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MODELLING, SIMULATION AND CONTROL FOR INTEGRO-DIFFERENTIAL SYSTEMS WITH DELAYS BY USING ASSOCIATED SYSTEMS

Carlos F. ALASTRUEY, José R. GONZÁLEZ DE MENDÍVIL and Farshad ARJOMANDI

Department of Electricity and Electronics
University of the Basque Country
P.O. Box 644–48080 Bilbao, Spain

Abstract. Control laws' design strategies in order to stabilize integro-differential systems with delays are developed by using an extended system and the delay measure.

Key words: integro-differential systems, distributed delays, stabilizability.

1. Introduction and problem statement. Mathematical models with time-delay constitute a natural way to represent a wide class of physical systems such as transportation problems, population growth laws and economic systems. The presence of time-delay in dynamical discrete linear equations can be overcome, when necessary, by using extended systems (Franklin and Powell, 1981). The stabilizability conditions for systems with general delays in state were extended by Pandolfi (1975) and also by Bhat and Koivo (1976). In a work due to Olbrot (1978) open-loop stabilizability problems for systems with control and state delays were defined. A characterization of trajectory-stabilizable systems, and of the relations between state- and trajectory-stabilizability were given Tadmor (1988); trajectory stabilizability is an appropriate notion in the presence of delays. Conditions for the delay-independent stabilization of linear systems were given by Amemiya *et al.* (1986), being the upper bound or the lower bound of the decay rates assignable, and Akazawa *et al.* (1987), by using in the proof matrices with some of their elements being arbitrary. In addition, Fiagbedzi and Pearson (1986, 1990) introduced techniques for the feedback and output feedback stabilization of delay systems by using a generalization of the transformation method. Furthermore, Mori *et al.* (1983) developed a way to stabilize linear systems with delayed state.

The stability of a linear delay-differential system with a point delay in its state has been studied in different works (Mori *et al.*, 1982; Hmamed, 1985, 1986 a-b; Mori, 1986; Bourlès, 1987; Mori and Kokame, 1989). In this note, several criteria in order to design stabilizing control laws for integro-differential systems with a distributed delay in their state are introduced by using the delay measure and an associated extended system under the form of a linear-differential system with a point delay in its state.

In particular, the problem of stability for a scalar differential system with two point delays in its state has been considered by several authors (see, for instance, Juri and Mansour, 1982). However, the exact delay-dependent algebraic stability conditions of such a system were not known until a recent result (Schoen and Geering, 1993), which was obtained by using an instability criterion together with the D-decomposition method.

The paper is organized as follows: Section 2 introduces the concepts of matrix measure and delay measure, and points out some stability results that will be used in the sequel. Section 3 introduces the main stabilizability results by using the delay measure notation and the associated extended system. Section 4 presents a result for stability of systems with two point delays. Section 5 points out the main stabilizability results for an integro-differential system with two distributed delays in its state. Section 6 rewrites some stability results under the presence of two state-point delays by using the delay measure. Finally, conclusions end the paper.

2. Matrix measure, delay measure and stability results. Matrix measure has been widely used in the literature when dealing with stability of delay-differential systems (see for instance Mori *et al.*, 1982). The matrix measure μ for matrix X is defined as follows:

$$\mu(X) \equiv \lim_{\varepsilon \longrightarrow 0} \frac{\|I + \varepsilon X\| - 1}{\varepsilon}. \tag{1}$$

The matrix measure defined in Eq. 1 can be subdefined in different ways according to the norm utilized in its definition. For example, if one considers 1-norms, the matrix measure can be computed as follows:

$$\mu_1(X) = \max_k \left(Re(x_{kk}) + \sum_{i=1_{\neq k}}^{n} |x_{ik}| \right). \tag{2}$$

Consider the following class of linear delay-differential systems with two point delays in the state and in the control variables:

$$\dot{x}(t) = Ax(t) + A_0 x(t-h) + Bu(t) + B_0 u(t-q), \quad h, q \in R^+, \tag{3}$$

where $A,\ A_0,\ B,\ B_0 \in W \subset R^{n\times n}$, being W the set of n-matrices Q such that $\|Q\| < \infty$.

DEFINITION 2.1 (Alastruey and González de Mendívil, 1993) the delay measure for system (3) is defined as follows:

$$\xi(h,q) \equiv \frac{\|A_0\|h + \|B_0\|q}{\mu(A) + \mu(B)}. \tag{4}$$

REMARK 2.1. If there is no delay (i.e., $h, q = 0$, or A_0 and B_0 are null matrices), then the delay measure is zero. On the other hand, if the point delays h and q verify $0 < h < \infty,\ 0 < q < \infty$, and there is not a delay-free term (i.e., A, B are matrices of zeros) then the delay measure is infinite. Therefore, the delay measure can be considered, intuitively, as a way to evaluate the effect of delay terms in a system compared with its delay free terms.

Let's introduce some stability results by using delay-measure notation. This representation will be useful in order to deduce the main stabilizability results that are to be presented in Section 3.

Consider the free linear delay-differential system:

$$\dot{x}(t) = Ax(t) + A_0(t-h), \quad \text{with} \quad A, A_0 \in W, \tag{5}$$

where $A, A_0 \in W \subset R^{n\times n}$, being W the set of n-matrices Q such that $\|Q\| < \infty$.

