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SELF-GUESSING FUZZY CLASSIFIERS

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Abstract. A possible interpretation, in terms of fuzzy classification models (fuzzy classifiers), of one of the general principles of choosing a scientific theory – a consistency principle – is considered. Supervised self-guessing fuzzy classifiers are determined. A theorem on character of restrictions induced on a set of supervised fuzzy classifiers by a self-guessing requirement is proved. Feasible alternatives of using the self-guessing property while constructing supervised fuzzy classifiers are analyzed.

Key words: supervised fuzzy classifier, criterion of internal justification, self-consistency principle, self-guessing property, learning set, training sequence, extension of learning set.

Introduction. Classification models (classifications)represent an effective means of interpretation of facts and are applied practically in all branches of science. In essence, an adequate classification is the necessary initial condition of any scientific research (Rozova, 1986).

However, the problem of forming the classification model is not so simple, as it may seem at first sight. Just as in building up models of other types, a question of choice of that classification which interprets the experimental facts more naturally arises here. The real problem lies in the fact that any model (including a classification one) not agreed with experience may be "corrected" with the help of additional assumptions. That is why, the criterion of model (theory) agreement with observed facts was called a criterion of external justification by Einstein. Such criterion gives only the necessary but not sufficient condition of "rightness" of the model (theory).

The second criterion was defined as a criterion of internal justification. This criterion deals with the prerequisites of theory itself, with what could be called briefly, though not quite clearly, "the naturality" or "a logical simplicity" of the prerequisites (Einstein And Modern Physics, 1956), rather than with the attitude to the experimental material.

Developing the internal justification conception, Einstein introduced several nonempirical principles of verification of correctness of physical models (theories). They include, first of all, the principles of invariance and consistency. The invariance principle postulates the independence of the scientific knowledge on concrete conditions and on a cognizing subject. The consistency principle requires that models whose validity for a certain group of events was established previously should not be rejected with the emergence of a new more general theory (model), but they should be preserved as its limiting (local) form or a particular case.

The internal justification criterion was formulated by Einstein during his work at the creation of the special relativity theory. However, the further evolution of science manifested a much greater extent of generality of verification principles of scientific theories introduced by him. At present, one of the established scientific canons is the belief in that whatever reality is modelled the quality of the its models must be evaluated with the help of the same principles which Einstein formulated for the evaluation of physical theories (Barker, 1957).

In the given series of papers a possible interpretation of one of the above principle – the consistency principle – is considered in terms of fuzzy classification models. The specificity of such models lies in the fact that they reflect one of the first stages of the formation of scientific knowledge (Rozova, 1986). At this stage there are no other theories (models), as a rule. The classification models themselves are just the basis for creating such theories. As a result the direct interpretation of the consistency principle as a property of preserving meanings of previous theoretical structures as limiting (local) forms of new similar structures proves to be impossible here.

The primary character of (fuzzy) classification models with respect to other forms of the scientific knowledge dictates the necessity of interpreting the principle under consideration as a self-consistency principle. From the operational point of view it is most convenient to represent the self-consistency principle either in the form of a self-guessing property (for supervised fuzzy classifiers) or in the form of stability property (for unsupervised fuzzy classifiers).

On an informal level the property of self-guessing of a supervised fuzzy classifier means that having learned on any part of a learning set it (the classifier) is capable of reproducing (recognizing) correctly the elements of the remaining part of this set. In its turn, the stability of unsupervised fuzzy classifiers implies that slight (and admissible) variations of the initial sample cause only slight variations in the corresponding classification model.

In this paper, supervised fuzzy classifiers meeting the property of self-guessing, are considered. In Section 1 the necessary preliminary information and a system of definitions, used throughout the paper, are formulated. In Section 2 a theorem on the character of constraints imposed on the set of supervised fuzzy classifiers by the requirement of self-guessing is presented.

In this work the basic theoretical constructions, introduced first in (Wolpert, 1989), are used.

1. Basic notions and definitions. Let us assume that descriptions of objects, feeding the fuzzy classifiers input may be encoded with the help of $\mathbf{R}^m$ space elements, and the description of its outputs (responses, labels of the corresponding classes) – with the help of elements μ of the $M_{\mathbf{R}}$ set such that $\mu : \mathbf{R} \to [0,1]$. In terms of theory of fuzzy sets μ is a membership function of the fuzzy set on $\mathbf{R}$.

Then, m-dimensional supervised fuzzy classifier is an infinite set of continuous functional mappings (functions) translating

1. Set $\mathbf{R}^m * M_{\mathbf{R}} * \mathbf{R}^m$ into set $M_{\mathbf{R}}$.
2. Set $\mathbf{R}^m * M_{\mathbf{R}} * \mathbf{R}^m * M_{\mathbf{R}} * \mathbf{R}^m$ into set $M_{\mathbf{R}}$.

