

CONSERVATIVE DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS WITH TIME-ADAPTIVE GRIDS

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Abstract. This paper is devoted to research aspects of the convergence rate of conservative difference schemes (d.s.) with time-adaptive grids in cases, where a space grid is irregular and the third boundary-value problem is considered for one-dimensional linear parabolic equations. The unconditional convergence of created d.s. is proved in a C-metric at the rate $O(h^2 + \tau_0^{1/2})$

Key words: difference schemes (d.s.), time-adaptive grid, convergence, stability, approximation.

Introduction. Unconditional converged conservative and non-conservative difference schemes (d.s.) with time-adaptive grids were constructed (Matus, 1990; 1991; 1993) for the large-scale set of mathematical physics problems. Numerical simulation of the problems with singularities in a solution demonstrates a high-level efficacy of numerical methods of this type.

The analysis of theoretical aspects (stability, convergence) of d.s. with time-adaptive grids is a non-trivial problem even in the linear case (Matus, 1991; 1993), because in this case it is impossible to use well-known a priori estimates of Samarskii (1977). The latter circumstance is explained by the fact that methods, discussed above, may be transformed to d.s. with variable (and, in addition, discontinuous) weights defined for the whole grid of nodes.

The aim of this paper is to generalize the results obtained by Matus (1993) both to the case of an arbitrary grid of nodes $\hat{\omega}_h$

(with weaker requirements to smoothness of a differential problem solution) and to the case of parabolic equations with boundary conditions of the third type.

1. Statement of the problem. Let us preliminarily introduce necessary designations. There is a finite number of lines $x = x_\nu$, $\nu = 0, 1, \dots, \nu_0$, in the $\bar{Q}_{t_0} = \{0 \leq x \leq L, 0 \leq t \leq t_0\}$ region, which are parallel to the axis Ot , and $x_{\nu_1} < x_{\nu_2}$ for $\forall \nu_1 < \nu_2$. Designate

$$\begin{aligned} \Delta_\nu &= \Omega_\nu \times (0, t_0] = (x_\nu < x < x_{\nu+1}, \quad 0 < t \leq t_0), \\ \nu &= 0, 1, \dots, \nu_0 - 1, \quad x_0 = 0, \quad x_{\nu_0} = L; \\ Q_{t_0} &= \sum_{\nu=0}^{\nu_0-1} \Delta_\nu, \quad \bar{\Delta}_\nu = (x_\nu \leq x \leq x_{\nu+1}, \quad 0 \leq t \leq t_0). \end{aligned}$$

Now we can formulate the first boundary-value problem for a linear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x, t) \frac{\partial u}{\partial x} \right) + f(x, t), \quad (x, t) \in Q_{t_0}, \quad (1.1)$$

$$0 < k_1 \leq k(x, t) \leq k_2, \quad k_1, k_2 = \text{const}, \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad u(0, t) = \mu_1(t), \quad u(L, t) = \mu_2(t). \quad (1.3)$$

We suppose the next sentences about the exact solution of the problem (1.1) – (1.3) $u(x, t)$ and functions $f(x, t), k(x, t)$ are true:

1⁰. The functions $k(x, t), f(x, t)$ can have the first type breaks for lines $x = x_\nu$, $\nu = 1, 2, \dots, \nu_0 - 1$. Conjugation conditions, fulfilled for each break line, are:

$$\begin{aligned} [u]_{x_\nu} &= u(x_\nu + 0, t) - u(x_\nu - 0, t) = 0, \\ \left[k \frac{\partial u}{\partial x} \right]_{x_\nu} &= 0, \quad \nu = 1, 2, \dots, \nu_0 - 1. \end{aligned}$$

2⁰. Outside of break lines $x = x_\nu$ the functions $u(x, t), k(x, t), f(x, t)$ have all necessary bounded derivations that will be necessary for future discussions.

Let us remark that some aspects of an existence and uniqueness of the solution of the problem (1.1) – (1.3) under given as-

sumptions for a smoothness have been investigated by Samarskii and Fryazinov (Samarskii, 1960; Samarskii and Fryazinov, 1961).

2. Creation and realization of d.s. Let us introduce an irregular (non-uniform) space grid

$$\hat{\omega}_h = \{x_i = x_{i-1} + h_i, i = 1, 2, \dots, N, x_0 = 0, x_N = L\},$$

that all break lines $x = x_\nu$ pass over nodes of $\hat{\omega}_h$. Besides, we consider the following two types of a time grid – with time steps equal to τ and $\tau = \tau/p$, respectively ($p \geq 1$ is an integer):

$$\begin{aligned} \bar{\omega}_\tau &= \{t_j = j\tau, j = 0, 1, \dots, j_0, \tau j_0 = t_0\}, \\ \bar{\omega}_{\tau_0} &= \{t_{j+\alpha/p} = (j + \alpha/p)\tau, \alpha = 0, 1, \dots, p, j = 0, 1, \dots, j_0 - 1\}. \end{aligned}$$

We suppose a priori that we can find a good enough approximate solution in the domain

$$\hat{\omega}_2^j = \{(x_i, t_j) : m_1^j \leq i \leq m_2^j, \alpha = 0, 1, \dots, p, m_1^j > 1, m_2^j < N - 1\}$$

by applying the small time step τ_0 only. But it is possible to use the big time step τ in the area $\hat{\omega}_1^j = \hat{\omega}_{h\tau_0} \setminus \hat{\omega}_2^j$ without an essential precision reduction (see Fig.1).