Lemma 2.1 (Alastruey and González de Mendívil, 1993). *Provided $h \geqslant 1$, a sufficient condition for system (5) to be stable is*

$$\xi(h) < -1. \tag{6}$$

REMARK 2.2. Observe that for system (6) the delay measure is reduced to

$$\xi(h,q) = \xi(h) = \frac{\|A_0\|h}{\mu(A)}. \tag{7}$$

Alternatively, one of the simplest conditions for stability in system (5) is given in terms of the matrix measure as follows.

Lemma 2.2 (Mori *et al.*, 1982). *A sufficient condition for system (5) to be stable is given by*

$$\mu(A) < -\|A_0\| \Rightarrow \mu(A) + \|A_0\| < 0. \tag{8}$$

The Lemmas introduced in this section will enable us to deduce the main stabilizability results in the sequel.

Some properties of the delay measure are outlined.

PROPERTY 2.1. Lower bounds for the first derivatives of the delay measure:

$$\rho_h \equiv \frac{\partial \xi(h,q)}{\partial h} = \frac{\|A_0\|}{\mu(A)+\mu(B)} \geqslant \frac{\|A_0\|}{\|A\|+\|B\|}, \tag{9}$$

$$\rho_q \equiv \frac{\partial \xi(h,q)}{\partial q} = \frac{\|B_0\|}{\mu(A)+\mu(B)} \geqslant \frac{\|B_0\|}{\|A\|+\|B\|}. \tag{10}$$

PROPERTY 2.2. Absolute lower bound for the delay measure (supposing h, q variables):

$$\xi(h,q) = \frac{\|A_0\|h + \|B_0\|q}{\mu(A)+\mu(B)} \geqslant \frac{\|A_0\|\widehat{h} + \|B_0\|\widehat{q}}{\mu(A)+\mu(B)}, \quad \text{with } \widehat{h} = \min h \text{ and } \widehat{q} = \min q. \tag{11}$$

REMARK 2.3. Observe that property 2.1 helps to estimate boundedness conditions for the variations in value of the delay measure. Property 2.2 gives absolute boundedness conditions for the delay measure, provided that n-matrices appearing in (7) belong to the class W.

3. Stabilizability of integro-differential systems with one distributed delay. In this section conditions for a control law to stabilize an integro-differential system with one distributed delay in its state will be discussed by using an associated extended system. Several results are to be introduced.

Consider the following integro-differential system with one distributed delay in its state, and containing a control law with a point delay and a control law with a distributed delay.

$$\begin{aligned}\dot{x}(t) =& Ax(t) + \int_0^t A_0 x(t'-h)dt' \\ &+ \frac{1}{2}\Bigg[B^{11}u_1(t) + B_0^{11}u_1(t-h) \\ &+ \int_0^t \left[B^{22}u_2(t') + B_0^{22}u_2(t'-h)\right]dt'\Bigg], \qquad (12)\end{aligned}$$

with $x(t) = g(t)$ for all $t < 0$, where $A, A_0, B^{11}, B_0^{11}, B^{22}, B_0^{22} \in W \subset R^{n\times n}$, $u_1(t), u_2(t) = 0\ \forall t < 0$ and $h \in R^+$

Result 3.1. Consider two control laws $u_1(t), u_2(t)$ defined by the delay-differential equations

$$\begin{aligned}\begin{bmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix} =& \begin{bmatrix}D^{11} & \bar{0}\\ \bar{0} & D^{22}\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \begin{bmatrix}E^{11} & \bar{0}\\ \bar{0} & E^{22}\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} \\ &+ \begin{bmatrix}D_0^{11} & \bar{0}\\ \bar{0} & D_0^{22}\end{bmatrix}\begin{bmatrix}x(t-h)\\ \dot{x}(t-h)\end{bmatrix} \\ &+ \begin{bmatrix}E_0^{11} & \bar{0}\\ \bar{0} & E_0^{22}\end{bmatrix}\begin{bmatrix}u_1(t-h)\\ u_2(t-h)\end{bmatrix}. \qquad (13)\end{aligned}$$

A sufficient condition for control laws (13) to stabilize system (12) is given by

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{-1}\left[\left\| I_{4n}+\varepsilon\begin{bmatrix}\bar{0} & I_n & B^{11} & \bar{0}\\ \bar{0} & A & \bar{0} & B^{22}\\ D^{11} & \bar{0} & E^{11} & \bar{0}\\ \bar{0} & D^{22} & \bar{0} & E^{22}\end{bmatrix}\right\| -1\right]\right) + \left\|\begin{bmatrix}\bar{0} & \bar{0} & B_0^{11} & \bar{0}\\ A_0 & \bar{0} & \bar{0} & B_0^{22}\\ D_0^{11} & \bar{0} & E_0^{11} & \bar{0}\\ \bar{0} & D_0^{22} & \bar{0} & E_0^{22}\end{bmatrix}\right\| < 0, \tag{14}$$

where $\bar{0}$ are blocks of $n \times n$ zeros.

