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ON THE STABILITY OF LOD DIFFERENCE SCHEMES WITH RESPECT TO BOUNDARY CONDITIONS

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Abstract. The convergence properties of some LOD schemes are considered. New stability estimates with respect to boundary conditions are proved. These results are used to investigate the accuracy of LOD schemes when no special boundary correction technique is used for the realization of LOD schemes. The accuracy of LOD schemes with corrected boundary conditions is also investigated. Results of the computational experiment are given.

Key words: LOD schemes, stability, correction of boundary conditions, unconditional convergence, computational experiment.

1. Introduction. In this paper the accuracy of some locally one dimensional (LOD) difference schemes will be analyzed. The schemes are used for the solution of parabolic problems in p space dimensions ($p \geqslant 2$). Let $\bar{G}$ be a cube in R^p

$$\overline{G} = \{x:\ x = (x_1, x_2, \cdots, x_p)_1,\ 0 \leqslant x_j \leqslant 1,\ j = 1, 2, \cdots, p\},$$

γ its boundary, $(0, T]$ be a bounded half open interval in R, and let $Q_T = G \times (0, T]$. We consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\left(k_i(x)\frac{\partial u}{\partial x_i}\right) + f(x,t), \ (x,t) \in Q_T, \quad (1.1a)$$

$$u(x,t) = u_\gamma(x,t), \qquad (x,t) \in \Gamma = \gamma \times (0,T), \qquad (1.1b)$$

$$u(x,0) = u_0(x), \qquad x \in \overline{G}. \qquad (1.1c)$$

Splitting methods are used for the numerical solution of multidimensional parabolic problems (see, e.g., Samarskij, 1974; Yanenko, 1971). The accuracy of such schemes essentially depends on the approximation of boundary conditions at fractional time moments. The simplest approximation method is to use exact boundary conditions. It is well known that for such an approximation LOD schemes may suffer from accuracy order reduction (see Yanenko, 1971; Hundsdorfer, 1992). Many authors investigated boundary correction techniques to restore the order of approximation near the boundary (see, e.g., Fryazinov, 1968; Samarskij, 1974; Sommeijer, Van der Houwen and Verwer, 1981; Stoyan, 1970; Swayne, 1987). Formulas for such a correction appear at the stage of elimination of intermediate solutions, or when homogeneous boundary conditions are stated for auxiliary boundary value problems used to estimate local discretization errors.

However in many cases we can not implement these correction techniques due to the complicated form of the region G or the differential operator. Hence in applications it is very important to know the accuracy of LOD schemes when the simplest method of boundary conditions approximation is considered. In fact we must study the stability of LOD schemes solution with respect to boundary conditions. For two and three dimensional LOD schemes stability was investigated by Yanenko (1971), Stoyan (1971). A new viewpoint to this problem was given by Hundsdorfer, Verwer (1989), Hundsdorfer (1992). They investigated a two dimensional model linear problem and estimated the order reduction of the discretization error in time, which may be affected by small meshwidths.

We study a general problem of LOD scheme solution stability with respect to boundary conditions. The main convergence results are obtained in the L_2 norm, but the C norm is also used in our analysis.

For the completeness of the analysis we also stated very briefly the main results of our papers Kiškis and Čiegis (1994a, 1994b).

2. Difference schemes. Let ω_τ, ω_h be difference grids

$$\begin{aligned}\omega_\tau = \{t_{j+\alpha/p} = (j+\alpha/p)\tau,\ \alpha = 0,1,\cdots,p,\\ j = 1,2,\cdots,K,\quad K\tau = T\},\\ \omega_h = \{(x_1{}^{(i_1)},\ x_2{}^{(i_2)},\cdots,x_p{}^{(i_p)})|\ x_\alpha{}^{(i_\alpha)} = i_\alpha\, h,\\ i_\alpha = 1,2,\cdots,N-1,\quad Nh = 1\}.\end{aligned}$$

Boundary points of ω_h are denoted by γ_h. We consider the following LOD scheme

$$y_{\bar{t}_\alpha} = \Lambda_\alpha y_\alpha^\sigma + \varphi_\alpha, \qquad (x,t)\in\omega_h\times\omega_\tau, \tag{2.1a}$$

$$y_\alpha = g_\alpha, \qquad (x,t)\in\gamma_h\times\omega_\tau, \tag{2.1b}$$

$$y(0) = u_0(x),\quad y(t_{j+1}) = y_p,\quad \alpha = 1,2,\cdots,p, \tag{2.1c}$$

where

$$\Lambda_\alpha y = (a y_{\bar{x}_\alpha})_{x_\alpha},\quad a = k(x_1,x_2,\cdots,x_\alpha - \frac{h}{2},\cdots,x_p).$$

There we have used the following notations for grid functions

$$\begin{aligned}&y_j = y(t_j),\quad y_\alpha = y_{j,\alpha} = y(t_{j+\alpha/p}),\\ &y_{\bar{t}_\alpha} = (y_\alpha - y_{\alpha-1})/\tau,\quad y_\alpha^\sigma = \sigma y_\alpha + (1-\sigma)y_{\alpha-1},\\ &y_{\bar{x}_\alpha} = \left(y^{(i_\alpha)} - y^{(i_\alpha-1)}\right)/h,\quad y_{x_\alpha} = \left(y^{(i_\alpha+1)} - y^{(i_\alpha)}\right)/h,\\ &y_0(t_{j+1}) = y_{j,p} = y_p(t_j).\end{aligned}$$

The main objective of the present investigation is the order of convergence of the LOD scheme when the simplest approximation of the boundary condition and the source term is used:

$$\begin{aligned}y_\alpha(x) &= u_\gamma(x, t_{j+\alpha/p}),\quad x\in\gamma_h,\\ \varphi_\alpha(x) &= \frac{1}{p}f(x, t_{j+(2\alpha-1)/2p}),\quad x\in\omega_h.\end{aligned} \tag{2.2}$$

We also consider the symetric LOD scheme. Symmetry can be restored by interchanging the direction of splitting after each step (see, e.g., Fryazinov, 1968; Samarskij, 1974; Marchuk, 1988; Hundsdorfer, 1992)

$$y_{\bar{t}_\alpha} = \Lambda_{\beta(\alpha)} y_\alpha^\sigma + \varphi_\alpha, \quad (x,t) \in \omega_h \times \omega_\tau, \qquad (2.3a)$$

$$y_\alpha = g_\alpha, \qquad (x,t) \in \gamma_h \times \omega_\tau, \qquad (2.3b)$$

$$\beta(\alpha) = 2p + 1 - \alpha, \qquad \alpha = p+1, p+2, \cdots, 2p.$$

Throughout the paper we denote by $||y||$ the discrete L_2 norm of a grid function y. As usual, $\overset{o}{W}{}_2^1(\overline{\omega}_h)$ is the space of grid functions on $\overline{\omega}_h$ which are zero at the boundary points of ω_h with the norm

$$||y_{\bar{x}_\alpha}]|^2 = \sum_{i_1=1}^{N-1} \cdots \sum_{i_{p-1}=1}^{N-1} \sum_{i_\alpha=1}^{N-1} y_{\bar{x}_\alpha}^2 h^p.$$

Evidently, this is a seminorm on $W_2^1(\overline{\omega}_h)$ but a norm on $\overset{o}{W}{}_2^1(\overline{\omega}_h)$.