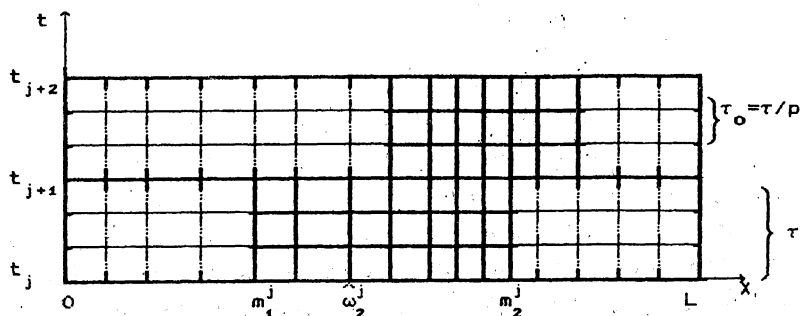


Fig. 1. Time-adaptive grid.

We want to construct conservative d.s. of unconditional stability which would permit to calculate a numerical solution of the initial problem for $\alpha = 1, 2, \dots, p - 1$ in the domains $\hat{\omega}_2^j$ only, $j = 0, 1, \dots, j_0 - 1$, and outside the adaptation zone – for integer time

layers t_j only. That makes it possible to substantially reduce the total amount of computation in cases $m_2^j - m_1^j \ll N$. Remark that practical approaches of interpolation or extrapolation in solving this problem lead to correlations between h and t (conditional stability).

Using the integral interpolation method the initial problem is approximated by the difference one

$$y_{\bar{i},\alpha} = ((ay_{\bar{x}})^{(\sigma_\alpha)})_{\bar{x}} + \varphi^{(\sigma_\alpha)}, \quad i = 1, \dots, N-1, \quad (2.1)$$

$$y_i^0 = u_{0,i}, \quad y_{(\alpha),0} = \mu_1(t_{j+\alpha/p}), \quad y_{(\alpha),N} = \mu_1(t_{j+\alpha/p}), \quad (2.2)$$

where:

$$\begin{aligned} y_{\bar{i},\alpha} &= (y_{(\alpha)} - y_{(\alpha-1)})/\tau_0, \quad y_{(\alpha)} = y_i^{j+\alpha/p} = y(x_i, t_{j+\alpha/p}), \\ \sigma_\alpha &= \begin{cases} \alpha, & i = 1, 2, \dots, m_1^j + 1, m_2^j, m_2^j + 1, \dots, N-1, \\ \sigma, & i = m_1^j + 2, \dots, m_2^j - 1, \end{cases} \quad (2.3) \\ v^{(\sigma_\alpha)} &= \sigma_\alpha v_{(\alpha)} + (1 - \sigma_\alpha)v_{(\alpha-1)}, \quad \sigma = \text{const} > 0, \end{aligned}$$

and template functionals a, φ are defined in the usual way (Samarskii, 1977)

$$a = k(x_{i-0.5}, t_j), \quad \varphi = \frac{h_i f(x_i - 0, t_j) + h_{i+1} f(x_i + 0, t_j)}{2\bar{h}_i},$$

$$\bar{h} = 0.5(h_i + h_{i+1}).$$

Other designations are taken from Samarskii (1977), too.

The realization of d.s. with a time-adaptive grid in the case of a regular space grid ω_h is detailed in (Matus, 1993). Now we will briefly describe our computation process organization. Owing to Lemma 1 (the lemma on the algebraic equivalence of d.s., Matus, 1990), difference equations (2.1) may be written in the form

$$(y_{(\alpha)} - y^j)/(\alpha\tau_0) = ((ay_{\bar{x}})_{\bar{x}} + \varphi)_{(\alpha)}, \quad (x, t) \in \hat{\omega}_1^j,$$

$$y_{\bar{i},\alpha} = ((ay_{\bar{x}})^{(\sigma_\alpha)})_{\bar{x}} + \varphi^{(\sigma_\alpha)}, \quad (x, t) \in \hat{\omega}_2^j.$$

This d.s. amounts to a system of three-point algebraic equations for each fractional time layer $\alpha = 1, 2, \dots, p$; the coefficients

of these equations are independent of $y_{(\alpha-1)}$ values for $(x, t) \in \hat{\omega}_1^j$; sufficient conditions of the pivot method stability and d.s. conservatism conditions are satisfied. Therefore, by using the opposed pivot algorithm, we may calculate an unknown function $y_{(\alpha)}$ in the adaptation zone $\hat{\omega}_2^j$ only for $\alpha = 1, 2, \dots, p-1$. When $\alpha = p$ the numerical solution must be found for all $x_i \in \hat{\omega}_h$ by the usual mode.

REMARK 1. If we apply ordinary implicit conservative d.s. to the general case, $8Np$ arithmetical operations are necessary to compute an approximate solution for $t_j < t \leq t_{j+1}$, $t \in \omega_{\tau_0}$ (Hockney and Estwood, 1987). Since we don't calculate a solution of a difference problem for $\alpha = 1, 2, \dots, p-1$ in the region $\hat{\omega}_1^j$, that makes it possible to save $2N(p-1)$ operations, if $m_2^j - m_1^j \ll N$. For instance, if $p \geq 5$, the total economy of arithmetical operations is 20%. For the non-linear case this economy may be even more significant, because of the iteration process used (Matus, 1990).