Proof. Consider the free part of system (12)

$$\dot{x}(t) = Ax(t) + \int_0^t A_0 x(t'-h)dt', \tag{14}$$

with $x(t) = g(t)$ for all $t < 0$. By differentiating system (14) one gets

$$\ddot{x}(t) = A\dot{x}(t) + A_0 x(t-h), \tag{15}$$

with $\dot{x}(t) = \dot{g}(t)\ \forall t < 0$. Define the $2n$-vector $\tilde{x}(t) \equiv \begin{bmatrix} x(t)\\ \dot{x}(t)\end{bmatrix}$. From Eq. 14 and Eq. 15 one gets

$$\dot{\tilde{x}}(t) = \begin{bmatrix}\dot{x}(t)\\ \ddot{x}(t)\end{bmatrix} = \begin{bmatrix}\bar{0} & I_n\\ \bar{0} & A\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \begin{bmatrix}\bar{0} & \bar{0}\\ A_0 & \bar{0}\end{bmatrix}\begin{bmatrix}x(t-h)\\ \dot{x}(t-h)\end{bmatrix}. \tag{16}$$

By defining

$$\tilde{A} \equiv \begin{bmatrix}\bar{0} & I_n\\ \bar{0} & A\end{bmatrix}, \quad 2n \times 2n\text{-matrix},$$

$$\tilde{A}_0 \equiv \begin{bmatrix}\bar{0} & \bar{0}\\ A_0 & \bar{0}\end{bmatrix}, \quad 2n \times 2n\text{-matrix},$$

one gets finally

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{A}_0\tilde{x}(t-h). \tag{17}$$

System (17) is called "associated extended system " of the free system (14). Firstly let us investigate stabilizability for system (14) by using control laws (15). The controlled associated extended system becomes

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{A}_0\tilde{x}(t-h) + \tilde{B}\tilde{u}(t) + \tilde{B}_0\tilde{u}(t-h), \tag{18}$$

where

$$\tilde{B} \equiv \begin{bmatrix} B^{11} & \bar{0} \\ \bar{0} & B^{22} \end{bmatrix}, \quad \tilde{B}_0 \equiv \begin{bmatrix} B_0^{11} & \bar{0} \\ \bar{0} & B_0^{22} \end{bmatrix}, \quad \tilde{u}(t) \equiv \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Observe that Eq. 15 and Eq. 17 can be rewritten as one single delay-differential equation as follows

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{\tilde{u}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{D} & \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix} + \begin{bmatrix} \tilde{A}_0 & \tilde{B}_0 \\ \tilde{D}_0 & \tilde{E}_0 \end{bmatrix} \begin{bmatrix} \tilde{x}(t-h) \\ \tilde{u}(t-h) \end{bmatrix}. \tag{19}$$

Define

$$z(t) \equiv \left[\tilde{x}^T(t) \vdots \tilde{u}^T(t)\right]^T, \quad \hat{A} \equiv \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{D} & \tilde{E} \end{bmatrix}, \quad \hat{A}_0 \equiv \begin{bmatrix} \tilde{A}_0 & \tilde{B}_0 \\ \tilde{D}_0 & \tilde{E}_0 \end{bmatrix},$$

where $z(t)$ is a $4n$-vector and $\hat{A}, \hat{A}_0$ are $4n \times 4n$ matrices. Then, Eq. 19 can be rewritten as

$$\dot{z}(t) = \hat{A}z(t) + \hat{A}_0 z(t-h). \tag{20}$$

Observe that

$$\begin{aligned} \hat{A} &= \begin{bmatrix} \bar{0} & I_n & B^{11} & \bar{0} \\ \bar{0} & A & \bar{0} & B^{22} \\ D^{11} & \bar{0} & E^{11} & \bar{0} \\ \bar{0} & D^{22} & \bar{0} & E^{22} \end{bmatrix}, \\ \hat{A}_0 &= \begin{bmatrix} \bar{0} & \bar{0} & B_0^{11} & \bar{0} \\ A_0 & \bar{0} & \bar{0} & B_0^{22} \\ D_0^{11} & \bar{0} & E_0^{11} & \bar{0} \\ \bar{0} & D_0^{22} & \bar{0} & E_0^{22} \end{bmatrix}. \end{aligned} \tag{21}$$

If, by hypothesis, condition (14) holds, then

$$\lim_{\varepsilon \to 0} \left(\frac{\|I_{4n} + \varepsilon \widehat{A}\| - 1}{\varepsilon} \right) + \|\widehat{A}_0\| < 0. \tag{22}$$

And therefore

$$\mu(\widehat{A}) + \|\widehat{A}_0\| < 0. \tag{23}$$

Then by Lemma 2.2, system (20) is stable, i.e., system (14) is stabilizable by control law (13). Observe that the controlled associated extended system (18) becomes

$$\begin{aligned} \dot{x}(t) =& Ax(t) + \int_0^t A_0 x(t' - h) dt' \\ &+ B^{11} u_1(t) + B_0^{11} u_1(t-h), \end{aligned} \tag{24}$$

$$\begin{aligned} \ddot{x}(t) =& A\dot{x}(t) + A_0 x(t-h) + B^{22} u_2(t) \\ &+ B_0^{22} u_2(t-h). \end{aligned} \tag{25}$$

Integrating Eq. 25 yields

$$\begin{aligned} \dot{x}(t) =& Ax(t) + \int_0^t A_0 x(t' - h) dt' \\ &+ \int_0^t B^{22} u_2(t') dt' + \int_0^t B_0^{22} u_2(t' - h) dt'. \end{aligned} \tag{26}$$

Finally, by adding Eqns. (24) and (26) one gets

$$\begin{aligned} \dot{x}(t) =& Ax(t) + \int_0^t A_0 x(t' - h) dt' \\ &+ \frac{1}{2} \Big[B^{11} u_1(t) + B_0^{11} u_1(t-h) \\ &+ \int_0^t B^{22} u_2(t') dt' + \int_0^t B_0^{22} u_2(t' - h) dt' \Big], \end{aligned} \tag{27}$$

and the controlled system (27) – which coincides with (12) – is stable. Result 3.1 can be rewritten in terms of the delay measure as follows.

