Novel Entropy Measure Definitions and Their Uses in a Modified Combinative Distance-Based Assessment (CODAS) Method Under Picture Fuzzy Environment

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Abstract. From the perspective of multiple attribute decision analysis, the evaluation of decision alternatives should be based on the performance scores determined with respect to more than one attribute. Fuzzy logic concepts can equip the evaluation process with different scales of linguistic terms to let the decision-makers point out their ideas and preferences. A more recent one of fuzzy sets is the picture fuzzy set which covers three separately allocable elements: positive, neutral, and negative membership degrees. The novel and distinctive element included by a picture fuzzy set is the refusal degree which is equal to the difference between 1 and the sum of the other three. In this study, we aim to contribute to the literature of the picture fuzzy sets by (i) proposing two novel entropy measures that can be used in objective attribute weighting and (ii) developing a novel picture fuzzy version of CODAS (COmbinative Distance-based ASsessment) method which is empowered with entropy-based attribute weighting. The applicability of the method is shown in a green supplier selection problem. To clarify the differences of the proposed method, a comparative analysis is provided by considering traditional CODAS, spherical fuzzy CODAS, and spherical fuzzy TOPSIS with different entropy-based scenarios.

Key words: picture fuzzy sets, CODAS, entropy, information measures, expert judgment, green supply chain management, supplier selection.

1. Introduction

Many Multiple Attribute Decision Making (MADM) problems do not involve appropriate data which are directly measurable such as cost, net profit, any financial ratio, volume or weight of an item, etc. To deal with the quantification problems, the decision-makers can benefit from linguistic terms in stating their thoughts and preferences regarding the problem. Linguistic terms are often defined in different fuzzy environments. Zadeh (1965)

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launched the domain of fuzzy sets for symbolizing human judgments. In conventional fuzzy set definition, the opinions can be represented by a single membership degree (μ) which ranges between 0 and 1. This membership degree measures the optimism or agreement level and possesses a positive perspective.

To smooth the representation challenge of the uncertainty in human judgments, fuzzy set domain has been extended by researchers from different fields. Atanassov (1986) introduced the intuitionistic fuzzy sets (IFS) by defining a negative membership (or non-membership) degree (v). This new element in the fuzzy set definition brings resilience to the uncertainty representation issue because the experts are able to specify their pessimistic views or disagreements in this way. Hence, non-membership degree points out a negative perspective. Atanassov (1986) also defined a new element regarding the indeterminacy (or hesitancy) which shows the experts' neutral preference: $\pi = 1 - \mu - v$. Thus, IFS was the first fuzzy concept that can cope with three dimensions of judgments (positive membership, negative membership, and indeterminacy). In real life, these degrees are the equivalents of yes, no, and abstain, respectively, in a voting environment. Nonetheless, the drawback in IFS is that the indeterminacy degree cannot be independently assigned by the experts.

After the introduction of the horizon widening features of fuzzy sets and IFSs, new fuzzy concepts have been proposed in the literature. Pythagorean fuzzy sets (Yager, 2013), q-Rung orthopair fuzzy sets (Yager, 2017), and Fermatean fuzzy sets (Senapati and Yager, 2020) have extended the representation domain of the expert by just considering the independently assignable positive and negative membership degrees. The independently assignable hesitancy degree is considered in neutrosophic sets (Smarandache, 1999) and spherical fuzzy sets (Kutlu Gündoğdu and Kahraman, 2019a). Many researchers work on developing aggregation operators, information measures such as entropy, distance, inclusion (subsethood), and knowledge measures for easing the uncertainty handling issue in decision-making problems.

The most recent fuzzy set concept considering all the three independently assignable membership degrees was presented by Cuong and Kreinovich (2013). This novel concept is called picture fuzzy sets (PFS) as a logical reasoning of fuzzy sets and IFSs. A PFS is characterized by three independently assignable degrees expressing the positive (μ), the neutral (η – which means hesitancy), and the negative (v – which is an equivalent to non-membership) membership degrees. The sole constraint defined in PFS enforces that their sum must not exceed 1. The gap between their sum and 1 is called refusal degree. The expert's choice of refusing the idea sharing is quantified in PFS by this novel element. The refusal degree is here defined as $\pi = 1 - \mu - \eta - v$.

PFS exhibits its importance in voting environments. Cuong (2014) clarifies the items in PFS for the voting case. The voters can be divided into four groups: vote for the candidate, abstain, vote against the candidate, and refusal of the voting, i.e. casting a veto. It is obvious that PFS has a broader representation power than previous extensions of fuzzy sets since it involves a fourth component called refusal degree. PFS is the only fuzzy set definition handling this issue. Son (2016) introduced an example showing the importance of PFS for MADM. The personnel selection activity needs information about the candidates for

understanding whether they are eligible for the job or not. The result of this selection could be one of the 4 classes: true positive, true negative, false negative, and false positive which can be accepted as the equivalents to the membership degrees of PFS. Each candidate is evaluated by considering 4 classes. The selection is based on these evaluations. Assume that two candidates are evaluated: *A* took (50%, 20%, 20%, 10%) and *B* took (40%, 10%, 30%, 20%). The most appropriate candidate can be selected by applying the score function defined for PFS. As given in Definition 3, score value of *A* is 50% − 20% = 30% and score value of *B* is 40% − 30% = 10%. Thus, candidate *A* will be selected as a result.

Most of the business and management issues are MADM problems because the companies, institutions, even societies, and governments are enforced to take a lot of features of the decisions they should take into account. The problem definition of any MADM problem involves the determination of three basic elements: the attributes which can possess a potential impact on the results, the decision-makers who will be consulted due to their expertise, and the alternatives that are the potential solutions.

In MADM understanding, each alternative is evaluated by the decision-makers with respect to attributes, and the very first issue that should be addressed in decision analysis emerges after this data collection activity: how will we process these data to obtain the overall performance of each alternative? The motivation behind this question is that the decision analyst has many methodologies that can be used in reflecting the importance of the attributes to the decision. Each attribute has its own mean with discrete and changing significance for the problem at hand.

The requirement explained above is also called attribute weighting and it can be handled via applying one of two basic methods or a mixture of them: (1) subjective methods such as AHP-Analytic Hierarchy Process, SWARA-Stepwise Weight Assessment Ratio Analysis and Simos' procedure are based on the experts' evaluations; (2) objective methods do not demand these individualistic preferences and thoughts. In calculating the attribute weights, they just examine the performances of the alternatives with the aim of removing or at least limiting the risk of manipulative preferential actions of decision-makers or too long data collection periods. Particularly, in auditing the companies in terms of the quality assurance performance of their business processes, occupational health issues, and their financial strength, the subjectivity in attribute weighting may be misleading. Therefore, objective attribute weighting tools can be used to handle these issues.

Entropy measure which is based on the performance scores can support the objective attribute weighting effort. In this study, two new entropy measures are developed for PFSs and their applicability in MADM is manifested in integration to a novel extension of CODAS (COmbinative Distance-based ASsessment) method under picture fuzzy environment. The main contributions of the study are listed as follows:

1. Two novel entropy measures for PFSs are proposed as a contribution to the literature of PFS and their validity is demonstrated via comparisons with previously developed entropy measures.
2. CODAS method which is established by Keshavarz-Ghorabaee *et al.* (2016) is modified with the operational rules of PFSs for allowing the MADM process to handle the picture fuzzy type data. This is the first attempt at this topic. The attribute weights are

calculated in this version via a methodology that is based on newly defined entropy measures, but it is applicable with subjective attribute weights.

There are 7 sections in the study. After this first section regarding the introduction, Section 2 covers the preliminaries of PFS and the results of an extensive literature review on the recent fuzzy extension of CODAS. Novel entropy measures for PFS are demonstrated and proved in Section 3. Section 4 presents the novel picture fuzzy extension of CODAS (PF-CODAS) and the integration of entropy measures into the model. In section 5, the newly proposed PF-CODAS version with entropy-based objective weighting is implemented in an example that was previously studied by Meksavang *et al.* (2019). The results of various implementations are compared in order to show the validity of the novel entropy-based PF-CODAS approach in Section 6. Section 7 concludes the study with some remarks and the future research possibilities are also mentioned.

2. Preliminaries

First, the PFS concept is introduced and the defined operations on PFS are specified in this section. Then, CODAS is introduced, and the results of a comprehensive literature overview on CODAS are summarized in order to understand and clarify the state-of-the-art. We limit our research on CODAS here in order not to distort the flow of the paper.

2.1. *Picture Fuzzy Sets*

PFS theory is introduced by Cuong and Kreinovich (2013) as a generalization and synthesis of Zadeh's fuzzy set theory and Atanassov's intuitionistic fuzzy set theory. A PFS has three independently assignable informative elements, i.e. the degree of positive membership (μ), the degree of neutral membership (η), the degree of negative membership (ν). Due to the definition's sole limitation forcing that their sum should be smaller than or equal to 1, the remaining part is presented as a novelty and a distinctive feature of PFS, which is called the degree of refusal membership (π). These four elements comprise the potential vote types such as yes, no, abstain, and refuse.

DEFINITION 1. Let X be a universal set. Then a PFS A on X is defined as follows:

$$A = \big\{ \big(x, \mu_A(x), \eta_A(x), \nu_A(x) \big) \,\big|\, x \in X \big\}, \tag{1}$$

where μ_A, η_A, ν_A are mapping from X to [0, 1]. For all $x \in X$, $\mu_A(x)$ is called positive membership degree of $x \in A$, $\eta_A(x)$ is called neutral membership degree of $x \in A$ and $\nu_A(x)$ is negative membership degree of $x \in A$. Also, μ_A, η_A, ν_A must satisfy the following condition:

$$0 \leqslant \mu_A(x) + \eta_A(x) + \nu_A(x) \leqslant 1, \quad \forall x \in X \tag{2}$$

and $\pi(x) = 1 - \mu_A(x) - \eta_A(x) - \nu_A(x)$ is called refusal membership degree of x in A (Cuong, 2014). $PFS(X)$ denotes all the picture fuzzy sets on a universal set X.

Cuong (2014) defined the subsethood, equality, union, intersection, and complement for every two PFSs A and B as follow:

1. $A \subseteq B$ if $\forall x \in X$, $\mu_A(x) \leqslant \mu_B(x)$, $\eta_A(x) \leqslant \eta_B(x)$, $\nu_A(x) \geqslant \nu_B(x)$;
2. $A = B$ iff $A \subseteq B$ and $B \subseteq A$;
3. $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$;
4. $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in X\}$;
5. $A^c = \{\langle x, \nu_A(x), \eta_A(x), \mu_A(x)\rangle \mid x \in X\}$.

DEFINITION 2. *Let $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ are two PF numbers (PFN), then their operations are as follows (Wang et al., 2017):*

$$A \oplus B = \left(1 - (1 - \mu_A)(1 - \mu_B), \eta_A\eta_B, (\eta_A + \nu_A)(\eta_B + \nu_B) - \eta_A\eta_B\right), \qquad (3)$$

$$A \otimes B = \left((\mu_A + \eta_A)(\mu_B + \eta_B) - \eta_A\eta_B, \eta_A\eta_B, 1 - (1 - \nu_A)(1 - \nu_B)\right), \qquad (4)$$

$$\tau A = \left(1 - (1 - \mu_A)^\tau, (\eta_A)^\tau, (\eta_A + \nu_A)^\tau - (\eta_A)^\tau\right), \qquad (5)$$

$$A^\tau = \left((\mu_A + \eta_A)^\tau - (\eta_A)^\tau, (\eta_A)^\tau, 1 - (1 - \nu_A)^\tau\right), \qquad (6)$$

$$A^C = (\nu_A, \eta_A, \mu_A). \qquad (7)$$

DEFINITION 3. *Let $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ are two PFNs. In order to compare them, the algorithm given below is used where score function is defined as $sc(A) = \mu_A - \nu_A$ and accuracy function is $ac(A) = \mu_A + \eta_A + \nu_A$ (Wang et al., 2017):*

(i) *If $sc(A) > sc(B)$, then $A > B$ means A is superior to B.*
(ii) *If $sc(A) = sc(B)$, then*

(1) *$ac(A) > ac(B)$ implies that $A > B$, which means A is superior to B;*
(2) *$ac(A) = ac(B)$ implies that $A = B$, which means A is equivalent to B.*

DEFINITION 4. *Let $A_i = (\mu_i, \eta_i, \nu_i)$ $(i = 1, \ldots, n)$ is a collection of PFNs in X. The picture fuzzy weighted averaging (PFWA) operator is given as follows (Wei, 2017a):*

$$PFWA_w(A_i) = \left(1 - \prod_{i=1}^{n}(1 - \mu_i)^{w_i}, \prod_{i=1}^{n}(\eta_i)^{w_i}, \prod_{i=1}^{n}(\nu_i)^{w_i}\right), \qquad (8)$$

where $w = (w_1, \ldots, w_n)$ is the weight vector of A_i, and satisfies $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$.

The distance, similarity, and subsethood measures are important tools to compare two fuzzy sets. Therefore, many researchers studied the concept of these measures for PFSs.