3. Stability. It is possible to show that a conservative d.s. with homogeneous boundary conditions belongs to the so-called "initial family of d.s." (Samarskii, 1977) for any coefficient $k(x, t)$ that is Lipschitz-continuous with respect to time with some constant c_0 . In that case the sufficient stability condition

$$B(t) \geq 0.5\tau_0 A(t), \quad \forall t \in \omega_{\tau_0},$$

is true for

$$t_0 < c_0^{-1}, \quad \sigma_\alpha \geq \sigma_0, \quad \sigma_0 = 0.5 + \tau_0 c_0 / 2.$$

And by virtue of Theorem 12 (Samarskii, 1977, page 377), the a priori estimate

$$\|y_{(\alpha)}\|_{A_{\alpha-1}} \leq M_1 \left(\|y_0\|_{A_{0,0}} + \|\varphi_0\|_{A_{0,0}^{-1}} + \|\varphi_{(\alpha-1)}\|_{A_{\alpha-1}^{-1}} + \sum_{j'=0}^{j-1} \sum_{k=1}^{p-1} \tau_0 \|(A^{-1}\varphi)_{\bar{r}_{j',k}}\| + \sum_{k=1}^{\alpha-1} \tau_0 \|(A^{-1}\varphi)_{\bar{r}_{j,k}}\| \right) \quad (3.1)$$

holds.

Here

$$\|y_{(\alpha)}\|_{A_{\alpha-1}}^2 = (A_{\alpha-1}y_{(\alpha)}, y_{(\alpha)})_* = -((a^{(\sigma_\alpha)}y_{(\alpha)\bar{x}})_{\bar{x}}, y_{(\alpha)})_*,$$

$$(y, v)_* = \sum_{i=1}^{N-1} \bar{h}_i y_i v_i, \quad \|y\|^2 = (y, y)_*,$$

$$A_{j,k} = A(t_{j+k/p}), \quad M_1 = e^{0.5c_0 t_0}.$$

4. Approximation error and convergence. Substituting $y = z + u$ into (2.1) – (2.2), we obtain the problem for the method error

$$z_{\bar{i},\alpha} = ((az_{\bar{x}})^{(\sigma_\alpha)})_{\bar{x}} + \psi_{(\alpha-1)}, \quad (4.1)$$

$$z_i^0 = 0, \quad z_{(\alpha),0} = z_{(\alpha),N} = 0, \quad (4.2)$$

where (Samarskii, 1977) we represent the approximation error $\psi_{(\alpha-1)}$ in the form

$$\psi_{(\alpha-1)} = \eta_{1(\alpha-1)\bar{x}} + \eta_{2(\alpha-1)\bar{x}} + \psi_{1(\alpha-1)}, \quad (4.3)$$

$$\eta_{1(\alpha-1)} = \tau_0 ((\sigma_\alpha - 0.5)(au_{\bar{x}})_{\bar{i},\alpha}) = O(\tau_0), \quad (4.4)$$

$$\eta_{2(\alpha-1)} = (au_{\bar{x},i})_{(\alpha)}^{(0.5)} - (ku')_{(i-0.5)}^{j+(\alpha-0.5)/p} + 0.125h_i^2(u' - f')_{(i-0.5)}^{j+(\alpha-0.5)/p} = O(h_i^2 + t_0^2), \quad (4.5)$$

$$\psi_{1(\alpha-1)} = O(\bar{h}_i^2 + \tau_0). \quad (4.6)$$

Unfortunately, we cannot use the a priori estimate in negative norm on the right-hand side of (3.1) to find the accuracy of our d.s., owing to the dependence of σ_α on a grid node $(x_i, t_{j+\alpha/p})$, $\eta_{1\bar{i},\alpha} = O(1)$. On the other hand, since the initial solution is not smooth, we cannot apply all the variety of a priori estimates of Matus (1993). Matus (1993) investigated the convergence of conservative d.s. with time-adaptive grids in the case when the space grid ω_h is regular and there exist high order limited derivations. In addition, it ought to be noted that, because of a discontinuity of the weight function σ_α in the nodes where the regions $\hat{\omega}_1^j$ and $\hat{\omega}_2^j$ join, both the local approximation

$$\psi_{(\alpha-1)} = O(\bar{h} + \tau_0 + \tau_0 h^{-1}),$$

and the error in the norm L_2

$$\|\psi_{(\alpha-1)}\|_{L_2} = O(h + \tau_0 + \tau_0 h^{-1/2})$$

are only of conditional type.

We apply both the scalar product $(y, v)_*$ (see above) and next definitions for the proof of convergence:

$$\begin{aligned} (y, v) &= \sum_{i=1}^N h_i y_i v_i, \quad \|y\| = (y, y), \quad \|z_{(\alpha)}\|_1^2 = (a_{(\alpha)}, z_{(\alpha)}^2), \\ Q_{(\alpha)}(v) &= \|a_{(\alpha)}^{1/2} v_{(\alpha)\bar{x}} + a_{(\alpha)}^{-1/2} \eta_{2(\alpha)}\|^2, \quad Q_{(0)} = Q^j, \quad Q_{(p)} = Q^{j+1}, \\ \|v\|_c &= \max_{0 \leq i \leq N} |v_i|, \quad h = \max_{1 \leq i \leq N} h_i. \end{aligned}$$

Theorem 4.1. *Let $u(x, t), k(x, t), f(x, t)$ satisfy conditions $1^0 - 2^0$, then, for $\sigma \geq 0.5 + \varepsilon$, $0 < \varepsilon \leq 1$, conservative d.s. (2.1) – (2.2) unconditionally converges in the c -metric such that for small enough $\tau_0 < \tau_0^*$ and all $\alpha = 1, \dots, p$, $j = 0, 1, \dots, j_0 - 1$ the following estimate*

$$\|z_{(\alpha)}\|_c \leq c(h^2 + \tau_0^{1/2}),$$

holds, $c = \text{const} > 0$ is independent of $h_i, \tau_0, y_{(\alpha)}$.