Result 3.2. Consider two control laws $u_1(t), u_2(t)$ defined by the delay-differential equations

$$\begin{aligned}\begin{bmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix} &= \begin{bmatrix}D^{11} & \bar{0}\\ \bar{0} & D^{22}\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \begin{bmatrix}E^{11} & \bar{0}\\ \bar{0} & E^{22}\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}\\ &+ \begin{bmatrix}D_0^{11} & \bar{0}\\ \bar{0} & D_0^{22}\end{bmatrix}\begin{bmatrix}x(t-h)\\ \dot{x}(t-h)\end{bmatrix}\\ &+ \begin{bmatrix}E_0^{11} & \bar{0}\\ \bar{0} & E_0^{22}\end{bmatrix}\begin{bmatrix}u_1(t-h)\\ u_2(t-h)\end{bmatrix},\end{aligned} \tag{28}$$

and consider also the auxiliary system

$$\dot{w}(t) = \widehat{A}w(t) + \widehat{A}_0 w(t-h), \tag{29}$$

where

$$\begin{aligned}\widehat{A} &= \begin{bmatrix}\bar{0} & I_n & B^{11} & \bar{0}\\ \bar{0} & A & \bar{0} & B^{22}\\ D^{11} & \bar{0} & E^{11} & \bar{0}\\ \bar{0} & D^{22} & \bar{0} & E^{22}\end{bmatrix},\\ \widehat{A}_0 &= \begin{bmatrix}\bar{0} & \bar{0} & B_0^{11} & \bar{0}\\ A_0 & \bar{0} & \bar{0} & B_0^{22}\\ D_0^{11} & \bar{0} & E_0^{11} & \bar{0}\\ \bar{0} & D_0^{22} & \bar{0} & E_0^{22}\end{bmatrix}.\end{aligned} \tag{21}$$

A sufficient condition for control laws (28) to stabilize system (12) is given by

$$\xi(h) < -1, \tag{31}$$

where $\xi(h)$ is the delay measure referred to the auxiliary system (29).

Proof (outline). The proof follows immediately by considering Lemma 2.1 and taking into account that condition (31) for the auxiliary system (29) is equivalent to condition (22) of Result 3.1.

4. Stability with two point delays. The following theorems will be useful in the next section in order to deduce stabilizability conditions for an integro-differential system with two distributed delays in its state given in terms of algebraic relations.

Consider the following scalar system with two point delays in its state

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h), \tag{32}$$

where a_0, a_1 and a_2 are constant coefficients and $h > 0$.

Theorem 4.1 (Schoen and Geering, 1993). *The time-delay system* (32) *with* $|a_2| < \pi/2h$ *is asymptotically stable if and only if the following three conditions hold for some* $y \in [0, \pi/h)$

(i) $a_0 + a_1 + a_2 < 0,$ (33)

(ii) $a_0 = \dfrac{y \cdot \cos(yh)}{\sin(yh)} + a_2,$ (34)

(iii) $a_1 > -\dfrac{y}{\sin(yh)} - 2a_2 \cos(yh).$ (35)

Now we provided an extension of Theorem 4.1 for the linear multivariable case with diagonal matrices.

Theorem 4.2. *Consider the following linear MIMO system with two point delays in its vector-state*

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h), \tag{36}$$

where A_0, A_1 *and* A_2 *are real constant diagonal* $n \times n$*-matrices and* $h > 0$. *Consider the following* $n \times n$*-matrices*

$$D = \left[-\left(|a_{ij}^2| - \frac{\pi}{2h} \right) \right], \tag{37}$$

$$E = -(A_0 + A_1 + A_2), \tag{38}$$

$$F = \left[\frac{y_{ij} \cos(y_{ij} h)}{\sin(y_{ij} h)} \right], \tag{39}$$

$$G = \left[\frac{y_{ij}}{\sin(y_{ij} h)} + 2a_{ij}^2 \cos(y_{ij} h) \right], \tag{40}$$

where a^2_{ij} are the elements of matrix A_2 and $y_{ij} \in [0, \frac{\pi}{h}]$ $i, j = 1, \ldots, n$ are real numbers. Then the time-delay system (36) with D positive is asymptotically stable if the following three conditions hold for some set of values $y_{ij} \in \left[0, \frac{\pi}{h}\right]$ $i, j = 1, \ldots, n$:

(i) E is positive (Golub and Van Loen, 1986), (41)

(ii) $A_0 = F + A_2$, (42)

(iii) $A_1 + G$ is positive. (43)

Proof. Let's rewrite system (36) as follows

$$\begin{aligned}\dot{x}_k =& a^0_{k1}x_1(t) + \ldots + a^0_{kk}x_k(t) + \ldots + a^0_{kn}x_n(t)\\ &+ a^1_{k1}x_1(t-h) + \ldots + a^1_{kk}x_k(t-h)\\ &\qquad + \ldots + a^1_{kn}x_n(t-h)\\ &+ a^2_{k1}x_1(t-2h) + \ldots + a^2_{kk}x_k(t-2h)\\ &\qquad + \ldots + a^2_{kn}x_n(t-2h), \end{aligned} \tag{44}$$

where $k = 1, \ldots, n$. Therefore. Eq. 44 contains the scalar differential equations representing the time-evolution of all the state variables of system (36). If asymptotic stability is demonstrated for Eq. 44, the proof will be done, because of the diagonal hypothesis on the system matrices. Again, it is possible to rewrite (44) as follows