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n. Set $\mathbf{R}^{nm} * M_{\mathbf{R}}^{n} * \mathbf{R}^{m}$ into set $M_{\mathbf{R}}$.

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i. Set $\mathbf{R}^{im} * M_{\mathbf{R}}^{i} * \mathbf{R}^{m}$ into set $M_{\mathbf{R}}$.

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In other words, the m-dimensional supervised fuzzy classifier is a set of continuous functions $g\{i\} : \mathbf{R}^{im} * M_{\mathbf{R}}^{i} * \mathbf{R}^{m} \rightarrow M_{\mathbf{R}}$, $i \in I$, I is at most a countable set of indices. The last Cartesian product $\mathbf{R}^{m}$ in the argument part of the $g\{i\}$ functions corresponds to the input object with the unknown output characteristic (to a question), and the Cartesian product $\mathbf{R}^{nm} * M_{\mathbf{R}}^{n}$ – to the learning set θ. Further, subsets of the kind $\mathbf{R}^{m} * M_{\mathbf{R}}$ will be called data spaces, and number n will be called the order of the learning set.

We will assume that supervised fuzzy classifiers $G = [g\{i\}(\cdot)]_{i\in I}^{n}$, defined in such a way, always satisfy the following collection of properties:

1. Each function $g\{i\} \in G$ is invariant to all possible permutations of the data spaces.
2. For any $g\{i\} \in G$ if two data spaces have the same value for some elements from $\mathbf{R}^{m}$, they have the same values for the corresponding elements from $M_{\mathbf{R}}$,
3. (The criterion of consistency of the supervised fuzzy classifier with the experimental data). For any $g\{i\} \in G$, if the value of the question coincides with the value of some element from an $\mathbf{R}^{m}$ component of the learning set, then the output of $g\{i\}$ will coincide with the value of the corresponding element from an $M_{\mathbf{R}}$ component of this set.
4. The supervised m – dimensional classifier $G = [g\{i\}(\cdot)]_{i\in I}^{n}$ is defined iff $n > m$.

For exact formulation of the property of self-guessing we also need the following definitions.

Let $\theta \subset \mathbf{R}^{nm} * M_{\mathbf{R}}^{n}$ be a fixed learning set. For every supervised fuzzy classifier operating with the above set, let's define an arbitrary combination $z_{\theta} = (x_1, \mu_{x_1}, x_2, \mu_{x_2}, \ldots, x_i, \mu_{x_i}; x, \mu_x^{*})$, $i \in \{1, 2, \ldots, n\}$ of elements of its data spaces, the question and the response of the corresponding function $g\{i\}$ as a training sequence.

The order of such sequence is the number of data spaces in it.

If $\theta \subset \mathbf{R}^{nm} * M_{\mathbf{R}}^{n}$ is an arbitrary learning set, then as an initial extension of this set we shall define the collection Z_θ^H of all those and only those training sequences $z_\theta^H = (x_1, \mu_{x_1}, x_2, \mu_{x_2}, \ldots, x_k, \mu_{x_k}; x_j, \mu_{x_j})$, that $x_j = x_k$, $\mu_{x_j} = \mu_{x_k}$, $k = \overline{1,i}$, $i \in \{1, 2, \ldots, n\}$.

Let us fix the family of mapping $\Pi = \{\tau\}$ so that

$$\tau\colon\ \mathbf{R}^{km} * M_{\mathbf{R}}^{k} * \mathbf{R}^{m} * M_{\mathbf{R}} \to \mathbf{R}^{km} * M_{\mathbf{R}}^{k} * \mathbf{R}^{m} * M_{\mathbf{R}}, \quad k \in \{1, 2, \ldots, n\},$$

and all τ satisfy properties 1–3, characteristic of functions $g\{i\} \in G$.

The complete extension θ is a set Z_θ^Π including those and only those training sequences, which may be obtained from any z_θ^H with the help of an arbitrary mapping $\tau \in \Pi$.

For any element $z_\theta^\Pi \in Z_\theta^\Pi$ we shall define a set of its commutative extensions $Z = \{z_l\}_{l=1}^{i}$, $i \in \{1, 2, \ldots, n\}$ as follows:

$$\begin{aligned}
\text{if}\quad & z_\theta^\Pi = (x_1, \mu_{x_1}, x_2, \mu_{x_2}, \ldots, x_i, \mu_{x_i}; x, \mu_x), \quad \text{then} \\
& z_1 = (x, \mu_x, x_2, \mu_{x_2}, \ldots x_i, \mu_{x_i}; x_1, \mu_{x_1}), \\
& z_2 = (x_1, \mu_{x_1}, x, \mu_x, \ldots, x_i, \mu_{x_i}; x_2, \mu_{x_2}), \\
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& z_i = (x_1, \mu_{x_1}, x_2, \mu_{x_2}, \ldots, x_{i-1}, \mu_{x_{i-1}}, x, \mu_x; x_i, \mu_{x_i}).
\end{aligned}$$

Let us consider a set of training sequences obtained as a result of constructing all possible commutative extensions for all elements from Z_θ^Π. We will denote the above set by $\widehat{Z}_\theta^\Pi$ and call it a commutative extension of the learning set θ (by the fixed family of mapping $\Pi = \{\tau\}$).