Proof. Multiply (as scalar) both parts of equation (4.1) by $2\tau_0 z_{\bar{i}, \alpha}$ and consider the scalar products separately. It is evident that

$$2\tau_0(z_{\bar{i}, \alpha}, z_{\bar{i}, \alpha})_* = 2\tau_0 \|z_{\bar{i}, \alpha}\|^2.$$

Applying Greens' formula, modify the expression

$$\begin{aligned} 2\tau_0(z_{\bar{i}, \alpha}, ((az_{\bar{x}})^{\sigma_\alpha})_{\bar{x}})_* &= -2\tau_0(z_{\bar{i}\bar{x}, \alpha}, (az_{\bar{x}})^{\sigma_\alpha}) \\ &= -\|z_{(\alpha)}\|_1^2 + \|z_{(\alpha-1)}\|_1^2 - 2\tau_0^2((\sigma_\alpha - 0.5)a_{(\alpha)}, z_{\bar{i}\bar{x}, \alpha}^2) \\ &\quad + \tau_0(a_{\bar{i}, \alpha}, z_{(\alpha-1)\bar{x}}^2) + 2\tau_0^2(z_{\bar{i}\bar{x}, \alpha}, (1 - \sigma_\alpha)a_{\bar{i}, \alpha}z_{(\alpha-1)\bar{x}}). \end{aligned}$$

Using Cauchy-Bunyakovskii's inequality, and the ε -inequality, we have

$$\begin{aligned} 2\tau_0 \|z_{\bar{i}, \alpha}\|^2 + 2\tau_0^2((\sigma_\alpha - 0.5 - \varepsilon/4)a_{(\alpha)}, z_{\bar{i}\bar{x}, \alpha}^2) + \|z_{(\alpha)}\|_1^2 \\ \leq (1 + \tau_0 c_1) \|z_{(\alpha-1)}\|_1^2 + 2\tau_0(z_{\bar{i}, \alpha}, \psi_{(\alpha-1)})_*, \\ c_1 = c_0(1 + 4c_0\tau_0^* \varepsilon^{-1}(p - 1)^2). \end{aligned}$$

Taking into account (4.3) – (4.6) and using the summation by parts formula, Cauchy-Bunyakovskii inequality, and the ε -inequality, we arrive at

$$\begin{aligned} & 2\tau_0(z_{\bar{i},\alpha}, \psi_{(\alpha-1)*}) = -2\tau_0(z_{\bar{i}\bar{x}\alpha}, \eta_{1(\alpha-1)}) + \\ & \quad + 2\tau_0(z_{\bar{i},\alpha}, \psi_{1(\alpha-1)*}) - 2\tau_0(z_{\bar{i}\bar{x},\alpha}, \eta_{2(\alpha-1)}), \\ & -2\tau_0(z_{\bar{i}\bar{x},\alpha}, \eta_{1(\alpha-1)}) + 2\tau_0(z_{\bar{i},\alpha}, \psi_{1(\alpha-1)*}) \leq (\varepsilon/4)\tau_0^2(a_{(\alpha)}, z_{\bar{i}\bar{x},\alpha}^2) \\ & \quad + 2\tau_0\|z_{\bar{i},\alpha}\|^2 + \tau_0(4(\tau\varepsilon k_1)^{-1}\|\eta_{1(\alpha-1)}\|^2 + 0.5\|\psi_{1(\alpha-1)}\|^2), \\ & -2\tau_0(z_{\bar{i}\bar{x},\alpha}, \eta_{2(\alpha-1)}) \leq -2(z_{(\alpha)\bar{x}}, \eta_{2(\alpha)}) + 2(z_{(\alpha-1)\bar{x}}, \eta_{2(\alpha-1)}) \\ & \quad + 2\tau_0\|z_{(\alpha-1)}\|_1^2 + (\varepsilon/4)\tau_0^2(a_{(\alpha)}, z_{\bar{i}\bar{x},\alpha}^2) + \tau_0 c_3\|\eta_{2\bar{i},\alpha}\|^2, \\ & c_3 = (1 + 8\tau_0^*)/(2k_1). \end{aligned}$$

By the use of

$$\|\psi_{2(\alpha-1)}\|^2 = 4(\tau_0\varepsilon k_1)^{-1}\|\eta_{1(\alpha-1)}\|^2 + 0.5\|\psi_{1(\alpha-1)}\|^2 + c_3\|\eta_{2\bar{i},\alpha}\|^2,$$

we obtain the following inequality

$$\begin{aligned} & 2\tau_0^2((\sigma_\alpha - 0.5 - \varepsilon/2)a_{(\alpha)}, z_{\bar{i}\bar{x},\alpha}^2) + Q_{(\alpha)}(z) \\ & \leq Q_{(\alpha-1)}(z) + \tau_0 c_2\|z_{(\alpha-1)}\|_1^2 + \|a_{(\alpha)}^{-1/2}\eta_{2(\alpha)}\|^2 \\ & \quad - \|a_{(\alpha-1)}^{-1/2}\eta_{2(\alpha-1)}\|^2 + \tau_0\|\psi_{2(\alpha-1)}\|^2, \quad c_2 = c_1 + 2. \end{aligned}$$