$$\dot{x}_k(t) = \dot{\tilde{x}}_1(t) + \ldots + \dot{\tilde{x}}_{k-1}(t) + \dot{\hat{x}}_k(t) + \dot{\tilde{x}}_{k+1}(t) + \ldots + \dot{\tilde{x}}_n(t), \tag{45}$$

where

$$\begin{aligned}\dot{\tilde{x}}_1(t) =& a^0_{k1}x_1(t) + a^1_{k1}x_1(t-h) + a^2_{k1}x_1(t-2h),\\ &\ldots\\ \dot{\tilde{x}}_{k-1}(t) =& a^0_{k(k-1)}x_{k-1}(t) + a^1_{k(k-1)}x_{k-1}(t-h)\\ &+ a^2_{k(k-1)}x_{k-1}(t-2h),\\ \dot{\hat{x}}_k(t) =& a^0_{kk}x_k(t) + a^1_{kk}x_k(t-h) + a^2_{kk}x_k(t-2h),\\ \dot{\tilde{x}}_{k+1}(t) =& a^0_{k(k+1)}x_{k+1}(t) + a^1_{k(k+1)}x_{k+1}(t-h)\\ &+ a^2_{k(k+1)}x_{k+1}(t-2h),\end{aligned}$$

$$\dots$$

$$\dot{\tilde{x}}_n(t) = a^0_{kn} x_n(t) + a^1_{kn} x_1(t-h) + a^2_{kn} x_n(t-2h), \quad (46)$$

but, by using (41), subsystems (46) satisfy the following coefficient relations

$$\begin{aligned}
&a^0_{k1} + a^1_{k1} + a^2_{k1} = -e_{k1} < 0, \\
&\dots \\
&a^0_{kk} + a^1_{kk} + a^2_{kk} = -e_{kk} < 0, \\
&\dots \\
&a^0_{kn} + a^1_{kn} + a^2_{kn} = -e_{kn} < 0.
\end{aligned} \quad (47)$$

Furthermore, by using (42) one gets

$$\begin{aligned}
&a^0_{k1} = f_{k1} + a^2_{k1} = \frac{y_{k1}\cos(y_{k1}h)}{\sin(y_{k1}h)} + a^2_{k1}, \\
&\dots \\
&a^0_{kk} = f_{kk} + a^2_{kk} = \frac{y_{kk}\cos(y_{kk}h)}{\sin(y_{kk}h)} + a^2_{kk}, \\
&\dots \\
&a^0_{kn} = f_{kn} + a^2_{kn} = \frac{y_{kn}\cos(y_{kn}h)}{\sin(y_{kn}h)} + a^2_{kn}.
\end{aligned} \quad (48)$$

Finally, by using (43)

$$\begin{aligned}
&a^1_{k1} + g_{k1} = a^1_{k1} + \frac{y_{k1}}{\sin(y_{k1}h)} + 2a^2_{k1}\cos(y_{k1}) > 0, \\
&\dots \\
&a^1_{kk} + g_{kk} = a^1_{kk} + \frac{y_{kk}}{\sin(y_{kk}h)} + 2a^2_{kk}\cos(y_{kk}) > 0, \\
&\dots \\
&a^1_{kn} + g_{kn} = a^1_{kn} + \frac{y_{kn}}{\sin(y_{kn}h)} + 2a^2_{kn}\cos(y_{kn}) > 0.
\end{aligned} \quad (49)$$

Therefore, scalar subsystems (46) satisfy the three conditions of Theorem 1, and then they are asymptotically stable. As system (44) is a linear combination of systems (46) because of diagonal hypothesis on system matrices, asymptotic stability for system (36) is deduced.

5. Main stabilizability results for a system with two distributed delays. In this section conditions for a control law to stabilize an integro-differential system with two distributed delays in its state will be discussed by using an associated extended system. A main result is introduced.

Consider the following integro-differential system with two distributed delays in its state, and containing a control law with two point delays and a control law with two distributed delays.

$$\begin{aligned}\dot{x}(t) =& Ax(t) + \int_0^t A_1 x(t'-h)dt' + \int_0^t A_2 x(t'-2h)dt' \\ &+ \frac{1}{2}\Big[B^{11}u_1(t) + B_1^{11}u_1(t-h) + B_2^{11}u_2(t)(t-2h)\Big] \\ &+ \frac{1}{2}\int_0^t \Big[B^{22}u_2(t') + B_1^{22}u_2(t'-h) \\ &\qquad + B_2^{22}u_2(t'-2h)\Big]dt', \qquad (50)\end{aligned}$$

with $x(t) = g(t)$ for all $t < 0$, where $A, A_1, A_2, B^{11}, B_1^{11}, B_2^{11}, B^{22}, B_1^{22}, B^{22} \in W \subset R^{n \times n}$, $u_1(t), u_2(t) = 0\ \forall t < 0$ and $h \in R^+$, where W is the class of diagonal real matrices.