In next sections we will consider two methods for the summation of local approximation errors.

3. Recursions for the global discretization errors. This method is very often used for the investigation of LOD schemes (see Yanenko, 1971; Marchuk, 1988; Hundsdorfer, 1992). In this section we will restrict our analysis to two dimensional ($p = 2$) model problem (2.1) with $k_\alpha(x) = 1$. We suppose that Λ_α act's as operator in

$$H_0 = \{\varphi |\ \varphi \in L_2,\ \varphi(x,t) = 0, \quad (x,t) \in \gamma_h \times \omega_\tau\}.$$

First we will obtain the recursion equations for global discretization errors $z_\alpha = y_\alpha - u(t_{j+\alpha/p})$, $p \geqslant 2$. By substituting $y\alpha = z_\alpha + u_\alpha$ into (2.1) and eliminating intermediate solutions z_α we get that global errors of the LOD scheme (2.1) satisfy the recursion

$$z_{j+1} = R_p z_j + \tau \delta_{j,p}, \qquad (3.1)$$

$$\delta_{j,1} = (E - \tau\sigma\Lambda_1)^{-1}\psi_{j1},$$
$$\delta_{j,\alpha} = r(\tau\Lambda_\alpha)\delta_{j,\alpha-1} + (E - \tau\sigma\Lambda_\alpha)^{-1}\psi_{j,\alpha},$$
$$\alpha = 2, 3, \cdots, p.$$

There $\psi_{j\alpha}$ is the local discretization error

$$\psi_{j,\alpha} = -\frac{u_\alpha - u_{\alpha-1}}{\tau} + \Lambda_\alpha u_\alpha^\sigma + \frac{1}{p} f\left(t_{j+(2\alpha-1)/2p}\right),$$

and $r(z, \sigma), R_p$ are given by the following formulas

$$r(z,\sigma) = (1 - \sigma z)^{-1}\left(1 + (1-\sigma)z\right),\ R_p = \prod_{\alpha=1}^{p} r(\tau\Lambda_\alpha).$$

By using the commutativity of Λ_α we obtain

$$\delta_{j,p} = \prod_{\alpha=1}^{p}(E - \sigma\tau\Lambda_\alpha)^{-1} \times \left(\sum_{\alpha=1}^{p}\prod_{l=1}^{\alpha-1}(E - \tau\sigma\Lambda_l)\prod_{l=\alpha+1}^{p}(E + (1-\sigma)\tau\Lambda_l)\psi_{j,\alpha}\right). \tag{3.2}$$

By a Taylor expansion around $\bar{t} = t_{j+0.5}$, and from (2.1) it follows that

$$\psi_{j,\alpha} = \psi_{j,\alpha}^0(\bar{t}) + \frac{\tau(2\alpha - 1 - p)}{2p}\frac{\partial}{\partial t}\psi_{j,\alpha}^0(\bar{t}) - (\sigma - 0.5)\tau L_\alpha \frac{\partial u}{\partial t}(\bar{t}) + O(\tau^2 + h^2), \tag{3.3}$$
$$\alpha = 1, 2, \cdots, p,$$

where we denoted

$$\psi_{j,\alpha}^0(t) = -\frac{1}{p}\frac{\partial u}{\partial t}(t) + L_\alpha u(t) + \frac{1}{p}f(t) = \frac{1}{p}\sum_{j=1}^{p} L_j u(t) - L_\alpha u(t). \tag{3.4}$$

All these expressions become more simple for two dimensional problems ($p = 2$), since we have a relation $\psi_{j1}^0(t) = -\psi_{j2}^0(t) = \nu(t)$, where

$$\nu(t) = -\frac{1}{2}\frac{\partial u}{\partial t}(t) + L_1 u + \frac{1}{2}f(t) = \frac{1}{2}(L_1 - L_2)u(t). \qquad (3.5)$$

Using $\nu(t)$ we obtain from (3.2) after some calculations

$$\begin{aligned}\delta_{j,2} = &-(E - \tau\sigma\Lambda_1)^{-1}(E - \tau\sigma\Lambda_2)^{-1}\Big(\frac{\tau}{2}\frac{\partial \nu}{\partial t}(\bar{t}) \\ &- \frac{\tau}{2}(\Lambda_1 + \Lambda_2)\nu(\bar{t}) - \frac{\tau^2}{8}\Big((\Lambda_2 - \Lambda_1)\frac{\partial \nu}{\partial t}(\bar{t}) \\ &+ (\sigma - 0.5)\tau(\Lambda_2 - \Lambda_1)\nu(\bar{t})\Big) \\ &- \tau^2\big(\sigma\Lambda_1 L_2 \frac{\partial u}{\partial t}(\bar{t}) - (1-\sigma)\Lambda_2 L_1 \frac{\partial u}{\partial t}(\bar{t})\big)\Big) \\ &+ O(\tau^2 + h^2).\end{aligned}$$

The uniform in h condition

$$\big\|(\Lambda_1 + \Lambda_2)\nu(t)\big\| \leqslant C \qquad (3.6)$$

does not hold in general ($\nu(t)$ need not be zero near the boundaries) and therefore the estimate $||\delta_{j2}|| \leqslant C(\tau + h^2)$ cannot be proved. Apparently this fact was first emphasized by Dyakonov (1962). Hundsdorfer (1992) proved the uniform upper bound for the local discretization errors $\delta_{j2} \leqslant C(\tau^{0.25} + h^2)$ (see, also Kiškis and Čiegis, 1994a). Combining this estimate with stability estimates

$$\begin{aligned}&\big\|r(\tau\Lambda_\alpha)\big\| \leqslant 1, \qquad ||E - \sigma\tau\Lambda_\alpha|| < 1, \\ &\big\|(E - \sigma\tau\Lambda_\alpha)^{-1}\tau\Lambda_\alpha\big\| < \frac{1}{\sigma},\end{aligned} \qquad (3.7)$$

we obtain the accuracy estimates for the global discretization error of LOD and SAS schemes

$$||y - u|| \leqslant C(\tau^{0.25} + h^2).$$

We will prove in the next section that this convergence result is not optimal.

REMARK 3.1. The given problem of convergence order reduction does not occur, when a strictly implicit LOD scheme $\sigma = 1$ is investigated. Using the maximum principle we can prove that a solution of such LOD scheme converges unconditionally in the uniform norm C with a convergence rate $O(\tau + h^2)$ (see Samarskij, 1974). Therefore we will assume that $0 < \sigma < 1$, unless noted otherwise.