Applying the condition of the theorem $\sigma \geq 0.5 + \varepsilon$ and (4.5), we find

$$Q_{(\alpha)}(z) \leq (1 + \tau_0 c_4)Q_{(\alpha-1)}(z) + \tau_0\|\psi_{3(\alpha-1)}\|^2, \quad (4.7)$$

where

$$\begin{aligned} \|\psi_{3(\alpha-1)}\|^2 &= 4(\tau_0\varepsilon k_1)^{-1}\|\eta_{1(\alpha-1)}\|^2 + 0.5\|\psi_{1(\alpha-1)}\|^2 \\ & \quad + c_5\|\eta_{2\bar{i},\alpha}\|^2 + c_6\|\eta_{2(\alpha)}\|^2 + c_7\|\eta_{2(\alpha-1)}\|^2, \quad (4.8) \\ c_4 &= 4c_2, \quad c_5 = (1 + 2\varepsilon + 8\tau_0^*)/(2k_1), \quad c_6 = (2k_1 + c_0 k_2)/(4k_1^2), \\ c_7 &= (4k_1 c_4 + 2k_1 + 3c_0 k_2)/(4k_1^2). \end{aligned}$$

Summing (4.7) over $\alpha = 1, \dots, p$, we obtain

$$Q^{j+1}(z) \leq e^{\tau c_4}(Q^j(z) + \tau \max_{1 \leq \alpha \leq p} \|\psi_{3(\alpha-1)}\|^2),$$

because $j = 0, 1, \dots, j_0 - 1$ is arbitrary:

$$Q^j(z) \leq t_j e^{c_4 t_j} \max_{0 \leq j' \leq j-1} \max_{1 \leq \alpha \leq p} \|\psi_{3(\alpha-1)}\|^2.$$

Thus, for arbitrary $j = 0, 1, \dots, j_0 - 1$, $\alpha = 1, \dots, p$ we derive from (4.7)

$$Q_{(\alpha)}(z) \leq e^{\alpha \tau_0 c_4} (t_j e^{c_4 t_j} \max_{0 \leq j' \leq j-1} \max_{1 \leq \alpha \leq p} \|\psi_{3(\alpha-1)}\|^2 + \alpha \tau_0 \max_{1 \leq \alpha' \leq \alpha} \|\psi_{3(\alpha'-1)}\|^2).$$

We obtain from the last expression and (4.3) - (4.6), (4.8) that

$$\|a_{(\alpha)}^{1/2} z_{(\alpha)\bar{x}} + a_{(\alpha)}^{-1/2} \eta_{2(\alpha)}\| \leq c_8 (h^2 + \tau_0^{1/2}),$$

$c_8 = \text{const} > 0$ is independent of $h_i, \tau_0, y_{(\alpha)}$. Using the triangle inequality, we find

$$\begin{aligned} \|a_{(\alpha)}^{1/2} z_{(\alpha)\bar{x}} + a_{(\alpha)}^{-1/2} \eta_{2(\alpha)}\| &\geq \|z_{(\alpha)}\|_1 - \|a_{(\alpha)}^{-1/2} \eta_{2(\alpha)}\|, \\ \|z_{(\alpha)}\|_1^2 &\leq c_8 (h^2 + \tau_0^{1/2}) + \|a_{(\alpha)}^{-1/2} \eta_{2(\alpha)}\| \\ &\leq c_8 (h^2 + \tau_0^{1/2}) + k_1^{-1/2} \|\eta_{2(\alpha)}\|. \end{aligned}$$

With the help of difference analogs of imbedding theorems (Samaraskii, 1977) it is easy to show that

$$\|z_{(\alpha)}\|_c \leq 0.5 k_1^{-1/2} L^{1/2} \|z_{(\alpha)}\|_1 \leq c (h^2 + \tau_0^{1/2}),$$

$c = \text{const} > 0$ is independent of $h_i, \tau_0, y_{(\alpha)}$.

5. The third boundary-value problem. Let us show that the idea of a time-adaptive grid can be used in the case of the third type boundary conditions too. Let functions $u(x, t), k(x, t), f(x, t)$ be smooth enough in

$$Q_{t_0} = \{(x, t) : 0 < t \leq t_0, x \in \Omega\}, \quad \Omega = \{x : 0 < x < L\}.$$

We consider the third boundary-value problem for the linear heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x, t) \frac{\partial u}{\partial x} \right) + f(x, t), \quad (x, t) \in Q_{\tau_0}, \quad (5.1)$$

$$0 < k_1 \leq k(x, t) \leq k_2, \quad k_1, k_2 = \text{const}, \quad (5.2)$$

$$u(x, 0) = u_0(x), \quad (5.3)$$

$$k(0, t) \frac{\partial u(0, t)}{\partial x} = \beta_1 u(0, t) - \mu_1(t), \quad \beta_1 = \text{const} > 0, \quad (5.4)$$

$$-k(L, t) \frac{\partial u(L, t)}{\partial x} = \beta_2 u(L, t) - \mu_2(t), \quad \beta_2 = \text{const} > 0.$$