DEFINITION 5. *Let $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ are two PFNs.*

Euclidean ($d_E(A, B)$) and Hamming ($d_H(A, B)$) distances are defined by Cuong and Kreinovich (2013) as follows:

$$d_E(A, B) = \left[\frac{1}{n} \sum_{i=1}^{n} ((\mu_A - \mu_B)^2 + (\eta_A - \eta_B)^2 + (v_A - v_B)^2) \right]^{\frac{1}{2}}, \qquad (9)$$

$$d_H(A, B) = \frac{1}{n} \sum_{i=1}^{n} (|\mu_A - \mu_B| + |\eta_A - \eta_B| + |v_A - v_B|). \qquad (10)$$

PF distance measures $d_{DT_1}, d_{DT_2}, d_{DT_3}, d_{DT_4}$ were introduced by Dinh and Thao (2018) as follow:

$$d_{DT_1}(A, B) = \frac{1}{3n} \sum_{i=1}^{n} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |v_A - v_B|], \qquad (11)$$

$$d_{DT_2} = \frac{1}{n} \left\{ \sum_{i=1}^{n} [(\mu_A - \mu_B)^2 + (\eta_A - \eta_B)^2 + (v_A - v_B)^2] \right\}^{\frac{1}{2}}, \qquad (12)$$

$$d_{DT_3} = (A, B) = \frac{1}{n} \sum_{i=1}^{n} \max[|\mu_A - \mu_B|, |\eta_A - \eta_B|, |v_A - v_B|], \qquad (13)$$

$$d_{DT_4} = (A, B) = \frac{1}{n} \left\{ \sum_{i=1}^{n} \max[|\mu_A - \mu_B|^2, |\eta_A - \eta_B|^2, |v_A - v_B|^2] \right\}^{\frac{1}{2}}. \qquad (14)$$

Dutta (2018) also defined the following distance measures $d_{D_1}, d_{D_2}, d_{D_3}, d_{D_4}, d_{D_5}$ for PFSs:

$$d_{D_1}(A, B) = \frac{1}{2} \sum_{i=1}^{n} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |v_A - v_B| + |\pi_A - \pi_B|], \qquad (15)$$

$$d_{D_2}(A, B) = \frac{1}{2n} \sum_{i=1}^{n} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |v_A - v_B| + |\pi_A - \pi_B|], \qquad (16)$$

$$d_{D_3}(A, B) = \left\{ \frac{1}{2} \sum_{i=1}^{n} [(\mu_A - \mu_B)^2 + (\eta_A - \eta_B)^2 + (v_A - v_B)^2 + (\pi_A - \pi_B)^2] \right\}^{\frac{1}{2}}, \qquad (17)$$

$$d_{D_4}(A, B) = \left\{ \frac{1}{2n} \sum_{i=1}^{n} [(\mu_A - \mu_B)^2 + (\eta_A - \eta_B)^2 + (v_A - v_B)^2 + (\pi_A - \pi_B)^2] \right\}^{\frac{1}{2}}, \qquad (18)$$

$$d_{D_5}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left[\begin{array}{l} \frac{1}{4}[|\mu_A - \mu_B| + |\eta_A - \eta_B| + |v_A - v_B| + |\pi_A - \pi_B|] \\ + \frac{1}{2} \max[|\mu_A - \mu_B|, |\eta_A - \eta_B|, |v_A - v_B|, |\pi_A - \pi_B|] \end{array} \right]. \qquad (19)$$

PF generalized distance measures d_{S_1}, d_{S_2} are proposed by Singh *et al.* (2018) as follow:

$$d_{S_1}(A, B) = \left\{ \frac{1}{4n} \sum_{i=1}^{n} \left[\begin{array}{c} |\mu_A - \mu_B|^{\lambda} + |\eta_A - \eta_B|^{\lambda} \\ + |\nu_A - \nu_B|^{\lambda} + |\pi_A - \pi_B|^{\lambda} \end{array} \right] \right\}^{\frac{1}{\lambda}}, \quad \lambda > 0, \quad (20)$$

$$d_{S_2}(A, B) = \left\{ \frac{1}{4n} \sum_{i=1}^{n} \max \left[\begin{array}{c} |\mu_A - \mu_B|^{\lambda}, |\eta_A - \eta_B|^{\lambda}, \\ |\nu_A - \nu_B|^{\lambda}, |\pi_A - \pi_B|^{\lambda} \end{array} \right] \right\}^{\frac{1}{\lambda}}, \quad \lambda > 0. \quad (21)$$

PF distance measure d_W is defined by Wei (2018a) as follows:

$$d_W(A, B) = \frac{1}{2n} \sum_{i=1}^{n} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |\nu_A - \nu_B|]. \quad (22)$$

PF distance measure d_{GS} is introduced by Ganie and Singh (2020) as follows:

$$d_{GS}(A, B) = 1 - \frac{1}{4n} \sum_{i=1}^{n} \left[\begin{array}{c} 3\sqrt{\mu_A \mu_B} + 3\sqrt{\nu_A \nu_B} + 3\sqrt{\eta_A \eta_B} + 3\sqrt{\pi_A \pi_B} \\ + \sqrt{(1 - \mu_A - \nu_A)(1 - \mu_B - \nu_B)} \\ + \sqrt{(1 - \mu_A - \eta_A)(1 - \mu_B - \eta_B)} \\ + \sqrt{(1 - \nu_A - \eta_A)(1 - \nu_B - \eta_B)} \end{array} \right]. \quad (23)$$

Wei (2017b) introduced some similarity measures based on cosine function for PFSs as follow:

$$S_{WC_1}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi}{2} \max \left(|\mu_A - \mu_B|, |\eta_A - \eta_B|, |\nu_A - \nu_B| \right) \right], \quad (24)$$

$$S_{WC_2}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi}{4} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |\nu_A - \nu_B|] \right], \quad (25)$$

$$S_{WC_3}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi}{2} \max \left(|\mu_A - \mu_B|, |\eta_A - \eta_B|, |\nu_A - \nu_B|, |\pi_A - \pi_B| \right) \right], \quad (26)$$

$$S_{WC_4}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi}{4} [|\mu_A - \mu_B| + |\eta_A - \eta_B| + |\nu_A - \nu_B| + |\pi_A - \pi_B|] \right], \quad (27)$$

$$S_{WC_5}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \max \left(|\mu_A - \mu_B|, |\eta_A - \eta_B|, |\nu_A - \nu_B| \right) \right], \quad (28)$$

$$S_{WC_6}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \max \left(|\mu_A - \mu_B|, |\eta_A - \eta_B|, \right. \right.$$

$$\left. \left. |\nu_A - \nu_B|, |\pi_A - \pi_B| \right) \right]. \quad (29)$$

PF similarity measures S_{W_1}, S_{W_2}, S_{W_3} are also introduced by Wei (2018b) as follow:

$$S_{W_1}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B}{\sqrt{\mu_A^2 + \eta_A^2 + \nu_A^2}\sqrt{\mu_B^2 + \eta_B^2 + \nu_B^2}}, \tag{30}$$

$$S_{W_2}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B}{\max(\mu_A^2 + \eta_A^2 + \nu_A^2, \mu_B^2 + \eta_B^2 + \nu_B^2)}, \tag{31}$$

$$S_{W_3}(A, B) = \frac{1}{3n} \sum_{i=1}^{n} \left(\begin{array}{c} \frac{\min(|\mu_A - \mu_B|) + \max(|\mu_A - \mu_B|)}{|\mu_A - \mu_B| + \max(|\mu_A - \mu_B|)} \\ + \frac{\min(|\eta_A - \eta_B|) + \max(|\eta_A - \eta_B|)}{|\eta_A - \eta_B| + \max(|\eta_A - \eta_B|)} \\ + \frac{\min(|\nu_A - \nu_B|) + \max(|\nu_A - \nu_B|)}{|\nu_A - \nu_B| + \max(|\nu_A - \nu_B|)} \end{array} \right). \tag{32}$$

PF similarity measures S_{S_1}, S_{S_2}, S_{S_3} are proposed by Singh *et al.* (2018) as follow:

$$S_{S_1}(A, B) = 1 - \left\{ \frac{1}{4n} \sum_{i=1}^{n} \left[\begin{array}{c} |\mu_A - \mu_B|^\lambda + |\eta_A - \eta_B|^\lambda \\ + |\nu_A - \nu_B|^\lambda + |\pi_A - \pi_B|^\lambda \end{array} \right] \right\}^{\frac{1}{\lambda}}, \quad \lambda > 0, \tag{33}$$

$$S_{S_2}(A, B) = 1 - \left\{ \frac{1}{4n} \sum_{i=1}^{n} \max \left[\begin{array}{c} |\mu_A - \mu_B|^\lambda, |\eta_A - \eta_B|^\lambda, \\ |\nu_A - \nu_B|^\lambda, |\pi_A - \pi_B|^\lambda \end{array} \right] \right\}^{\frac{1}{\lambda}}, \quad \lambda > 0, \tag{34}$$

$$S_{S_3}(A, B) = \frac{1}{4n} \sum_{i=1}^{n} \frac{\min \left(\begin{array}{c} |\mu_A - \mu_B|, |\eta_A - \eta_B|, \\ |\nu_A - \nu_B|, |\pi_A - \pi_B| \end{array} \right)}{\max \left(\begin{array}{c} |\mu_A - \mu_B|, |\eta_A - \eta_B|, \\ |\nu_A - \nu_B|, |\pi_A - \pi_B| \end{array} \right)}. \tag{35}$$

Thao (2020) defined similarity measures S_{T_1}, S_{T_2} for PFSs as follow:

$$S_{T_1}(A, B) = \sum_{i=1}^{n} \omega_i \frac{1}{4 \ln 4} \left\{ \begin{array}{c} (|\mu_A - \mu_B| - 1) \ln \frac{1 - |\mu_A - \mu_B|}{4} \\ + (|\eta_A - \eta_B| - 1) \ln \frac{1 - |\eta_A - \eta_B|}{4} \\ + (|\nu_A - \nu_B| - 1) \ln \frac{1 - |\nu_A - \nu_B|}{4} \\ - \left(\begin{array}{c} |\mu_A - \mu_B| \\ + |\eta_A - \eta_B| \\ + |\nu_A - \nu_B| + 1 \end{array} \right) \\ \times \ln \frac{|\mu_A - \mu_B| + |\eta_A - \eta_B| + |\nu_A - \nu_B| + 1}{4} \end{array} \right\}, \tag{36}$$

$$S_{T_2}(A, B) = \sum_{i=1}^{n} \omega_i \left\{ 1 - \frac{|\mu_A - \mu_B| + |\eta_A - \eta_B| + |\nu_A - \nu_B|}{3} \right\}. \tag{37}$$

PF Dice-similarity measures S_{WG_1}, S_{WG_2}, S_{WG_1} are introduced by Wei and Gao (2018) as follow:

$$S_{WG_1}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{2(\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B)}{\mu_A^2 + \eta_A^2 + \nu_A^2 + \mu_B^2 + \eta_B^2 + \nu_B^2}, \tag{38}$$

$$S_{WG_2}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{2(\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B + \pi_A \pi_B)}{\mu_A^2 + \eta_A^2 + \nu_A^2 + \pi_A^2 + \mu_B^2 + \eta_B^2 + \nu_B^2 + \pi_B^2}, \tag{39}$$

$$S_{WG_3}(A, B) = \frac{\sum_{i=1}^{n} 2(\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B)}{\sum_{i=1}^{n} (\mu_A^2 + \eta_A^2 + \nu_A^2) + \sum_{i=1}^{n} (\mu_B^2 + \eta_B^2 + \nu_B^2)}, \tag{40}$$

$$S_{WG_3}(A, B) = \frac{\sum_{i=1}^{n} 2(\mu_A \mu_B + \eta_A \eta_B + \nu_A \nu_B + \pi_A \pi_B)}{\sum_{i=1}^{n} (\mu_A^2 + \eta_A^2 + \nu_A^2 + \pi_A^2) + \sum_{i=1}^{n} (\mu_B^2 + \eta_B^2 + \nu_B^2 + \pi_B^2)}. \tag{41}$$

The only subsethood measures for PFSs are presented by Aydoğdu (2020) as follow:

$$c_{A_1}(A, B) = 1 - \frac{1}{2n} \sum_{i=1}^{n} \big[|\mu_A - \mu_{A \cap B}| + |\eta_A - \eta_{A \cap B}| + |\nu_A - \nu_{A \cap B}| \big], \tag{42}$$

$$c_{A_2}(A, B) = 1 - \left[\frac{1}{n} \sum_{i=1}^{n} \big((\mu_A - \mu_{A \cap B})^2 + (\eta_A - \eta_{A \cap B})^2 + (\nu_A - \nu_{A \cap B})^2 \big) \right]^{\frac{1}{2}}. \tag{43}$$

In some MADM applications, the weights of the attributes which are required for aggregation of the performance evaluations and/or the weights of the decision-makers representing their expertise level which are used in combining their evaluations can be unknown. Also, the subjectivity in the evaluations is required to be limited. For the purposes mentioned, objective weighting methods which are based on entropy measures, OWA operator, maximizing deviation methods, SECA (Simultaneous Evaluation of Criteria and Alternatives), and MEREC (Method based on the Removal Effects of Criteria) can be employed in the domain of MADM (Keshavarz-Ghorabaee *et al.*, 2018, 2021). Entropy is a very common manner of doing it and some entropy measures defined under PF environment are given below.

Wang *et al.* (2018) introduced the definition of entropy for PFSs in the following.

