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A Group Decision Making Method with Interval-Valued Intuitionistic Fuzzy Preference Relations and Its Application in the Selection of Cloud Computing Vendors for SMEs

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Abstract. To solve the problem of choosing the appropriate cloud computing vendors in small and medium-sized enterprises (SMEs), this paper boils it down to a group decision making (GDM) problem. To facilitate the judgment, this paper uses preference relation as the decision making technology. Considering the situation where uncertain positive and negative judgments exist simultaneously, interval-valued intuitionistic fuzzy preference relations (IVIFPRs) are employed to express the decision makers' judgments. In view of the multiplicative consistency and consensus analysis, a new GDM algorithm with IVIFPRs is offered. To accomplish this goal, a new multiplicative consistency is first defined, which can avoid the limitations of the previous ones. Then, a programming model is built to check the consistency of IVIFPRs. To deal with incomplete IVIFPRs, two programming models are constructed to determine the missing values with the goal of maximizing the level of multiplicative consistency and minimizing the total uncertainty. To achieve the minimum adjustment of original preference information, a programming model is established to repair inconsistent IVIFPRs. In addition, programming models for getting the decision makers (DMs)' weights and improving the consensus degree are offered. Finally, a practical decision making example is given to illustrate the effectiveness of the proposed method and to compare it with previous methods.

Key words: group decision making, IVIFPR, multiplicative consistency, programming model, consensus.

1. Introduction

In the era of economic globalization, the increasingly fierce competitive environment is a common challenge faced by global small and medium-sized enterprises (SMEs). SMEs must actively participate in the network economy to survive and develop. Compared with the sizeable enterprises, SMEs lack funds and technology in enterprise informatization construction (IC). To a certain extent, this limits the implementation effect of enterprise

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informatization. In recent years, with the rapid development of cloud computing (CC) technology, it opens up a new avenue for the application of information technology in SMEs. Many information processing-related businesses in CC model no longer install and execute on local computers. They run on remote servers, known as the cloud, and the results are offered to users as professional services. In this way, CC technology can effectively alleviate the contradiction between the demand of SMEs' personalized information processing and the higher IC investment. This revolutionary technology is prompting more and more information technology (IT) service companies to adjust their service patterns by serving customers in the cloud. Facing the numerous CC vendors in the market, how to choose the most appropriate one has become an important decision-making problem for SMEs, which will greatly affect their business performance and competitiveness.

CC vendors selection can be treated as a typical multi-criteria decision-making (MCDM) problem since it deals with the task of ranking a finite number of alternatives under different criteria. At present, the research on MCDM has achieved fruitful results. Among of which technique for order of preference by similarity to ideal solution (TOPSIS) and analytic hierarchy process (AHP) are the most of widely used decision making technologies. Based on the transformation between interval-valued intuitionistic fuzzy variables (IVIFVs) and linguistic variables (LVs), Büyüközkan *et al.* (2018) proposed a hybrid method consisting of AHP, complex proportional assessment (COPRAS), multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA), TOPSIS, and višekriterijumsko kompromisno rangiranje (VIKOR) to rank CC vendors. Jatoth *et al.* (2018) presented a hybrid MCDM approach containing TOPSIS and AHP for the selection of cloud services. Onar *et al.* (2018) provided a Pythagorean fuzzy TOPSIS method to select the proper cloud service provider by combining a variety of tangible and intangible criteria. Based on AHP scale mapping methodology, Meesariganda and Ishizaka (2017) provided an approach to evaluate cloud storage strategies for a company. Repschlaeger *et al.* (2014) described an AHP-based approach to support IT organization for selecting an appropriate CC vendor. By integrating AHP, quality function deployment (QFD) and multi-choice goal programming technologies, Liao and Kao (2014) presented a method to address the CC vendor selection problem in information service. Garg *et al.* (2013) proposed a framework that measures the quality of cloud service and prioritizes CC vendors by applying AHP. Moreover, Sohaib *et al.* (2019) utilized a TOPSIS based 2-tuple fuzzy linguistic multi-criteria group decision-making method to facilitate assessing and choosing the CC models for SMEs.

From the above literature review, one can see that previous research about the CC vendor selection problem mainly limits to decision matrix that needs DMs to offer the absolute evaluation of each object for every criterion or attribute. However, it is difficult or impossible for DMs due to various reasons such as the pressure of time, the limitations of expertise, and the complexity of the selection of CC vendors. In addition, it is difficult to establish a perfect index system to carry out quantitative evaluation on the CC vendors. Methods with preference relations (PRs) which only require the DMs to compare two objects at one time and permit the existence of missing judgments provide a convenient tool to solve complex MCDM problems. Since Saaty (1977) first proposed multiplicative PRs

(MPRs), decision making based on PRs has made great success. In view of the DMs' indeterminacies, different types of PRs are introduced such as interval multiplicative PRs (IMPRs) (Krejčí, 2017), fuzzy interval PRs (FIPRs) (Barrenechea *et al.*, 2014) and interval linguistic PRs (ILPRs) (Tapia García *et al.*, 2012). The common feature of PRs mentioned above is to consider only the preferred information of one object over another. Sometimes this might be insufficient. In addition to the preferred information, non-preferred judgments are also needed to fully express the recognition of DMs. For example, a DM may think that one object has a preferred degree of 0.6 over another. However, the judgment of non-preferred degree is 0.3 rather than 0.4, namely, the hesitancy degree between them is 0.1. In this situation, intuitionistic fuzzy values (IFVs) proposed by Szmidt and Kacprzyk (2002) are powerful tools. However, IFVs only allow the DM to employ numeric values [0, 1] to express their exact preferred and non-preferred judgments, which may be still insufficient. In the above example, if the DM's preferred degree is between 0.6 and 0.8 and the non-preferred degree is between 0.1 and 0.2, IFV is useless. To denote the uncertainty of DMs' judgments, Xu (2007a) further proposed interval-valued intuitionistic fuzzy values (IVIFVs) and introduced interval-valued intuitionistic fuzzy preference relations (IVIFPRs). Meanwhile, considering the complexity of decision-making problems, group decision making (GDM) is necessary to avoid the possible decision bias caused by a single DM. Considering the advantages of PRs and GDM, this paper introduces a GDM method based on IVIFPRs to solve the problem of CC vendors selection for SMEs through pairwise comparison among objects under the experts' comprehensive consideration of the evaluation criteria.

In decision making with PRs, the lack of consistency will lead to unreasonable conclusions. Therefore, it is critical to study the consistency of IVIFPRs. Motivated by Xu and Chen's method (2008) for deriving the priority vector from FIPRs, Wang *et al.* (2009) used the normalized interval weight vector to propose an additive consistency concept for IVIFPRs. According to the additive consistency concept for FIPRs, Wan *et al.* (2017) defined an additive consistency concept for IVIFPRs and then presented a GDM method. Wan *et al.* (2018) defined another additive consistency concept for IVIFPRs by transforming an IVIFPR into an IFPR. Recently, Tang *et al.* (2018) analysed the limitations of previous additive consistency concepts for IVIFPRs. Using 2-tuple preferred FIPRs (2TPFIPRs) and quasi 2TPFIPRs (Q2TPFIPRs), the authors proposed a new definition for additive consistent IVIFPRs.

In addition to the above additive consistency concepts for IVIFPRs, there are some researches on the multiplicative consistency of IVIFPRs. For instance, based on IVIFVs' operational laws, Xu and Cai (2009) proposed a multiplicative consistency concept for IVIFPRs and developed a method for estimating missing values. Later, according to the multiplicative transitivity for fuzzy PRs (Tanino, 1984), Xu and Cai (2015) introduced another multiplicative consistency concept for IVIFPRs and developed an approach to GDM with incomplete IVIFPRs. Similar to the multiplicative consistency concept in Xu and Cai (2015), Liao *et al.* (2014) provided a multiplicative consistency concept for IVIFPRs and investigated inconsistent IVIFPRs. To improve the consistency and consensus level, the authors proposed two iterative algorithms. On the basis of Liao and Xu's multiplicative consistency definition for IFPRs (Liao and Xu, 2014) and the induced matrices obtained from IVIFPRs, Wan *et al.* (2016) proposed another definition for multiplicative consistent IVIFPRs and introduced a GDM method. Meng *et al.* (2018) analysed the issues of multiplicative consistency concepts for IVIFPRs and proposed a new definition for multiplicatively consistent IVIFPRs. Based on the discussions on the multiplicative consistency and consensus, an algorithm for GDM with IVIFPRs is proposed. Other researches on decision making with IVIFPRs can be found in the literature (Wu and Chiclana, 2012; Zhou *et al.*, 2018).

Through the above literature review about IVIFPRs, we find that there are some aspects of research on decision making with IVIFPRs that can be further improved. Some methods do not calculate the priority vector based on the consistency analysis, which may lead to illogical ranking results. Although some methods are based on the consistency analysis, the consistency conclusion is dependent on compared orders of objects. Due to the complexity of decision-making problems and the limitation of DMs' subjective cognition, incomplete information may occur in IVIFPRs, which is not covered in some of the literature. Moreover, the iterative method for consistency and consensus improvement may change the initial preference information of DMs greatly, and the procedure for calculating the priority vector is complex in some of the literature.

Considering these facts, this paper offers a new method for GDM with IVIFPRs and provides an effective way for SMEs to choose the appropriate CC products. The main contributions of this paper include:

- (i) A new multiplicative consistency definition for IVIFPRs is proposed and a programming model is built to check the consistency.
- (ii) For incomplete IVIFPRs, two programming models are constructed to determine unknown preference information, which aim at both maximizing the multiplicative consistency level and minimizing the total uncertain degree.
- (iii) To repair inconsistent IVIFPRs, a goal programming model for deriving associated multiplicatively consistent IVIFPRs is established, which minimizes the total adjustment.
- (iv) In GDM, a programming model is established to improve individual consensus level, which endows different IVIFVs with different adjustments and minimizes the adjustment of individual IVIFPRs to retain more original information.
- (v) A GDM algorithm with IVIFPRs based on the multiplicative consistency and consensus analysis is proposed, which can avoid the limitations of previous ones.
- (vi) The application of the new method in selecting the appropriate CC vendors for SMEs is discussed, which provides a new path for solving this important problem.

This paper is arranged as follows. Section 2 introduces some related basic concepts and analyses several multiplicative consistency definitions. Section 3 proposes a new multiplicative consistency definition for IVIFPRs that can avoid the issues of existing ones. Meanwhile, a programming model is built to check the multiplicative consistency of IVIFPRs. Section 4 tackles incomplete and inconsistent IVIFPRs. First, two programming models are constructed to determine missing values in incomplete IVIFPRs, and then a programming model is presented for repairing inconsistent IVIFPRs. Section 5 discusses GDM with incomplete and inconsistent IVIFPRs. It first gives a programming model to determine the DMs' weights and offers a formula to measure the consensus degree. Then, a programming model is provided to reach the given consensus threshold. After that, the concrete algorithm for GDM with IVIFPRs is put forward. Section 6 applies the proposed method to solve the selection of CC products for SMEs and to compare the new method with previous ones. Conclusions and future remarks are conducted in Section 7.

2. Basic Concepts

To make the following discussions easily, the section reviews some related concepts.

Definition 1 (*See* Xu, 2004). A FIPR $\overline{A} = (\overline{a}_{ij})_{n\times n}$ on the finite set $X = \{x_1, x_2, \ldots, x_n\}$ is an interval fuzzy binary relation, characterized by an interval fuzzy subset of $X \times X$, i.e. $\overline{\mu}_{\overline{A}} : X \times X \to D[0,1]$ such that $D[0,1]$ is the set of all possible intervals in $[0,1]$, where the interval $\overline{a}_{ij} = \overline{\mu}_{\overline{A}}(x_i, x_j)$ is the interval preferred degree of the object x_i over x_j. Furthermore, its elements satisfy

$$\begin{cases} \bar{a}_{ij} = [a_{ij}^L, a_{ij}^U] \subseteq [0,1], \\ a_{ij}^L \leqslant a_{ij}^U, \\ a_{ij}^L + a_{ji}^U = a_{ij}^U + a_{ji}^L = 1, \end{cases} \tag{1}$$

where $i, j = 1, 2, \ldots, n$.

To ensure the rationality of decision-making results, many scholars devoted themselves to the investigation of the consistency of FIPRs. After comparing and analysing the previous multiplicative consistency concepts, Meng *et al.* (2017) introduced the below multiplicative consistency concept for FIPRs.

Definition 2 (*See* Meng *et al.*, 2017). Let $\overline{A} = (\overline{a}_{ij})_{n\times n}$ be a FIPR. $\overline{A}$ is multiplicatively consistent if there is an associated multiplicatively consistent quasi FIPR (QFIPR) $\overline{B} = (\overline{b}_{ij})_{n\times n}$, namely,

$$\bar{b}_{ij} \otimes \bar{b}_{jk} \otimes \bar{b}_{ki} = \bar{b}_{ji} \otimes \bar{b}_{ik} \otimes \bar{b}_{kj} \tag{2}$$

for all $i, k, j = 1, 2, \ldots, n$, where $\begin{cases} \bar{b}_{ij} = [a_{ij}^L, a_{ij}^U] \\ \bar{b}_{ji} = [a_{ji}^U, a_{ji}^L] \end{cases}$ or $\begin{cases} \bar{b}_{ij} = [a_{ij}^U, a_{ij}^L] \\ \bar{b}_{ji} = [a_{ji}^L, a_{ji}^U] \end{cases}$ for all $i, j = 1, 2, \ldots, n$.

From Definition 2, we can see that Meng *et al.*'s concept requires the endpoints of interval judgments to satisfy the multiplicative transitivity. In contrast to Definition 2, Krejčí (2019) offered another multiplicative consistency concept for FIPRs based on the constrained operations on intervals (Lodwick and Jenkins, 2013) that does not restrict to the endpoints of interval judgments.