Theorem 5.1 (Asymptotic stabilizability). Consider two control laws $u_1(t), u_2(t)$ defined by the delay-differential equations

$$\begin{aligned}\begin{bmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\end{bmatrix} &= \begin{bmatrix}D^{11} & \bar{0}\\ \bar{0} & D^{22}\end{bmatrix}\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} + \begin{bmatrix}E^{11} & \bar{0}\\ \bar{0} & E^{22}\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} \\ &+ \begin{bmatrix}D_1^{11} & \bar{0}\\ \bar{0} & D_1^{22}\end{bmatrix}\begin{bmatrix}x(t-h)\\ \dot{x}(t-h)\end{bmatrix} + \begin{bmatrix}E_1^{11} & \bar{0}\\ \bar{0} & E_1^{22}\end{bmatrix}\begin{bmatrix}u_1(t-h)\\ u_2(t-h)\end{bmatrix}\end{aligned}$$

$$+\begin{bmatrix} D_2^{11} & \bar{0} \\ \bar{0} & D_2^{22} \end{bmatrix} \begin{bmatrix} x(t-2h) \\ \dot{x}(t-2h) \end{bmatrix}$$

$$+\begin{bmatrix} E_2^{11} & \bar{0} \\ \bar{0} & E_2^{22} \end{bmatrix} \begin{bmatrix} u_1(t-2h) \\ u_2(t-2h) \end{bmatrix}. \tag{51}$$

Define the following $2n \times 2n$-matrices

$$\widetilde{A} \equiv \begin{bmatrix} \bar{0} & I_n \\ \bar{0} & A \end{bmatrix}, \quad \widetilde{A}_1 \equiv \begin{bmatrix} \bar{0} & \bar{0} \\ A_1 & \bar{0} \end{bmatrix}, \quad \widetilde{A}_2 \equiv \begin{bmatrix} \bar{0} & \bar{0} \\ A_2 & \bar{0} \end{bmatrix}; \tag{52a}$$

$$\widetilde{B} \equiv \begin{bmatrix} B^{11} & \bar{0} \\ \bar{0} & B^{22} \end{bmatrix}, \quad \widetilde{B}_1 \equiv \begin{bmatrix} B_1^{11} & \bar{0} \\ \bar{0} & B_1^{22} \end{bmatrix},$$

$$\widetilde{B}_2 \equiv \begin{bmatrix} B_2^{11} & \bar{0} \\ \bar{0} & B_2^{22} \end{bmatrix}; \tag{52b}$$

$$\widetilde{D} \equiv \begin{bmatrix} D^{11} & \bar{0} \\ \bar{0} & D^{22} \end{bmatrix}, \quad \widetilde{D}_1 \equiv \begin{bmatrix} D_1^{11} & \bar{0} \\ \bar{0} & D_1^{22} \end{bmatrix},$$

$$\widetilde{D}_2 \equiv \begin{bmatrix} D_2^{11} & \bar{0} \\ \bar{0} & D_2^{22} \end{bmatrix}; \tag{53}$$

$$\widetilde{E} \equiv \begin{bmatrix} E^{11} & \bar{0} \\ \bar{0} & E^{22} \end{bmatrix}, \quad \widetilde{E}_1 \equiv \begin{bmatrix} E_1^{11} & \bar{0} \\ \bar{0} & E_1^{22} \end{bmatrix},$$

$$\widetilde{E}_2 \equiv \begin{bmatrix} E_2^{11} & \bar{0} \\ \bar{0} & E_2^{22} \end{bmatrix}. \tag{54}$$

Define the following $4n \times 4n$-matrices

$$\widehat{A} \equiv \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{D} & \widetilde{E} \end{bmatrix}, \quad \widehat{A}_1 \equiv \begin{bmatrix} \widetilde{A}_1 & \widetilde{B}_1 \\ \widetilde{D}_1 & \widetilde{E}_1 \end{bmatrix}, \quad \widehat{A}_2 \equiv \begin{bmatrix} \widetilde{A}_2 & \widetilde{B}_2 \\ \widetilde{D}_2 & \widetilde{E}_2 \end{bmatrix}; \tag{55}$$

$$\widehat{D} \equiv \left[-\left(\left|\widehat{a}_{ij}^2\right| - \frac{\pi}{h}\right)\right] \text{ where } \widehat{a}_{ij}^2 \text{ are the elements of } \widehat{A}_2; \tag{56}$$

$$\widehat{E} \equiv -(\widehat{A} + \widehat{A}_1 + \widehat{A}_2); \tag{57}$$

$$\widehat{F} \equiv \left[\frac{y_{ij}\cos(y_{ij}h)}{\sin(y_{ij}h)}\right]; \tag{58}$$

$$\widehat{G} \equiv \left[\frac{y_{ij}}{\sin(y_{ij}h)} + 2\widehat{a}_{ij}^2\cos(y_{ij}h)\right], \tag{59}$$

where $y_{ij} \in [0, \frac{\pi}{h}]$ $i, j = 1, \ldots, n$ are real numbers.

Control laws (51) provide asymptotic stability for system (50) if the following four conditions hold for some set of values $y_{ij} \in [0, \frac{\pi}{h}]$, $i, j = 1, \ldots, n$:

(i) $\widehat{D}$ positive, (60)

(ii) $\widehat{E}$ positive, (61)

(iii) $\widehat{A} = \widehat{F} + \widehat{A}_2$, (62)

(iv) $\widehat{A}_1 + \widehat{G}$ positive. (63)