DEFINITION 6. Let $A, B \in PFS(X)$. A function $E_w : PFS(X) \to [0, 1]$ is an entropy on PFS, if E_w satisfies the following axioms:

1. $E_w(A) = 0$ if A is a crisp set,
2. $E_w(A) = 1$, if $\mu_A = \eta_A = \nu_A$,
3. $E_w(A) = E(A^c)$,
4. $E_w(A) \leqslant E_w(B)$, if A is less fuzzy then B, that is,

$$|\mu_A - \eta_A| + |\mu_A - \nu_A| + |\eta_A - \nu_A| \geqslant |\mu_B - \eta_B| + |\mu_B - \nu_B| + |\eta_B - \nu_B|.$$

PF entropy measure E_W is presented by Wang *et al.* (2018) as follows:

$$E_W(A) = 1 - \frac{1}{2n} \sum_{i=1}^{n} \big(\big| \mu_A(x_i) - \eta_A(x_i) \big| + \big| \mu_A(x_i) - \nu_A(x_i) \big|$$
$$+ \big| \eta_A(x_i) - \nu_A(x_i) \big| \big). \tag{44}$$

Thao (2020) presented another concept of PF entropy that extends the concept of intuitionistic fuzzy entropy and proposed PF entropies E_{T_1}, E_{T_2}, E_{T_3}.

DEFINITION 7. For any $C, D \in PFS(X)$, the function $E : PFS(X) \to [0, 1]$ is an entropy measure for PFS, if E_T satisfies the following axioms:

1. $E(C) = 0$ if $\mu_C(x_i), \eta_C(x_i), \nu_C(x_i) \in \{0, 1\}$ for all $x_i \in X$,
2. $E(C) = 1$ if $\mu_C = \eta_C = \nu_C = 0.25$,
3. $E(C) = E(C^c)$,
4. $E(C) \leqslant E(D)$ if C is less fuzzy than D, that is,

$$\mu_C(x_i) \leqslant \mu_D(x_i), \qquad \eta_C(x_i) \leqslant \eta_D(x_i), \qquad \nu_C(x_i) \leqslant \nu_D(x_i)$$
$$\text{if } \max\{\mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\} \leqslant 0.25$$

or

$$\mu_C(x_i) \geqslant \mu_D(x_i), \qquad \eta_C(x_i) \geqslant \eta_D(x_i), \qquad \nu_C(x_i) \geqslant \nu_D(x_i)$$
$$\text{if } \min\{\mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\} \geqslant 0.25$$

for all $x_i \in X$.

$$E_{T_1}(C) = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{\ln 4} \left\{ \begin{array}{c} -\mu_C(x_i) \ln \mu_C(x_i) - \eta_C(x_i) \ln \eta_C(x_i) \\ - \nu_C(x_i) \ln \nu_C(x_i) - \pi_C(x_i) \ln \pi_C(x_i) \end{array} \right\} \right], \tag{45}$$

$$E_{T_2}(C) = 1 - \frac{4}{3n} \sum_{i=1}^{n} \max \left\{ \begin{array}{c} |\mu_C(x_i) - 0.25|, |\eta_C(x_i) - 0.25|, \\ |\nu_C(x_i) - 0.25|, |\pi_C(x_i) - 0.25| \end{array} \right\}, \tag{46}$$

$$E_{T_3}(C) = 1 - \frac{4}{3n} \sum_{i=1}^{n} \max \left\{ \begin{array}{c} |\mu_C(x_i) - 0.25|, |\eta_C(x_i) - 0.25|, \\ |\nu_C(x_i) - 0.25|, |\pi_C(x_i) - 0.25| \end{array} \right\}. \tag{47}$$

The following entropy measures E_{J_1}, E_{J_2} are introduced by Joshi (2020a; 2020b):

$$E_{J_1}(A) = \begin{cases} \frac{1}{n(1-r)} \sum_{i=1}^{n} \left[\begin{array}{c} \mu_A^r(x_i) + \eta_A^r(x_i) + \nu_A^r(x_i) \\ + \pi_A^r(x_i) - 1 \end{array} \right], & r > 0, \ r \neq 1, \\ \frac{1}{n(1-r)} \sum_{i=1}^{n} \left[\mu_A^r(x_i) + \nu_A^r(x_i) + \pi_A^r(x_i) - 1 \right], & \text{if } \eta_A(x_i) = 0, \\ -\left(\begin{array}{c} \mu_A(x_i) \log(\mu_A(x_i)) + \nu_A(x_i) \log(\nu_A(x_i)) \\ + \pi_A(x_i) \log(\pi_A(x_i)) \end{array} \right), & \begin{array}{c} \text{if } \eta_A(x_i) = 0, \\ r = 0, \end{array} \\ \frac{1}{n(1-r)} \sum_{i=1}^{n} \left[\mu_A^r(x_i) + \left(1 - \mu_A(x_i)\right)^r - 1 \right], & \begin{array}{c} \text{if } \eta_A(x_i) \\ = \pi_A(x_i) = 0, \end{array} \end{cases} \tag{48}$$

$$E_{J_2}(A) = \begin{cases} \frac{r}{n(r-1)} \sum_{i=1}^{n} \left[1 - \left(\begin{matrix} \mu_A^r(x_i) + \eta_A^r(x_i) \\ + \nu_A^r(x_i) + \pi_A^r(x_i) \end{matrix} \right)^{\frac{1}{r}} \right], & r > 0, \ r \neq 1, \\ \frac{r}{n(r-1)} \sum_{i=1}^{n} \left[1 - \left(\mu_A^r(x_i) + \nu_A^r(x_i) + \eta_A^r(x_i) \right)^{\frac{1}{r}} \right], & \text{if } \pi_A(x_i) = 0, \\ \frac{r}{n(r-1)} \sum_{i=1}^{n} \left[1 - \left(\mu_A^r(x_i) + \left(1 - \mu_A(x_i) \right)^r \right)^{\frac{1}{r}} \right], & \text{if } \eta_A(x_i) \\ & = \pi_a(x_i) = 0. \end{cases} \quad (49)$$

Arya and Kumar (2020a; 2020b) proposed fuzzy entropy measures E_{K_1}, E_{K_2} for PFS as follow:

$$E_{K_1}(A) = -\frac{1}{n} \sum_{i=1}^{n} \left[\begin{matrix} \mu_A(x_i) \log_2(\mu_A(x_i)) + \eta_A(x_i) \log_2(\eta_A(x_i)) \\ + \nu_A(x_i) \log_2(\nu_A(x_i)) + \pi_A(x_i) \log_2(\pi_A(x_i)) \end{matrix} \right], \quad (50)$$

$$E_{K_2}(A) = \frac{1}{n(1-\lambda)} \sum_{i=1}^{n} \left[\left(\mu_A(x_i)^\lambda + \eta_A(x_i)^\lambda + \nu_A(x_i)^\lambda + \pi_A(x_i)^\lambda \right) - 1 \right], \quad \lambda \neq 1. \quad (51)$$

DEFINITION 8. Let $C = (\mu_C, \eta_C, \nu_C)$ be a PFN. Its knowledge measure is computed as follows:

$$K(C) = 1 - 0.5(En_C + \pi_C) \quad \text{with } En_C = \frac{\min(d_H(C, C_{\min}), d_H(C, C_{\max}))}{\max(d_H(C, C_{\min}), d_H(C, C_{\max}))}, \quad (52)$$

where En_C denotes the entropy value of C and $d_H(.,.)$ is the Hamming distance (Eq. (10)) between the given two PFNs. As seen from the definition, the summation operation is removed from Eq. (10) for this case because of $n = 2$ (Lin *et al.*, 2020).

2.2. CODAS and Its Fuzzy Extensions

Keshavarz-Ghorabaee *et al.* (2016) developed CODAS method as a novel MADM tool in order to support decision-analysts in their efforts of analysing the alternatives which are evaluated with respect to the appropriate attributes in the decision problem at hand. The distinctive feature of the method is the simultaneous consideration of two well-known distance measures. The performance of an alternative is evaluated by the Euclidean and Taxicab distances from the negative-ideal point. The CODAS utilizes the Euclidean distance as the primary measure for assessing the alternatives. When the Euclidean distances of two alternatives to the negative-ideal point are very close to each other, the Taxicab distance is also considered in comparing them. The degree of closeness of Euclidean distances is set by a threshold parameter. The CODAS algorithm is as follows:

1. $X = [x_{ij}]$ decision matrix is constructed where $i \ (= 1, \ldots, m)$ represents alternatives and $j \ (= 1, \ldots, n)$ represents attributes.
2. Linear normalization is conducted to standardize the decision matrix. If attribute j is a cost type attribute, $\min_i x_{ij}/x_{ij}$ is performed. For a benefit attribute j, the normalization equation is $x_{ij}/\max_i x_{ij}$. Normalized decision matrix is shown as $N = [n_{ij}]$.

3. In order to consider the importance of attributes, the normalized values are multiplied by the weights: $r_{ij} = w_j n_{ij}$. The output of this step is the weighted normalized decision matrix that can be represented by $R = [r_{ij}]$.

4. As a comparison measure, CODAS uses the distances to the negative-ideal point. This is derived from $R = [r_{ij}]$ via taking the minimum values of each attribute: $ns_j = \min_i r_{ij}$. The negative-ideal solution is a row vector: $ns = [ns_j]$.

5. Euclidean and Taxicab distances of each alternative i to ns are computed via $E_i = \sqrt{\sum_{j=1}^{n}(r_{ij} - ns_j)^2}$ and $T_i = \sum_{j=1}^{n} |r_{ij} - ns_j|$.

6. The square relative assessment matrix $Ra = [h_{ik}]$ is built by considering the pairwise comparisons of alternatives' Euclidean and Taxicab distances. To compute the h_{ik} values, the threshold parameter of $\Theta \in [0.01, 0.05]$ should be set by the decision-maker. If the difference between E_i and E_k is less than Θ, $h_{ik} = (E_i - E_k)$. Otherwise (if $(E_i - E_k) \geqslant \Theta$), both the Euclidean and Taxicab distances are considered: $h_{ik} = (E_i - E_k) + (T_i - T_k)$.

7. The assessment score of alternative i is calculated via $H_i = \sum_{k=1}^{n} h_{ik}$.

8. The alternatives are ranked in ascending order of their H_i values. The best alternative will naturally be the alternative with the highest assessment score.

When we search for fuzzy extensions of CODAS in the SCOPUS platform, it is seen that 32 studies have appeared since 2017. Table A1 in Appendix A summarizes the findings on the state-of-the-art of fuzzy versions of CODAS. In the second column of the table, the fuzzy set concepts used in studies are given. The most utilized version is intuitionistic fuzzy sets with 11 publications. 2-tuple, Pythagorean, spherical, neutrosophic, hesitant, and linguistic term-based fuzzy versions are the other concepts used. From this finding, it is understood that any picture fuzzy version of CODAS has not been developed yet. Even though spherical and neutrosophic versions handle the hesitancy representation issue in CODAS, both hesitancy and refusal degree representation power of PFS has not already been considered. The current study aims at filling this gap.

While the third column shows the hybridized MADM methods for different purposes, the fourth column shows the application fields of the studies. The versions of AHP (Analytic Hierarchy Process) are usually employed for subjectively weighting of the attributes. Another important finding is that no study applies for the option of objective weighting of the attributes. To show the applicability of objective weighting methods with CODAS, entropy-based weighting is proposed and used in the study. First, two entropy measures developed in Section 3 are respectively integrated into the modified CODAS version for PFS. Then, some entropy measures given in Section 2.1 and the knowledge measure given in Definition 7 are comparatively exploited for monitoring the differences or the similarities among them. Also, some MADM methods in the literature are used for comparison of the results to show the validity of the propositions.

3. Novel Entropy Measures for PFSs

In this section, we propose two new entropy measures for PFSs.

DEFINITION 9. Let $A, B \in PFS(X)$. A function $E : PFS(X) \to [0, 1]$ is an entropy on PFS, if E satisfies the following axioms:

1. $E(A) = 0$ if A is a crisp set,
2. $E(A) = 1$, if $\mu_A = \eta_A = \nu_A$,
3. $E(A) = E(A^c)$,
4. $E(A) \leqslant E(B)$, if A is less fuzzy then B, that is,

$$|\mu_A - \nu_A| + |\eta_A - \pi_A| \geqslant |\mu_B - \nu_B| + |\eta_B - \pi_B|.$$

For any $A \in PFS(X)$, we define a function $E_{SA_1} : PFS(X) \to [0, 1]$ by

$$E_{SA_1}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - |\mu_A(x_i) - \nu_A(x_i)| - |\eta_A(x_i) - \pi_A(x_i)|}{1 + |\mu_A(x_i) - \nu_A(x_i)| + |\eta_A(x_i) - \pi_A(x_i)|}, \quad \forall x_i \in X. \quad (53)$$

Theorem 1. *The function E_{SA_1} is an entropy measure for a PFS.*

Proof. To prove that Eq. (53) is an entropy measure for PFSs, we show that it satisfies the axioms shown in Definition 8.