4. Unconditional global error bounds. In this section we will use another method of summation of local discretization errors. This method is a modification of Samarskij's method (see Samarskij, 1974). Firstly it was proposed by Kiškis and Čiegis (1994a). The global discretization error z_α satisfies a boundary value problem

$$\begin{aligned} &z_{\bar{t}_\alpha} = \Lambda_\alpha z_\alpha^\sigma + \psi_\alpha, \qquad \alpha = 1, 2, \cdots, p, \quad x \in \omega_h, \qquad (4.1)\\ &z_\alpha(x, t_\alpha) = 0, \qquad x \in \gamma_h, \end{aligned}$$

where ψ_α is defined in (3.3).

Consider a more general problem

$$\begin{aligned} &z_{\bar{t}_\alpha} = \Lambda_{\beta(\alpha)} z_\alpha^\sigma + \phi_\alpha, \quad \alpha = 1, 2, \cdots, P, \quad P = np, \qquad (4.2)\\ &z_\alpha(x, t_\alpha) = 0, \quad x \in \gamma_h, \end{aligned}$$

where $n \geqslant 1, \ \ n \in N, \ \ \beta(\alpha)$ defines a direction of splitting. We obviously assume, that

$$\{\beta(\alpha)\} = \{1, 2, \cdots, p\}, \ \alpha = (k-1)p+1, (k-1)p+2, \cdots, kp,$$

for all $k = 1, 2, \cdots, n$. For the LOD scheme we have (4.2) with $n = 1$, $\beta(\alpha) = \alpha$, and for SAS, we obtain (4.2) with $n = 2$, $\beta(\alpha) = \alpha$, $\alpha = 1, 2, \cdots, p$; $\beta(\alpha) = 2p + 1 - \alpha$, $\alpha = p+1, p+2, \cdots, 2p$.

Consider the following statements (see, also (3.3))

$$\begin{aligned} &\phi_\alpha = \tau^s w_{1\alpha} + w_{2\alpha}, \quad \alpha = 1, 2, \cdots, P, \qquad (4.3a)\\ &||w_{1\alpha}|| \leqslant C, \quad ||w_\alpha|| \leqslant C(\tau^{s+1} + h^2), \quad s \geqslant 0,\\ &\sum_{\alpha=1}^{p} w_{1\alpha}(x) = 0. \qquad (4.3b) \end{aligned}$$

Lemma 4.1. *Assume* (4.3a) *and* $\sigma \geqslant 0.5$. *Then we have for a solution of* (4.2) *the following estimate*

$$||z_\alpha - z_{\alpha-1}|| \leqslant C(\tau^{s+1} + \tau h^2). \tag{4.4}$$

Proof. Multiplying (4.2) by $\tau(z_\alpha - z_{\alpha-1})$ and using Green's formula we have

$$\begin{aligned} &||z_\alpha - z_{\alpha-1}||^2 + \tau^3(\sigma - \frac{1}{2})||z_{\alpha t,\bar{x}_\beta}|| \\ &\quad = \frac{\tau}{2}\left(||z_{\alpha-1,\bar{x}_\beta}||^2 - ||z_{\alpha\bar{x}_\beta}||^2 + 2(\phi_\alpha, z_\alpha - z_{\alpha-1})\right). \end{aligned}$$

The difference scheme (4.1) is stable in the $\overset{o}{W}{}_2^1(\overline{\omega}_h)$ norm, hence

$$||z_{\alpha\bar{x}_\beta}]|^2 \leqslant ||z_{\alpha-1,\bar{x}_\beta}]|^2 + \tau||\phi_\alpha||^2.$$

This leads in the standard way to the estimation

$$\begin{aligned} ||z_\alpha - z_{\alpha-1}||^2 &\leqslant \frac{\tau^2}{2}||\phi_\alpha||^2 + \tau\varepsilon||\phi_\alpha||^2 + \frac{\tau}{4\varepsilon}||z_\alpha - z_{\alpha-1}||^2 \\ &\leqslant \tau^2||\phi_\alpha||^2 + \frac{1}{2}||z_\alpha - z_{\alpha-1}||^2. \end{aligned} \tag{4.5}$$

Then (4.4) follows directly from (4.5) and the assumption (4.3a). The lemma is proved.

Lemma 4.2. *Assume* (4.3) *and* $\sigma \geqslant 0.5$. *Then the following unconditional error estimate in the* L_2 *norm*

$$||z(t_j)|| \leqslant C(\tau^{(s+1)/2} + h^2) \tag{4.6}$$

holds.

Proof. Multiplying (4.2) by $2\tau z_\alpha^\sigma$ and using Green's formula we have

$$\begin{aligned} &||z_\alpha||^2 - ||z_{\alpha-1}||^2 + \tau^2(2\sigma - 1)||z_{t_\alpha}||^2 + 2\tau||z^\sigma_{\alpha\bar{x}_\beta}]|^2 \\ &\quad = 2\tau(\phi_\alpha, z_\alpha^\sigma). \end{aligned} \tag{4.7}$$

Function z_α^σ can be written as

$$z_\alpha^\sigma = z_j + \sum_{l=1}^{\alpha-1}(z_{j,l} - z_{j,l-1}) + \sigma(z_\alpha - z_{\alpha-1}).$$

We obtain after summation of (4.7)

$$||z_{j+1}||^2 + 2\tau \sum_{\alpha=1}^{P} ||z^\sigma_{\alpha\bar{x}_\beta}||^2$$
$$\leqslant ||z_j||^2 + 2\tau \left| \left(\sum_{\alpha=1}^{P} w_{1\alpha}, z_j \right) + \sum_{\alpha=1}^{P} (w_{2\alpha}, z_\alpha^\sigma) \right.$$
$$\left. + \sum_{\alpha=1}^{P} \left(w_{1\alpha}, \sum_{l=1}^{\alpha-1}(z_{j,l} - z_{j,l-1}) + \sigma(z_\alpha - z_{\alpha-1}) \right) \right| \quad (4.8a)$$

Using (4.3b) we get

$$\left(\sum_{\alpha=1}^{P} w_{1\alpha}, z_j \right) = 0.$$

Next we estimate the term

$$\sum_{\alpha=1}^{P} (w_{2\alpha}, z_\alpha^\sigma) \leqslant \sum_{\alpha=1}^{P} ||z^\sigma_{\alpha\bar{x}_\alpha}||^2 + C \sum_{\alpha=1}^{P} ||W_{2\alpha}||^2, \qquad (4.8b)$$

where a new function $W_{2\alpha}$ is defined

$$W_{2\alpha}\left(x_1, \cdots, x_\alpha^{(k)}, \cdots, x_p\right) = -\sum_{i=k}^{N-1} w_{2\alpha} x\alpha^{(i)} h,$$
$$W_{2\alpha}\left(x_\alpha^{(N)}\right) = 0.$$

It follows from (4.3a), that $||W_{2\alpha}|| \leqslant C(\tau^{s+1}+h^2)$. Using the bound (4.4), we obtain after some calculations

$$\left|\sum_{\alpha=1}^{P}\left(w_{1\alpha}, \sum_{l=1}^{\alpha-1}(z_{j,l}-z_{j,l-1})+\sigma(z_\alpha - z_{\alpha-1})\right)\right|$$
$$\leqslant \sum_{\alpha=1}^{P}\sum_{l=1}^{P}\|w_{1\alpha}\| \, \big\|(z_{j,l}-z_{j,l-1})\big\| \leqslant C\tau^{s+1}. \quad (4.8c)$$

Then the error estimate (4.6) follows directly from (4.8). The lemma is proved.