Definition 3 (*See* Krejčí, 2019). Let $\overline{A} = (\overline{a}_{ij})_{n\times n}$ be a FIPR. $\overline{A}$ is multiplicatively consistent if

$$\forall a_{ij} \in \bar{a}_{ij}, \exists a_{ik} \in \bar{a}_{ik} \wedge a_{kj} \in \bar{a}_{kj} \quad \Rightarrow \quad a_{ij} = \frac{a_{ik}a_{kj}}{1 - a_{ik} - a_{kj} + 2a_{ik}a_{kj}} \tag{3}$$

is true for all $i, k, j = 1, 2, \dots, n$.

By the reciprocity of FIPRs, one can check that formula (3) is equivalent to

$$\begin{cases} \dfrac{a_{ij}^L}{1 - a_{ij}^L} \geqslant \dfrac{a_{ik}^L a_{kj}^L}{(1 - a_{ik}^L)(1 - a_{kj}^L)}, \\ \dfrac{a_{ij}^U}{1 - a_{ij}^U} \leqslant \dfrac{a_{ik}^U a_{kj}^U}{(1 - a_{ik}^U)(1 - a_{kj}^U)} \end{cases} \tag{4}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

From formulae (2) and (4), one can conclude that Definition 2 can be seen as a special case of Definition 3. Just as Meng *et al.* (2019) noted, Definition 3 is more flexible than Definition 2.

To simultaneously express both the preferred and non-preferred information of one object over the other, Szmidt and Kacprzyk (2002) proposed the concept for IFVs. Following this, Xu (2007b) gave the definition for IFPRs.

Definition 4 (*See* Xu, 2007b). An IFPR R on the finite set $X = \{x_1, x_2, \dots, x_n\}$ is presented by a matrix $R = (r_{ij})_{n\times n}$ such that $r_{ij} = (\mu_{ij}, v_{ij})$ is an IFV, which denotes the intuitionistic fuzzy preference of the object x_i over x_j, $i, j = 1, 2, \dots, n$. Furthermore, its elements satisfy the following conditions:

$$\begin{cases} \mu_{ij}, v_{ij} \geqslant 0, \quad \mu_{ij} + v_{ij} \leqslant 1, \\ \mu_{ji} = v_{ij}, \quad v_{ji} = \mu_{ij}, \quad \mu_{ii} = v_{ii} = 0.5, \end{cases} \tag{5}$$

where $i, j = 1, 2, \dots, n$.

To reflect the uncertainty of DMs' judgments on preferred and non-preferred information, Atanassov and Gargov (1989) further introduced IVIFSs, and Xu (2007a) proposed IVIFVs for facilitating the application. Based on IVIFVs, Xu (2007c) offered the concept of IVIFPRs.

Definition 5 (*See* Xu, 2007c). An IVIFPR $\tilde{R}$ on the finite set $X = \{x_1, x_2, \dots, x_n\}$ is represented by a matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ such that $\tilde{r}_{ij} = (\overline{\mu}_{ij}, \overline{v}_{ij})$ is an IVIFV, which denotes the interval-valued intuitionistic fuzzy preference of the object x_i over x_j, $i, j = 1, 2, \dots, n$. Furthermore, its elements satisfy the following conditions:

$$\begin{cases} \bar{\mu}_{ij} = [\mu_{l,ij}, \mu_{u,ij}] \subseteq [0, 1], \quad \bar{v}_{ij} = [v_{l,ij}, v_{u,ij}] \subseteq [0, 1], \\ \bar{\mu}_{ji} = \bar{v}_{ij}, \quad \bar{v}_{ji} = \bar{\mu}_{ij}, \quad \mu_{u,ij} + v_{u,ij} \leqslant 1, \\ \bar{\mu}_{ii} = \bar{v}_{ii} = [0.5, 0.5], \end{cases} \tag{6}$$

where $i, j = 1, 2, \dots, n$.

Regarding IVIFPRs' multiplicative consistency, Liao *et al.* (2014) introduced a concept on the basis of multiplicative transitivity for fuzzy PRs (Tanino, 1984). By applying Liao and Xu's multiplicative consistency definition for IFPRs (Liao and Xu, 2014), Wan *et al.* (2016) also offered a multiplicative consistency concept for IVIFPRs. The common limitation of the two concepts is that they do not satisfy the robustness, which means contradictory conclusions about consistency might be obtained under different comparison orders between objects. Furthermore, different ranking orders might be derived too.

Example 1. Let the IVIFPR $\tilde{R}$ on $X = \{x_1, x_2, x_3\}$ be defined as follows:

$$\tilde{R} = \begin{array}{c} \begin{matrix} \quad x_1 \quad & \quad x_2 \quad & \quad x_3 \quad \end{matrix} \\ \begin{pmatrix} ([1/2, 1/2], [1/2, 1/2]) & ([1/4, 1/3], [2/5, 1/2]) & ([1/5, 1/3], [2/17, 1/5]) \\ ([2/5, 1/2], [1/4, 1/3]) & ([1/2, 1/2], [1/2, 1/2]) & ([3/7, 1/2], [1/6, 1/5]) \\ ([2/17, 1/5], [1/5, 1/3]) & ([1/6, 1/5], [3/7, 1/2]) & ([1/2, 1/2], [1/2, 1/2]) \end{pmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \end{array}.$$

One can check that $\tilde{R}$ is multiplicatively consistent according to the consistency definition in Liao *et al.* (2014). On the other hand, let σ be a permutation on the labels of objects, where $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 2$. Then, we get the below IVIFPR

$$\tilde{R}^{\sigma} = \begin{array}{c} \begin{matrix} \quad x_2 \quad & \quad x_3 \quad & \quad x_1 \quad \end{matrix} \\ \begin{pmatrix} ([1/2, 1/2], [1/2, 1/2]) & ([3/7, 1/2], [1/6, 1/5]) & ([2/5, 1/2], [1/4, 1/3]) \\ ([1/6, 1/5], [3/7, 1/2]) & ([1/2, 1/2], [1/2, 1/2]) & ([2/17, 1/5], [1/5, 1/3]) \\ ([1/4, 1/3], [2/5, 1/2]) & ([1/5, 1/3], [2/17, 1/5]) & ([1/2, 1/2], [1/2, 1/2]) \end{pmatrix} \begin{matrix} x_2 \\ x_3 \\ x_1 \end{matrix} \end{array},$$

which is inconsistent following Liao *et al.*'s consistency definition. However, the IVIFPR $\tilde{R}$ and $\tilde{R}^{\sigma}$ are identical for the compared objects x_1, x_2 and x_3. Thus, the multiplicative consistency definition in Liao *et al.* (2014) is insufficient to judge the consistency of IVIFPRs. Similarly, we can illustrate the limitation of the consistency definition proposed in Wan *et al.* (2016).

In contrast to the above two multiplicative consistency concepts, Meng *et al.* (2018) defined 2TPFIPRs by which the authors derived a new multiplicative consistency concept.

Definition 6 (*See* Meng *et al.*, 2018). Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIFPR, where $\tilde{r}_{ij} = ([\mu_{l,ij}, \mu_{u,ij}], [v_{l,ij}, v_{u,ij}])$ for all $i, j = 1, 2, \dots, n$. $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is called a 2TPFIPR, where $\tilde{p}_{ij} = ([\mu_{l,ij}, 1 - v_{l,ij}], [\mu_{u,ij}, 1 - v_{u,ij}])$ denotes the interval possible preferred degree of the object x_i over x_j for all $i, j = 1, 2, \dots, n$.

Corresponding to 2TPFIPRs, the concept for Q2TPFIPRs is offered as follows:

Definition 7 (*See* Meng *et al.*, 2018). Let $\tilde{R}$ be an IVIFPR and $\tilde{P}$ be its 2TPFIPR as shown in Definition 6. $\tilde{S} = (\tilde{s}_{ij})_{n\times n}$ is called a Q2TPFIPR if its elements satisfy one of the following four cases:

$$\begin{array}{ll}\text{(i)} & \begin{cases}\tilde{s}_{ij} = ([\mu_{l,ij}, 1-v_{l,ij}], [\mu_{u,ij}, 1-v_{u,ij}]),\\ \tilde{s}_{ji} = ([1-\mu_{l,ij}, v_{l,ij}], [1-\mu_{u,ij}, v_{u,ij}]),\end{cases}\\ \text{(ii)} & \begin{cases}\tilde{s}_{ij} = ([1-v_{l,ij}, \mu_{l,ij}], [\mu_{u,ij}, 1-v_{u,ij}]),\\ \tilde{s}_{ji} = ([v_{l,ij}, 1-\mu_{l,ij}], [1-\mu_{u,ij}, v_{u,ij}]),\end{cases}\\ \text{(iii)} & \begin{cases}\tilde{s}_{ij} = ([\mu_{l,ij}, 1-v_{l,ij},], [1-v_{u,ij}, \mu_{u,ij}]),\\ \tilde{s}_{ji} = ([1-\mu_{l,ij}, v_{l,ij}], [v_{u,ij}, 1-\mu_{u,ij}]),\end{cases}\\ \text{(iv)} & \begin{cases}\tilde{s}_{ij} = ([1-v_{l,ij}, \mu_{l,ij}], [1-v_{u,ij}, \mu_{u,ij}]),\\ \tilde{s}_{ji} = ([v_{l,ij}, 1-\mu_{l,ij}], [v_{u,ij}, 1-\mu_{u,ij}])\end{cases}\end{array} \tag{7}$$

for all $i, j = 1, 2, \dots, n$.

Definition 7 indicates that the Q2TPFIPR $\tilde{S} = (\tilde{s}_{ij})_{n\times n}$ is composed by two QFIPRs $\overline{\eta} = (\overline{\eta}_{ij})_{n\times n}$ and $\overline{\lambda} = (\overline{\lambda}_{ij})_{n\times n}$, where

$$\begin{array}{l}\begin{cases}\bar{\eta}_{ij} = [\mu_{l,ij}, 1-v_{l,ij}]\\ \bar{\eta}_{ji} = [1-\mu_{l,ij}, v_{l,ij}]\end{cases} \vee \begin{cases}\bar{\eta}_{ij} = [1-v_{l,ij}, \mu_{l,ij}]\\ \bar{\eta}_{ji} = [v_{l,ij}, 1-\mu_{l,ij}],\end{cases}\\ \begin{cases}\bar{\lambda}_{ij} = [\mu_{u,ij}, 1-v_{u,ij}]\\ \bar{\lambda}_{ji} = [1-\mu_{u,ij}, v_{u,ij}]\end{cases} \vee \begin{cases}\bar{\lambda}_{ij} = [1-v_{u,ij}, \mu_{u,ij}]\\ \bar{\lambda}_{ji} = [v_{u,ij}, 1-\mu_{u,ij}]\end{cases}\end{array} \tag{8}$$

for all $i, j = 1, 2, \dots, n$.

Based on the above fact and Definition 2, Meng *et al.* (2018) gave the below multiplicative consistency concept for Q2TPFIPRs:

Definition 8 (*See* Meng *et al.*, 2018). Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIFPR and $\tilde{S} = (\tilde{s}_{ij})_{n\times n}$ be its Q2TPFIPR. $\tilde{S}$ is multiplicatively consistent if the QFIPRs $\overline{\eta} = (\overline{\eta}_{ij})_{n\times n}$ and $\overline{\lambda} = (\overline{\lambda}_{ij})_{n\times n}$ as shown in formula (8) are both multiplicatively consistent, namely,

$$\begin{cases}\bar{\eta}_{ij} \otimes \bar{\eta}_{jk} \otimes \bar{\eta}_{ki} = \bar{\eta}_{ji} \otimes \bar{\eta}_{ik} \otimes \bar{\eta}_{kj},\\ \bar{\lambda}_{ij} \otimes \bar{\lambda}_{jk} \otimes \bar{\lambda}_{ki} = \bar{\lambda}_{ji} \otimes \bar{\lambda}_{ik} \otimes \bar{\lambda}_{kj}\end{cases} \tag{9}$$

for all $i, k, j = 1, 2, \dots, n$.

Based on the multiplicative consistency of Q2TPFIPRs, Meng *et al.* (2018) further proposed the following multiplicative consistency concept for IVIFPRs:

Definition 9 (*See* Meng *et al.*, 2018). Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIFPR and $\tilde{S} = (\tilde{s}_{ij})_{n\times n}$ be its Q2TPFIPR. If $\tilde{S}$ is multiplicatively consistent following Definition 8, then $\tilde{R}$ is multiplicatively consistent.

For example, the IVIFPR:

$$\tilde{R}=\begin{pmatrix} ([0.50,0.50],[0.50,0.50]) & ([0.46,0.46],[0.52,0.52]) & ([0.57,0.60],[0.35,0.36]) & ([0.70,0.70],[0.29,0.29]) \\ ([0.52,0.52],[0.46,0.46]) & ([0.50,0.50],[0.50,0.50]) & ([0.59,0.62],[0.32,0.33]) & ([0.73,0.73],[0.27,0.27]) \\ ([0.35,0.36],[0.57,0.60]) & ([0.32,0.33],[0.59,0.62]) & ([0.50,0.50],[0.50,0.50]) & ([0.55,0.56],[0.35,0.38]) \\ ([0.29,0.29],[0.70,0.70]) & ([0.27,0.27],[0.73,0.73]) & ([0.35,0.38],[0.55,0.56]) & ([0.50,0.50],[0.50,0.50]) \end{pmatrix}$$

is multiplicatively consistent according to Definition 9, since there are two corresponding multiplicatively consistent QFIPRs as follows:

$$\bar{\eta}=\begin{pmatrix} [0.50,0.50] & [0.48,0.46] & [0.57,0.65] & [0.71,0.70] \\ [0.52,0.54] & [0.50,0.50] & [0.59,0.68] & [0.73,0.73] \\ [0.43,0.35] & [0.41,0.32] & [0.50,0.50] & [0.65,0.55] \\ [0.29,0.30] & [0.27,0.27] & [0.35,0.45] & [0.50,0.50] \end{pmatrix},$$

$$\bar{\lambda}=\begin{pmatrix} [0.50,0.50] & [0.46,0.48] & [0.64,0.60] & [0.70,0.71] \\ [0.54,0.52] & [0.50,0.50] & [0.67,0.62] & [0.73,0.73] \\ [0.36,0.40] & [0.33,0.38] & [0.50,0.50] & [0.56,0.62] \\ [0.30,0.29] & [0.27,0.27] & [0.44,0.38] & [0.50,0.50] \end{pmatrix}.$$

From Definition 9, one can find that Meng *et al.*'s multiplicative consistency concept is based on Definition 2. Although it can avoid the limitations of the multiplicative concepts in Liao *et al.* (2014), Wan *et al.* (2016), the main issue of Definition 9 is not flexible enough (Meng *et al.*, 2019).