1. Suppose that A is a crisp set. This refers either
   $\mu_A(x_i) = 1, \nu_A(x_i) = 0, \eta_A(x_i) = 0, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 1, \eta_A(x_i) = 0, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 0, \eta_A(x_i) = 1, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 0, \eta_A(x_i) = 0, \pi_A(x_i) = 1$.
   If $A = \{\langle x, 1, 0, 0 \rangle\}$, then

   $$E_{SA_1}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - |1 - 0| - |0 - 0|}{1 + |1 - 0| + |0 - 0|} = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - 1}{1 + 1} = 0.$$

   When we substitute other values in Eq. (53), we find that $E_{SA_1}(A) = 0$.

2. If $\mu_A(x_i) = \eta_A(x_i) = \nu_A(x_i)$, then

   $$\frac{1}{n} \sum_{i=1}^{n} \frac{1 - |\mu_A(x_i) - \nu_A(x_i)| - |\eta_A(x_i) - \pi_A(x_i)|}{1 + |\mu_A(x_i) - \nu_A(x_i)| + |\eta_A(x_i) - \pi_A(x_i)|} = \frac{1}{n} \sum_{i=1}^{n} 1 = 1.$$

3. Due to the definition of the complement of A, it is clear that $E_{SA_1}(A) = E_{SA_1}(A^c)$.
4. For any two PFSs A and B, if

   $$|\mu_A - \nu_A| + |\eta_A - \pi_A| \geqslant |\mu_B - \nu_B| + |\eta_B - \pi_B|,$$
   $$\text{then } 1 + |\mu_A - \nu_A| + |\eta_A - \nu_A| \geqslant 1 + |\mu_B - \nu_B| + |\eta_B - \nu_B|$$

   and

   $$1 - |\mu_A - \nu_A| - |\eta_A - \nu_A| \leqslant 1 - |\mu_B - \nu_B| - |\eta_B - \nu_B|.$$

   Thus, we get $E_{SA_1}(A) \leqslant E_{SA_1}(B)$. □

DEFINITION 10. For any $A \in PFS(X)$, we define a function $E_{SA_2} : PFS(X) \to [0, 1]$ by

$$
E_{SA_2}(A)
= 1 - 2\left\{\frac{1}{3n}\sum_{i=1}^{n}\left[\begin{array}{c}(\mu_A(x_i) - 0.25)^2 + (\eta_A(x_i) - 0.25)^2 \\ + (\nu_A(x_i) - 0.25)^2 + (\pi_A(x_i) - 0.25)^2\end{array}\right]\right\}^{\frac{1}{2}}, \quad \forall x_i \in X.
\tag{54}
$$

Lemma 1 (Joshi, 2020a). *If we have either* $\mu_C(x_i) \leqslant \mu_D(x_i)$, $\eta_C(x_i) \leqslant \eta_D(x_i)$, $\nu_C(x_i) \leqslant \nu_D(x_i)$ *if* $\max\{\mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\} \leqslant 0.25$ *or* $\mu_C(x_i) \geqslant \mu_D(x_i)$, $\eta_C(x_i) \geqslant \eta_D(x_i)$, $\nu_C(x_i) \geqslant \nu_D(x_i)$ *if* $\min\{\mu_C(x_i), \eta_C(x_i), \nu_C(x_i)\} \geqslant 0.25$ *for all* $x_i \in X$, *then we get*

$$
\begin{aligned}
&\left(\mu_A(x_i) - 0.25\right)^2 + \left(\eta_A(x_i) - 0.25\right)^2 + \left(\nu_A(x_i) - 0.25\right)^2 + \left(\pi_A(x_i) - 0.25\right)^2 \\
&\geqslant \left(\mu_B(x_i) - 0.25\right)^2 + \left(\eta_B(x_i) - 0.25\right)^2 + \left(\nu_B(x_i) - 0.25\right)^2 \\
&+ \left(\pi_B(x_i) - 0.25\right)^2.
\end{aligned}
$$

Theorem 2. *The function* $E_{SA_2}(A)$ *is an entropy measure for a PFS.*

Proof. To prove Eq. (54) is an entropy measure for PFSs, we show that it satisfies the axioms in Definition 6.

1. Suppose that A is a crisp set. This refers either
   $\mu_A(x_i) = 1, \nu_A(x_i) = 0, \eta_A(x_i) = 0, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 1, \eta_A(x_i) = 0, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 0, \eta_A(x_i) = 1, \pi_A(x_i) = 0$;
   or $\mu_A(x_i) = 0, \nu_A(x_i) = 0, \eta_A(x_i) = 0, \pi_A(x_i) = 1$.
   If $A = \{\langle x, 1, 0, 0\rangle\}$, then

$$
\begin{aligned}
E_{SA_2}
&= 1 - 2\left\{\frac{1}{3n}\sum_{i=1}^{n}\left[(1 - 0.25)^2 + (0 - 0.25)^2 + (0 - 0.25)^2 + (0 - 0.25)^2\right]\right\}^{\frac{1}{2}} \\
&= 1 - 2\left\{\frac{1}{3n}\sum_{i=1}^{n}\left[(0.75)^2 + (0.25)^2 + (0.25)^2 + (0.25)^2\right]\right\}^{\frac{1}{2}} \\
&= 1 - 2\left\{\frac{1}{3n}\sum_{i=1}^{n}[0, 75]\right\}^{\frac{1}{2}} = 0.
\end{aligned}
$$

Substituting other values in Eq. (54), we find that $E_{SA_2}(A) = 0$.

2. If $A = \{\langle x_i, \ 0.25, \ 0.25, \ 0.25 \rangle \mid x_i \in X\}$, then $\pi_A(x_i) = 0.25$. We get

$$E_{SA_2}(A) = 1 - 2 \left\{ \frac{1}{3n} \sum_{i=1}^{n} \left[\begin{array}{c} (\mu_A(x_i) - 0.25)^2 + (\eta_A(x_i) - 0.25)^2 \\ + (\nu_A(x_i) - 0.25)^2 + (\pi_A(x_i) - 0.25)^2 \end{array} \right] \right\}^{\frac{1}{2}}$$

$$= E_{SA_2}(A)$$

$$= 1 - 2 \left\{ \frac{1}{3n} \sum_{i=1}^{n} \left[\begin{array}{c} (0.25 - 0.25)^2 + (0.25 - 0.25)^2 \\ + (0.25 - 0.25)^2 + (0.25 - 0.25)^2 \end{array} \right] \right\}^{\frac{1}{2}}$$

$$= 1.$$

3. It is clear that $E_{SA_2}(A) = E_{SA_2}(A^c)$ for all $A \in PFS(X)$.
4. This property follows Lemma 1. $\qquad\square$

4. CODAS Extension Under Picture Fuzzy Environment (PF-CODAS)

The study has extended CODAS method into the picture fuzzy environment as a contribution to the literature. In this novel proposition, PFNs provide a better opportunity of independency to the decision-makers since it is allowed to express independent degrees for positive, negative, and hesitant preferences. The refusal degrees can also be calculated as the fourth element in PFS. Especially, the consideration of the refusal degree in decision analysis is scarce in the literature of MADM. Moreover, entropy-based objective weighting process is joint to the method in case that the initial problem definition does not have the attribute weights. The algorithm is given as follows:

Step 1. Decision-makers ($e = 1, \ldots, k$) express their judgments about alternatives' ($i = 1, \ldots, m$) performances with respect to attributes ($j = 1, \ldots, n$) via linguistic evaluations. The linguistic term set (Table 1) having PFN correspondences can be used for this purpose.

After gathering evaluations from decision-makers, there will be k decision matrices ($\tilde{X}^1, \tilde{X}^2, \ldots, \tilde{X}^k$). The judgments are combined through an aggregation operator defined in PFS. In this aggregation, we consider the decision-makers' weights representing their

Table 1
Linguistic terms for expert evaluations (Meksavang
et al., 2019).

Linguistic term	PFN correspondence
Very Poor (VP)	(0.10, 0.00, 0.85)
Poor (P)	(0.25, 0.05, 0.60)
Moderately Poor (MP)	(0.30, 0.00, 0.60)
Fair (F)	(0.50, 0.10, 0.40)
Moderately Good (MG)	(0.60, 0.00, 0.30)
Good (G)	(0.75, 0.05, 0.10)
Very Good (VG)	(0.90, 0.00, 0.05)

expertise (ω_e). $\tilde{X}^e = [\tilde{x}_{ij}^e]$ is the eth decision-maker's evaluation matrix where $\tilde{x}_{ij}^e = \langle \mu_{ij}^e, \eta_{ij}^e, v_{ij}^e \rangle$ depicts the PFN correspondence of the linguistic evaluation and $\tilde{X} = [\tilde{x}_{ij}]$ is the aggregated decision matrix where $\tilde{x}_{ij} = \langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle$. For obtaining $\tilde{X}^e$ given in Eq. (55), the picture fuzzy weighted averaging (PFWA) operator (Eq. (56)) is performed (Wei, 2017a).

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} = \langle \mu_{11}, \eta_{11}, v_{11} \rangle & \cdots & \tilde{x}_{1n} = \langle \mu_{1n}, \eta_{1n}, v_{1n} \rangle \\ \vdots & \ddots & \vdots \\ \tilde{x}_{m1} = \langle \mu_{m1}, \eta_{m1}, v_{m1} \rangle & \cdots & \tilde{x}_{mn} = \langle \mu_{mn}, \eta_{mn}, v_{mn} \rangle \end{bmatrix}, \quad (55)$$

$$PFWA_\omega\left(\tilde{X}^1, \tilde{X}^2, \ldots, \tilde{X}^k\right) = \bigoplus_{e=1}^{k} \omega_e \tilde{X}^e = \langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle$$

$$= \left\{ 1 - \prod_{e=1}^{k} \left(1 - \mu_{ij}^e\right)^{\omega_e}, \prod_{e=1}^{k} \left(\eta_{ij}^e\right)^{\omega_e}, \prod_{e=1}^{k} \left(v_{ij}^e\right)^{\omega_e} \right\}. \quad (56)$$

Step 2. The attribute considered in any decision problem can be a cost or a benefit type attribute. To convert cost attributes to benefit ones, the positive (μ_{ij}) and negative (v_{ij}) membership degrees should be replaced while the neutral membership degrees (η_{ij}) keep their values. This is called normalization.

Step 3. After normalization, the weights of attributes representing their importance and significance should be considered. 4 possibilities might be thought of in weighting:

  (I) If the weights are already known as prior information, we can directly use them;
 (II) If the decision-makers' preferences are important for the analyst, their expertise can be gathered, and the subjective weights are computed via various MADM tools such as AHP, Analytic Network Process (ANP), SWARA, Simos' procedure, etc.;
(III) When the subjectivity is not desired with the purpose of eliminating manipulation risk or when there is not enough time for data collection or when the analyst does not have the weights of any kind, the weights can be objectively calculated from the current data by referring to the methods such as entropy-based approaches or maximizing standard deviation method;
(IV) When required, a mixture of objective and subjective methods can be exploited (Kabak and Ruan, 2011; Çalışkan *et al.*, 2013; Li *et al.*, 2014; Freeman and Chen, 2015).

This proposition aims at showing the applicability of entropy-based objective weighting with the integration of CODAS under PF environment. Therefore, the method requires first the calculation of the entropies of each attribute via Eq. (57) or Eq. (58). Then, the

weights are obtained as formulated in Eq. (59) (Aydoğdu and Gül, 2020)

$$En_j = \frac{1}{m} \sum_{i=1}^{m} \frac{1 - |\mu_{ij} - v_{ij}| - |\eta_{ij} - \pi_{ij}|}{1 + |\mu_{ij} - v_{ij}| + |\eta_{ij} - \pi_{ij}|}, \tag{57}$$

$$En_j = 1 - 2 \left\{ \frac{1}{3m} \sum_{i=1}^{m} \left[\begin{array}{c} (\mu_{ij} - 0.25)^2 + (\eta_{ij} - 0.25)^2 \\ + (v_{ij} - 0.25)^2 + (\pi_{ij} - 0.25)^2 \end{array} \right] \right\}^{\frac{1}{2}}, \tag{58}$$

$$w_j = \frac{1 - En_j}{n - \sum_{j=1}^{n} En_j}. \tag{59}$$

The attribute weights obtained in Eq. (59) are used in constructing the weighted normalized decision matrix $\tilde{R} = [\tilde{r}_{ij}]$ via Eq. (60). This equation is a reorganization of the multiplication operation defined in Eq. (5)

$$\tilde{r}_{ij} = \langle \mu_{ij}^w, \eta_{ij}^w, v_{ij}^w \rangle = w_j * \langle \mu_{ij}, \eta_{ij}, v_{ij} \rangle$$
$$= \langle 1 - (1 - \mu_{ij})^{w_j}, \eta_{ij}^{w_j}, (\eta_{ij} + v_{ij})^{w_j} - (\eta_{ij})^{w_j} \rangle. \tag{60}$$

Step 4. The distinctive feature of CODAS is the consideration of the distance of each alternative from the negative-ideal solution which can be obtained via Eq. (60)

$$\tilde{ns}_j = \langle \min(\mu_{ij}^w), \min(\eta_{ij}^w), \max(v_{ij}^w) \rangle. \tag{61}$$

Step 5. Euclidean and Hamming distances of each alternative i to $\tilde{ns} = [\tilde{ns}_j]$ are computed

$$E_i = \left[\frac{1}{n} \sum_{i=1}^{n} ((\mu_{ij}^w - \min(\mu_{ij}^w))^2 + (\eta_{ij}^w - \min(\eta_{ij}^w))^2 + (v_{ij}^w - \max(v_{ij}^w))^2) \right]^{\frac{1}{2}}, \tag{62}$$

$$T_i = \frac{1}{n} \sum_{i=1}^{n} (|\mu_{ij}^w - \min(\mu_{ij}^w)| + |\eta_{ij}^w - \min(\eta_{ij}^w)| + |v_{ij}^w - \max(v_{ij}^w)|). \tag{63}$$

The distance values are crisp numbers now. So, the remaining three steps are the same as the original CODAS method.