REMARK 4.1. The estimate of Lemma 4.2 does not depend on the splitting order, therefore it holds for both, the LOD and SAS schemes.

We have proved in Sect. 3 that for the LOD scheme (and SAS) the estimates (4.3) are valid with $s = 0$. Therefore the following result follows directly.

Theorem 4.1. *Assume* $\sigma \geqslant 0.5$. *Then a solution of the LOD scheme* (2.1) *(or SAS) converges unconditionally to the solution of* (1.1) *and the following accuracy estimate in the* L_2 *norm holds*

$$\big\|y(t_j)-u(t_j)\big\| \leqslant C(\tau^{0.5}+h^2). \quad (4.8)$$

To give an illustration of this convergence result, we present some numerical results for a simple model problem (1.1) with $k(x)=1$, $p=2$, $T=1$ and the exact solution $u(x_1,x_2,t)=\exp((2{x_1}^2+x_1x_2-0.5{x_2}^2)/(1+t))$.

In Table 4.1 global errors in the L_2 and C norms are given for SAS with $h=2\tau=1/N$.

The errors in the discreteL_2 norm illustrate the estimate of Theorem 4.1. It appears that there is no convergence for SAS in the maximum norm. We think that this fact can be explained by the Gibbs phenomenon, since boundary conditions on split time steps are discontinuous if no boundary correction formulas are used. More computational examples are given by Hundsdorfer (1992).

Table 4.1. Global errors (L_2 and C) norm for SAS with $h = 2\tau$

N	10	20	40	80
L_2 norm	0.0995	0.0682	0.0476	0.0334
C norm	0.2978	0.2978	0.2992	0.2917

5. Improved global error estimates. In some cases it is possible to improve the results obtained in Sect. 4 by taking into account certain cancellation effects of local discretization error. Pioneering work in this area for two-dimensional LOD scheme with $\sigma = 0.5$ has been done by Samarskij (1962). He have proved the following stability estimate

$$||z_{j+1}||^2 \leqslant (1 + C\tau)||z_j||^2 + C\tau Q_{j+1} + \tau^2(\nu_j, z_j)_t, \qquad (5.1)$$

where ν_j defined by (3.5) and

$$Q_j = \tau^2 t_j \left\|(\nu_j)_{\bar{t}}\right\|^2 + O(\tau + h^2)^2.$$

It follows directly from (5.1) that

$$||z_j|| \leqslant C(\tau + h^2).$$

Inequality (5.1) cannot be used to investigate the convergence of SAS, because for this scheme the estimate $||\nu_{\bar{t}}|| \leqslant C$ is not valid and there is no cancellation in the last term of (5.1). This method of analysis is not useful for LOD schemes with $p \geqslant 3$. Hundsdorfer (1992) augmented this approach by new techniques for the estimation of the global discretization error by taking into account cancellation effects. He investigated two-dimensional symmetric ($\sigma = 0.5$) LOD and SAS schemes using the following statement.

Lemma 5.1. *Assume, that an error recursion of the form*

$$\begin{aligned} &z_{j+1} = Sz_j + \tau d_j, \quad ||S|| \leqslant 1, \\ &j = 0,1,2,\cdots,N-1, \qquad z_0 = 0 \end{aligned} \tag{5.2}$$

holds, and the local discretization error can be represented as

$$d_j = (E - S)\mu_j + \eta_j, \;\; j = 0,1,\cdots,N-1$$

with μ_j, η_j *such that*

$$||\mu_j|| \leqslant C\tau^{\alpha-1}, \;\; ||\eta_j|| \leqslant C\tau^{\alpha}, \;\; ||\mu_j - \mu_{j-1}|| \leqslant C\tau^{\alpha}.$$

Then unconditional global error estimate

$$||z_j|| \leqslant C\tau^{\alpha}, \;\; j = 1,2,\cdots,N \tag{5.3}$$

holds.

In case the local errors are constant, the reverse implication also holds (see Hundsdorfer, 1992).

We will use this lemma for the convergence analysis of the two-dimensional LOD scheme with $\sigma \geqslant 0.5$.

An error recursion is obtained in Sect. 3 (see (3.1)). Simple calculations give us the equality

$$\begin{aligned} E - R_2 =& -\tau(E - \sigma\tau\Lambda_1)^{-1}(E - \sigma\tau\Lambda_2)^{-1} \\ &\times \big(\Lambda_1 + \Lambda_2 - (2\sigma - 1)\tau\Lambda_1\Lambda_2\big), \\ \mu_j =& -\frac{1}{2}(\Lambda_1 + \Lambda_2 - (2\sigma - 1)\tau\Lambda_1\Lambda_2)^{-1} \\ &\times \big(\Lambda_1 + \Lambda_2 - (2\sigma - 1)(\Lambda_2 - \Lambda_1)\big)\nu(\bar{t}), \\ \eta_j =& \delta_{j2} - (E - R_2)\mu_j. \end{aligned}$$

We note that

$$-(\Lambda_1 + \Lambda_2) + (2\sigma - 1)\Lambda_1\Lambda_2 \geqslant -(\Lambda_1 + \Lambda_2) \geqslant cE. \tag{5.4}$$

Smoothness of $u(x_1, x_2, t)$, stability estimates (3.4) and inequalities (5.4) imply

$$||\mu_j|| \leqslant C, \quad ||\eta_j|| \leqslant C(\tau + h^2), \quad ||\mu_j - \mu_{j-1}|| \leqslant C\tau.$$

Then it follows from Lemma 5.1 that

$$||z_j|| \leqslant C(\tau + h^2).$$

There is no such cancellation effect for SAS scheme with $\sigma = 0.5$ (see also Sect. 7).