For simplicity, we assume that a space step h is constant and coefficient $k(x, t)$ depends only on the variable x ($k_1 \leq k(x) \leq k_2$). A time-adaptive grid ω_{τ_0} is defined like in Part 2. Using an integral interpolation method we proceed from the differential problem (5.1) – (5.4) to the conservative d.s. with the grid $\omega_{h\tau_0}$

$$y_{\bar{i}, \alpha} = ((ay_{\bar{x}})^{(\sigma_\alpha)})_x + \varphi^{(\sigma_\alpha)}, \quad i = 1, \dots, N-1, \quad (5.5)$$

$$y_i^0 = u_0, \quad (5.6)$$

$$(a_1 y_{\bar{x}, 1} - \beta_1 y_0)^{(\sigma_\alpha)} = 0.5h(y_{\bar{i}, \alpha})_0 - \bar{\mu}_1 - 0.5h\bar{f}_0, \quad (5.7)$$

$$(-a_N y_{\bar{x}, N} - \beta_2 y_N)^{(\sigma_\alpha)} = 0.5h(y_{\bar{i}, \alpha})_N - \bar{\mu}_2 - 0.5h\bar{f}_N,$$

$$\bar{v} = v(x, \bar{t}), \quad \bar{t} = t_{j+(\alpha-0.5)/p}.$$

This scheme is realized likewise as (2.1) – (2.2). Referring to the existence and uniqueness of a solution of the initial differential problem (5.1) – (5.4), this proof may be found, for instance, in (Ladyzhenskaya *et al.*, 1967). Let us denote

$$\varphi_0^{(\sigma_\alpha)} = \bar{\mu}_1 / (0.5h) + \bar{f}_0, \quad \varphi_N^{(\sigma_\alpha)} = \bar{\mu}_2 / (0.5h) + \bar{f}_N$$

and redefine (5.5) for $i = 0, N$ respectively:

$$(y_{\bar{i}, \alpha})_0 = (a_1 y_{\bar{x}, 1} - \beta_1 y_0)^{(\sigma_\alpha)} / (0.5h) + \varphi_0^{(\sigma_\alpha)},$$

$$(y_{\bar{i}, \alpha})_N = (-a_N y_{\bar{x}, N} - \beta_2 y_N)^{(\sigma_\alpha)} / (0.5h) + \varphi_N^{(\sigma_\alpha)}.$$

Now write (5.5), (5.7) in a more convenient form (Samarskii and Gulin, 1973)

$$y_{\bar{i}, \alpha} = \tilde{A}_{(\alpha)} y + \varphi^{(\sigma_\alpha)}, \quad i = 0, 1, \dots, N. \quad (5.8)$$

From (5.8), (5.6) we have the problem for the method error z

$$\begin{aligned} z_{\bar{i},\alpha} &= \tilde{A}_{(\alpha)}z + \psi_{(\alpha-1)}, \\ i &= \overline{0,N}, \quad a = \overline{1,p}, \quad j = \overline{0,j-1}, \\ z(x, 0) &= 0, \quad x \in \omega_h; \end{aligned} \tag{5.10}$$

with

$$\begin{aligned} \psi_{(\alpha-1)} &= \eta_{(\alpha-1)x} + \psi_{1(\alpha-1)}, \quad i = 1, \dots, N-1, \\ \eta_{(\alpha-1)} &= \tau_0 \sigma_\alpha (au_{\bar{x}})_{\bar{i},\alpha} = O(\tau_0), \\ \psi_{1(\alpha-1)} &= O(h^2 + \tau_0); \\ \psi_{(\alpha-1)} &= O(h^2 + \tau_0)/(0.5h), \quad i = 0, N. \end{aligned}$$

We introduce the following scalar products and notation (Samarskii and Gulin, 1973, page 40):

$$\begin{aligned} \eta_{1(\alpha-1)} &= 0.5h\psi_{(\alpha-1),0}, \quad \eta_{2(\alpha-1)} = 0.5h\psi_{(\alpha-1),N}, \\ \eta_{i(\alpha-1)} &= O(h^2 + \tau_0), \quad i = 1, 2, \\ [y, v] &= \sum_{i=1}^{N-1} y_i v_i h + 0.5h(y_0 v_0 + y_N v_N), \\ (y, v) &= \sum_{i=1}^{N-1} y_i v_i h, \quad (y, v) = \sum_{i=1}^N y_i v_i h. \end{aligned}$$

We perform a scalar multiplication of (5.9) by $2\tau_0 z_{\bar{i},\alpha}$:

$$2\tau_0 [z_{\bar{i},\alpha}, z_{\bar{i},\alpha}] = 2\tau_0 [\tilde{A}_{(\alpha)}z, z_{\bar{i},\alpha}] + 2\tau_0 [z_{\bar{i},\alpha}, \psi_{(\alpha-1)}].$$

Applying the form of $\tilde{A}_{(\alpha)}$ and Greens' formula, we arrive at to

$$\begin{aligned} &2\tau_0 [\tilde{A}_{(\alpha)}z, z_{\bar{i},\alpha}] \\ &= -2\tau_0^2 ((\sigma_\alpha - 0.5)a, z_{\bar{i}\bar{x},\alpha}^2] + (a, z_{(\alpha-1)\bar{x}}^2) - (a, z_{(\alpha)\bar{x}}^2) \\ &\quad - 2\tau_0^2 (\sigma_\alpha - 0.5)_0 \beta_1 (z_{\bar{i},\alpha}_0)^2 - 2\tau_0^2 (\sigma_\alpha - 0.5)_N \beta_2 (z_{\bar{i},\alpha}_N)^2 \\ &\quad - \beta_1 (z_{(\alpha),0}^2 - z_{(\alpha-1),0}^2) - \beta_2 (z_{(\alpha),N}^2 - z_{(\alpha-1),N}^2). \end{aligned} \tag{5.11}$$