Let $\widehat{Z}_\theta^\Pi$ be a commutative extension of θ. The fuzzy classifier $G = [g\{i\}(\cdot)]_{i \in I}^{n}$, operating with θ, will be called a self-guessing fuzzy classifier in that and only that case if all its functions $g\{i\}$ are consistent (in a sense of equality of values of the outputs) with the training sequences of the corresponding orders from $\widehat{Z}_\theta^\Pi$ (Wolpert, 1989).

2. Character of constraints, imposed on a set of supervised fuzzy classifiers by the requirement of self-guessing. Let's consider the set of all fuzzy self-guessing classifiers, operating with the fixed learning set θ. We will denote this set by $\mathrm{Sg}(\theta)$. Let also W_0 denote an arbitrary family of fuzzy classifiers obeying the system of constraints U_0, not including the requirement of self-guessing. Then $W_0 \cap \mathrm{Sg}(\theta)$ will correspond to the set of all self-guessing fuzzy classifiers, operating with θ and obeying the constraint system U_0. Under these conditions the following theorem holds:

Theorem. (A fuzzy analog of the corresponding assertion from (Wolpert, 1989)). *There is no such system of restrictions U_0, under which the set $W_0 \cap \mathrm{Sg}(\theta)$ contains exactly one element for any learning set θ.*

Proof. Let's fix some learning set θ and consider any $\theta' \supset \theta$. Let's denote by $G = [g\{i\}(\cdot)]_{i\in I}^{n}$ and $G' = [g'\{i\}(\cdot)]_{i\in I}^{n}$ arbitrary elements of sets $W_0 \cap \mathrm{Sg}(\theta)$ and $W_0 \cap \mathrm{Sg}(\theta')$ respectively (n and n' are the orders of learning sets θ and θ'). Let $(x, \mu_x) \in \theta'/\theta$ and $G'(\theta, x)$ $(G(\theta, x))$ denote the value of output of that function $g\{i\}$ $(g'\{i\})$ which corresponds to the order of learning set θ. Then, we can write $G'(\theta, x) = \mu_x, G(\theta, x) = \mu$ where μ is an unknown response of the G classifier to question x (in view of $(x, \mu_x) \notin \theta$).

Since $\theta' \supset \theta$ and θ' is arbitrary, we can always choose a pair (x, μ_x) so that $\mu_x \neq \mu$ at any μ. This means that we can always point out such $\theta' \supset \theta$ that

$$W_0 \cap \mathrm{Sg}(\theta) \neq W_0 \cap \mathrm{Sg}(\theta'), \quad \forall W_0. \tag{1}$$

Suppose now that in the condition of the theorem its conclusion doesn't hold. Then for any $\theta' \supset \theta$ according to the definition of commutative extension we have $\widehat{Z}_{\theta}^{\Pi} \subset \widehat{Z}_{\theta'}^{\Pi}$ and, therefore,

$$\mathrm{Sg}(\theta') \subset \mathrm{Sg}(\theta').$$

Hence

$$W_0 \cap \mathrm{Sg}(\theta') \subset W_0 \cap \mathrm{Sg}(\theta), \quad \forall W_0. \tag{2}$$

At the same time, according to our assumption the sets $W_0 \cap \mathrm{Sg}(\theta)$ and $W_0 \cap \mathrm{Sg}(\theta')$ contain exactly one element for any $\theta' \supset \theta$ and any W_0. Then from (2) it immediately follows:

$$W_0 \cap \mathrm{Sg}(\theta') = W_0 \cap \mathrm{Sg}(\theta), \quad \forall \theta' \supset \theta, \ \forall W_0. \tag{3}$$

Joint consideration of relationships (1) and (3) results in a contradiction, which proves the theorem.

Conclusion. The given theorem indicates, that adding some other different properties of fuzzy classifiers to the classifier property of being self-guessing leads to either overrestrictions (Card $(W_0 \cap \mathrm{Sg}(\theta)) = 0$) or underrestrictions (Card $(W \cap \mathrm{Sg}(\theta)) > 1$) from the point of the general problem of choosing the most valid classification model. The above theorem also specifies three possible alternatives of using the property of self-guessing in practice. In the first one, constructing the self-guessing classifier which is closest to obeying the restrictions system U_0. In the second one, constructing the fuzzy classifier obeying the restrictions system U_0 and which is closest to meet the self-guessing property. Finally, in the third one, constructing the fuzzy classifier neither perfectly self-guessing not perfectly obeying the restrictions system U_0, but whose distance (in some predefined sense) from the two above classifiers is minimal.

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