Analogous investigation can be done for the 3D LOD scheme. We restrict ourselves to a symmetric scheme $\sigma = 0.5$. By observing that

$$\begin{aligned} E - R_3 = &- \prod_{\alpha=1}^{3} (E - \frac{\tau}{2}\Lambda_\alpha)^{-1} \\ &\times \left(\tau(\Lambda_1 + \Lambda_2 + \Lambda_3) + \frac{\tau^3}{4}\Lambda_1\Lambda_2\Lambda_3 \right), \end{aligned}$$

we define

$$\begin{aligned} \mu_j =& - \frac{1}{3}(\Lambda_1 + \Lambda_2 + \Lambda_3 + \frac{\tau^2}{4}\Lambda_1\Lambda_2\Lambda_3)^{-1} \\ &\times \Big((\Lambda_2 + \Lambda_3)\nu_{0.5}(\bar{t}) + (\Lambda_3 - \Lambda_1)\nu_{1.5}(\bar{t}) - (\Lambda_1 + \Lambda_2)\nu_{2.5}(\bar{t}) \\ &+ \frac{\tau}{2}\big(\Lambda_2\Lambda_3\nu_{0.5}(\bar{t}) - \Lambda_1\Lambda_3\nu_{1.5}(\bar{t}) + \Lambda_1\Lambda_2\nu_{2.5}(\bar{t})\big)\Big), \\ \eta_j =& \delta_{j3} - (E - R_3)\mu_j. \end{aligned}$$

Since there is no uniform unconditional bound for the function

$$\begin{aligned} &C(\lambda_1, \lambda_2, \lambda_3) = \frac{0.5\tau(\lambda_2\lambda_3 - \lambda_1\lambda_3 + \lambda_1\lambda_2)}{\lambda_1 + \lambda_2 + \lambda_3 + 0.25\tau^2\lambda_1\lambda_2\lambda_3}, \\ &C_1 \leqslant \lambda_j \leqslant C_2 h^{-2}, \end{aligned}$$

Table 5.1. Global errors (L_2 and C norms) for the LOD scheme with $h = 2\tau$

N	10	20	40	80
L_2	0.0857	0.0464	0.0242	0.0123
C	0.2125	0.1216	0.0650	0.0336

the results of Lemma 5.1 cannot be used. Therefore for the LOD scheme with $\sigma = 0.5,\ \ p \geqslant 3$ a possibility to improve the convergence result of Theorem 4.1 is an open problem.

At the end of this section in Table 5.1 we present numerical results for the model problem from Sect. 3, obtained for the LOD scheme.

6. The LOD scheme with σ=0.5 + σ_0, σ_0>0. The other case, when better accuracy estimates can be proved is the LOD scheme (or SAS) with $\sigma = 0.5 + \sigma_0,\ \ \sigma_0 > 0$. We have proved in Sect. 4 that a reduction of the convergence order is dependent on imbedding theorems used in the stability analysis. Now we propose a modification of spectral method, which enables us to obtain a better error estimate (see, Kiškis and Čiegis, 1994b). Let assume that (1.1) is

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\left(k_i(x_i)\frac{\partial u}{\partial x_i}\right) - q_i(x_i)u + f(x,t).$$

As it follows from the analysis given in Sect. 4, the convergence rate of the LOD scheme (and SAS) depends on the optimality of the stability estimation in the L_2 norm for the 1 D problem

$$\frac{\hat{y} - \widetilde{y}}{\tau} = \Lambda\big(\sigma\hat{y} + (1-\sigma)\widetilde{y}\big), \quad x \in \omega_h, \qquad (6.1a)$$
$$\widetilde{y} = \rho y, \quad -1 \leqslant \rho \leqslant 1,$$

$$y_0 = \mu_0(t_j), \quad y_N = \mu_1(t_j), \quad x \in \gamma_h, \tag{6.1b}$$
$$\widetilde{y}_0 = \mu_0^*(t_j), \quad \widetilde{y}_N = \mu_1^*(t_j),$$
$$y(x,0) = 0, \quad x \in \overline{\omega}_h, \tag{6.1c}$$

where the boundary conditions satisfy the following conditions

$$\begin{aligned} &|\mu_k(t)| \leqslant C, \quad |\mu_k^*(t)| \leqslant C, \\ &|\mu_k(t_j) - \mu_k^*(t_{j-1})| \leqslant C, \quad k = 0, 1. \end{aligned} \tag{6.2}$$

The last inequality in (6.2) means that functions $\mu_k(t),\ \mu_k{}^*(t)$ are unrelated bounded functions.

For any function $v(x), x \in \overline{\omega}_h$ we define a new function

$$\overset{o}{v}(x_i) = v(x_i), \quad x \in \omega_h, \quad \overset{o}{v}(x_i) = 0, \quad x \in \gamma_h.$$

Then an operator A defined as $Av = -\Lambda \overset{o}{v}$ is symmetric and positively definite, hence a system of its eigenvectors $\xi_l(x_i)$ is orthonormal and complete. We first consider two auxiliary boundary value problems

$$\begin{aligned} &\Lambda W^n = 0, \quad x \in \omega_h, \quad n = 0, 1, \\ &W_0^n = \delta_n^0, \quad W_N^n = \delta_n^N, \end{aligned} \tag{6.3}$$

where δ_j^i is the Kronecker function. Now we can express W^n in the form

$$W^n(x_i) = \overset{o}{W}{}^{\mathrm{n}}(x_i) + \overset{*}{W}{}^{\mathrm{n}}(x_i), \quad \overset{*}{W}{}^{\mathrm{n}}(x_i) = (1-n)\delta_i^0 + n\delta_i^{\mathrm{n}}.$$

For $\overset{o}{W}{}^{\mathrm{n}}(x_i)$ we have the Fourier sum

$$\overset{o}{W}{}^{\mathrm{n}}(x_i) = \sum_{l=1}^{N-1} s_l^n \xi_l(x_i), \quad s_l^n = (\overset{o}{W}{}^{\mathrm{n}}, \xi_l). \tag{6.4}$$

We also represent a solution of the difference scheme (6.1) in the form

$$\begin{aligned} &y = \overset{o}{y} + \mu_0 \overset{*}{W}{}^{\circ} + \mu_1 \overset{*}{W}{}^{1}, \\ &\widetilde{y} = \rho \overset{o}{y} + \mu_0^* \overset{*}{W}{}^{\circ} + \mu_1^* \overset{*}{W}{}^{1}, \ x \in \overline{\omega}_h. \end{aligned} \tag{6.5}$$

Our main problem is to find Fourier coefficients of $\overset{o}{y}_i(t_j)$

$$\overset{o}{y}_i = \sum_{l=1}^{N-1} v_l(t_j)\xi_l(x_i).$$

Lemma 6.1. *The following formula is valid for the Fourier coefficients of* $\overset{o}{y}_i(t_j)$

$$\begin{aligned} v_l(t_j) =& p_l \sum_{m=1}^{j} \big((1-p_l)\rho\big)^{j-m} \sum_{n=0}^{1} \big(\sigma\mu_n(t_m) \\ &+ (1-\sigma)\mu_n^*(t_{m-1})\big) s_l^n, \end{aligned} \tag{6.6}$$

where $p_l = \tau\lambda_l/(1+\sigma\tau\lambda_l)$.