Using the summing by parts formula, the ε -inequality, and Cauchy-Bunyakovskii's inequality we have

$$\begin{aligned} 2\tau_0[z_{\bar{i},\alpha}, \psi_{(\alpha-1)}] &= 2\tau_0(z_{\bar{i},\alpha}, \psi_{(\alpha-1)}) \\ &\quad + 2\tau_0\eta_{1(\alpha-1)}(z_{\bar{i},\alpha})_0 + 2\tau_0\eta_{2(\alpha-1)}(z_{\bar{i},\alpha})_N, \\ 2\tau_0(z_{\bar{i},\alpha}, \psi_{(\alpha-1)}) &= -2\tau_0(z_{\bar{i}\bar{x},\alpha}, \eta_{(\alpha-1)}) + 2\tau_0(z_{\bar{i},\alpha}, \psi_{1(\alpha-1)}) \\ &\leq 2\tau_0^2\varepsilon(a, z_{\bar{i}\bar{x}}^2\alpha) + 2\tau_0\|z_{\bar{i},\alpha}\|^2 + \tau_0\|\psi_{2(\alpha-1)}\|^2, \end{aligned} \quad (5.12)$$

where the next designation is used

$$\|\psi_{2(\alpha-1)}\|^2 = (2\tau_0\varepsilon k_1)^{-1}\|\eta_{(\alpha-1)}\|^2 + 0.5\|\psi_{1(\alpha-1)}\|^2 = O(h^2 + \tau_0^{1/2})^2.$$

In its turn,

$$\begin{aligned} 2\tau_0\eta_{1(\alpha-1)}(z_{\bar{i},\alpha})_0 &= 2\eta_{1(\alpha)}z_{(\alpha),0} - 2\eta_{1(\alpha-1)}z_{(\alpha-1),0} \\ &\quad - 2\tau_0^2(z_{\bar{i},\alpha})_0\eta_{1\bar{i},\alpha} - 2\tau_0\eta_{1\bar{i},\alpha}z_{(\alpha-1),0}, \\ 2\tau_0^2(z_{\bar{i},\alpha})_0\eta_{1\bar{i},\alpha} &\leq 2\tau_0^2\varepsilon\beta_1(z_{\bar{i},\alpha})_0^2 + \tau_0(\tau_0/(2\varepsilon\beta_1))\eta_{1\bar{i},\alpha}^2, \\ 2\tau_0\eta_{1\bar{i},\alpha}z_{(\alpha-1),0} &\leq \tau_0(3\beta_1/4)z_{(\alpha-1),0}^2 + \tau_0(1/(3\beta_1))\eta_{1\bar{i},\alpha}^2. \end{aligned} \quad (5.13)$$

The following expression is discussed similarly

$$2\tau_0\eta_{2(\alpha-1)}(z_{\bar{i},\alpha})_N.$$

Taking into account

$$\eta_{i(\alpha)}^2 - \eta_{i(\alpha-1)}^2 = \tau_0\eta_{i\bar{i},\alpha}(\eta_{i(\alpha)} + \eta_{i(\alpha-1)}), \quad i = 1, 2,$$

and using

$$\begin{aligned} Q_{(\alpha)} &= (a, z_{(\alpha)\bar{x}}^2) + (3\beta_1/4)z_{(\alpha),0}^2 + (0.5\beta_1^{1/2}z_{(\alpha),0} - 2\beta_1^{-1/2}\eta_{1(\alpha)})^2 \\ &\quad + (3\beta_2/4)z_{(\alpha),N}^2 + (0.5\beta_2^{1/2}z_{(\alpha),N} - 2\beta_2^{-1/2}\eta_{2(\alpha)})^2, \\ \|\psi_{3(\alpha-1)}\|^2 &= \|\psi_{2(\alpha-1)}\|^2 + (4/\beta_1)\eta_{1\bar{i},\alpha}(\eta_{1(\alpha)} + \eta_{1(\alpha-1)}) \\ &\quad + (4/\beta_2)\eta_{2\bar{i},\alpha}(\eta_{2(\alpha)} + \eta_{2(\alpha-1)}) + (1/(3\beta_1)) \\ &\quad + \tau_0^*/(2\varepsilon\beta_1)\eta_{1\bar{i},\alpha}^2 + (1/(3\beta_2) + \tau_0^*/(2\varepsilon\beta_2))\eta_{2\bar{i},\alpha}^2 \\ &= O(h^2 + \tau_0^{1/2})^2, \quad \tau_0^* = \text{const}, \quad \tau_0 < \tau_0^*, \end{aligned}$$

we derive from (5.11) – (5.13) the recurrent inequality

$$Q_{(\alpha)} \leq (1 + \tau_0)Q_{(\alpha-1)} + \tau_0 \|\psi_{3(\alpha-1)}\|^2, \quad j = 0, \dots, j_0 - 1, \quad \alpha = 1, \dots, p;$$

hence

$$(a, z_{(\alpha)\bar{x}}^2] + 0.5(\beta_1 z_{(\alpha),0}^2 + \beta_2 z_{(\alpha),N}^2) = O(h^2 + \tau_0^{1/2})^2.$$

According to (Samarskii, 1977)

$$\|y\|_{\mathbb{C}}^2 \leq 2(L\|y\|)^2 + y_0^2,$$

we draw a conclusion that for $\sigma \geq 0.5 + \varepsilon$, $0 < \varepsilon \leq 1$, $\max_{t \in \omega_{\tau_0}} \|z\|_{\mathbb{C}} \leq c(h^2 + \tau_0^{1/2})$, $c = \text{const} > 0$, i.e., d.s. (5.5) – (5.7) converges in the \mathbb{C} -metric with the rate $O(h^2 + \tau_0^{1/2})$.