Proof. Consider the free part of system (50)

$$\dot{x}(t) = Ax(t) + \int_0^t A_1 x(t' - h)dt' + \int_0^t A_2 x(t' - 2h)dt', \quad (64)$$

with $x(t) = g(t)$ for all $t < 0$. By differentiating system (64) one gets

$$\ddot{x}(t) = A\dot{x}(t) + A_1 x(t - h) + A_2 x(t - 2h), \quad (65)$$

with $\dot{x}(t) = \dot{g}(t)$ $\forall t < 0$. Define the $2n$-vector $\widetilde{x}(t) \equiv \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$. From Eq. 64 and Eq. 65 one gets

$$\begin{aligned} \dot{\widetilde{x}}(t) = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} &= \begin{bmatrix} \bar{0} & I_n \\ \bar{0} & A \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &+ \begin{bmatrix} \bar{0} & \bar{0} \\ A_1 & \bar{0} \end{bmatrix} \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \end{bmatrix} \\ &+ \begin{bmatrix} \bar{0} & \bar{0} \\ A_2 & \bar{0} \end{bmatrix} \begin{bmatrix} x(t-2h) \\ \dot{x}(t-2h) \end{bmatrix}. \end{aligned} \quad (66)$$

By using definitions in Eq. 65, Eq. 66 can be rewritten as follows

$$\dot{\widetilde{x}}(t) = \widetilde{A}\widetilde{x}(t) + \widetilde{A}_1\widetilde{x}(t - h) + \widetilde{A}_2\widetilde{x}(t - 2h). \quad (67)$$

System (67) is called "associated extended system" of the free system (64). By applying the proposed control laws system (67) one gets

$$\begin{aligned}\dot{\widetilde{x}}(t) =& \widetilde{A}\widetilde{x}(t) + \widetilde{A}_1\widetilde{x}(t-h) + \widetilde{A}_2\widetilde{x}(t-2h)\\ &+ \widetilde{B}\widetilde{u}(t) + \widetilde{B}_1\widetilde{u}(t-h) + \widetilde{B}_2\widetilde{u}(t-2h), \qquad (68)\end{aligned}$$

where

$$\widetilde{u}(t) \equiv \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \qquad (69)$$

Observe that Eq. 51 and Eq. 68 can be rewritten as one single delay-differential equation as follows

$$\begin{aligned}\begin{bmatrix} \dot{\widetilde{x}}(t) \\ \dot{\widetilde{u}}(t) \end{bmatrix} =& \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{D} & \widetilde{E} \end{bmatrix} \begin{bmatrix} \widetilde{x}(t) \\ \widetilde{u}(t) \end{bmatrix} + \begin{bmatrix} \widetilde{A}_1 & \widetilde{B}_1 \\ \widetilde{D}_1 & \widetilde{E}_1 \end{bmatrix} \begin{bmatrix} \widetilde{x}(t-h) \\ \widetilde{u}(t-h) \end{bmatrix}\\ &+ \begin{bmatrix} \widetilde{A}_2 & \widetilde{B}_2 \\ \widetilde{D}_2 & \widetilde{E}_2 \end{bmatrix} \begin{bmatrix} \widetilde{x}(t-2h) \\ \widetilde{u}(t-2h) \end{bmatrix}. \qquad (70)\end{aligned}$$

Define

$$z(t) \equiv \left[\widetilde{x}^T(t) \vdots \widetilde{u}^T(t)\right]^T, \qquad (71)$$

where $z(t)$ is a $4n$-vector and $\widehat{A}, \widehat{A}_1, \widehat{A}_2$, are $4n \times 4n$ matrices defined in (65). Then, Eq. 70 can be rewritten as

$$\dot{z}(t) = \widehat{A}z(t) + \widehat{A}_1 z(t-h) + \widehat{A}_2 z(t-2h). \qquad (72)$$

If, by hypothesis, conditions (50)–(53) hold, then by Theorem 5.1 system (72) is asymptotically stable, i.e., system (57) is asymptotically stabilizable by control law (59).

Observe that the controlled associated extended system (68) becomes

$$\begin{aligned}\dot{x}(t) =& Ax(t) + \int_0^t A_1 x(t'-h)dt'\\ &+ B^{11}u_1(t) + B_1^{11}u_1(t-h) + B_1^{11}u_1(t-2h), \qquad (73)\\ \ddot{x}(t) =& A\dot{x}(t) + A_1 x(t-h)\\ &+ B^{22}u_2(t) + B_1^{22}u_2(t-h) + B_2^{22}u_2(t-2h). \qquad (74)\end{aligned}$$

Integrating Eq. 74 yields

$$\begin{aligned}\dot{x}(t) =&Ax(t) + \int_0^t A_1 x(t'-h)dt' + \int_0^t A_2 x(t'-2h)dt' \\ &+ \int_0^t B^{22} u_2(t')dt' + \int_0^t B_1^{22} u_2(t'-h)dt' \\ &+ \int_0^t B_2^{22} u_2(t'-2h)dt'.\end{aligned} \tag{75}$$

Finally, by adding Eq. 73 and Eq. 75 one gets

$$\begin{aligned}\dot{x}(t) =&Ax(t) + \int_0^t A_1 x(t'-h)dt' + \int_0^t A_2 x(t'-2h)dt' \\ &+ \frac{1}{2}\Big[B^{11}u_1(t) + B_1^{11}u_1(t-h) + B_2^{11}u_2(t)(t-2h)\Big] \\ &+ \frac{1}{2}\int_0^t \Big[B^{22}u_2(t') + B_1^{22}u_2(t'-h) \\ &+ B_2^{22}u_2(t'-2h)\Big]dt',\end{aligned} \tag{76}$$

and the controlled system (76) – which coincides with (50) – is asymptotically stable.

Theorem 5.1 provides a way to evaluate asymptotic stabilizability of a delay-differential system with two distributed delays in its state. The evaluation is made on the context of a set of algebraic relations, which are very suitable for computer applications.

6. Stability by using delay-measure approach. In this section, some stability results for a class of free linear differential systems with two point delays in the state vector are introduced under delay-measure notation.