Proof. By substituting (6.5) into the difference scheme we obtain for $x \in \omega_h$

$$\begin{aligned} (E+\tau\sigma A)\overset{o}{y}(t_{j+1}) =& \big(E-\tau(1-\sigma)A\big)\rho\overset{o}{y} \\ &+ \tau\sum_{n=0}^{1}\big(\sigma\mu_n(t_{j+1}) + (1-\sigma)\mu_n^*\big)A\overset{*}{W}{}^{n}. \end{aligned}$$

It follows from (6.3) that $\Lambda\overset{*}{W}{}^{n} = -\Lambda\overset{o}{W}{}^{n}$, $x \in \omega_h$. The system of eigenvectors $\xi_l(x_i)$ is complete and orthonormal, hence

$$\begin{aligned} (1+\tau\sigma\lambda_l)v_l(t_{j+1}) =& \big(1-\tau(1-\sigma)\lambda_l\big)\rho v_l \\ &+ \tau\lambda_l\sum_{n=0}^{1}\big(\sigma\mu_n(t_{j+1}) + (1-\sigma)\mu_n^*\big)s_l^n, \end{aligned}$$

or after simple computations we obtain

$$\begin{aligned} v_l(t_{j+1}) =& (1-p_l)\rho v_l \\ &+ p_l\sum_{n=0}^{1}\big(\sigma\mu_n(t_{j+1}) + (1-\sigma)\mu_n^*\big)s_l^n. \end{aligned} \tag{6.7}$$

Combining (6.7) and $v_l(0) = 0$ we prove the lemma.

Theorem 6.2. *Assume that* $\sigma = 0.5 + \sigma_0,\ \sigma_0 > 0$, *then the solution of* (6.1) *is stable with respect to boundary conditions and the following estimate holds:*

$$\|\overset{o}{y}\|^2 \leqslant \frac{1}{2\sigma_0^2}\sum_{n=0}^{1}\left(\sigma M_n^2 + (1-\sigma)M_n^{*2}\right), \quad M_n = \max_j |\mu_n(t_j)|.$$

Proof. Recalling that $|\rho| \leqslant 1$ we find from (6.6)

$$\begin{aligned}|v_l(t_j)| \leqslant & p_l \sum_{n=1}^{j} |1-p_l|^{j-m} \\ & \times \sum_{n=0}^{1} |s_l^n|\Big(\sigma|\mu_n(t_m)| + (1-\sigma)|\mu_n^*(t_{m-1})|\Big) \\ \leqslant & \sum_{n=0}^{1} |s_l^n|\left(\sigma M_n + (1-\sigma)M_n^*\right)p_l/\left(1-|1-p_l|^j\right).\end{aligned}$$

We first note that $p_l/(1-|1-p_l|) = 1$ for $p_l \leqslant 1$. On the other hand for $p_l > 1,\ \sigma = 0.5 + \sigma_0,\ \sigma_0 > 0$ we have

$$\begin{aligned}\frac{p_l}{1-|1-p_l|} = \frac{p_l}{2-p_l} &= \frac{\tau\lambda_l}{2 + 2\tau\sigma\lambda_l - \tau\lambda_l} \\ &\leqslant \frac{\tau\lambda_l}{2 + 2\tau\lambda_l\sigma_0} \leqslant \frac{1}{2\sigma_0}.\end{aligned}$$

Hence we obtain an estimate

$$\begin{aligned}\|\overset{o}{y}\|^2 = \sum_{l=1}^{N-1} v_l^2 &\leqslant \frac{1}{2\sigma_0^2}\sum_{n=0}^{1}\left((\sigma M_n + (1-\sigma)M_n^*)^2 \sum_{l=1}^{N-1}|s_l^n|^2\right) \\ &\leqslant \frac{1}{2\sigma_0^2}\sum_{n=0}^{1}\left(\sigma M_n^2 + (1-\sigma)M_n^{*2}\right)\|\overset{o}{W}{}^n\|^2.\end{aligned}$$

It remains to use the maximum principle to get a bound $|\overset{o}{W}^{n}(x_i)| \leqslant 1$. The theorem is proved.

An obvious consequence of Theorem 3.4 is that for $\sigma = 0.5 + \sigma_0$ the convergence rate of the LOD method (and SAS) is $O(\tau + h^2)$.

In the case of $0 \leqslant \rho \leqslant 1$, $\mu_k = \mu_k^*$, $k = 0, 1$, the difference scheme (6.1) solution is stable with respect to boundary conditions for $\sigma \geqslant 0.5$. We note that only the apriori bounds (6.1b) are used in the proof.

Theorem 6.3. *Assume that* $\sigma \geqslant 0.5$, $0 \leqslant \rho \leqslant 1$, $\mu_k = \mu_k^*$, $k = 0, 1$, *then the stability inequality is valid*

$$||\overset{o}{y}||^2 \leqslant \frac{2(2-\sigma)^2}{\sigma^2}\left(M_0^2 + M_1^2\right).$$

Proof. It is sufficient to investigate the case $p_l > 1$. We have from the definition of p_l that $p_l \leqslant 1/\sigma \leqslant 2$. Let define

$$r_l = -\rho(1 - p_l),$$

then $0 < r_l \leqslant 1$, $l = 1, 2, \cdots, N-1$. In order to estimate the Fourier coefficients we have first

$$\begin{aligned} v_l(t_j) =& p_l \sum_{n=0}^{1} s_l^n \left(\left(\sigma - \frac{1-\sigma}{r_l} \right) \sum_{m=1}^{j-1} (-r_l)^{j-m} \mu_n(t_m) \right. \\ & \left. + \sigma \mu_n(t_j) + (-r_l)^{j-1}(1-\sigma)\mu_n(0) \right), \end{aligned}$$

and therefore

$$\begin{aligned} |v_l(t_j)| \leqslant& |p_l| \sum_{n=0}^{1} |s_l^n| M_n \left(|\sigma r_l - 1 + \sigma| \frac{1 - r_l^{j-1}}{1-r} \right. \\ & \left. + \sigma + (1-\sigma) r_l^{j-1} \right) \\ \leqslant& |p_l| \sum_{n=0}^{1} |s_l^n| M_n \left(1 + \frac{|\sigma r_l - 1 + \sigma|}{1 - r_l} \right). \end{aligned}$$

We shall define a set $R(\sigma) = \{r|0 \leqslant r \leqslant 1/\sigma - 1\}$. It is easy to verify that $\sigma r - 1 + \sigma \leqslant 0$ for $r \in R(\sigma)$. Hence we shall consider the variational problem

$$\max_{r \in R(\sigma)} g(r) = g(r^*), \quad g(r) = \frac{1 - \sigma - \sigma r}{1 - r}.$$

Using straighforward variational methods we obtain that $g'(r) \leqslant 0$, therefore $r^* = 0$, $g(r^*) = 1 - \sigma$. Now we are able to derive the uniform error estimate

$$|v_l(t_j)| \leqslant (2-\sigma)|p_l| \sum_{n=0}^{1} |s_l^n| M_n \leqslant \frac{2-\sigma}{\sigma} \sum_{n=0}^{1} |s_l^n| M_n.$$

Summation over l then yields the desired estimate and completes the proof.