Step 6. Identical to Step 6 of CODAS given in Section 2.2.

Step 7. Identical to Step 7 of CODAS given in Section 2.2.

Step 8. Identical to Step 8 of CODAS given in Section 2.2.

5. An Application of Green Supplier Selection in the Beef Industry

In the study, we have developed a novel PFS version of CODAS with the integration of entropy-based objective attribute weighting. We have also tried to keep the computations

picture fuzzy until the very end of the method. The proposed PF-CODAS is here applied in a supplier selection problem for the beef industry previously defined and analysed by Meksavang *et al.* (2019).

Büyüközkan and Çifçi (2012) stated that even if material, funds, and information flows establish a supply chain system, due to governmental rules and growing consciousness in the society about keeping the environment safe, organizations must be more sensitive to environmental issues, particularly if they want to keep their existence in global markets. Supplier selection issue has gained greater attraction today because organizations focus on improving their core competence and they need to outsource less profitable activities to supply chain partners for this reason (Govindan *et al.*, 2015). In this selection process, environmental issues have been emphasized from a perspective of green supply chain management. In the literature, green supplier selection problem is very fruitful. Govindan *et al.* (2015) presented a very extensive literature review on MADM applications on green supplier selection problem. Liou *et al.* (2021) integrated support vector machines, fuzzy best-worst method, and fuzzy TOPSIS method and presented the model's applicability in a real case of a Taiwanese electronics company. Wei *et al.* (2021) developed a probabilistic uncertain linguistic version of CODAS and applied it to a green supplier selection problem. Kumar and Barman (2021) applied and compared the results of fuzzy VIKOR and fuzzy TOPSIS methods in the green supplier selection issue of India's small-scale iron and steel industry. Çalık (2021) proposed a Pythagorean fuzzy extension of AHP and TOPSIS integration for green supply chain management in the Industry 4.0 era and made a trial for agricultural tool manufacturers in Turkey.

After a brief explanation about MADM in green supply chain management field, we can return to the application of the proposed method in a case that was previously defined by Meksavang *et al.* (2019). The authors stated that carbon footprint reduction received great attention throughout the world, and the agriculture sector is one of the main contributors to global carbon emission. They also mentioned the increasing pressure on the beef industry from the government and clients to reduce carbon emissions in its supply chain. For mitigating carbon emission, the proposed entropy-based PF-CODAS approach is here applied to the selection of a supplier for a beef abattoir company.

Step 1. Ten potential beef farmers are considered as supplier alternatives and denoted as A_i $(i = 1, \ldots, 10)$. Their green supply performance is assessed following seven attributes $(j = 1, \ldots, 7)$: quality of meat (C_1), age of cattle (C_2), diet fed to cattle (C_3), average weight (C_4), traceability (C_5), carbon footprint (C_6), and price (C_7). Three decision-makers (DM_1, DM_2, DM_3) make the evaluations of the performance ratings of suppliers. The weight set for the decision-makers' is assumed as $(0.3, 0.4, 0.3)$ due to the differences in their technical knowledge and expertise levels. The linguistic evaluations given in Table 1 are used for this purpose and the corresponding PFNs are found in the same table. Table A2 and Table A3 in Appendix A show the linguistic evaluations of decision-makers and the corresponding PFNs ($\tilde{X}^e = [\tilde{x}_{ij}^e]$), respectively. In order to build the aggregated picture fuzzy decision matrix, Eq. (56) is performed.

Table 2 shows the aggregated decision matrix (Eq. (55)). As an illustration, the aggregated performance value ($\tilde{x}_{31}$) of A_3 with respect to C_1 is computed as follows where the

Table 2
The aggregated picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.900 0.000 0.050	0.300 0.000 0.600	0.900 0.000 0.050	0.750 0.050 0.100	0.868 0.000 0.062	0.483 0.000 0.414	0.500 0.100 0.400
A_2	0.543 0.000 0.357	0.750 0.050 0.100	0.500 0.100 0.400	0.781 0.000 0.113	0.900 0.000 0.050	0.629 0.000 0.235	0.447 0.000 0.452
A_3	0.581 0.000 0.266	0.354 0.000 0.531	0.781 0.000 0.113	0.600 0.000 0.300	0.629 0.000 0.235	0.600 0.000 0.300	0.354 0.000 0.531
A_4	0.600 0.000 0.300	0.500 0.100 0.400	0.781 0.000 0.113	0.250 0.050 0.600	0.856 0.000 0.066	0.856 0.000 0.066	0.388 0.000 0.510
A_5	0.718 0.000 0.191	0.900 0.000 0.050	0.810 0.000 0.081	0.265 0.000 0.600	0.224 0.000 0.666	0.750 0.050 0.100	0.435 0.081 0.452
A_6	0.750 0.050 0.100	0.354 0.000 0.531	0.750 0.050 0.100	0.750 0.050 0.100	0.900 0.000 0.050	0.629 0.000 0.235	0.428 0.000 0.470
A_7	0.551 0.000 0.298	0.300 0.000 0.600	0.483 0.000 0.414	0.354 0.000 0.531	0.265 0.000 0.600	0.483 0.000 0.414	0.428 0.000 0.470
A_8	0.817 0.000 0.105	0.629 0.000 0.235	0.629 0.000 0.235	0.354 0.000 0.531	0.250 0.050 0.600	0.781 0.000 0.113	0.428 0.000 0.470
A_9	0.484 0.000 0.403	0.354 0.000 0.531	0.483 0.000 0.414	0.781 0.000 0.113	0.600 0.000 0.300	0.500 0.100 0.400	0.500 0.100 0.400
A_{10}	0.653 0.000 0.216	0.483 0.000 0.414	0.600 0.000 0.300	0.900 0.000 0.050	0.900 0.000 0.050	0.900 0.000 0.050	0.435 0.081 0.452

Table 3
The normalized picture fuzzy decision matrix.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
A_1	0.900 0.000 0.050	0.300 0.000 0.600	0.900 0.000 0.050	0.750 0.050 0.100	0.868 0.000 0.062	0.483 0.000 0.414	0.400 0.100 0.500
A_2	0.543 0.000 0.357	0.750 0.050 0.100	0.500 0.100 0.400	0.781 0.000 0.113	0.900 0.000 0.050	0.629 0.000 0.235	0.452 0.000 0.447
A_3	0.581 0.000 0.266	0.354 0.000 0.531	0.781 0.000 0.113	0.600 0.000 0.300	0.629 0.000 0.235	0.600 0.000 0.300	0.531 0.000 0.354
A_4	0.600 0.000 0.300	0.500 0.100 0.400	0.781 0.000 0.113	0.250 0.050 0.600	0.856 0.000 0.066	0.856 0.000 0.066	0.510 0.000 0.388
A_5	0.718 0.000 0.191	0.900 0.000 0.050	0.810 0.000 0.081	0.265 0.000 0.600	0.224 0.000 0.666	0.750 0.050 0.100	0.452 0.081 0.435
A_6	0.750 0.050 0.100	0.354 0.000 0.531	0.750 0.050 0.100	0.750 0.050 0.100	0.900 0.000 0.050	0.629 0.000 0.235	0.470 0.000 0.428
A_7	0.551 0.000 0.298	0.300 0.000 0.600	0.483 0.000 0.414	0.354 0.000 0.531	0.265 0.000 0.600	0.483 0.000 0.414	0.470 0.000 0.428
A_8	0.817 0.000 0.105	0.629 0.000 0.235	0.629 0.000 0.235	0.354 0.000 0.531	0.250 0.050 0.600	0.781 0.000 0.113	0.470 0.000 0.428
A_9	0.484 0.000 0.403	0.354 0.000 0.531	0.483 0.000 0.414	0.781 0.000 0.113	0.600 0.000 0.300	0.500 0.100 0.400	0.400 0.100 0.500
A_{10}	0.653 0.000 0.216	0.483 0.000 0.414	0.600 0.000 0.300	0.900 0.000 0.050	0.900 0.000 0.050	0.900 0.000 0.050	0.452 0.081 0.435

evaluations are P (0.25, 0.05, 0.60), MG (0.60, 0.00, 0.30), G (0.75, 0.05, 0.10):

$$\tilde{x}_{31} = 0.3 * (0.25, 0.05, 0.60) + 0.4 * (0.60, 0.00, 0.30) + 0.3 * (0.75, 0.05, 0.10)$$
$$= \left(1 - (1 - 0.25)^{0.3} * (1 - 0.60)^{0.4} * (1 - 0.75)^{0.3}, 0.05^{0.3} * 0.00^{0.4} * 0.05^{0.3},\right.$$
$$\left. 0.60^{0.3} * 0.30^{0.4} * 0.10^{0.3}\right) = (0.581, 0.000, 0.266).$$

Step 2. In normalization step, the normalized decision matrix is constructed as given in Table 3. There is only one cost attribute in this problem: price (C_7). Hence, this last attribute's positive (μ_{i7}) and negative membership (v_{i7}) degrees are replaced.

Step 3. Entropy-based objective weights of the attributes are calculated in this step. For each attribute j, Eq. (57) or Eq. (58) is performed for computing entropies and the weights are gathered after making a normalization which is formulated in Eq. (59). The related entropies and weights of the attributes are given in Table 4. Generating from the definitions of the entropy measures, there are few differences between the importance rankings of the attributes as well as the weights. Therefore, in further steps, we will see the impact of this difference on the solution of the problem.

By considering the weights of the attributes, the normalized matrix is weighted by using the formula in Eq. (60). For simplicity, the application of the further steps will be explained for the weight set which is found by Eq. (57): [0.156, 0.130, 0.154, 0.164, 0.184, 0.150, 0.062]. Table 5 shows the weighted normalized picture fuzzy decision matrix.

Table 4
The weights of the attributes which are based on two novel entropy measures.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
En_j (Eq. 51)	0.330	0.441	0.339	0.297	0.211	0.355	0.732
$1- En_j$	0.670	0.559	0.661	0.703	0.789	0.645	0.268
w_j	0.156	0.130	0.154	0.164	0.184	0.150	0.062
Importance ranking	3	6	4	2	1	5	7
En_j (Eq. 52)	0.391	0.436	0.365	0.374	0.291	0.375	0.532
$1- En_j$	0.609	0.564	0.635	0.626	0.709	0.625	0.468
w_j	0.144	0.133	0.150	0.148	0.167	0.147	0.111
Importance ranking	5	6	2	3	1	4	7
Difference between weights	0.012	−0.003	0.004	0.016	0.016	0.003	−0.048

Table 5
The weighted normalized picture fuzzy decision matrix.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
A_1	0.302 0.000 0.627	0.045 0.000 0.936	0.298 0.000 0.631	0.203 0.613 0.121	0.311 0.000 0.599	0.094 0.000 0.876	0.031 0.866 0.102
A_2	0.115 0.000 0.851	0.165 0.677 0.104	0.101 0.702 0.197	0.220 0.000 0.700	0.345 0.000 0.577	0.138 0.000 0.805	0.037 0.000 0.951
A_3	0.127 0.000 0.813	0.055 0.000 0.921	0.209 0.000 0.715	0.139 0.000 0.821	0.166 0.000 0.766	0.129 0.000 0.835	0.046 0.000 0.937
A_4	0.133 0.000 0.829	0.086 0.741 0.173	0.209 0.000 0.715	0.046 0.613 0.319	0.299 0.000 0.607	0.252 0.000 0.665	0.044 0.000 0.943
A_5	0.179 0.000 0.772	0.259 0.000 0.677	0.226 0.000 0.679	0.049 0.000 0.920	0.046 0.000 0.928	0.188 0.638 0.114	0.037 0.855 0.105
A_6	0.194 0.627 0.117	0.055 0.000 0.921	0.192 0.631 0.116	0.203 0.613 0.121	0.345 0.000 0.577	0.138 0.000 0.805	0.039 0.000 0.948
A_7	0.117 0.000 0.828	0.045 0.000 0.936	0.096 0.000 0.873	0.069 0.000 0.902	0.055 0.000 0.910	0.094 0.000 0.876	0.039 0.000 0.948
A_8	0.233 0.000 0.704	0.121 0.000 0.828	0.141 0.000 0.800	0.069 0.000 0.902	0.051 0.577 0.347	0.204 0.000 0.721	0.039 0.000 0.948
A_9	0.098 0.000 0.868	0.055 0.000 0.921	0.096 0.000 0.873	0.220 0.000 0.700	0.155 0.000 0.802	0.099 0.708 0.193	0.031 0.866 0.102
A_{10}	0.152 0.000 0.787	0.082 0.000 0.892	0.132 0.000 0.831	0.314 0.000 0.613	0.345 0.000 0.577	0.292 0.000 0.638	0.037 0.855 0.105
$\tilde{n}s$	0.098 0.000 0.868	0.045 0.000 0.936	0.096 0.000 0.873	0.046 0.000 0.920	0.046 0.000 0.928	0.094 0.000 0.876	0.031 0.000 0.951

Step 4. In CODAS, the origin point for the comparison of alternatives is the negative-ideal solution. By performing Eq. (60), $\tilde{n}s = [\tilde{n}s_j]$ is found as a row vector and shown in the last row of Table 5.