3. A New Multiplicative Consistency Concept for IVIFPRs

Considering the limitations of previous multiplicative consistency concepts for IVIFPRs, this section introduces a new one based on Krejčí's multiplicative consistency concept (Krejčí, 2019) for FIPRs.

Let $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ be an IVIFPR, and let $\tilde{P}$ be its 2TPFIPR as shown in Definition 6. One can find that the 2TPFIPR $\tilde{P}$ is composed by the matrices $\overline{P}^L=(\overline{p}^L_{ij})_{n\times n}$ and $\overline{P}^U=(\overline{p}^U_{ij})_{n\times n}$, where

$$\begin{cases} \bar{p}^L_{ij}=[\mu_{l,ij},1-v_{l,ij}], \\ \bar{p}^U_{ij}=[\mu_{u,ij},1-v_{u,ij}] \end{cases} \tag{10}$$

for all $i,j=1,2,\dots,n$.

Following the concept for IVIFPRs and formula (10), we have

$$\begin{cases} \bar{p}^L_{ij}=[\mu_{l,ij},1-v_{l,ij}]\subseteq[0,1], \\ \bar{p}^L_{ji}=[\mu_{l,ji},1-v_{l,ji}]\subseteq[0,1], \\ \mu_{l,ji}+1-v_{l,ij}=v_{l,ij}+1-v_{l,ij}=1, \\ \mu_{l,ij}+1-v_{l,ji}=\mu_{l,ij}+1-\mu_{l,ij}=1, \end{cases} \quad \text{and}$$

$$\begin{cases} \bar{p}_{ij}^U = [\mu_{u,ij}, 1 - \upsilon_{u,ij}] \subseteq [0, 1], \\ \bar{p}_{ji}^U = [\mu_{u,ji}, 1 - \upsilon_{u,ji}] \subseteq [0, 1], \\ \mu_{u,ji} + 1 - \upsilon_{u,ij} = \upsilon_{u,ij} + 1 - \upsilon_{u,ij} = 1, \\ \mu_{u,ij} + 1 - \upsilon_{u,ji} = \mu_{u,ij} + 1 - \mu_{u,ij} = 1 \end{cases} \tag{11}$$

for all $i, j = 1, 2, \ldots, n$. Therefore, $\overline{P}^L = (\overline{p}_{ij}^L)_{n\times n}$ and $\overline{P}^U = (\overline{p}_{ij}^U)_{n\times n}$ are FIPRs.

Based on the above relationship, we employ the multiplicative consistent FIPRs to define multiplicative consistent 2TPFIPRs.

Definition 10. Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be the 2TPFIPR of the IVIFPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$. $\tilde{P}$ is multiplicatively consistent if the FIPRs $\overline{P}^L = (\overline{p}_{ij}^L)_{n\times n}$ and $\overline{P}^U = (\overline{p}_{ij}^U)_{n\times n}$ shown in formula (10) are both multiplicatively consistent, namely,

$$\begin{cases} \forall p_{ij}^L \in \bar{p}_{ij}^L, \exists p_{ik}^L \in \bar{p}_{ik}^L \wedge p_{kj}^L \in \bar{p}_{kj}^L \quad \Rightarrow \quad p_{ij}^L = \dfrac{p_{ik}^L p_{kj}^L}{1 - p_{ik}^L - p_{kj}^L + 2p_{ik}^L p_{kj}^L}, \\ \forall p_{ij}^U \in \bar{p}_{ij}^U, \exists p_{ik}^U \in \bar{p}_{ik}^U \wedge p_{kj}^U \in \bar{p}_{kj}^U \quad \Rightarrow \quad p_{ij}^U = \dfrac{p_{ik}^U p_{kj}^U}{1 - p_{ik}^U - p_{kj}^U + 2p_{ik}^U p_{kj}^U} \end{cases} \tag{12}$$

for all $i, k, j = 1, 2, \ldots, n$.

Following formula (4) and Definition 10, one can easily derive the below theorem:

Theorem 1. *Let $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ be the* 2TPFIPR *of the* IVIFPR $\tilde{R}$, *$\tilde{P}$ is multiplicatively consistent according to Definition 10 if and only if the following is true, where*

$$\begin{cases} \dfrac{\mu_{l,ij}}{1 - \mu_{l,ij}} \geqslant \dfrac{\mu_{l,ik}\mu_{l,kj}}{(1 - \mu_{l,ik})(1 - \mu_{l,kj})}, \\ \dfrac{\upsilon_{l,ij}}{1 - \upsilon_{l,ij}} \geqslant \dfrac{\upsilon_{l,ik}\upsilon_{l,kj}}{(1 - \upsilon_{l,ik})(1 - \upsilon_{l,kj})}, \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\mu_{u,ij}}{1 - \mu_{u,ij}} \geqslant \dfrac{\mu_{u,ik}\mu_{u,kj}}{(1 - \mu_{u,ik})(1 - \mu_{u,kj})}, \\ \dfrac{\upsilon_{u,ij}}{1 - \upsilon_{u,ij}} \geqslant \dfrac{\upsilon_{u,ik}\upsilon_{u,kj}}{(1 - \upsilon_{u,ik})(1 - \upsilon_{u,kj})} \end{cases} \tag{13}$$

for all $i, k, j = 1, 2, \ldots, n$ with $k \neq i, j \wedge i < j$.

Theorem 1 shows that the multiplicative consistency of 2TPFIPRs is based on the multiplicative transitivity of the endpoints of IVIFVs in associated IVIFPRs. Thus, we can adopt Definition 10 to further define the multiplicative consistency of IVIFPRs.

Definition 11. Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an IVIFPR. $\tilde{R}$ is multiplicatively consistent if its 2TPFIPR $\tilde{P} = (\tilde{p}_{ij})_{n\times n}$ is multiplicatively consistent according to Definition 10, namely, its elements satisfy formula (13).

According to the independence of Definition 3 for the compared orders, it can be easily checked that Definition 11 is invariant for the permutation of objects.

In the following, we study the relationship between Definitions 9 and 11 to show the flexibility of the new concept.

Theorem 2. *Let* $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ *be an* IVIFPR. *When* $\tilde{R}$ *is multiplicatively consistent following Definition* 9*, then it is multiplicatively consistent based on Definition* 11.

Proof. When $\tilde{R}$ is multiplicatively consistent following Definition 9, according to formulae (3.4) and (3.5) in Meng *et al.* (2018), we have:

$$\begin{aligned}
&\text{(i)} \begin{cases} \dfrac{\mu_{l,ji}\mu_{l,ik}\mu_{l,kj}}{(1-\mu_{l,ji})(1-\mu_{l,ik})(1-\mu_{l,kj})} = 1, \\ \dfrac{\upsilon_{l,ji}\upsilon_{l,ik}\upsilon_{l,kj}}{(1-\upsilon_{l,ji})(1-\upsilon_{l,ik})(1-\upsilon_{l,kj})} = 1, \end{cases} \\
&\text{(ii)} \begin{cases} \dfrac{\mu_{l,ij}}{(1-\mu_{l,ij})} = \dfrac{\mu_{l,ik}\mu_{l,kj}}{(1-\mu_{l,ik})(1-\mu_{l,kj})}, \\ \dfrac{\upsilon_{l,ij}}{(1-\upsilon_{l,ij})} = \dfrac{\upsilon_{l,ik}\upsilon_{l,kj}}{(1-\upsilon_{l,ik})(1-\upsilon_{l,kj})}, \end{cases} \\
&\text{(iii)} \begin{cases} \dfrac{\mu_{l,ik}}{(1-\mu_{l,ik})} = \dfrac{\mu_{l,ij}\mu_{l,jk}}{(1-\mu_{l,ij})(1-\mu_{l,jk})}, \\ \dfrac{\upsilon_{l,ik}}{(1-\upsilon_{l,ik})} = \dfrac{\upsilon_{l,ij}\upsilon_{l,jk}}{(1-\upsilon_{l,ij})(1-\upsilon_{l,jk})}, \end{cases} \\
&\text{(iv)} \begin{cases} \dfrac{\mu_{l,kj}}{(1-\mu_{l,kj})} = \dfrac{\mu_{l,ki}\mu_{l,ij}}{(1-\mu_{l,ki})(1-\mu_{l,ij})}, \\ \dfrac{\upsilon_{l,kj}}{(1-\upsilon_{l,kj})} = \dfrac{\upsilon_{l,ki}\upsilon_{l,ij}}{(1-\upsilon_{l,ki})(1-\upsilon_{l,ij})} \end{cases}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\text{(i)} \begin{cases} \dfrac{\mu_{u,ji}\mu_{u,ik}\mu_{u,kj}}{(1-\mu_{u,ji})(1-\mu_{u,ik})(1-\mu_{u,kj})} = 1, \\ \dfrac{\upsilon_{u,ji}\upsilon_{u,ik}\upsilon_{u,kj}}{(1-\upsilon_{u,ji})(1-\upsilon_{u,ik})(1-\upsilon_{u,kj})} = 1, \end{cases} \\
&\text{(ii)} \begin{cases} \dfrac{\mu_{u,ij}}{(1-\mu_{u,ij})} = \dfrac{\mu_{u,ik}\mu_{u,kj}}{(1-\mu_{u,ik})(1-\mu_{u,kj})}, \\ \dfrac{\upsilon_{u,ij}}{(1-\upsilon_{u,ij})} = \dfrac{\upsilon_{u,ik}\upsilon_{u,kj}}{(1-\upsilon_{u,ik})(1-\upsilon_{u,kj})}, \end{cases} \\
&\text{(iii)} \begin{cases} \dfrac{\mu_{u,ik}}{(1-\mu_{u,ik})} = \dfrac{\mu_{u,ij}\mu_{u,jk}}{(1-\mu_{u,ij})(1-\mu_{u,jk})}, \\ \dfrac{\upsilon_{u,ik}}{(1-\upsilon_{u,ik})} = \dfrac{\upsilon_{u,ij}\upsilon_{u,jk}}{(1-\upsilon_{u,ij})(1-\upsilon_{u,jk})}, \end{cases} \\
&\text{(iv)} \begin{cases} \dfrac{\mu_{u,kj}}{(1-\mu_{u,kj})} = \dfrac{\mu_{u,ki}\mu_{u,ij}}{(1-\mu_{u,ki})(1-\mu_{u,ij})}, \\ \dfrac{\upsilon_{u,kj}}{(1-\upsilon_{u,kj})} = \dfrac{\upsilon_{u,ki}\upsilon_{u,ij}}{(1-\upsilon_{u,ki})(1-\upsilon_{u,ij})} \end{cases}
\end{aligned} \tag{15}$$

for all $i, k, j = 1, 2, \dots, n$.

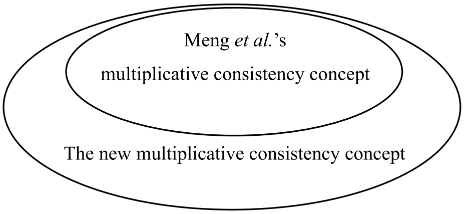

Fig. 1. The relationship between two multiplicative consistency concepts.

According to formulae (14) and (15), one can check that formula (13) is true. Therefore, $\tilde{R}$ is multiplicatively consistent based on Definition 11. □

Remark 1. When an IVIFPR is multiplicatively consistent following Definition 11, we cannot conclude that it is multiplicatively consistent in accordance with Definition 9.

Considering the below IVIFPR:

$$\tilde{R} = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.60], [0.30, 0.40]) & ([0.40, 0.60], [0.40, 0.40]) \\ ([0.30, 0.40], [0.40, 0.60]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.50], [0.45, 0.50]) \\ ([0.40, 0.40], [0.40, 0.60]) & ([0.45, 0.50], [0.35, 0.50]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

One can check that $\tilde{R}$ is multiplicatively consistent according to Definition 11. Nevertheless, it is inconsistent following Definition 9.

The relationship between the new and Meng *et al.*'s multiplicative consistency concepts for IVIFPRs is listed in Fig. 1.