We note that the stability estimate proved by Stoyan (1971) can be applied only for problems with continuous boundary conditions $|\mu_{nt}(t_j)| \leqslant C$.

7. Boundary correction method. We have proved in previous sections that LOD schemes with $\sigma = 0.5$ may suffer from order reduction. For such problems boundary correction methods are used to restore the order of consistency (see Samarskij, 1983; Sommeijer *et al.*, 1981; Muchinsky and Tsurko, 1992).

Following the method proposed by Samarskij (1974) we obtain the corrected boundary conditions (see the basic LOD scheme (2.1))

$$y_\alpha = u_\gamma(x, t_{j+\alpha/p}) + \tau\, \zeta_\alpha, \quad \alpha = 1, 2, \cdots, p, \quad x \in \gamma_h, \quad (7.1)$$

where we denoted

$$\zeta_\alpha = \sum_{i=1}^{\alpha} \psi_i{}^0(\bar{t}),$$

$$\psi_\alpha{}^0(\bar{t}) = -1/p\, \frac{\partial}{\partial t}(\bar{t}) + L_\alpha u(\bar{t}) + 1/p\, f(\bar{t}),$$

$$\bar{t} = t_{j+0.5}.$$

The equality ${\psi_1}^0 + {\psi_2}^0 + \cdots + {\psi_p}^0 = 0$ is used for the implementation of (7.1).

Then we obtain from (3.2) that a local discretization error satisfies estimation $||\delta_{j,p}|| \leqslant C(\tau + h^2)$. Substitution of this inequality into (3.1) leads to the unconditional convergence result in the L_2 norm for both schemes, the LOD scheme and SAS

$$||z_j|| \leqslant C(\tau + h^2), \quad j = 0, 1, 2, \cdots, K. \tag{7.2}$$

As it follows from Sect. 5 such corrections are not necessary for the two dimensional LOD scheme with $\sigma = 0.5$. Still, boundary corrections may be useful to obtain smaller error constants (see, e.g., Muchinsky and Tsurko, 1992). To give an illustration of the last remark, we present numerical results for the model problem from Sect. 4. In Table 7.1 global errors are given for the LOD scheme (2.1) when boundary conditions are given by (7.1).

Table 7.1. Global errors (L_2 and C norms) for the LOD scheme with corrected boundary conditions

N	10	20	40	60
L_2	0.0130	0.00826	0.00467	0.00248
C	0.0282	0.0169	0.00919	0.00478

Next we will estimate more exactly the accuracy of SAS with corrected boundary conditions. It is reasonable to expect some improvement of the accuracy due to symmetry of SAS. First we consider the 2 D SAS. Recursions for the global errors z_j are easily obtained from the results for the LOD scheme. For the SAS we thus have

$$\begin{aligned} z_{j+1} &= R_2 z_j + \tau \delta_{j,2}, \\ z_{j+2} &= R_2 z_{j+1} + \tau {\delta^*}_{j+1,2}, \end{aligned} \tag{7.3}$$

where $\delta^*{}_{j+1,2}$ is the local approximation error

$$\delta^*_{j+1,2} = \left(E - \frac{\tau}{2}\Lambda_1\right)^{-1}\left(E - \frac{\tau}{2}\Lambda_2\right)^{-1}\left(\frac{\tau}{2}\frac{\partial \nu}{\partial t}(t_{j+1.5}) - \frac{\tau}{2}(\Lambda_1 + \Lambda_2)\nu(t_{j+1.5}) - \frac{\tau^2}{8}(\Lambda_2 - \Lambda_1)\frac{\partial \nu}{\partial t}(t_{j+1.5})\right) + O(\tau^2 + h^2).$$

Taking the two steps together, it follows that

$$z_{j+2} = R_2^2 z_j + \tau\tilde{\delta}_{j,2}, \quad j = 0,1,2,\cdots,K-2, \tag{7.4}$$

where

$$\tilde{\delta}_{j,2} = R_2\delta_{j,2} + \delta^*_{j+1,2} = -\frac{\tau^2}{4}\left(E - \frac{\tau}{2}\Lambda_1\right)^{-2}\left(E - \frac{\tau}{2}\Lambda_2\right)^{-2} \times \left((5\Lambda_2 + 3\Lambda_1)\frac{\partial \nu}{\partial t}(t_j) - 2(\Lambda_1 + \Lambda_2)^2\nu(t_j) - \tau^2\Lambda_1\Lambda_2(3\Lambda_2 + \Lambda_1)\frac{\partial \nu}{\partial t}(t_j)\right) + O(\tau^2 + h^2). \tag{7.5}$$

Theorem 7.1. *Consider 2D SAS with corrected boundary conditions* (7.1). *There is a constant C, depending only on T and the smoothness of $u(x,t)$, such that*

$$||u(t_j) - y_j|| \leqslant C(\tau^2 + h^2).$$

Proof. We intend to use Lemma 5.1 for (7.4). First we prove that

$$E - {R_2}^2 = -2\tau\left(E - \frac{\tau}{2}\Lambda_1\right)^{-2}\left(E - \frac{\tau}{2}\Lambda_2\right)^{-2} \times (\Lambda_1 + \Lambda_2)\left(E + \frac{\tau^2}{4}\Lambda_1\,\Lambda_2\right).$$

Now we rewrite $\tilde{\delta}_{j,2}$ in the following way

$$\tilde{\delta}_{j,2} = (E - R_2^2)\left(P_1 \frac{\partial \nu}{\partial t} + P_2 \nu\right)(t_j) + \eta_j,$$

where we have denoted

$$\begin{aligned} P_1 =&(\Lambda_1 + \Lambda_2)^{-1}\left(E + \frac{\tau^2}{4}\Lambda_1\,\Lambda_2\right)^{-1}(5\Lambda_2 + 3\Lambda_1) \\ &- \tau^2\Lambda_1\Lambda_2(3\Lambda_2 + \Lambda_1), \\ P_2 =&(\Lambda_1 + \Lambda_2)^{-1}\left(E + \frac{\tau^2}{4}\Lambda_1\,\Lambda_2\right)^{-1}(\Lambda_1 + \Lambda_2), \end{aligned}$$

It follows from (7.5) that $||\eta_j|| \leqslant C(\tau^2 + h^2)$. Note that both P_1 and P_2 are uniformly bounded in the L_2 norm. Let $\mu_j = (P_1 \frac{\partial \nu}{\partial t} + P_2\nu)(t_j)$. Smoothness of $u(x,t)$ implies $||\mu_j|| \leqslant C\tau$, $||\mu_j - \mu_{j-1}|| \leqslant C\tau^2$. The proof follows from Lemma 5.1.

Table 7.2 nicely illustrates the theory.