Step 5. The ranking of the alternatives is based on the distances between the alternatives and the negative-ideal solution. Euclidean and Hamming distances are found via Eq. (62) and Eq. (63), respectively, and shown in Table 6. These distances are now crisp numbers.

Step 6. The pairwise comparison matrix including the combinative distances is built by performing the procedure explained by the algorithm in Section 2.2. Table 7 gives the comparison matrix. The required parameter of Θ is set to 0.05.

Step 7. The assessment score generated by summing the row values of the comparison matrix is computed for each alternative. These H_i assessment values are represented in Table 7.

Step 8. The highest separation measure refers to the best alternative. So, the alternatives are ranked in descending order of H_i values as given in the last column of Table 7. Hence, the best alternative is the 6th supplier while the worst one is the 7th supplier of beef in terms of green supply performance. The full rank is $A_6 \succ A_1 \succ A_5 \succ A_2 \succ A_9 \succ A_4 \succ A_{10} \succ A_8 \succ A_3 \succ A_7$.

Table 6
Euclidean and Hamming distances to the
negative-ideal solution.

	E_i	T_i
A_1	0.642	0.681
A_2	0.587	0.601
A_3	0.122	0.137
A_4	0.554	0.581
A_5	0.612	0.596
A_6	0.677	0.761
A_7	0.022	0.020
A_8	0.333	0.297
A_9	0.603	0.538
A_{10}	0.526	0.523

Table 7
Comparison of distances and the ranking of alternatives.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	H_i	Rank
A_1	0.000	0.135	1.064	0.188	0.030	−0.034	1.282	0.693	0.039	0.274	3.671	2
A_2	−0.055	0.000	0.930	0.033	−0.025	−0.089	1.147	0.558	−0.016	0.139	2.622	4
A_3	−0.520	−0.465	0.000	−0.432	−0.490	−0.555	0.217	−0.211	−0.481	−0.404	−3.341	9
A_4	−0.088	−0.033	0.876	0.000	−0.059	−0.123	1.094	0.505	−0.049	0.028	2.150	6
A_5	−0.030	0.025	0.950	0.074	0.000	−0.064	1.167	0.579	0.010	0.159	2.871	3
A_6	0.034	0.249	1.179	0.303	0.229	0.000	1.396	0.807	0.297	0.388	4.882	1
A_7	−0.621	−0.565	−0.100	−0.532	−0.591	−0.655	0.000	−0.311	−0.581	−0.505	−4.461	10
A_8	−0.309	−0.254	0.371	−0.221	−0.279	−0.344	0.589	0.000	−0.270	−0.193	−0.910	8
A_9	−0.039	0.016	0.882	0.049	−0.010	−0.074	1.099	0.511	0.000	0.091	2.525	5
A_{10}	−0.116	−0.061	0.791	−0.028	−0.086	−0.150	1.008	0.419	−0.077	0.000	1.701	7

In Step 3, there are two sets of attribute weights and we have shown the calculations for the first weight set until now for simplicity. The application of PF-CODAS with the second weight set determined via Eq. (58) is not shown in detail but the tables constituted are given in Appendix A. While Table A4 shows the weighted normalized picture fuzzy decision matrix, Table A5 depicts the distances obtained. Table A6 summarizes the comparison results, assessment scores, and the rankings of the alternatives which is $A_6 \succ A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_9 \succ A_{10} \succ A_8 \succ A_3 \succ A_7$. A comparison of the rankings will be made in Section 6.

6. Comparison of Results

In order to check the validity of the proposed entropy-based PF-CODAS method, we have analysed two cases. In the first analysis, the rankings of the alternatives which are found by different entropy measures are compared. In the second one, a similar comparison is done for the applications with various versions of CODAS and TOPSIS methods.

In terms of entropy-based comparison, the weight sets are determined. Table 8 gives the results of the comparison of weights and Table 9 shows the rankings of the alterna-

Table 8
The weights of the attributes which are based on several measures.

	En_j with Eq. (44)	w_j with Eq. (44)	Rank	En_j with Eq. (48)	w_j with Eq. (48)	Rank	w_j with Eq. (66)	Rank	En_j with Eq. (57)	w_j with Eq. (57)	Rank	En_j with Eq. (58)	w_j with Eq. (58)	Rank
C_1	0.346	0.143	5	0.472	0.149	4	0.147	4	0.330	0.156	3	0.391	0.144	5
C_2	0.409	0.132	6	0.511	0.134	6	0.136	6	0.441	0.130	6	0.436	0.133	6
C_3	0.343	0.149	2	0.448	0.149	3	0.148	3	0.339	0.154	4	0.365	0.150	2
C_4	0.332	0.147	3	0.456	0.151	2	0.152	2	0.297	0.164	2	0.374	0.148	3
C_5	0.253	0.170	1	0.373	0.169	1	0.165	1	0.211	0.184	1	0.291	0.167	1
C_6	0.354	0.147	4	0.457	0.147	5	0.146	5	0.355	0.150	5	0.375	0.147	4
C_7	0.555	0.112	7	0.586	0.101	7	0.106	7	0.732	0.062	7	0.532	0.111	7

Table 9
The comparison of assessment scores and alternative rankings.

	H_i	Ranks with Eq. (44)	H_i	Ranks with Eq. (48)	H_i	Ranks with Eq. (66)	H_i	Ranks with Eq. (57)	H_i	Ranks with Eq. (58)
A_1	3.327	2	3.203	2	3.274	2	3.671	2	3.271	2
A_2	2.712	3	2.751	3	2.713	3	2.622	4	2.731	3
A_3	−3.279	9	−3.255	9	−3.273	9	−3.341	9	−3.260	9
A_4	2.377	5	2.469	5	2.360	5	2.150	6	2.447	5
A_5	2.627	4	2.547	4	2.606	4	2.871	3	2.558	4
A_6	5.305	1	5.436	1	5.330	1	4.882	1	5.406	1
A_7	−4.349	10	−4.324	10	−4.331	10	−4.461	10	−4.326	10
A_8	−0.658	8	−0.639	8	−0.591	8	−0.910	8	−0.615	8
A_9	2.266	6	2.186	6	2.240	6	2.525	5	2.192	6
A_{10}	1.277	7	1.170	7	1.224	7	1.701	7	1.180	7

tives. The entropy measures used are represented on the columns. The last two are the novel entropy measures proposed in the study and the first three are the existing ones in the literature: Eq. (44) defined by Wang *et al.* (2018), Eq. (48) defined by Joshi (2020a), and the knowledge measure in Eq. (52) defined by Lin *et al.* (2020). The third one provides a knowledge measure which is based on entropy measure and hesitancy degrees. The stepwise methodology is summarized as follows:

(a) The entropy matrix showing the entropies for each pair of alternative and attribute is derived as follows:

$$\left[En(\tilde{x}_{ij})\right] = \begin{bmatrix} En(\tilde{x}_{11}) & \cdots & En(\tilde{x}_{1n}) \\ \vdots & \ddots & \vdots \\ En(\tilde{x}_{m1}) & \cdots & En(\tilde{x}_{mn}) \end{bmatrix}, \tag{64}$$

where $En(\tilde{x}_{ij}) = \frac{\min(d_H(\tilde{x}_{ij},C_{\min}),d_H(\tilde{x}_{ij},C_{\max}))}{\max(d_H(\tilde{x}_{ij},C_{\min}),d_H(\tilde{x}_{ij},C_{\max}))}$ and $C_{\min} = \langle 0, 0, 1 \rangle$ and $C_{\max} = \langle 1, 0, 0 \rangle$.
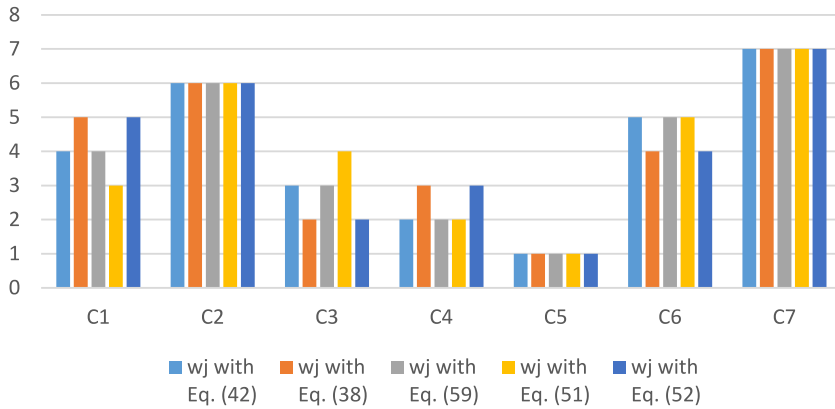
Comparison of attributes

Fig. 1. Comparison of the ranks of the attributes.

(b) The knowledge matrix is derived as follows:

$$[K(\tilde{x}_{ij})] = \begin{bmatrix} K(\tilde{x}_{11}) & \cdots & K(\tilde{x}_{1n}) \\ \vdots & \ddots & \vdots \\ K(\tilde{x}_{m1}) & \cdots & K(\tilde{x}_{mn}) \end{bmatrix}, \tag{65}$$

where $K(\tilde{x}_{ij}) = 1 - 0.5(En(\tilde{x}_{ij}) + \pi_{ij})$ and $\pi_{ij} = 1 - \mu_{ij} - \eta_{ij} - \nu_{ij}$.

(c) If the knowledge measure of a criterion is larger across the alternative, it means that the value of this criterion has a smaller variation. Hence, this one shows a greater impact on the overall ratings of the alternatives. From this understanding, Eq. (66) is used for attribute weighting:

$$w_j = \frac{\sum_{i=1}^{m} K(\tilde{x}_{ij})}{\sum_{j=1}^{n} \sum_{i=1}^{m} K(\tilde{x}_{ij})}. \tag{66}$$

The importance ranking of the attributes is presented in Fig. 1. It is seen that there are few differences among the weight sets. Throughout the five sets, while C_5, C_2, and C_7 keep their ranks, the most changing attributes are C_1 and C_3. Table 9 and Fig. 2 show the comparisons of the alternative rankings gathered by applying the entropy-based weightings determined in Table 8. As seen in Fig. 2, the rankings are slightly different for the alternative pairs (A_2, A_5) and (A_4, A_9), while the other six alternatives keep their rankings in all 5 applications. Another important finding is that the three applications which are based on the existing entropy measures in the literature give the same ranking as the proposed second entropy (Eq. (58)). The slightly different rankings are generated by the application based on the first novel entropy proposition given in Eq. (57). In conclusion, it is seen that although there are differences in entropy measures and attribute weights associated, the rankings of the alternatives are stable throughout the process.
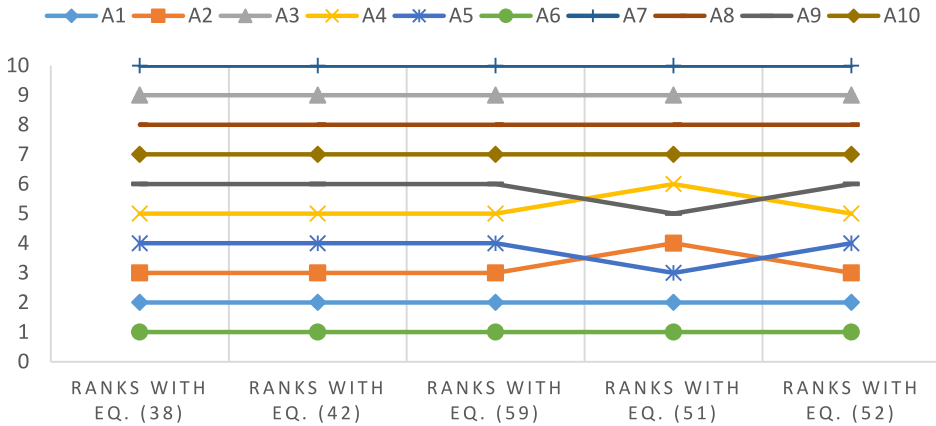
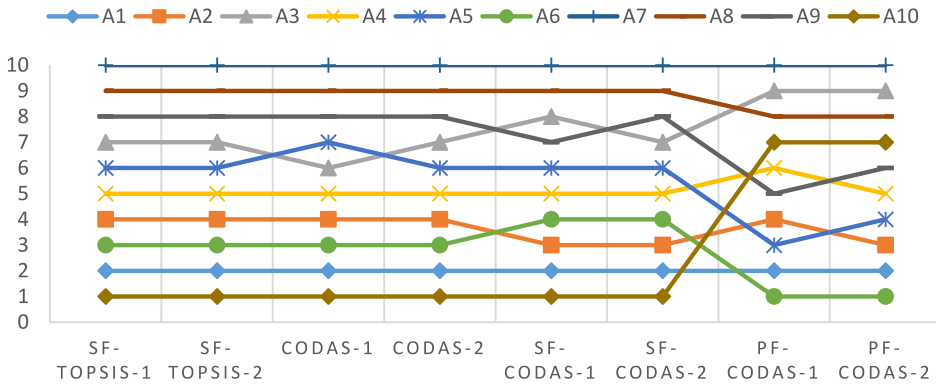Fig. 2. Comparison of the ranks of alternatives for different weight sets.