Although Definition 11 owns all properties of Definition 3, it is inefficient to judge the multiplicative consistency of IVIFPRs directly. For an IVIFPR $\tilde{R}$, there are two associated FIPRs based on formula (10). Then, $2n(n-1)(n-2)$ triples of (i, k, j) need to be compared by formula (13) for judging its multiplicative consistency. Therefore, this way is time consuming and infeasible. To solve this problem, we construct the following programming model to judge the IVIFPRs' multiplicative consistency:

$$\textbf{Model 1:}\ \Gamma^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\alpha_{l,ij} + \beta_{l,ij} + \alpha_{u,ij} + \beta_{u,ij})$$

$$\text{s.t.} \begin{cases} \log(\mu_{l,ij}) - \log(1-\mu_{l,ij}) + \alpha_{l,ij} \\ \quad \geqslant \log(\mu_{l,ik}) + \log(\mu_{l,kj}) - \log(1-\mu_{l,ij}) - \log(1-\mu_{l,kj}), \\ \log(v_{l,ij}) - \log(1-v_{l,ij}) + \beta_{l,ij} \\ \quad \geqslant \log(v_{l,ik}) + \log(v_{l,kj}) - \log(1-v_{l,ij}) - \log(1-v_{l,kj}), \\ \log(\mu_{u,ij}) - \log(1-\mu_{u,ij}) + \alpha_{u,ij} \\ \quad \geqslant \log(\mu_{u,ik}) + \log(\mu_{u,kj}) - \log(1-\mu_{u,ij}) - \log(1-\mu_{u,kj}), \\ \log(v_{u,ij}) - \log(1-v_{u,ij}) + \beta_{u,ij} \\ \quad \geqslant \log(v_{u,ik}) + \log(v_{u,kj}) - \log(1-v_{u,ij}) - \log(1-v_{u,kj}), \\ \quad i, k, j = 1, 2, \dots, n,\ k \neq i,\ j, i < j, \\ \alpha_{l,ij}, \beta_{l,ij}, \alpha_{u,ij}, \beta_{u,ij} \geqslant 0, \quad i, j = 1, 2, \dots, n,\ i < j, \end{cases} \tag{16}$$

where the first four constraints are obtained by taking the natural logarithm of the inequalities in formula (13) and adding the non-negative deviation variables $\alpha_{l,ij}, \beta_{l,ij}, \alpha_{u,ij}, \beta_{u,ij}$.

By solving Model 1, when the optimal objective value $\Gamma^* = 0$, $\tilde{R}$ is multiplicatively consistent according to Definition 11. Otherwise, it is inconsistent.

4. Programming Models for Dealing with Incomplete and Inconsistent IVIFPRs

Due to DMs' personal limitations and time pressure, DMs may not provide some preference information. In this section, two programming models are constructed to determine the missing values under the conditions of maximizing the multiplicative consistency level and minimizing the total uncertainty.

Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an incomplete IVIFPR. If it is multiplicatively consistent, formula (13) is true. When $\tilde{R}$ is inconsistent, we cannot derive formula (13). In this case, we relax formula (13) by adding the deviation variables, where

$$\begin{cases} \log(\mu_{l,ij}) - \log(1-\mu_{l,ij}) + \alpha_{l,ij} \\ \quad \geqslant \log(\mu_{l,ik}) + \log(\mu_{l,kj}) - \log(1-\mu_{l,ik}) - \log(1-\mu_{l,kj}), \\ \log(v_{l,ij}) - \log(1-v_{l,ij}) + \beta_{l,ij} \\ \quad \geqslant \log(v_{l,ik}) + \log(v_{l,kj}) - \log(1-v_{l,ik}) - \log(1-v_{l,kj}), \\ \log(\mu_{u,ij}) - \log(1-\mu_{u,ij}) + \alpha_{u,ij} \\ \quad \geqslant \log(\mu_{u,ik}) + \log(\mu_{u,kj}) - \log(1-\mu_{u,ik}) - \log(1-\mu_{u,kj}), \\ \log(v_{u,ij}) - \log(1-v_{u,ij}) + \beta_{u,ij} \\ \quad \geqslant \log(v_{u,ik}) + \log(v_{u,kj}) - \log(1-v_{u,ik}) - \log(1-v_{u,kj}) \end{cases} \tag{17}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$, and $\alpha_{l,ij}, \beta_{l,ij}, \alpha_{u,ij}, \beta_{u,ij} \geqslant 0$.

In addition, there are following 15 different types of missing values of an IVIFV $\tilde{r}_{ij} = ([\mu_{l,ij}, \mu_{u,ij}], [v_{l,ij}, v_{u,ij}])$.

$$\begin{cases} \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \notin U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \notin U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \notin U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \notin U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \in U_v^+, \\ \mu_{l,ij} \in U_\mu^- \wedge \mu_{u,ij} \in U_\mu^+ \wedge v_{l,ij} \in U_v^- \wedge v_{u,ij} \in U_v^+. \end{cases} \tag{18}$$

where

$U_\mu^- = \{\mu_{l,ij}$ is unknown for all $i, j = 1, 2, \ldots, n$ with $i < j\}$,
$U_\mu^+ = \{\mu_{u,ij}$ is unknown for all $i, j = 1, 2, \ldots, n$ with $i < j\}$,
$U_v^- = \{v_{l,ij}$ is unknown for all$i, j = 1, 2, \ldots, n$ with $i < j\}$,
$U_v^+ = \{v_{u,ij}$ is unknown for all $i, j = 1, 2, \ldots, n$ with $i < j\}$.

Each case in formula (18) corresponds to a constraint c_i, $(i = 1, 2, \ldots, 15)$ as listed in the following formula:

$$\begin{cases} c_1: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij},\ v_{l,ji} = \mu_{l,ij}, \\ c_2: \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ v_{u,ji} = \mu_{u,ij}, \\ c_3: 0 \leqslant v_{l,ij} \leqslant v_{u,ij},\ \mu_{l,ji} = v_{l,ij}, \\ c_4: v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_5: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ v_{l,ji} = \mu_{l,ij},\ v_{u,ji} = \mu_{u,ij}, \\ c_6: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij},\ v_{l,ji} = \mu_{l,ij},\ \mu_{l,ji} = v_{l,ij} \\ c_7: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij},\ v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ v_{l,ji} = \mu_{l,ij},\ \mu_{u,ji} = v_{u,ij} \\ c_8: \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij},\ v_{u,ji} = \mu_{u,ij},\ \mu_{l,ji} = v_{l,ij}, \\ c_9: \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ v_{l,ij} \leqslant v_{u,ij},\ v_{u,ji} = \mu_{u,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_{10}: 0 \leqslant v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ \mu_{l,ji} = v_{l,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_{11}: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij},\ v_{l,ji} = \mu_{l,ij}, \\ \quad v_{u,ji} = \mu_{u,ij},\ \mu_{l,ji} = v_{l,ij}, \\ c_{12}: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij},\ v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ v_{l,ji} = \mu_{l,ij}, \\ \quad v_{u,ji} = \mu_{u,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_{13}: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ v_{l,ji} = \mu_{l,ij}, \\ \quad \mu_{l,ji} = v_{l,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_{14}: \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij},\ v_{u,ji} = \mu_{u,ij}, \\ \quad \mu_{l,ji} = v_{l,ij},\ \mu_{u,ji} = v_{u,ij}, \\ c_{15}: 0 \leqslant \mu_{l,ij} \leqslant \mu_{u,ij} \leqslant 1 - v_{u,ij},\ 0 \leqslant v_{l,ij} \leqslant v_{u,ij} \leqslant 1 - \mu_{u,ij},\ v_{l,ji} = \mu_{l,ij}, \\ \quad v_{u,ji} = \mu_{u,ij},\ \mu_{l,ji} = v_{l,ij},\ \mu_{u,ji} = v_{u,ij}. \end{cases} \tag{19}$$

Let $C = \{c_1, c_2, \ldots, c_{15}\}$. The following programming model is constructed to obtain the values for missing information.

$$\textbf{Model 2: } f^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\alpha_{l,ij} + \beta_{l,ij} + \alpha_{u,ij} + \beta_{u,ij}) \tag{20}$$

$$\text{s.t.} \begin{cases} \log(\mu_{l,ij}) - \log(1 - \mu_{l,ij}) + \alpha_{l,ij} \\ \quad \geqslant \log(\mu_{l,ik}) + \log(\mu_{l,kj}) - \log(1 - \mu_{l,ik}) - \log(1 - \mu_{l,kj}), \\ \log(v_{l,ij}) - \log(1 - v_{l,ij}) + \beta_{l,ij} \\ \quad \geqslant \log(v_{l,ik}) + \log(v_{l,kj}) - \log(1 - v_{l,ik}) - \log(1 - v_{l,kj}), \\ \log(\mu_{u,ij}) - \log(1 - \mu_{u,ij}) + \alpha_{u,ij} \\ \quad \geqslant \log(\mu_{u,ik}) + \log(\mu_{u,kj}) - \log(1 - \mu_{u,ik}) - \log(1 - \mu_{u,kj}), \\ \log(v_{u,ij}) - \log(1 - v_{u,ij}) + \beta_{u,ij} \\ \quad \geqslant \log(v_{u,ik}) + \log(v_{u,kj}) - \log(1 - v_{u,ik}) - \log(1 - v_{u,kj}), \\ \alpha_{l,ij}, \beta_{l,ij}, \alpha_{u,ij}, \beta_{u,ij} \geqslant 0, \quad i, k, j = 1, 2, \ldots, n,\ k \neq i,\ j \wedge i < j, \\ c_i \in C,\ i = 1, 2, \ldots, 15, \end{cases}$$

where the first four constraints ensure the obtained IVIFPR to be multiplicatively consistent.

Considering the fact that the larger the uncertain degree is, the lesser the useful information will be. We further offer the following programming model:

$$\textbf{Model 3: } g^* = \min \sum_{\mu_{l,ij}\in U_\mu^- \vee \mu_{u,ij}\in U_\mu^+ \vee v_{l,ij}\in U_v^- \vee v_{u,ij}\in U_v^-} (\mu_{u,ij} - \mu_{l,ij} + v_{u,ij} - v_{l,ij})$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\alpha_{l,ij}+\beta_{l,ij}+\alpha_{u,ij}+\beta_{u,ij}) = f^*, \\ \log(\mu_{l,ij}) - \log(1-\mu_{l,ij}) + \alpha_{l,ij} \\ \quad \geqslant \log(\mu_{l,ik}) + \log(\mu_{l,kj}) - \log(1-\mu_{l,ik}) - \log(1-\mu_{l,kj}), \\ \log(v_{l,ij}) - \log(1-v_{l,ij}) + \beta_{l,ij} \\ \quad \geqslant \log(v_{l,ik}) + \log(v_{l,kj}) - \log(1-v_{l,ik}) - \log(1-v_{l,kj}), \\ \log(\mu_{u,ij}) - \log(1-\mu_{u,ij}) + \alpha_{u,ij} \\ \quad \geqslant \log(\mu_{u,ik}) + \log(\mu_{u,kj}) - \log(1-\mu_{u,ik}) - \log(1-\mu_{u,kj}), \\ \log(v_{u,ij}) - \log(1-v_{u,ij}) + \beta_{u,ij} \\ \quad \geqslant \log(v_{u,ik}) + \log(v_{u,kj}) - \log(1-v_{u,ik}) - \log(1-v_{u,kj}), \\ \quad \alpha_{l,ij}, \beta_{l,ij}, \alpha_{u,ij}, \beta_{u,ij} \geqslant 0,\ i,k,j = 1,2,\dots,n,\ k \neq i,\ j \wedge i < j. \\ \quad c_i \in C,\ i = 1,2,\dots,15, \end{cases} \tag{21}$$

where f^* is the optimal objective value of Model 2, and other constraints are the same as that in Model 2.

Here, we use the example in Meng *et al.* (2018) to show the effectiveness of the above methods.

Example 2 (*See* Meng *et al.*, 2018). Let $\tilde{R}$ be an incomplete IVIFPR on $X = \{x_1, x_2, \dots, x_n\}$ as follows:

$$\tilde{R} = \begin{pmatrix} ([1/2,1/2],[1/2,1/2]) & ([\mu_{l,12},\mu_{u,12}],[v_{l,12},v_{u,12}]) & ([1/3,1/2],[v_{l,13},1/5]) & ([1/4,\mu_{u,14}],[v_{l,14},2/7]) \\ ([v_{l,12},v_{u,12}],[\mu_{l,12},\mu_{u,12}]) & ([1/2,1/2],[1/2,1/2]) & ([\mu_{l,23},3/5],[1/6,1/4]) & ([\mu_{l,24},3/7],[v_{l,24},2/5]) \\ ([v_{l,13},1/5],[1/3,1/2]) & ([1/6,1/4],[\mu_{l,23},3/5]) & ([1/2,1/2],[1/2,1/2]) & ([1/5,1/2],[2/9,1/3]) \\ ([v_{l,14},2/7],[1/4,\mu_{u,14}]) & ([v_{l,24},2/5],[\mu_{l,24},3/7]) & ([2/9,1/3],[1/5,1/2]) & ([1/2,1/2],[1/2,1/2]) \end{pmatrix}.$$

By solving Models 2 and 3, we derive

$$\mu_{l,12} = 0.31,\quad \mu_{u,12} = 0.40,\quad v_{l,12} = 0.28,\quad v_{u,12} = 0.28;\quad v_{l,13} = 0.20;$$
$$\mu_{u,14} = 0.50,\quad v_{l,14} = 0.29;\quad \mu_{l,23} = 0.53;\quad \mu_{l,24} = 0.43,\quad v_{l,24} = 0.20.$$

Furthermore, the associated complete IVIFPR is obtained as follows:

$$\tilde{R} = \begin{pmatrix} ([0.50,0.50],[0.50,0.50]) & ([0.31,0.40],[0.28,0.28]) & ([0.33,0.50],[0.20,0.20]) & ([0.25,0.50],[0.29,0.29]) \\ ([0.28,0.28],[0.31,0.40]) & ([0.50,0.50],[0.50,0.50]) & ([0.53,0.60],[0.17,0.25]) & ([0.43,0.43],[0.20,0.40]) \\ ([0.20,0.20],[0.33,0.50]) & ([0.17,0.25],[0.53,0.60]) & ([0.50,0.50],[0.50,0.50]) & ([0.20,0.50],[0.22,0.33]) \\ ([0.29,0.29],[0.25,0.50]) & ([0.20,0.40],[0.43,0.43]) & ([0.22,0.33],[0.20,0.50]) & ([0.50,0.50],[0.50,0.50]) \end{pmatrix}.$$

