52. If $m < N_{\Delta t}^p$, we return to *Step* 37, otherwise – go to *Step* 53.

Step 53. Variable λ'_x is incremented by value α_l^x.

Step 54. If $l < N^p_{\Delta x}$, we return to *Step* 34, otherwise – go to *Step* 55.

Step 55. The parameters estimates are calculated by the equations (22)–(23).

Step 56. If $p < n_c$, we return to *Step* 31, otherwise – the calculations are ended.

We remind, that this algorithm can be used for the parameters estimation of the QH AR fields in the case of the rectangle FC, QH and PCNJ areas with the net type location.

4. Conclusions. It is possible to estimate the parameters of the QHQS, QHS and HQS AR fields by the help of the identification methods of the homogeneous AR fields, if they are used only inside every QH area of such field. The proposed parameters estimation method and algorithm permit to estimate the parameters of the QHQS, QHS and HQS AR fields in the case of rectangle FC, QH and PCNJ areas with the variable or constant step net type location.

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A. Kapustinskas in 1972 maintained a candidate thesis at the Kaunas Polytechnic Institute. At present he is a senior researcher of Adaptive System Department at the Lithuanian Energetic Institute. His research interests are identification of random processes and fields.