Consider the free linear delay-differential system:

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h) + a_2 x(t-2h), \tag{77}$$

where a_0, a_1 and a_2 are constant coefficients and $h > 0$. It is possible to extend the definition of delay measure for system (77) as follows:

DEFINITION 6.1. The delay measure for system (77) is defined as follows:

$$\xi(h) \equiv \frac{|a_1| + 2|a_2|}{|a_0|} h. \tag{78}$$

The stability criteria introduced in Theorem 4.2 can be rewritten by using the delay measure as follows.

Theorem 6.1. *Suppose that $a_0 \in R^-$, $a_1, a_2 \in R^+$. System (77) is asymptotically stable if the following condition holds for some $y \in [0, \pi/h)$:*

$$\frac{\xi(h)}{h} + \frac{y \cdot \cos(yh)}{\sin(yh)} = -\frac{a_1}{a_0} - 2, \tag{79}$$

where the delay measure function is that defined in (78).

Proof. Condition (79) implies that

$$\frac{\xi(h)}{h} + \frac{y \cdot \cos(yh)}{\sin(yh)} = \frac{|a_1|}{|a_0|} - 2, \tag{80}$$

therefore

$$\begin{aligned}
&\left[\frac{|a_1|}{|a_0|} - 2 - \frac{y \cdot \cos(yh)}{\sin(yh)}\right] h = \xi(h) \\
&\Rightarrow |a_1|h - |a_0|2h - \frac{y \cdot \cos(yh)}{\sin(yh)} 2h = |a_1|h + |a_2|2h \\
&\Rightarrow \frac{1}{2}|a_1| - |a_0| = \frac{y \cdot \cos(yh)}{\sin(yh)} + \frac{1}{2}|a_1| + |a_2| \\
&\Rightarrow -|a_0| = \frac{y \cdot \cos(yh)}{\sin(yh)} + |a_2|.
\end{aligned} \tag{81}$$

But expression (81) coincides with condition (ii) in Theorem 4.2.

The main utility of Theorem 6.1 is that it substitutes one of the system's parameters appearing in condition (ii), Theorem 4.2, (i.e., a_2) by another one (i.e., a_1). Thus, Theorem 6.1 can be useful for evaluating asymptotic stability of the system when a_2 is not available, or the use of a_1 is more suitable for some design reason.

Theorem 6.2. *Suppose that $a_0 \in R^-$, $a_1, a_2 \in R^+$. System (77) is asymptotically stable if the following condition holds for some $y \in [0, \pi/h)$:*

$$\frac{\xi(h)}{h} > \left[\cos(yh) - 1\right]\frac{2|a_2|}{|a_0|} - \frac{y}{|a_0|\sin(yh)}. \tag{82}$$

Proof. Condition (82) implies

$$\begin{aligned}\xi(h) &> -\frac{yh}{|a_0|\sin(yh)} + \left[\cos(yh) - 1\right]\frac{|a_2|}{|a_0|}2h \\ &\Rightarrow \frac{|a_1|h + |a_2|2h}{|a_0|} + \frac{|a_2|2h}{|a_0|}\left[1 - \cos(yh)\right] > -\frac{yh}{|a_0|\sin(yh)} \\ &\Rightarrow |a_1| > -\frac{y}{\sin(yh)} - 2|a_2|\cos(yh).\end{aligned} \tag{83}$$

But expression (83) coincide with condition (iii) in Theorem 4.2.

Similarly to Theorem 6.1, the main utility of Theorem 6.2 is that it substitutes one of the system's parameters appearing in condition (iii), Theorem 4.2, (i.e., a_1) by another one (i.e., a_0).

7. Conclusions. This note has introduced two important results for stabilizability of a class of integro-differential systems with one distributed delay, by using the delay-measure notation and an associated extended system. The concept of delay-measure allows to express stabilizability results in a very simple way. The delay-measure function can be implemented for computational purposes and permits to establish a study about in what measure the stability depends on the delay terms.

It also provides sufficient conditions for testing asymptotic stabilizability of a class of delay-differential systems with two point delays in their state. The conditions are given in terms of algebraic relations, useful for computer applications.

The control is given in terms of two independent control laws defined by using dynamic differential equations which contain point delays. The control laws are independent in the sense that each one of them are applied independently to the state equation (i.e., they are not interconnected).

Finally the note provides also two results for asymptotic stability of a class of linear delay-differential systems, with two point delays by using a delay-measure approach.

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C. F. Alastruey was born in the Basque Country in 1963. In 1981 he became Bahá'í. He received the degree of Physicist (area: Electronics and Control) from the University of the Basque Country in 1988, and the degree of Doctor in Physics from the same university in 1993. He is professor in the Department of Electricity and Electronics in the University of the Basque Country. His research interests are mathematical and computing modelling of functional differential equations, control and identification of delay-differential systems and control of industrial processes.

J. R. González de Mendívil was born in the Basque Country in 1963. He received the degree of Physicist (area: Electronics and Control) from the University of the Basque Country in 1987, and the degree of Doctor in Physics from the same university in 1993. He is professor in the Department of Electricity and Electronics in the University of the Basque Country. His research interests are the problems of dead-lock detection in computing systems and the simulation and modelling of dynamical systems.

F. Arjomandi was born in Iran in 1967 in a Bahá'í family. He received the degree of Technical Engineer (area: telecommunications) from the Polytechnical University of Catalonia. His research interests are modelling of electronic systems.