In addition, because of the inherent vagueness of human thinking, it is very difficult for the DMs to provide multiplicative consistent IVIFPRs. For inconsistent IVIFPRs, we need to adjust the DMs' original judgments. Meanwhile, the adjustment should be as small as possible to retain more original information. With these conditions, the following programming model is built:

$$\textbf{Model 4: } \Phi^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\gamma_{l,ij}^+ + \gamma_{l,ij}^- + \delta_{l,ij}^+ + \delta_{l,ij}^- + \gamma_{u,ij}^+ + \gamma_{u,ij}^- + \delta_{u,ij}^+ + \delta_{u,ij}^-\right)$$

$$\text{s.t.} \begin{cases} \log(\mu_{l,ij} - \gamma_{l,ij}^+ + \gamma_{l,ij}^-) - \log(1 - \mu_{l,ij} + \gamma_{l,ij}^+ - \gamma_{l,ij}^-) \\ \quad \geqslant \log(\mu_{l,ik} - \gamma_{l,ik}^+ + \gamma_{l,ik}^-) + \log(\mu_{l,kj} - \gamma_{l,kj}^+ + \gamma_{l,kj}^-) \\ \quad - \log(1 - \mu_{l,ik} + \gamma_{l,ik}^+ - \gamma_{l,ik}^-) - \log(1 - \mu_{l,kj} + \gamma_{l,kj}^+ - \gamma_{l,kj}^-), \\ \log(v_{l,ij} - \delta_{l,ij}^+ + \delta_{l,ij}^-) - \log(1 - v_{l,ij} + \delta_{l,ij}^+ - \delta_{l,ij}^-) \\ \quad \geqslant \log(v_{l,ik} - \delta_{l,ik}^+ + \delta_{l,ik}^-) + \log(v_{l,kj} - \delta_{l,kj}^+ + \delta_{l,kj}^-) \\ \quad - \log(1 - v_{l,ik} + \delta_{l,ik}^+ - \delta_{l,ik}^-) - \log(1 - v_{l,kj} + \delta_{l,kj}^+ - \delta_{l,kj}^-), \\ \log(\mu_{u,ij} - \gamma_{u,ij}^+ + \gamma_{u,ij}^-) - \log(1 - \mu_{u,ij} + \gamma_{u,ij}^+ - \gamma_{u,ij}^-) \\ \quad \geqslant \log(\mu_{u,ik} - \gamma_{u,ik}^+ + \gamma_{u,ik}^-) + \log(\mu_{u,kj} - \gamma_{u,kj}^+ + \gamma_{u,kj}^-) \\ \quad - \log(1 - \mu_{u,ik} + \gamma_{u,ik}^+ - \gamma_{u,ik}^-) - \log(1 - \mu_{u,kj} + \gamma_{u,kj}^+ - \gamma_{u,kj}^-), \\ \log(v_{u,ij} - \delta_{u,ij}^+ + \delta_{u,ij}^-) - \log(1 - v_{u,ij} + \delta_{u,ij}^+ - \delta_{u,ij}^-) \\ \quad \geqslant \log(v_{u,ik} - \delta_{u,ik}^+ + \delta_{u,ik}^-) + \log(v_{u,kj} - \delta_{u,kj}^+ + \delta_{u,kj}^-) \\ \quad - \log(1 - v_{u,ik} + \delta_{u,ik}^+ - \delta_{u,ik}^-) - \log(1 - v_{u,kj} + \delta_{u,kj}^+ - \delta_{u,kj}^-), \\ \quad i, k, j = 1, 2, \dots, n,\ k \neq i,\ j, i < j, \\ 0 \leqslant \mu_{l,ij} - \gamma_{l,ij}^+ + \gamma_{l,ij}^- \leqslant \mu_{u,ij} - \gamma_{u,ij}^+ + \gamma_{u,ij}^-,\ i, j = 1, 2, \dots, n, i < j, \\ 0 \leqslant v_{l,ij} - \delta_{l,ij}^+ + \delta_{l,ij}^- \leqslant v_{u,ij} - \delta_{u,ij}^+ + \delta_{u,ij}^-,\ i, j = 1, 2, \dots, n, i < j, \\ \mu_{u,ij} - \gamma_{u,ij}^+ + \gamma_{u,ij}^- + v_{u,ij} - \delta_{u,ij}^+ + \delta_{u,ij}^- \leqslant 1,\ i, j = 1, 2, \dots, n, i < j, \\ \gamma_{l,ij}^- - \gamma_{l,ij}^+ + \delta_{l,ji}^+ - \delta_{l,ji}^- = 0,\ i, j = 1, 2, \dots, n,\ i < j, \\ \delta_{l,ij}^- - \delta_{l,ij}^+ + \gamma_{l,ji}^+ - \gamma_{l,ji}^- = 0,\ i, j = 1, 2, \dots, n, i < j, \\ \gamma_{u,ij}^- - \gamma_{u,ij}^+ + \delta_{u,ji}^+ - \delta_{u,ji}^- = 0,\ i, j = 1, 2, \dots, n, i < j, \\ \delta_{u,ij}^- - \delta_{u,ij}^+ + \gamma_{u,ji}^+ - \gamma_{u,ji}^- = 0,\ i, j = 1, 2, \dots, n, i < j, \\ \gamma_{l,ij}^+, \gamma_{l,ij}^-, \delta_{l,ij}^+, \delta_{l,ij}^-, \gamma_{u,ij}^+, \gamma_{u,ij}^-, \delta_{u,ij}^+, \delta_{u,ij}^- \geqslant 0,\ i, j = 1, 2, \dots, n, i < j, \end{cases} \tag{22}$$

where the first four constraints are constructed from formula (13) by adding the non-negative deviation variables $\gamma_{l,ij}^+$, $\gamma_{l,ij}^-$, $\delta_{l,ij}^+$, $\delta_{l,ij}^-$, $\gamma_{u,ij}^+$, $\gamma_{u,ij}^-$, $\delta_{u,ij}^+$, $\delta_{u,ij}^-$, $i, j = 1, 2, \dots, n, i < j$, the fifth to seventh constraints are obtained from the construction of IVIFVs in IVIFPRs, and the eighth to eleventh constraints can guarantee the endpoints of corresponding IVIFVs to have the same adjustment.

With respect to the complete IVIFPR $\tilde{R}$ in Example 2, we have $\Gamma^* = 2.62$ following Model 1, which shows that the IVIFPR $\tilde{R}$ is inconsistent. In this case, we use Model 4 to adjust it, and the corresponding multiplicatively consistent IVIFPR is obtained as follows:

$$\tilde{R} = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.30, 0.40], [0.28, 0.28]) & ([0.33, 0.50], [0.20, 0.22]) & ([0.25, 0.50], [0.29, 0.29]) \\ ([0.28, 0.28], [0.30, 0.40]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.53, 0.53], [0.17, 0.25]) & ([0.43, 0.43], [0.20, 0.30]) \\ ([0.20, 0.22], [0.33, 0.50]) & ([0.17, 0.25], [0.53, 0.53]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.20, 0.40], [0.22, 0.33]) \\ ([0.29, 0.29], [0.25, 0.50]) & ([0.20, 0.30], [0.43, 0.43]) & ([0.22, 0.33], [0.20, 0.40]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

From the marked judgments, one can see that only five values in the upper triangular part are adjusted.

To overcome the limitation of a single DM, GDM has been widely applied in many real-life decision-making problems. Next, we further discuss GDM with IVIFPRs.

5. A New Method for GDM with IVIFPRs

Suppose that m DMs $E = \{e_1, e_2, \dots, e_m\}$ are invited to evaluate n objects $X = \{x_1, x_2, \dots, x_n\}$. Let $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n\times n}$ be the multiplicatively consistent IVIFPR of the individual IVIFPR $\tilde{R}_p$, where $\tilde{r}_{p,ij}^* = ([\mu_{l,ij}^{*p}, \mu_{u,ij}^{*p}], [v_{l,ij}^{*p}, v_{u,ij}^{*p}])$, $i, j = 1, 2, \dots, n$ and $p = 1, 2, \dots, m$. Let $w = (w_1, w_2, \dots, w_m)^T$ be weight vector on the DM set E, where w_p is the weight of DM e_p satisfying $w_p \geqslant 0$, $p = 1, 2, \dots, m$, and $\sum_{p=1}^m w_p = 1$. Using the symmetric interval-valued intuitionistic fuzzy weighted averaging (SIVIFWA) operator (Liao *et al.*, 2014) to calculate the collective PR $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n\times n}$, where

$$\begin{aligned}\tilde{r}_{C,ij}^* &= \left([\mu_{l,ij}^{*C}, \mu_{u,ij}^{*C}], [v_{l,ij}^{*C}, v_{u,ij}^{*C}]\right)\\ &= \left(\left[\frac{\prod_{p=1}^m(\mu_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^m(\mu_{l,ij}^{*p})^{w_p} + \prod_{p=1}^m(1-\mu_{l,ij}^{*p})^{w_p}}, \frac{\prod_{p=1}^m(\mu_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^m(\mu_{u,ij}^{*p})^{w_p} + \prod_{p=1}^m(1-\mu_{u,ij}^{*p})^{w_p}}\right],\right.\\ &\qquad \left.\left[\frac{\prod_{p=1}^m(v_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^m(v_{l,ij}^{*p})^{w_p} + \prod_{p=1}^m(1-v_{l,ij}^{*p})^{w_p}}, \frac{\prod_{p=1}^m(v_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^m(v_{u,ij}^{*p})^{w_p} + \prod_{p=1}^m(1-v_{u,ij}^{*p})^{w_p}}\right]\right)\end{aligned} \tag{23}$$

for all $i, j = 1, 2, \dots, n$.

One can check that the collective PR $\tilde{R}_C^*$ is an IVIFPR. Based on the multiplicative consistency of the collective IVIFPR $\tilde{R}_C^*$, we have the following theorem.

Theorem 3. *If all individual* IVIFPRs $\tilde{R}_p^*$, $p = 1, 2, \dots, m$, *are multiplicatively consistent, the collective* IVIFPR $\tilde{R}_C^*$ *derived from formula* (23) *is multiplicatively consistent.*

Proof. To prove the multiplicative consistency of $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n\times n}$, it only needs to prove the following equation

$$\begin{cases} \dfrac{\mu_{l,ij}^{*C}}{1-\mu_{l,ij}^{*C}} \geqslant \dfrac{\mu_{l,ik}^{*C}\mu_{l,kj}^{*C}}{(1-\mu_{l,ik}^{*C})(1-\mu_{l,kj}^{*C})},\\ \dfrac{v_{l,ij}^{*C}}{1-v_{l,ij}^{*C}} \geqslant \dfrac{v_{l,ik}^{*C}v_{l,kj}^{*C}}{(1-v_{l,ik}^{*C})(1-v_{l,kj}^{*C})},\\ \dfrac{\mu_{u,ij}^{*C}}{1-\mu_{u,ij}^{*C}} \geqslant \dfrac{\mu_{u,ik}^{*C}\mu_{u,kj}^{*C}}{(1-\mu_{u,ik}^{*C})(1-\mu_{u,kj}^{*C})},\\ \dfrac{v_{u,ij}^{*C}}{1-v_{u,ij}^{*C}} \geqslant \dfrac{v_{u,ik}^{*C}v_{u,kj}^{*C}}{(1-v_{u,ik}^{*C})(1-v_{u,kj}^{*C})} \end{cases} \tag{24}$$

holds for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

According to formula (23), the above inequalities can be transformed into:

$$\begin{cases} \dfrac{\prod_{p=1}^{m}(\mu_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-\mu_{l,ij}^{*p})^{w_p}} \geqslant \dfrac{\prod_{p=1}^{m}(\mu_{l,ik}^{*p})^{w_p}\prod_{p=1}^{m}(\mu_{l,kj}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-\mu_{l,ik}^{*p})^{w_p}\prod_{p=1}^{m}(1-\mu_{l,kj}^{*p})^{w_p}}, \\ \dfrac{\prod_{p=1}^{m}(v_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-v_{l,ij}^{*p})^{w_p}} \geqslant \dfrac{\prod_{p=1}^{m}(v_{l,ik}^{*p})^{w_p}\prod_{p=1}^{m}(v_{l,kj}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-v_{l,ik}^{*p})^{w_p}\prod_{p=1}^{m}(1-v_{l,kj}^{*p})^{w_p}}, \\ \dfrac{\prod_{p=1}^{m}(\mu_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-\mu_{u,ij}^{*p})^{w_p}} \geqslant \dfrac{\prod_{p=1}^{m}(\mu_{u,ik}^{*p})^{w_p}\prod_{p=1}^{m}(\mu_{u,kj}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-\mu_{u,ik}^{*p})^{w_p}\prod_{p=1}^{m}(1-\mu_{u,kj}^{*p})^{w_p}}, \\ \dfrac{\prod_{p=1}^{m}(v_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-v_{u,ij}^{*p})^{w_p}} \geqslant \dfrac{\prod_{p=1}^{m}(v_{u,ik}^{*p})^{w_p}\prod_{p=1}^{m}(v_{u,kj}^{*p})^{w_p}}{\prod_{p=1}^{m}(1-v_{u,ik}^{*p})^{w_p}\prod_{p=1}^{m}(1-v_{u,kj}^{*p})^{w_p}}, \end{cases} \tag{25}$$

where $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

Based on the multiplicative consistency of $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n\times n}$, where $\tilde{r}_{p,ij}^* = ([\mu_{l,ij}^{*p}, \mu_{u,ij}^{*p}], [v_{l,ij}^{*p}, v_{u,ij}^{*p}])$ for all $i, j = 1, 2, \dots, n$ and all $p = 1, 2, \dots, m$, we have the following inequality conditions:

$$\begin{cases} \dfrac{\mu_{l,ij}^{*p}}{1-\mu_{l,ij}^{*p}} \geqslant \dfrac{\mu_{l,ik}^{*p}\mu_{l,kj}^{*p}}{(1-\mu_{l,ik}^{*p})(1-\mu_{l,kj}^{*p})}, \\ \dfrac{v_{l,ij}^{*p}}{1-v_{l,ij}^{*p}} \geqslant \dfrac{v_{l,ik}^{*p}v_{l,kj}^{*p}}{(1-v_{l,ik}^{*p})(1-v_{l,kj}^{*p})}, \\ \dfrac{\mu_{u,ij}^{*p}}{1-\mu_{u,ij}^{*p}} \geqslant \dfrac{\mu_{u,ik}^{*p}\mu_{u,kj}^{*p}}{(1-\mu_{u,ik}^{*p})(1-\mu_{u,kj}^{*p})}, \\ \dfrac{v_{u,ij}^{*p}}{1-v_{u,ij}^{*p}} \geqslant \dfrac{v_{u,ik}^{*p}v_{u,kj}^{*p}}{(1-v_{u,ik}^{*p})(1-v_{u,kj}^{*p})} \end{cases} \tag{26}$$

for all $i, k, j = 1, 2, \dots, n$ such that $k \neq i, j \wedge i < j$, and $p = 1, 2, \dots, m$.