Fig. 3. Comparison of the ranks of alternatives for CODAS and TOPSIS versions.

The second analysis aims to compare the rankings of the alternatives of various approaches. Traditional CODAS developed by Keshavraz Ghorabaee *et al.* (2016), spherical fuzzy version of CODAS developed by Kutlu Gündoğdu and Kahraman (2019b), and spherical fuzzy extension of TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) developed by Kutlu Gündoğdu (2020) are utilized for this purpose. Each method is performed by considering two entropy measure propositions of the study, separately. To keep the flow of the study, the details of the methods are not given. The interested readers can look at the referred studies.

Fig. 3 shows the results of the comparisons. SF-TOPSIS, CODAS, and SF-CODAS give slightly different results while the proposed PF-CODAS generates different rankings of the alternatives. In terms of A_2 which is the best alternative obtained by PF-CODAS, CODAS and SF-TOPSIS rank it as the 3rd best while SF-CODAS sees it as the 4th best. A_{10} is found as the best alternative by the mentioned methods but PF-CODAS ranks it as the 7th best. A_1 and A_7 keep their 2nd and 10th positions, respectively, but it is obvious
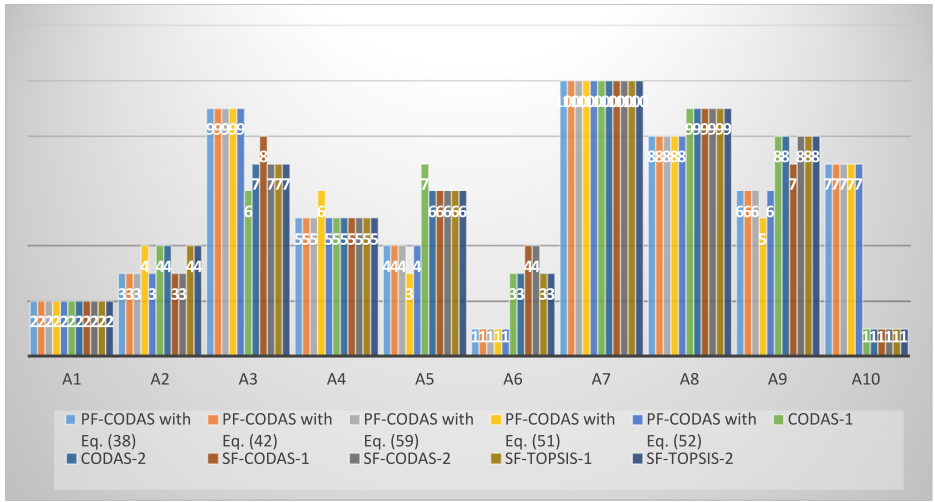
Fig. 4. A comprehensive comparison.

that there are significant differences for the other 8 alternatives. For a more comprehensive view, Fig. 2 and Fig. 3 are aggregated and the result is depicted in Fig. 4.

As Fig. 4 indicates, the most rank fluctuating alternatives are A_{10} and A_6 while the rankings of the other alternatives stay in an interval. For example, A_8 is ranked as either 8th or 9th best while the ranking interval of A_3 is [6–9]. The reason for these significantly different findings can be that the PFSs allow the decision analysts to take the refusal degree into account while aiding a MADM process. Spherical fuzzy versions of CODAS and TOPSIS are selected for this comparison due to their hesitancy consideration power but they do not cope with the refusal degree of the decision-makers. In short, it seems that the fourth element considered by PF-CODAS may generate different rankings from the existing CODAS and TOPSIS versions.

For a deeper understanding of the differences generated by the proposed entropy-based SF-CODAS, Spearman's rank correlation coefficients are also computed. Kahraman *et al.* (2009) proposed the usage of this statistical tool for revealing the differences between the rankings of various methods under group environment. Eq. (67) shows Spearman's rank correlation coefficient. A larger coefficient indicates a larger level of consensus among the results of the compared approaches.

$$\rho = 1 - \frac{6 \sum_j d^2}{m(m^2 - 1)}, \tag{67}$$

where d is the difference between the rank of the alternatives determined by each pair of approaches applied.

Table 10 shows the sum of the squared differences among the ranking results of CODAS and TOPSIS versions and PF-CODAS method while Table 11 shows the correlation coefficients of ρ. Except for 2 comparisons [CODAS-1 & PF-CODAS with Eq. (57)

Table 10
Sum of the squared differences between methods.

	PF-CODAS with Eq. (44)	PF-CODAS with Eq. (48)	PF-CODAS with Eq. (66)	PF-CODAS with Eq. (57)	PF-CODAS with Eq. (58)
CODAS-1	64	64	64	76	64
CODAS-2	54	54	54	64	54
SF-CODAS-1	52	52	52	62	52
SF-CODAS-2	58	58	58	70	58
SF-TOPSIS-1	54	54	54	64	54
SF-TOPSIS-2	54	54	54	64	54

Table 11
Spearman rank correlation coefficients between methods.

	PF-CODAS with Eq. (44)	PF-CODAS with Eq. (48)	PF-CODAS with Eq. (66)	PF-CODAS with Eq. (57)	PF-CODAS with Eq. (58)
CODAS-1	0.612	0.612	0.612	0.539	0.612
CODAS-2	0.673	0.673	0.673	0.612	0.673
SF-CODAS-1	0.685	0.685	0.685	0.624	0.685
SF-CODAS-2	0.648	0.648	0.648	0.576	0.648
SF-TOPSIS-1	0.673	0.673	0.673	0.612	0.673
SF-TOPSIS-2	0.673	0.673	0.673	0.612	0.673

and SF-CODAS-2 & PF-CODAS with Eq. (57)], ρ coefficients are higher than 0.60. For this case, it is concluded that there are medium to high level positive correlations among the rankings of the methods used.

7. Concluding Remarks

PFS has been recently accepted by the MADM domain as one of the useful fuzzy environments because of its extensive representation power of the preferences and opinions of decision-makers. PFS is defined by four elements, namely positive, negative, neutral, and refusal membership degrees and the first three elements can be independently assignable. The only rule that must be satisfied is that the sum of these four elements should be equal to 1.

Entropy is a very important information measure of fuzzy sets such as distance, inclusion, or similarity. In the literature, there are few entropy measures developed for PFSs. Entropy measures are exploited for determining the objective weights of attributes or the importance of decision-makers. These objective weights are found beneficial in case the subjective evaluation of weights is not desired or needed. The contributions of the study may be listed as follows:

- Two novel entropy measures for PFSs are developed and their proofs are given.
- CODAS, which is based on two different distance measures from the negative-ideal solution such as Euclidean and Hamming distance, is extended into PFS for the first time in the literature. Although spherical fuzzy and neutrosophic versions of CODAS

can handle the hesitancy degree of the decision-makers, none of the current versions of CODAS are capable of handling their refusal degrees. The most powerful aspect of the novel extension is the simultaneous consideration of both the hesitancy and refusal degrees of the decision-makers.

- To validate the novel PF-CODAS, a real green supplier selection application for the beef industry is conducted. The rankings are compared with different applications' rankings such as SF-TOPSIS, CODAS, and SF-CODAS. It is found that the proposed method generates different rankings of the alternatives due to the consideration of refusal degree.
- To understand the meaning of the differences in alternative rankings better, Spearman's rank correlation coefficient is used, and it is seen that there are medium to high correlations among the alternative ranking results of the methods compared in the study. In future applications, this situation should be investigated deeply.

In the proposed method, there are some limitations that should be handled. To cope with the disadvantageous parts of the study, the possible improvements are listed as follows:

- Rather than enforcing the decision-makers to use a fixed and not-flexible linguistic term set that has PFN correspondences, a future study may work on allowing the decision-makers to directly allocate positive, neutral, and negative membership degrees so that the data collection process becomes more realistic.
- Further studies should investigate the reason behind the finding of generating different rankings of alternatives by PF-based MADM methods. In order to make more comprehensive comparisons, novel fuzzy set definitions such as Fermatean fuzzy sets (Senapati and Yager, 2020), diophantine fuzzy sets (Riaz and Hashmi, 2019), and bipolar soft sets (Mahmood, 2020) may be utilized.
- The entropy-based attribute weighting technique has some drawbacks. In some cases requiring expert opinions about the importance of attributes, subjective and objective methods can be incorporated. In this manner, the subjectivity can be kept in control while respecting the expertise of the decision-makers.
- In the literature, a few studies (Wang, 2009; Han and Xiao, 2009) criticized the entropy definitions from a probability perspective and claimed that entropy measure is not enough to measure information in a data set. To deal with these sorts of problems, future works can focus on studying newer objective attribute weighting methods, such as MEREC, SECA, CRITIC, maximizing standard deviation, etc. under picture fuzzy environment.

A. Appendix

Table A1
The literature on fuzzy extensions of CODAS and their applications.

Paper	CODAS Version	Hybrid Method(s)	Application
Keshavarz-Ghorabaee et al. (2017)	Fuzzy sets	F-EDAS and F-TOPSIS for comparison	Numerical example on market segment evaluation
Panchal et al. (2017)	Fuzzy sets	F-AHP for attribute weighting	Selection of an optimal maintenance strategy for an Ammonia Synthesis Unit of a urea fertilizer industry located in North India
Peng and Garg (2018)	Interval-valued fuzzy soft sets	IVFS-MABAC and IVFS-WDBA for comparison	A numerical example of emergency decision-making issue of mine accidents
Karaşan et al. (2019c)	Interval-valued hesitant fuzzy sets	F-CODAS and HF-TOPSIS for comparison	Residential construction site selection
Büyüközkan and Göçer (2019)	Intuitionistic fuzzy sets	IF-TOPSIS and IF-VIKOR for comparison	Prioritization of the strategies for smart city logistics
Karagöz et al. (2020)	Intuitionistic fuzzy sets	IF-WASPAS and IF-TOPSIS for comparison	Locating an authorized dismantling centre in Turkey
Dahooei et al. (2018)	Interval-valued intuitionistic fuzzy sets	IVIF-TODIM, IVIF-COPRAS, IVIF-MABAC for comparison	Evaluation of business intelligence for enterprise systems
Bolturk and Kahraman (2018)	Interval-valued intuitionistic fuzzy sets	CODAS for comparison	Evaluation of wave energy technology investments in Turkey
Roy et al. (2019)	Interval-valued intuitionistic fuzzy sets	The linear programming model for attribute weighting CODAS, F-CODAS, IVIF-VIKOR, IVIF-TOPSIS for comparison	Numerical example on an automotive parts factory in India searching for the best material for the automotive instrument panel
Yeni and Özçelik (2019)	Interval-valued intuitionistic fuzzy sets	IVIF-TOPSIS, IVIF-VIKOR, IVIF-SAW for comparison	Personnel selection for an engineering position in a company
Dahooei et al. (2020)	Interval-valued intuitionistic fuzzy sets	–	Choosing the appropriate system for cloud computing implementation in Iran
Deveci et al. (2020)	Interval-valued intuitionistic fuzzy sets	–	Evaluation of renewable energy alternatives in Turkey
Ouhibi and Frikha (2020)	Interval-valued intuitionistic fuzzy sets	–	Sorting of natural resources in Tunisia
Remadi and Frikha (2020)	Triangular interval-valued intuitionistic fuzzy sets	TOPSIS, VIKOR, GRA, and CODAS for comparison	Green supplier selection problem for olive oil

(*continued on next page*)

Paper	CODAS Version	Hybrid Method(s)	Application
Seker and Aydin (2020)	Interval-valued intuitionistic fuzzy sets	IVIF-AHP for attribute weighting AHP&CODAS, F-AHP&F-CODAS for comparison	Determination of the most appropriate public transportation system to transfer people along the campus
Yalçın and Yapıcı Pehlivan (2019)	Hesitant fuzzy linguistic term sets	F-EDAS, F-TOPSIS, F-WASPAS, F-ARAS, F-COPRAS for comparison	Blue-collar personnel selection problem for a manufacturing firm in Turkey
Sansabas-Villalpando *et al.* (2019)	Hesitant fuzzy linguistic term sets	AHP for attribute weighting	Appraisal of the organizational culture of innovation and complex technological changing environments
Mukul *et al.* (2020)	Hesitant fuzzy linguistic term sets	HFL-AHP for attribute weighting	Evaluation of smart health technologies
Büyüközkan and Mukul (2020)	Hesitant fuzzy linguistic term sets	HFL-AHP for attribute weighting HFL-TOPSIS for comparison	Evaluation of smart health technologies
Bolturk (2018)	Pythagorean fuzzy sets	CODAS for comparison	Supplier selection in a manufacturing firm
Peng and Ma (2020)	Pythagorean fuzzy sets	TOPSIS and TODIM for comparison	Several hypothetical examples
Büyüközkan and Göçer (2020)	Pythagorean fuzzy sets	–	Selection of additive manufacturing technologies for the needs of the supply chain in Turkey
Bolturk and Kahraman (2019)	Interval-valued Pythagorean fuzzy sets	–	Selection of the best AS/RS technology
Peng and Li (2019)	Hesitant fuzzy soft sets	HFS-WDBA	A numerical example of an emergency decision-making issue of mine accidents
Wang *et al.* (2020)	2-tuple linguistic neutrosophic sets	–	A numerical example for the safety assessment of a construction project
He *et al.* (2020)	2-tuple linguistic Pythagorean fuzzy sets	2TLPF-TODIM for comparison	Numerical example on the assessment of financial management performance
Karaşan *et al.* (2019a)	Neutrosophic sets	IVIF-TOPSIS for comparison	Wind energy plant location selection problem
Karaşan *et al.* (2019b)	Neutrosophic sets	–	Assessment of livability index of urban districts in Turkey
Karaşan *et al.* (2020)	Neutrosophic sets	–	Evaluation of defense strategies for Turkey
Kutlu Gündoğdu and Kahraman (2019b)	Spherical fuzzy sets	IF-TOPSIS and IF-CODAS for comparison	A hypothetical example
Kutlu Gündoğdu and Kahraman (2020)	Spherical fuzzy sets	IF-TOPSIS for comparison	Warehouse site selection problem
Karaşan *et al.* (2021)	Spherical fuzzy sets	–	Assessment of livability index of suburban districts in Turkey

Table A2
Linguistic evaluations of the decision-makers.