Table 7.2. Global errors (L_2 and C norms) for SAS with corrected boundary conditions

N	10	20	40	80
L_2	0.02853	0.00876	0.00244	0.00065
C	0.05450	0.01723	0.00527	0.00156

For the analysis of the accuracy of p-dimensional ($p \geqslant 3$) SAS with corrected boundary conditions (7.1) we will use the method developed in

Sect. 4. The global error z_j can be found as a solution of the following problem

$$z_{\bar{t}_\alpha} = \Lambda_{\beta(\alpha)} \frac{z_\alpha + z_{\alpha-1}}{2} + \psi_\alpha, \tag{7.6}$$
$$z_\alpha(x) = \eta_\alpha, \quad x \in \gamma_h, \quad \alpha = 1, 2, \ldots, 2p,$$
$$z_0(x) = 0, \quad x \in \bar{\omega}_h,$$

where $\beta(\alpha) = \alpha$ for $\alpha = 1, 2, \ldots, p$ and $\beta(\alpha) = 2p + 1 - \alpha$ for $\alpha = p+1, p+2, \ldots, 2p$.
Note that boundary points are included into definition of Λ_α.
The discretization error ψ_α can be represented as a sum of three terms (see (3.3))

$$\psi_\alpha = \psi_\alpha^0 + \psi_\alpha^1 + \psi_\alpha^*, \quad \alpha = 1, 2, \ldots, p, \tag{7.7}$$

where we have denoted

$$\begin{aligned}
\psi_\alpha^0 =& -\frac{1}{p}\frac{\partial u}{\partial t}(t_{j+0.5}) + L_\alpha u(t_{j+0.5}) + \frac{1}{p} f(t_{j+0.5}),\\
&\alpha = 1, 2, \ldots, p,\\
\psi_\alpha^0 =& -\frac{1}{p}\frac{\partial u}{\partial t}(t_{j+1.5}) + L_{\beta(\alpha)} u(t_{j+1.5}) + \frac{1}{p} f(t_{j+1.5}),\\
&\alpha = p+1, p+2, \ldots, 2p,\\
\psi_\alpha^1 =& \tau \frac{2\alpha - 1 - p}{2p^2}\Big(-\frac{\partial u}{\partial t}(t_{j+1}) + pL_\alpha u(t_{j+1}) + f(t_{j+1})\Big),\\
&\alpha = 1, 2, \ldots, p,\\
\psi_\alpha^1 =& \tau \frac{2\alpha - 1 - 3p}{2p^2}\Big(-\frac{\partial u}{\partial t}(t_{j+1}) + pL_{\beta(\alpha)} u(t_{j+1}) + f(t_{j+1})\Big),\\
&\alpha = p+1, p+2, \ldots, 2p.
\end{aligned}$$

The last term $||\psi_\alpha^*||$ is uniformly bounded by $C(\tau^2 + h^2)$ for a sufficiently smooth solution $u(x,t)$. It follows from (7.7) that

$$\sum_{\alpha=1}^{p} \psi_\alpha^0 = 0. \qquad \sum_{\alpha=p+1}^{2p} \psi_\alpha^0. \tag{7.8a}$$

Observing that $\psi_\alpha^1 = -\psi_{\beta(2p+1-\alpha)}^1$ we obtain the equality

$$\sum_{\alpha=1}^{2p} \psi_\alpha^1 = 0. \tag{7.8b}$$

We look for a solution of (7.6) of the form $z_{j,\alpha} = v_{j,\alpha} + \eta_{j,\alpha}, \quad \alpha = 1, 2, \ldots, 2p$. Recall that we defined η_α as a solution of the problem

$$\eta_{\bar{t}_\alpha} = \psi_\alpha^0, \quad x \in \bar{\omega}_h, \quad \eta_0(0) = 0. \tag{7.9}$$

It follows from (7.9) that

$$\eta_{j,\alpha} = \eta_{j-2,2p} + \tau \sum_{k=1}^{\alpha} \psi_k^0.$$

Taking into account (7.8a) we obtain

$$\eta_{j,p} = \eta_{j-1,p} = \cdots = \eta_0(0) = 0.$$

Therefore, the following estimation follows directly

$$||\eta_{j,\alpha}|| \leqslant C\tau, \quad \alpha = 1, 2, \cdots, 2p.$$

We also use the equalities obtained by Taylor expansion

$$\begin{aligned}
\eta_\alpha(t_{j+0.5}) =& \eta_\alpha(t_{j+1}) - \frac{\tau}{2}\frac{\partial \eta}{\partial t}(\widetilde{t}_{1\alpha}), \\
& \alpha = 1, 2, \ldots, p, \\
\eta_\alpha(t_{j+1.5}) =& \eta_\alpha(t_{j+1}) + \frac{\tau}{2}\frac{\partial \eta}{\partial t}(\widetilde{t}_{2\alpha}), \\
& \alpha = p+1, p+2, \ldots, 2p.
\end{aligned}$$

Now we have a problem for v_α:

$$\begin{aligned}
& v_{\bar{t}_\alpha} = \Lambda_{\beta(\alpha)} \frac{v_\alpha + v_{\alpha-1}}{2} + \tilde{\psi}_\alpha^1 + \tilde{\psi}_\alpha^*, \quad x \in \omega_h, \\
& v_\alpha = 0, \quad x \in \gamma_h, \quad \alpha = 1, 2, \ldots, 2p, \\
& v(x, 0) = 0, \quad x \in \bar{\omega}_h,
\end{aligned}$$

where

$$\tilde{\psi}_\alpha^1 = \psi_\alpha^1 + \Lambda_{\beta(\alpha)}\Big(\frac{\eta_\alpha + \eta_{\alpha-1}}{2}\Big)(t_{j+1}),$$

$$\tilde{\psi}_\alpha^* = \psi_\alpha^* \pm \frac{\tau}{2}\Lambda_{\beta(\alpha)}\frac{\partial}{\partial t}\frac{\eta_\alpha + \eta_{\alpha-1}}{2}, \quad \alpha = 1, 2, \ldots, 2p.$$

Suppose that smoothness of $u(x, t)$ provides that

$$||\Lambda_m \psi_\alpha^0|| \leqslant C, \quad ||\frac{\partial}{\partial t}\Lambda_m \psi_\alpha^0|| \leqslant C, \quad m, \alpha = 1, 2, \ldots, 2p,$$

where C is independent on τ, h (compare with (3.7)). Then we have the following uniform estimates

$$||\tilde{\psi}_\alpha^1|| \leqslant C\tau, \quad ||\tilde{\psi}_\alpha^*|| \leqslant C(\tau^2 + h^2). \tag{7.10a}$$

Observing that $\eta_\alpha = -\eta_{2p-\alpha}$ from (7.7b) we obtain the equality

$$\sum_{\alpha=1}^{2p} \tilde{\psi}_\alpha^1 = 0. \tag{7.10b}$$

Theorem 7.2. *Consider p-dimensional $(p \geqslant 3)$ SAS with corrected boundary conditions (7.1). There is a constant C depending only on T and the smoothness of $u(x, t)$, such that*

$$||u(t_j) - y_j|| \leqslant C(\tau^{1.5} + h^2).$$

The proof follows from Lemma 4.2 and equalities (7.10).

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