Combining formula (26), it is apparent that formula (25) is true. The proof of Theorem 3 is completed. □

As aforementioned, the DMs' weights are needed in the process of calculating collective IVIFPRs. Next, we introduce a maximum consensus-based programming model to determine the weights of the DMs. Considering the fact that the higher the consensus degree is, the better the individual IVIFPRs will be, we establish the following programming model:

$$\textbf{Model 5: } J^* = \min \sum_{p=1}^{m} (\rho_p + \kappa_p + o_p + \pi_p)$$

$$\text{s.t.} \begin{cases} \sum\limits_{i,j=1,i<j}^{n} \left(\mu_{l,ij}^{*p} - \dfrac{\prod_{p=1}^{m} (\mu_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^{m} (\mu_{l,ij}^{*p})^{w_p} + \prod_{p=1}^{m} (1-\mu_{l,ij}^{*p})^{w_p}} \right)^2 - \rho_p = 0, \\ \sum\limits_{i,j=1,i<j}^{n} \left(\mu_{u,ij}^{*p} - \dfrac{\prod_{p=1}^{m} (\mu_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^{m} (\mu_{u,ij}^{*p})^{w_p} + \prod_{p=1}^{m} (1-\mu_{u,ij}^{*p})^{w_p}} \right)^2 - \kappa_p = 0, \\ \sum\limits_{i,j=1,i<j}^{n} \left(v_{l,ij}^{*p} - \dfrac{\prod_{p=1}^{m} (v_{l,ij}^{*p})^{w_p}}{\prod_{p=1}^{m} (v_{l,ij}^{*p})^{w_p} + \prod_{p=1}^{m} (1-v_{l,ij}^{*p})^{w_p}} \right)^2 - o_p = 0, \\ \sum\limits_{i,j=1,i<j}^{n} \left(v_{u,ij}^{*p} - \dfrac{\prod_{p=1}^{m} (v_{u,ij}^{*p})^{w_p}}{\prod_{p=1}^{m} (v_{u,ij}^{*p})^{w_p} + \prod_{p=1}^{m} (1-v_{u,ij}^{*p})^{w_p}} \right)^2 - \pi_p = 0, \\ \rho_p, \kappa_p, o_p, \pi_p \geqslant 0,\ p = 1, 2, \dots, m,\ \sum\limits_{p=1}^{m} w_p = 1, \end{cases} \tag{27}$$

where the constraints are constructed from the deviation between individual IVIFPRs and the collective IVIFPR.

In the process of GDM, prior to the selection of the best object, there should be a high consensus degree among all DMs. To measure the individual consensus degree, we use the distance measure between IVIFPRs.

Definition 12. Let $\tilde{R}_p = (\tilde{r}_{p,ij})_{n\times n}$, $p = 1, 2, \dots, m$, be m IVIFPRs, and $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n\times n}$, $p = 1, 2, \dots, m$, be the associated multiplicatively consistent IVIFPRs. Furthermore, let $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n\times n}$ be the collective IVIFPR shown as formula (23). Then, the consensus index of the individual IVIFPR $\tilde{R}_p^*$ is defined as

$$GCI(\tilde{R}_p^*) = 1 - \frac{1}{2n(n-1)} \sum_{i,j=1,i<j}^{n} \big(|\mu_{l,ij}^{*p} - \mu_{l,ij}^{*C}| + |\mu_{u,ij}^{*p} - \mu_{u,ij}^{*C}| + |v_{l,ij}^{*p} - v_{l,ij}^{*C}| + |v_{u,ij}^{*p} - v_{u,ij}^{*C}| \big). \tag{28}$$

Let θ^* be the threshold of the consensus index. If $GCI(\tilde{R}_p^*) < \theta^*$, we need to improve the consensus level of $\tilde{R}_p^*$. To do this, we construct the following programming model:

$$\textbf{Model 6: } \zeta^* = \max \sum_{i,j=1,i<j}^{n} (\alpha_{l,ij} + \alpha_{u,ij} + \beta_{l,ij} + \beta_{u,ij})$$

$$\text{s.t.} \begin{cases} \sum_{i,j=1,i<j}^{n} \Bigg(\left| \mu_{l,ij}^{**p} - \frac{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t}}{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t} + (1-(\mu_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{l,ij}^{*t})^{w_t}} \right| \\ \quad + \left| \mu_{u,ij}^{**p} - \frac{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t}}{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t} + (1-(\mu_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{u,ij}^{*t})^{w_t}} \right| \\ \quad + \left| v_{l,ij}^{**p} - \frac{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t}}{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t} + (1-(v_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{l,ij}^{*t})^{w_t}} \right| \\ \quad + \left| v_{u,ij}^{**p} - \frac{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t}}{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t} + (1-(v_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{u,ij}^{*t})^{w_t}} \right| \Bigg) \\ \quad \leqslant 2n(n-1)(1-\theta^*), \quad i,j = 1,2,\dots,n, i<j, \\ \log(\mu_{l,ij}^{**p}) - \log(1-\mu_{l,ij}^{**p}) \\ \quad \geqslant \log(\mu_{l,ik}^{**p}) + \log(\mu_{l,kj}^{**p}) - \log(1-\mu_{l,ik}^{**p}) - \log(1-\mu_{l,kj}^{**p}), \\ \log(\mu_{u,ij}^{**p}) - \log(1-\mu_{u,ij}^{**p}) \\ \geqslant \log(\mu_{u,ik}^{**p}) + \log(\mu_{u,kj}^{**p}) - \log(1-\mu_{u,ik}^{**p}) - \log(1-\mu_{u,kj}^{**p}), \\ \log(v_{l,ij}^{**p}) - \log(1-v_{l,ij}^{**p}) \\ \quad \geqslant \log(v_{l,ik}^{**p}) + \log(v_{l,kj}^{**p}) - \log(1-v_{l,ik}^{**p}) - \log(1-v_{l,kj}^{**p}), \\ \log(v_{u,ij}^{**p}) - \log(1-v_{u,ij}^{**p}) \\ \quad \geqslant \log(v_{u,ik}^{**p}) + \log(v_{u,kj}^{**p}) - \log(1-v_{u,ik}^{**p}) - \log(1-v_{u,kj}^{**p}), \\ 0 \leqslant \alpha_{l,ij}, \alpha_{l,ik}, \alpha_{l,kj}, \alpha_{u,ij}, \alpha_{u,ik}, \alpha_{u,kj}, \beta_{l,ij}, \beta_{l,ik}, \beta_{l,kj}, \beta_{u,ij}, \beta_{u,ik}, \beta_{u,kj} \\ \quad \leqslant 1, \\ i,k,j = 1,2,\dots,n, \ k \neq i, \ j \wedge i < j, \\ \mu_{l,ij}^{**p} \leqslant \mu_{u,ij}^{**p}, v_{l,ij}^{**p} \leqslant v_{u,ij}^{**p}, \ i,j = 1,2,\dots,n, \ i<j, \\ \alpha_{l,ij} = \beta_{l,ji}, \alpha_{u,ij} = \beta_{u,ji}, \beta_{l,ij} = \alpha_{l,ji}, \beta_{u,ij} = \alpha_{u,ji}, \\ i,j = 1,2,\dots,n, \ i<j, \end{cases} \tag{29}$$

where

$$\begin{cases} \mu_{l,ij}^{**p} = \alpha_{l,ij}\mu_{l,ij}^{*p} + (1-\alpha_{l,ij})\mu_{l,ij}^{*C}, \\ \mu_{u,ij}^{**p} = \alpha_{u,ij}\mu_{u,ij}^{*p} + (1-\alpha_{u,ij})\mu_{u,ij}^{*C}, \\ v_{l,ij}^{**p} = \beta_{l,ij}v_{l,ij}^{*p} + (1-\beta_{l,ij})v_{l,ij}^{*C}, \\ v_{u,ij}^{**p} = \beta_{u,ij}v_{u,ij}^{*p} + (1-\beta_{u,ij})v_{u,ij}^{*C}, \\ i,j = 1,2,\dots,n, i \neq j. \end{cases} \tag{30}$$

In Model 6, the first constraint is derived from formula (28) that ensures the adjusted IVIFPR to satisfy the consensus requirement, the second to fifth constraints ensure the adjusted IVIFPR to be multiplicative consistent, the sixth constraint defines the range of variables, the seventh constraint is obtained from the construction of IVIFVs in IVIFPR, and the last constraint guarantees the endpoints of corresponding IVIFVs to have the same adjustment.

To facilitate the solution of Model 6, let

$$\begin{cases} \mu_{l,ij}^{**p} - \dfrac{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t}}{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t} + (1-(\mu_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{l,ij}^{*t})^{w_t}} \\ \quad - \tau_{ij}^{+} + \tau_{ij}^{-} = 0, \\ \mu_{u,ij}^{**p} - \dfrac{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t}}{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t} + (1-(\mu_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{u,ij}^{*t})^{w_t}} \\ \quad - \eta_{ij}^{+} + \eta_{ij}^{-} = 0, \\ v_{l,ij}^{**p} - \dfrac{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t}}{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t} + (1-(v_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{l,ij}^{*t})^{w_t}} \\ \quad - \phi_{ij}^{+} + \phi_{ij}^{-} = 0, \\ v_{u,ij}^{**p} - \dfrac{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t}}{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t} + (1-(v_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{u,ij}^{*t})^{w_t}} \\ \quad - \varepsilon_{ij}^{+} + \varepsilon_{ij}^{-} = 0, \end{cases} \tag{31}$$

where $\tau_{ij}^{+}, \tau_{ij}^{-}, \eta_{ij}^{+}, \eta_{ij}^{-}, \phi_{ij}^{+}, \phi_{ij}^{-}, \varepsilon_{ij}^{+}, \varepsilon_{ij}^{-}$ are non-negative deviation variables such that $\tau_{ij}^{+} \times \tau_{ij}^{-} = \eta_{ij}^{+} \times \eta_{ij}^{-} = \phi_{ij}^{+} \times \phi_{ij}^{-} = \varepsilon_{ij}^{+} \times \varepsilon_{ij}^{-} = 0$.

Therefore, we have

$$\begin{cases} \left| \mu_{l,ij}^{**p} - \dfrac{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t}}{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t} + (1-(\mu_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{l,ij}^{*t})^{w_t}} \right| \\ \quad = \tau_{ij}^{+} + \tau_{ij}^{-}, \\ \left| \mu_{u,ij}^{**p} - \dfrac{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t}}{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t} + (1-(\mu_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{u,ij}^{*t})^{w_t}} \right| \\ \quad = \eta_{ij}^{+} + \eta_{ij}^{-}, \\ \left| v_{l,ij}^{**p} - \dfrac{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t}}{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t} + (1-(v_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{l,ij}^{*t})^{w_t}} \right| \\ \quad = \phi_{ij}^{+} + \phi_{ij}^{-}, \\ \left| v_{u,ij}^{**p} - \dfrac{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t}}{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t} + (1-(v_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{u,ij}^{*t})^{w_t}} \right| \\ \quad = \varepsilon_{ij}^{+} + \varepsilon_{ij}^{-}. \end{cases} \tag{32}$$

With formula (32), the Model 6 can be transformed as follows:

Model 7: $\zeta^* = \max \sum_{i,j=1,i<j}^{n} (\alpha_{l,ij} + \alpha_{u,ij} + \beta_{l,ij} + \beta_{u,ij})$

$$\text{s.t.}\begin{cases} \sum_{i,j=1,i<j}^{n} \left(\tau_{ij}^{+} + \tau_{ij}^{-} + \eta_{ij}^{+} + \eta_{ij}^{-} + \phi_{ij}^{+} + \phi_{ij}^{-} + \varepsilon_{ij}^{+} + \varepsilon_{ij}^{-}\right) \leqslant 2n(n-1)\left(1-\theta^*\right) \\ \mu_{l,ij}^{**p} - \dfrac{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t}}{(\mu_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{l,ij}^{*t})^{w_t} + (1-(\mu_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{l,ij}^{*t})^{w_t}} \\ \quad - \tau_{ij}^{+} + \tau_{ij}^{-} = 0, \\ \mu_{u,ij}^{**p} - \dfrac{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t}}{(\mu_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (\mu_{u,ij}^{*t})^{w_t} + (1-(\mu_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-\mu_{u,ij}^{*t})^{w_t}} \\ \quad - \eta_{ij}^{+} + \eta_{ij}^{-} = 0, \\ v_{l,ij}^{**p} - \dfrac{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t}}{(v_{l,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{l,ij}^{*t})^{w_t} + (1-(v_{l,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{l,ij}^{*t})^{w_t}} \\ \quad - \phi_{ij}^{+} + \phi_{ij}^{-} = 0, \\ v_{u,ij}^{**p} - \dfrac{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t}}{(v_{u,ij}^{**p})^{w_p} \prod_{t=1,t\neq p}^{m} (v_{u,ij}^{*t})^{w_t} + (1-(v_{u,ij}^{**p}))^{w_p} \prod_{t=1,t\neq p}^{m} (1-v_{u,ij}^{*t})^{w_t}} \\ \quad - \varepsilon_{ij}^{+} + \varepsilon_{ij}^{-} = 0, \\ \tau_{ij}^{+} \times \tau_{ij}^{-} = \eta_{ij}^{+} \times \eta_{ij}^{-} = \phi_{ij}^{+} \times \phi_{ij}^{-} = \varepsilon_{ij}^{+} \times \varepsilon_{ij}^{-} = 0, \\ \tau_{ij}^{+}, \tau_{ij}^{-}, \eta_{ij}^{+}, \eta_{ij}^{-}, \phi_{ij}^{+}, \phi_{ij}^{-}, \varepsilon_{ij}^{+}, \varepsilon_{ij}^{-} \geqslant 0,\ i, j = 1, 2, \dots, n,\ i < j, \\ \log\left(\mu_{l,ij}^{**p}\right) - \log\left(1-\mu_{l,ij}^{**p}\right) \geqslant \log\left(\mu_{l,ik}^{**p}\right) + \log\left(\mu_{l,kj}^{**p}\right) - \log\left(1-\mu_{l,ik}^{**p}\right) \\ \quad - \log\left(1-\mu_{l,kj}^{**p}\right), \\ \log\left(\mu_{u,ij}^{**p}\right) - \log\left(1-\mu_{u,ij}^{**p}\right) \geqslant \log\left(\mu_{u,ik}^{**p}\right) + \log\left(\mu_{u,kj}^{**p}\right) - \log\left(1-\mu_{u,ik}^{**p}\right) \\ \quad - \log\left(1-\mu_{u,kj}^{**p}\right), \\ \log\left(v_{l,ij}^{**p}\right) - \log\left(1-v_{l,ij}^{**p}\right) \geqslant \log\left(v_{l,ik}^{**p}\right) + \log\left(v_{l,kj}^{**p}\right) - \log\left(1-v_{l,ik}^{**p}\right) \\ \quad - \log\left(1-v_{l,kj}^{**p}\right), \\ \log\left(v_{u,ij}^{**p}\right) - \log\left(1-v_{u,ij}^{**p}\right) \geqslant \log\left(v_{u,ik}^{**p}\right) + \log\left(v_{u,kj}^{**p}\right) - \log\left(1-v_{u,ik}^{**p}\right) \\ \quad - \log\left(1-v_{u,kj}^{**p}\right) \\ 0 \leqslant \alpha_{l,ij}, \alpha_{l,ik}, \alpha_{l,kj}, \alpha_{u,ij}, \alpha_{u,ik}, \alpha_{u,kj}, \beta_{l,ij}, \beta_{l,ik}, \beta_{l,kj}, \beta_{u,ij}, \beta_{u,ik}, \beta_{u,kj} \leqslant 1, \\ i, k, j = 1, 2, \dots, n,\ k \neq i,\ j \wedge i < j, \\ \mu_{l,ij}^{**p} \leqslant \mu_{u,ij}^{**p}, v_{l,ij}^{**p} \leqslant v_{u,ij}^{**p},\ i, j = 1, 2, \dots, n,\ i < j. \\ \alpha_{l,ij} = \beta_{l,ji},\ \alpha_{u,ij} = \beta_{u,ji},\ \beta_{l,ij} = \alpha_{l,ji}, \beta_{u,ij} = \alpha_{u,ji}, \\ i, j = 1, 2, \dots, n,\ i < j. \end{cases} \tag{33}$$

Model 7 can not only guarantee the multiplicative consistency and consensus of the adjusted individual IVIFPRs, but also endow different IVIFVs with different adjustments. Meanwhile, the adjusted individual IVIFPR has the smallest total adjustment to retain more original information.

Based on the above analysis, this paper develops the following GDM method.

Algorithm 1.

Step 1: If the individual IVIFPRs $\tilde{R}_p = (\tilde{r}_{p,ij})_{n\times n},\ p = 1, 2, \dots, m$, are all complete, go to Step 2. Otherwise, Models 2 and 3 are adopted to determine the missing values.

Step 2: Model 1 is used to judge the multiplicative consistency of individual IVIFPRs. When individual IVIFPRs are all multiplicatively consistent, go to Step 3. Otherwise, Model 4 is applied to derive the multiplicatively consistent IVIFPRs, denoted as $\tilde{R}^*_p = (\tilde{r}^*_{p,ij})_{n\times n}$, $p = 1, 2, \dots, m$.

Step 3: Model 5 is used to determine the DMs' weights, and formula (23) is adopted to obtain the collective IVIFPR $\tilde{R}^*_C = (\tilde{r}^*_{C,ij})_{n\times n}$.

Step 4: Formula (28) is adopted to measure the consensus level of individual IVIFPRs. Let θ^* be the given consensus threshold. If we have $GCI(\tilde{R}^*_p) > \theta^*$ for all $p = 1, 2, \dots, m$, go to Step 5. Otherwise, Model 7 is applied to improve the consensus level of the corresponding individual IVIFPR.

Step 5: The symmetric interval-valued intuitionistic fuzzy averaging (SIVIFA) operator (Liao *et al.*, 2014) is used to fuse the comprehensive IVIFV $\tilde{r}_i$, where

$$\tilde{r}_i = \left(\left[\frac{\prod_{j=1}^n(\mu^{*C}_{l,ij})^{1/n}}{\prod_{j=1}^n(\mu^{*C}_{l,ij})^{1/n} + \prod_{j=1}^n(1-\mu^{*C}_{l,ij})^{1/n}}, \frac{\prod_{j=1}^n(\mu^{*C}_{u,ij})^{1/n}}{\prod_{j=1}^n(\mu^{*C}_{u,ij})^{1/n} + \prod_{j=1}^n(1-\mu^{*C}_{u,ij})^{1/n}}\right],\right.$$
$$\left.\left[\frac{\prod_{j=1}^n(v^{*C}_{l,ij})^{1/n}}{\prod_{j=1}^n(v^{*C}_{l,ij})^{1/n} + \prod_{j=1}^n(1-v^{*C}_{l,ij})^{1/n}}, \frac{\prod_{j=1}^n(v^{*C}_{u,ij})^{1/n}}{\prod_{j=1}^n(v^{*C}_{u,ij})^{1/n} + \prod_{j=1}^n(1-v^{*C}_{u,ij})^{1/n}}\right]\right) \tag{34}$$

for all $i = 1, 2, \dots, n$.

Step 6: For the comprehensive IVIFVs $\tilde{r}_i = ([\mu_{l,i}, \mu_{u,i}], [v_{l,i}, v_{u,i}])$, $i = 1, 2, \dots, n$, the following functions (Xu, 2007a)

$$S(\tilde{r}_i) = 0.5(\mu_{l,i} + \mu_{u,i} - v_{l,i} - v_{u,i}), \tag{35}$$
$$H(\tilde{r}_i) = 0.5(\mu_{l,i} + \mu_{u,i} + v_{l,i} + v_{u,i}) \tag{36}$$

are applied to calculate the score and accuracy values, by which we get the ranking of objects $x_1, x_2, \dots, x_n$.

6. Case Study and Comparison

To illustrate the application of the proposed algorithm and compare the new method with previous ones, the selection of CC vendor for a SME is provided.

EXAMPLE 3. Silver Swallow Clothing Co., LTD (SS for short) is a SME in Qingdao, China. With the rapid popularity of online shopping among consumers in China, the traditional sales model has been unable to adapt to consumer demands. To broaden the sales channels, the company needs to add e-commerce (EC) modules on the basis of the existing enterprise management information system (MIS). Due to the funds and technical conditions, as well as the high concurrent response requirements during the promotion, the company decides to adopt the EC order management system (OMS) based on the CC Software-as-a-Service (SaaS) model. Through the application program interface (API) provided by the system, data exchange between offline enterprise resource planning (ERP) and various third-party systems can be established. At present, there are more than 30 providers of such CC application for SMEs in Chinese market. After primary screening, four different CC vendors enter the final selection process, namely, Guan Yi Yun, Ju Shui Tan, Wang Dian Tong and Wand Dian Guan Jia, marked as $\{x_1, x_2, x_3, x_4\}$, respectively.

To select the most appropriate one, three experts (DMs) $\{e_1, e_2, e_3\}$ are commissioned to carry out the assessment. Unlike the choice of a physical product, CC applications belong to a kind of service with abstraction. At the same time, both the vendors and the internal conditions of the SMEs are related to the application effect of CC. That is to say, the selection of CC vendors should be combined with the enterprise's characteristics, which increases the difficulty of dealing with such problems. Generally speaking, there are several recognized factors in selecting CC vendors for SMEs, including economy, security, compatibility, technical stability and service quality. Due to the complexity of the problem, it is difficult for the experts to quantify these factors by constructing the indicator system. The method based on PRs as aforementioned is a good choice. Furthermore, due to the experts' expertise, experience and preferences, they are allowed to offer the uncertain preferred and non-preferred judgments simultaneously, which comes down to a GDM problem with IVIFPRs. By pairwise comparisons among these four vendors, the experts $e_p, (p = 1, 2, 3)$ give the following IVIFPRs.

$$\tilde{R}_1 = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.55], [0.25, 0.45]) & ([0.45, 0.60], [0.25, 0.40]) & ([0.60, 0.75], [0.10, 0.15]) \\ ([0.25, 0.45], [0.40, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.55], [0.25, 0.40]) & ([0.40, 0.60], [0.15, 0.40]) \\ ([0.25, 0.40], [0.45, 0.60]) & ([0.25, 0.40], [0.35, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.55], [0.25, 0.45]) \\ ([0.10, 0.15], [0.60, 0.75]) & ([0.15, 0.40], [0.40, 0.60]) & ([0.25, 0.45], [0.40, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix},$$

$$\tilde{R}_2 = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.25, 0.35], [0.40, 0.65]) & ([0.40, 0.55], [0.30, 0.35]) & ([0.30, 0.40], [0.35, 0.55]) \\ ([0.40, 0.65], [0.25, 0.35]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.50, 0.75], [0.15, 0.20]) & ([0.45, 0.55], [0.30, 0.45]) \\ ([0.30, 0.35], [0.40, 0.55]) & ([0.15, 0.20], [0.50, 0.75]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.20, 0.35], [0.45, 0.60]) \\ ([0.35, 0.55], [0.30, 0.40]) & ([0.30, 0.45], [0.45, 0.55]) & ([0.45, 0.60], [0.20, 0.35]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix},$$

$$\tilde{R}_3 = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.30, 0.40], [0.40, 0.55]) & ([0.35, 0.50], [0.30, 0.50]) & ([0.55, 0.70], [0.15, 0.20]) \\ ([0.40, 0.55], [0.30, 0.40]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.45, 0.55], [0.30, 0.35]) & ([0.65, 0.80], [0.05, 0.20]) \\ ([0.30, 0.50], [0.35, 0.50]) & ([0.30, 0.35], [0.45, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.55, 0.65], [0.20, 0.25]) \\ ([0.15, 0.20], [0.55, 0.70]) & ([0.05, 0.20], [0.65, 0.80]) & ([0.20, 0.25], [0.55, 0.65]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

It should be noted that Lingo software is used to solve the models involved in the processing of this problem.

Step 1: Since the individual IVIFPRs are all complete, go to Step 2.