	C_1			C_2			C_3			C_4			C_5			C_6			C_7		
A_1	VG	VG	VG	MP	MP	MP	VG	VG	VG	G	G	G	G	VG	VG	MP	F	MG	F	F	F
A_2	F	MG	F	G	G	G	F	F	F	MG	G	VG	VG	VG	VG	G	MG	F	F	F	MP
A_3	P	MG	G	P	MP	F	MG	G	VG	MG	MG	MG	G	MG	F	MG	MG	MG	P	MP	F
A_4	MG	MG	MG	F	F	F	VG	G	MG	P	P	P	VG	G	VG	VG	G	VG	MP	F	MP
A_5	F	MG	VG	VG	VG	VG	VG	G	G	P	P	MP	VP	P	MP	G	G	G	F	F	P
A_6	G	G	G	F	MP	P	G	G	G	G	G	G	VG	VG	VG	F	MG	G	F	MP	F
A_7	MP	F	G	MP	MP	MP	MP	F	MG	P	MP	F	P	P	MP	MP	F	MG	F	MP	F
A_8	P	VG	VG	G	MG	F	F	MG	G	P	MP	F	P	P	P	VG	G	MG	F	MP	F
A_9	P	MG	F	F	MP	P	MP	F	MG	MG	G	VG	MG	MG	MG	F	F	F	F	F	F
A_{10}	MG	MG	G	MP	F	MG	MG	MG	MG	VG	VG	VG	VG	VG	VG	VG	VG	VG	P	F	F

Table A3
The experts' decision matrices including PFNs.

DM_1	C_1			C_2			C_3			C_4			C_5			C_6			C_7		
A_1	0.9	0	0.05	0.3	0	0.6	0.9	0	0.05	0.75	0.05	0.1	0.75	0.05	0.1	0.3	0	0.6	0.5	0.1	0.4
A_2	0.5	0.1	0.4	0.75	0.05	0.1	0.5	0.1	0.4	0.6	0	0.3	0.9	0	0.05	0.75	0.05	0.1	0.5	0.1	0.4
A_3	0.25	0.05	0.6	0.25	0.05	0.6	0.6	0	0.3	0.6	0	0.3	0.75	0.05	0.1	0.6	0	0.3	0.25	0.05	0.6
A_4	0.6	0	0.3	0.5	0.1	0.4	0.9	0	0.05	0.25	0.05	0.6	0.9	0	0.05	0.9	0	0.05	0.3	0	0.6
A_5	0.5	0.1	0.4	0.9	0	0.05	0.9	0	0.05	0.25	0.05	0.6	0.1	0	0.85	0.75	0.05	0.1	0.5	0.1	0.4
A_6	0.75	0.05	0.1	0.5	0.1	0.4	0.75	0.05	0.1	0.75	0.05	0.1	0.9	0	0.05	0.5	0.1	0.4	0.5	0.1	0.4
A_7	0.3	0	0.6	0.3	0	0.6	0.3	0	0.6	0.25	0.05	0.6	0.25	0.05	0.6	0.3	0	0.6	0.5	0.1	0.4
A_8	0.25	0.05	0.6	0.75	0.05	0.1	0.5	0.1	0.4	0.25	0.05	0.6	0.25	0.05	0.6	0.9	0	0.05	0.5	0.1	0.4
A_9	0.25	0.05	0.6	0.5	0.1	0.4	0.3	0	0.6	0.6	0	0.3	0.6	0	0.3	0.5	0.1	0.4	0.5	0.1	0.4
A_{10}	0.6	0	0.3	0.3	0	0.6	0.6	0	0.3	0.9	0	0.05	0.9	0	0.05	0.9	0	0.05	0.25	0.05	0.6

DM_2	C_1			C_2			C_3			C_4			C_5			C_6			C_7		
A_1	0.9	0	0.05	0.3	0	0.6	0.9	0	0.05	0.75	0.05	0.1	0.9	0	0.05	0.5	0.1	0.4	0.5	0.1	0.4
A_2	0.6	0	0.3	0.75	0.05	0.1	0.5	0.1	0.4	0.75	0.05	0.1	0.9	0	0.05	0.6	0	0.3	0.5	0.1	0.4
A_3	0.6	0	0.3	0.3	0	0.6	0.75	0.05	0.1	0.6	0	0.3	0.6	0	0.3	0.6	0	0.3	0.3	0	0.6
A_4	0.6	0	0.3	0.5	0.1	0.4	0.75	0.05	0.1	0.25	0.05	0.6	0.75	0.05	0.1	0.75	0.05	0.1	0.5	0.1	0.4
A_5	0.6	0	0.3	0.9	0	0.05	0.75	0.05	0.1	0.25	0.05	0.6	0.25	0.05	0.6	0.75	0.05	0.1	0.5	0.1	0.4
A_6	0.75	0.05	0.1	0.3	0	0.6	0.75	0.05	0.1	0.75	0.05	0.1	0.9	0	0.05	0.6	0	0.3	0.3	0	0.6
A_7	0.5	0.1	0.4	0.3	0	0.6	0.5	0.1	0.4	0.3	0	0.6	0.25	0.05	0.6	0.5	0.1	0.4	0.3	0	0.6
A_8	0.9	0	0.05	0.6	0	0.3	0.6	0	0.3	0.3	0	0.6	0.25	0.05	0.6	0.75	0.05	0.1	0.3	0	0.6
A_9	0.6	0	0.3	0.3	0	0.6	0.5	0.1	0.4	0.75	0.05	0.1	0.6	0	0.3	0.5	0.1	0.4	0.5	0.1	0.4
A_{10}	0.6	0	0.3	0.5	0.1	0.4	0.6	0	0.3	0.9	0	0.05	0.9	0	0.05	0.9	0	0.05	0.5	0.1	0.4

DM_3	C_1			C_2			C_3			C_4			C_5			C_6			C_7		
A_1	0.9	0	0.05	0.3	0	0.6	0.9	0	0.05	0.75	0.05	0.1	0.9	0	0.05	0.6	0	0.3	0.5	0.1	0.4
A_2	0.5	0.1	0.4	0.75	0.05	0.1	0.5	0.1	0.4	0.9	0	0.05	0.9	0	0.05	0.5	0.1	0.4	0.3	0	0.6
A_3	0.75	0.05	0.1	0.5	0.1	0.4	0.9	0	0.05	0.6	0	0.3	0.5	0.1	0.4	0.6	0	0.3	0.5	0.1	0.4
A_4	0.6	0	0.3	0.5	0.1	0.4	0.6	0	0.3	0.25	0.05	0.6	0.9	0	0.05	0.9	0	0.05	0.3	0	0.6
A_5	0.9	0	0.05	0.9	0	0.05	0.75	0.05	0.1	0.3	0	0.6	0.3	0	0.6	0.75	0.05	0.1	0.25	0.05	0.6
A_6	0.75	0.05	0.1	0.25	0.05	0.6	0.75	0.05	0.1	0.75	0.05	0.1	0.9	0	0.05	0.75	0.05	0.1	0.5	0.1	0.4
A_7	0.75	0.05	0.1	0.3	0	0.6	0.6	0	0.3	0.5	0.1	0.4	0.3	0	0.6	0.6	0	0.3	0.5	0.1	0.4
A_8	0.9	0	0.05	0.5	0.1	0.4	0.75	0.05	0.1	0.5	0.1	0.4	0.25	0.05	0.6	0.6	0	0.3	0.5	0.1	0.4
A_9	0.5	0.1	0.4	0.25	0.05	0.6	0.6	0	0.3	0.9	0	0.05	0.6	0	0.3	0.5	0.1	0.4	0.5	0.1	0.4
A_{10}	0.75	0.05	0.1	0.6	0	0.3	0.6	0	0.3	0.9	0	0.05	0.9	0	0.05	0.9	0	0.05	0.5	0.1	0.4

Table A4
The weighted normalized picture fuzzy decision matrix for Eq. (58).

	c_1	c_2	c_3	c_4	c_5	c_6	c_7
A_1	0.282 0.000 0.650	0.046 0.000 0.934	0.292 0.000 0.638	0.185 0.642 0.113	0.288 0.000 0.627	0.093 0.000 0.878	0.055 0.775 0.170
A_2	0.106 0.000 0.862	0.169 0.671 0.106	0.099 0.708 0.193	0.201 0.000 0.724	0.320 0.000 0.606	0.136 0.000 0.808	0.064 0.000 0.915
A_3	0.117 0.000 0.826	0.057 0.000 0.919	0.204 0.000 0.721	0.127 0.000 0.837	0.153 0.000 0.785	0.126 0.000 0.837	0.080 0.000 0.892
A_4	0.123 0.000 0.841	0.088 0.736 0.176	0.204 0.000 0.721	0.042 0.642 0.296	0.277 0.000 0.634	0.248 0.000 0.670	0.076 0.000 0.901
A_5	0.166 0.000 0.788	0.264 0.000 0.671	0.220 0.000 0.686	0.045 0.000 0.927	0.042 0.000 0.934	0.185 0.643 0.113	0.064 0.758 0.172
A_6	0.181 0.650 0.111	0.057 0.000 0.919	0.188 0.638 0.114	0.185 0.642 0.113	0.320 0.000 0.606	0.136 0.000 0.808	0.068 0.000 0.910
A_7	0.109 0.000 0.840	0.046 0.000 0.934	0.094 0.000 0.876	0.063 0.000 0.911	0.050 0.000 0.918	0.093 0.000 0.878	0.068 0.000 0.910
A_8	0.217 0.000 0.724	0.124 0.000 0.825	0.138 0.000 0.805	0.063 0.000 0.911	0.047 0.606 0.325	0.201 0.000 0.725	0.068 0.000 0.892
A_9	0.091 0.000 0.877	0.057 0.000 0.919	0.094 0.000 0.876	0.201 0.000 0.724	0.142 0.000 0.817	0.097 0.712 0.191	0.055 0.775 0.170
A_{10}	0.141 0.000 0.802	0.084 0.000 0.889	0.128 0.000 0.835	0.288 0.000 0.642	0.320 0.000 0.606	0.288 0.000 0.643	0.064 0.758 0.172
$\bar{n}s$	0.091 0.000 0.877	0.046 0.000 0.934	0.094 0.000 0.876	0.042 0.000 0.927	0.042 0.000 0.934	0.093 0.000 0.878	0.055 0.000 0.915

Table A5
Euclidean and Hamming distances to the
negative-ideal solution for Eq. (58).

	E_i	T_i
A_1	0.608	0.647
A_2	0.583	0.592
A_3	0.116	0.132
A_4	0.558	0.583
A_5	0.575	0.568
A_6	0.687	0.766
A_7	0.020	0.019
A_8	0.346	0.303
A_9	0.564	0.504
A_{10}	0.472	0.480

Table A6
Comparison of distances and the ranking of alternatives for Eq. (58).

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	H_i	Rank
A_1	0.000	0.025	1.007	0.115	0.033	−0.079	1.216	0.606	0.044	0.302	3.271	2
A_2	−0.025	0.000	0.926	0.025	0.008	−0.104	1.135	0.526	0.019	0.222	2.731	3
A_3	−0.493	−0.467	0.000	−0.442	−0.459	−0.571	0.208	−0.230	−0.448	−0.357	−3.260	9
A_4	−0.050	−0.025	0.893	0.000	−0.017	−0.129	1.101	0.492	−0.006	0.188	2.447	5
A_5	−0.033	−0.008	0.895	0.017	0.000	−0.112	1.103	0.494	0.011	0.190	2.558	4
A_6	0.198	0.279	1.205	0.312	0.310	0.000	1.413	0.804	0.385	0.500	5.406	1
A_7	−0.588	−0.562	−0.095	−0.538	−0.555	−0.666	0.000	−0.326	−0.544	−0.452	−4.326	10
A_8	−0.262	−0.237	0.401	−0.212	−0.229	−0.341	0.609	0.000	−0.218	−0.126	−0.615	8
A_9	−0.044	−0.019	0.820	0.006	−0.011	−0.123	1.028	0.419	0.000	0.115	2.192	6
A_{10}	−0.136	−0.111	0.705	−0.086	−0.103	−0.215	0.913	0.304	−0.092	0.000	1.180	7

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