6. Construction of d.s. of a more general type. During the discussion on d.s. (2.1) – (2.2) we approximate of the second order derivative with respect to a spatial variable on the upper fractional time layer in the region $\hat{\omega}_1^j$, i.e., for $t = t_{(\alpha)} = t_{j+\alpha/p}$, whereas the time derivative is approximated with the help of values y_j and $y_{(\alpha)}$ (in the region $\hat{\omega}_1^j$). It would be natural to approximate the space operator from (1.1) as a linear combination of flow derivative approximations for $t = t_j$ and $t = t_{(\alpha)} = t_{j+\alpha/p}$ time layers.

Let the first boundary-value problem be formulated

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x, t) \frac{\partial u}{\partial x} \right) + f(x, t), \quad (\tau, t) \in Q_{t_0}, \quad (6.1)$$

$$0 < k_1 \leq k(x, t) \leq k_2, \quad k_1, k_2 = \text{const}, \quad (6.2)$$

$$u(x, 0) = u_0(x), \quad u(0, t) = \mu_1(t), \quad u(L, t) = \mu_2(t); \quad (6.3)$$

where

$$Q_{t_0} = \{(x, t) : 0 < t \leq t_0, x \in \Omega\}, \quad \Omega = \{x : 0 < x < L\}.$$

We assume that $u(x, t), k(x, t), f(x, t)$ have a sufficient number of bounded derivative with respect to x and t for all $(x, t) \in Q_{t_0}$. The existence and uniqueness of the solution of problem (6.1) – (6.3) are proved, for example, in (Ladyzhenskaya *et al.*, 1967).

A time-space grid is defined analogously as in Part 2 (see Fig.1).

Applying an integral interpolation method it is possible to approximate (6.1) – (6.3) by the difference scheme

$$y_{\bar{i},\alpha} = W_x + \varphi, \quad (6.4)$$

$$W = \begin{cases} \sigma_1 (ay_{\bar{x}})^{(\alpha)} + (1 - \sigma_1) (ay_{\bar{x}})^j, & i = 1, \dots, m_1^j + 1, m_2^j, \dots, N, \\ \sigma (ay_{\bar{x}})_{(\alpha)} + (1 - \sigma) (ay_{\bar{x}})_{(\alpha-1)}, & i = m_1^j + 2, \dots, m_2^j - 1, \end{cases}$$

$$\varphi = \begin{cases} \sigma_1 f^{(\alpha)} + (1 - \sigma_1) f^j, & i = 1, \dots, m_1^j + 1, m_2^j, \dots, N, \\ \sigma f_{(\alpha)} + (1 - \sigma) f_{(\alpha-1)}, & i = m_1^j + 2, \dots, m_2^j - 1. \end{cases}$$

$$y_i^0 = u_{0i}, \quad y_{(\alpha),0} = \mu_1(t_{j+\alpha/p}), \quad y_{(\alpha),N} = \mu_1(t_{j+\alpha/p}). \quad (6.5)$$

It is not difficult to see, if $\sigma_1 = 1$, that (6.4) is an analog of (2.1) with a regular space grid. But, choosing $\sigma_1 = 0.5$, we can create a scheme with the second order approximation with respect to both x and t variables in the regions $\omega_1^j \forall j = 0, \dots, j_0 - 1$. As a result of attempts of numerical simulation, there is a wide class of problems such that calculations for $\sigma_1 = 0.5$ provide a more exact numerical solution a for $\sigma_1 = 1$.

Unfortunately, now we can prove the convergence of d.s. (6.4) – (6.5) in the C-metric for $\sigma_1 \geq 0.5 + \epsilon$, $0 < \epsilon \leq 1$, $\epsilon = \text{const}$ only.

REMARK 2. The achieved results may be generalized to non-linear equations, too. Let us consider, for instance, the first boundary-value problem for a quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(k(x, t, u) \frac{\partial u}{\partial x} \right) + f(x, t, u),$$

$$u(x, 0) = u_0(x), \quad u(0, t) = \mu_1(t), \quad u(L, t) = \mu_2(t).$$

A conservative d.s. with time-adaptive grid (see Fig.1) has the following form

$$y_{\bar{i},\alpha} = ((ay_{\bar{x}})^{(\sigma_\alpha)})_{\bar{x}} + \varphi^{(\sigma_\alpha)}, \quad (x, t) \in \hat{\omega}_{h\tau_0},$$

where

$$a_1 = a(y), \quad i \neq m_1^j + 2, \dots, m_2^j - 1 \text{ and } a \neq p,$$

$$a_1 = a(y_{(\alpha)}) - \text{for all the remaining of indexes } i, \alpha;$$

template functionals $a(y)$, $\varphi(y)$ may be defined according to Samarskii (1977)

$$a(y) = 0.5(k(x_{i-1}, t_j, y_{i-1}^j) + k(x_i, t_j, y_i^j)),$$

$$\varphi(y) = f(x, t, y), \quad (x, t) \in \hat{\omega}_{h\tau_0},$$

and the weight σ_α is given by (2.3).

This d.s. is non-linear, therefore it is necessary to use an iteration process (Matus, 1991) for its realization.

If some conditions on the smoothness of functions $u(x, t)$, $k(x, t)$, $f(x, t)$ are true then it is possible to prove an unconditional convergence of the last d.s. solution in the uniform metric (C-matrix) at the rate $O(h^{3/2} + \tau_0^{1/2})$. This proof will be described more in detail in a separate paper.

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