Step 2: By Model 1, we have the optimal objective values $\Gamma_1^* = 3.30$, $\Gamma_2^* = 2.05$, $\Gamma_3^* = 1.66$. Thus, none of them is consistent. In this case, Model 4 is used to derive multiplicatively consistent IVIFPRs as follows:

$$\tilde{R}_1^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.55], [0.25, 0.45]) & ([0.45, 0.60], [0.25, 0.40]) & ([0.60, 0.65], [0.10, 0.35]) \\ ([0.25, 0.45], [0.40, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.55], [0.25, 0.45]) & ([0.40, 0.60], [0.15, 0.40]) \\ ([0.25, 0.40], [0.45, 0.60]) & ([0.25, 0.45], [0.35, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.55], [0.25, 0.45]) \\ ([0.10, 0.35], [0.60, 0.65]) & ([0.15, 0.40], [0.40, 0.60]) & ([0.25, 0.45], [0.40, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix},$$

$$\tilde{R}_2^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.25, 0.35], [0.40, 0.65]) & ([0.40, 0.55], [0.30, 0.35]) & ([0.30, 0.40], [0.35, 0.55]) \\ ([0.40, 0.65], [0.25, 0.35]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.50, 0.69], [0.15, 0.23]) & ([0.45, 0.55], [0.30, 0.40]) \\ ([0.30, 0.35], [0.40, 0.55]) & ([0.15, 0.23], [0.50, 0.69]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.20, 0.31], [0.45, 0.60]) \\ ([0.35, 0.55], [0.30, 0.40]) & ([0.30, 0.40], [0.45, 0.55]) & ([0.45, 0.60], [0.20, 0.31]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix},$$

$$\tilde{R}_3^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.30, 0.40], [0.40, 0.55]) & ([0.35, 0.50], [0.30, 0.45]) & ([0.55, 0.70], [0.15, 0.21]) \\ ([0.40, 0.55], [0.30, 0.40]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.45, 0.55], [0.30, 0.35]) & ([0.65, 0.78], [0.10, 0.18]) \\ ([0.30, 0.45], [0.35, 0.50]) & ([0.30, 0.35], [0.45, 0.55]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.55, 0.65], [0.20, 0.25]) \\ ([0.15, 0.21], [0.55, 0.70]) & ([0.10, 0.18], [0.65, 0.78]) & ([0.20, 0.25], [0.55, 0.65]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

Step 3: Based on the above multiplicatively consistent IVIFPRs and Model 5, we derive three experts' weights

$$w_1 = 0.342, \qquad w_2 = 0.342, \qquad w_3 = 0.316.$$

By formula (23), the collectively multiplicatively consistent IVIFPR $\tilde{R}_C^*$ is aggregated.

$$\tilde{R}_C^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.31, 0.43], [0.34, 0.55]) & ([0.40, 0.55], [0.28, 0.40]) & ([0.48, 0.58], [0.18, 0.36]) \\ ([0.34, 0.55], [0.31, 0.43]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.43, 0.60], [0.23, 0.33]) & ([0.50, 0.65], [0.17, 0.32]) \\ ([0.28, 0.40], [0.40, 0.55]) & ([0.23, 0.33], [0.43, 0.60]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.37, 0.50], [0.29, 0.43]) \\ ([0.18, 0.36], [0.48, 0.58]) & ([0.17, 0.32], [0.50, 0.65]) & ([0.29, 0.43], [0.37, 0.50]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

Step 4: Let the consensus threshold be $\theta^* = 0.9$. Using formula (28), we calculate the consensus indices of the individual IVIFPRs, where

$$GCI(\tilde{R}_1^*) = 0.939, \qquad GCI(\tilde{R}_2^*) = 0.897, \qquad GCI(\tilde{R}_3^*) = 0.920.$$

Because $GCI(\tilde{R}_2^*) < 0.9$, Model 7 is applied to improve the consensus level of $\tilde{R}_2^*$. The adjusted IVIFPR $\tilde{R}_2'^*$ is shown below.

$$\tilde{R}_2'^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.25, 0.38], [0.40, 0.56]) & ([0.40, 0.55], [0.30, 0.40]) & ([0.30, 0.43], [0.35, 0.40]) \\ ([0.40, 0.56], [0.25, 0.38]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.50, 0.63], [0.15, 0.33]) & ([0.45, 0.55], [0.30, 0.33]) \\ ([0.30, 0.40], [0.40, 0.55]) & ([0.15, 0.33], [0.50, 0.63]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.20, 0.38], [0.45, 0.46]) \\ ([0.35, 0.40], [0.30, 0.43]) & ([0.30, 0.33], [0.45, 0.55]) & ([0.45, 0.46], [0.20, 0.38]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

Again, following Step 3, we derive the experts' weights $w_1' = 0.335$, $w_2' = 0.345$, $w_3' = 0.320$. The new collective IVIFPR $\tilde{R}_C'^*$ is obtained by fusing $\tilde{R}_1^*$, $\tilde{R}_2'^*$, $\tilde{R}_3^*$.

$$\tilde{R}_C'^* = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.31, 0.44], [0.35, 0.52]) & ([0.40, 0.55], [0.28, 0.41]) & ([0.48, 0.59], [0.18, 0.32]) \\ ([0.35, 0.52], [0.31, 0.44]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.43, 0.58], [0.22, 0.38]) & ([0.50, 0.65], [0.17, 0.30]) \\ ([0.28, 0.41], [0.40, 0.55]) & ([0.22, 0.38], [0.43, 0.58]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.37, 0.53], [0.29, 0.38]) \\ ([0.18, 0.32], [0.48, 0.59]) & ([0.17, 0.30], [0.50, 0.65]) & ([0.29, 0.38], [0.37, 0.53]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

Based on $\tilde{R}_C'^*$, we use formula (28) to get the individual consensus indices, where $GCI(\tilde{R}_1^*) = 0.940$, $GCI(\tilde{R}_2'^*) = 0.920$, $GCI(\tilde{R}_3^*) = 0.927$, which meet the consensus threshold.

Table 1
The ranking results based on different methods.

Methods	Subjective parameter 1	Comprehensive IVIFVs	Subjective parameter 2	Ranking values	Ranking order
New method	None	$\tilde{r}_1 = ([0.56, 0.57], [0.41, 0.41])$ $\tilde{r}_2 = ([0.59, 0.59], [0.38, 0.39])$ $\tilde{r}_3 = ([0.43, 0.43], [0.50, 0.52])$ $\tilde{r}_4 = ([0.35, 0.35], [0.62, 0.63])$	None	$S(\tilde{r}_2) > S(\tilde{r}_1) >$ $S(\tilde{r}_3) > S(\tilde{r}_4)$	$x_2 \succ x_1 \succ x_3 \succ x_4$
Method in Meng *et al.* (2018)	None	$\tilde{r}_1 = ([0.55, 0.56], [0.40, 0.40])$ $\tilde{r}_2 = ([0.58, 0.58], [0.38, 0.38])$ $\tilde{r}_3 = ([0.42, 0.43], [0.49, 0.51])$ $\tilde{r}_4 = ([0.34, 0.35], [0.61, 0.61])$	None	$S(\tilde{r}_2) > S(\tilde{r}_1) >$ $S(\tilde{r}_3) > S(\tilde{r}_4)$	$x_2 \succ x_1 \succ x_3 \succ x_4$
Method in Liao *et al.* (2014)	None	$\tilde{r}_1 = ([0.29, 0.52], [0.21, 0.43])$ $\tilde{r}_2 = ([0.38, 0.58], [0.26, 0.37])$ $\tilde{r}_3 = ([0.27, 0.41], [0.37, 0.54])$ $\tilde{r}_4 = ([0.19, 0.38], [0.30, 0.55])$	None	$S(\tilde{r}_2) > S(\tilde{r}_1) >$ $S(\tilde{r}_3) > S(\tilde{r}_4)$	$x_2 \succ x_1 \succ x_3 \succ x_4$
	$\xi = 0$	$\tilde{r}_1 = ([0.18, 0.24], [0.54, 0.72])$ $\tilde{r}_2 = ([0.22, 0.32], [0.62, 0.63])$ $\tilde{r}_3 = ([0.10, 0.17], [0.78, 0.80])$ $\tilde{r}_4 = ([0.06, 0.16], [0.78, 0.80])$	$q \in [0, 1]$	$T(\tilde{r}_2) > T(\tilde{r}_1) >$ $T(\tilde{r}_3) > T(\tilde{r}_4)$	$x_2 \succ x_1 \succ x_3 \succ x_4$
Method in Wan *et al.* (2016)	$\xi = 1$	$\tilde{r}_1 = ([0.19, 0.29], [0.55, 0.70])$ $\tilde{r}_2 = ([0.17, 0.30], [0.68, 0.68])$ $\tilde{r}_3 = ([0.10, 0.20], [0.74, 0.78])$ $\tilde{r}_4 = ([0.05, 0.16], [0.77, 0.83])$	$q \in [0, 1]$	$T(\tilde{r}_1) > T(\tilde{r}_2) >$ $T(\tilde{r}_3) > T(\tilde{r}_4)$	$x_1 \succ x_2 \succ x_3 \succ x_4$
	$\xi = 0.5$	$\tilde{r}_1 = ([0.18, 0.27], [0.58, 0.70])$ $\tilde{r}_2 = ([0.18, 0.31], [0.67, 0.67])$ $\tilde{r}_3 = ([0.09, 0.18], [0.75, 0.80])$ $\tilde{r}_4 = ([0.06, 0.17], [0.76, 0.80])$	$q \in [0, 0.555)$ $q = 0.555$ $q = (0.555, 1]$	$T(\tilde{r}_1) > T(\tilde{r}_2) >$ $T(\tilde{r}_3) > T(\tilde{r}_4)$ $T(\tilde{r}_1) = T(\tilde{r}_2) >$ $T(\tilde{r}_3) > T(\tilde{r}_4)$ $T(\tilde{r}_2) > T(\tilde{r}_1) >$ $T(\tilde{r}_3) > T(\tilde{r}_4)$	$x_1 \succ x_2 \succ x_3 \succ x_4$ $x_1 \sim x_2 \succ x_3 \succ x_4$ $x_2 \succ x_1 \succ x_3 \succ x_4$

Step 5: With $\tilde{R}_C'^{*}$ and formula (5), the following overall IVIFVs of the four CC vendors are derived:

$$\tilde{r}_1 = \big([0.56, 0.57], [0.41, 0.41]\big), \qquad \tilde{r}_2 = \big([0.59, 0.60], [0.38, 0.39]\big),$$
$$\tilde{r}_3 = \big([0.43, 0.43], [0.50, 0.52]\big), \qquad \tilde{r}_4 = \big([0.35, 0.35], [0.62, 0.63]\big).$$

Step 6: By the score function for IVIFVs, we obtain

$$S(\tilde{r}_1) = 0.155, \qquad S(\tilde{r}_2) = 0.210, \qquad S(\tilde{r}_3) = -0.080, \qquad S(\tilde{r}_4) = -0.275.$$

Therefore, the ranking order is $x_2 \succ x_1 \succ x_3 \succ x_4$.

From the ranking results, the object x_2 should be chosen as the best CC vendor.

To facilitate the comparison between the proposed method and previous ones, we further explore the solution for this example by using several other methods (Liao *et al.*, 2014; Wan *et al.*, 2016; Meng *et al.*, 2018). The ranking results based on different methods are listed in Table 1.

As can be seen from Table 1, the ranking orders of four CC vendors in the first three methods are the same. Due to the different values of two subjective parameters, diverse ranking orders of the four objects are derived by Wan *et al.*'s method (Wan *et al.*, 2016).

In addition, all of the above four GDM methods are based on the multiplicative consistency of IVIFPRs. Through the application process of these methods in Example 3, the comparison of them can be seen in Table 2.

Table 2
Comparison of different methods.

Methods	Is it independent of the object labels?	Dose it study the incomplete case?	Does it consider how to determine the weights of the DMs?	Does it study the consensus?
New method	Yes	Yes	Yes	Yes
Method in Meng *et al.* (2018)	Yes	Yes	Yes	Yes
Method in Liao *et al.* (2014)	No	Yes	No	Yes
Method in Wan *et al.* (2016)	No	No	Yes	No

Table 2 shows that there are many similarities between the new method and Meng *et al.*'s method (Meng *et al.*, 2018), and both methods can avoid some limitations of other existing methods. However, since the two methods are based on different multiplicative consistency concepts for IVIFPRs, there are also some differences between them.

Remark 2. By the comparison between the proposed method and Meng *et al.*'s method (Meng *et al.*, 2018) in solving Example 3, we further summarize the similarities and differences between them.

The similarities between Meng *et al.*'s method (Meng *et al.*, 2018) and the proposed method include:

(i) Both of them are based on consistency and consensus analysis to solve GDM problems with IVIFPRs, which can ensure the rationality of ranking results.
(ii) The multiplicative consistency definitions proposed by the two methods are independent of compared objects.
(iii) Both of them can deal with IVIFPRs with missing values, especially the situation where preference information of some objects is completely unknown.

Compared with Meng *et al.*'s method (Meng *et al.*, 2018), the proposed method has the following advantages:

(i) Without the transformation between IVIFPRs and the associated QFIPRs, the new method is simpler than Meng *et al.*'s method.
(ii) The multiplicative consistency definition proposed in the new method is more flexible than that in Meng *et al.*'s method. It is noticeable that the former is based on Definition 3, while the latter is based on Definition 2.
(iii) Programming-model-based methods for improving the consistency and consensus level permit different judgments to have different adjustments, while Meng *et al.*'s method adjusts all judgments without considering their differences. Besides, the proposed method can ensure the minimum total adjustment, while Meng *et al.*'s method cannot.

7. Conclusions

To solve the problem of choosing the appropriate CC vendors in SMEs, this paper proposed a multiplicative consistency and consensus-based method for GDM with IVIFPRs.

The consistency concept defined in this paper is flexible and is independent of the comparison order of objects. The situation that there is incomplete information about DMs' preference is also discussed. Based on the new multiplicative consistency concept, a programming model is established to deal with inconsistent IVIFPRs. Compared with the iterative method, the programming model-based method can achieve the minimum adjustment of original preference information. In addition, programming models are constructed to determine the DMs' weights and improve individual consensus in the circumstance of GDM. The advantage of the programming model-based method for consensus improvement is to endow different IVIFVs with different adjustments and retain more original preference information. A practical decision-making example of a SME in China is given to show the feasibility and efficiency of the proposed method. However, there are also some limitations of the proposed method. The pairwise comparison method based on the subjective cognition of experts is directly adopted to obtain the priority of objects, which makes the decision result lack the objective explanation under the specific evaluation index. In addition, some models involve non-linear constraints which may increase the processing time of the software.

In this paper, PRs with IVIFVs is studied. The main characteristic is that this type of PRs reflects the DMs' uncertain preferred and non-preferred degrees of one object over the other. In future research, we can further study PRs with other types of fuzzy information such as hesitate fuzzy information (Narayanamoorthy *et al.*, 2019), picture fuzzy information (Si *et al.*, 2019), neutrosophic information (Liu *et al.*, 2018) and interval-valued intuitionistic linguistic information (Tang *et al.*, 2019, 2020). Moreover, this paper only analyses the application of the proposed method in the selection of CC vendors. In general, this method is suitable for dealing with complex MCDM problems with IVIFPRs such as QFD (Liu *et al.*, 2017; Yu *et al.*, 2018), project investment scheme selection (Wu *et al.*, 2019) and information system development program selection (Kirmizi and Kocaoglu, 2019).

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