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A Note on Reconstruction of Bandlimited Signals of Several Variables Sampled at Nyquist Rate

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Abstract. A standard problem in certain applications requires one to find a reconstruction of an analogue signal f from a sequence of its samples $f(t_k)_k$. The great success of such a reconstruction consists, under additional assumptions, in the fact that an analogue signal f of a real variable $t \in \mathbb{R}$ can be represented equivalently by a sequence of complex numbers $f(t_k)_k$, i.e. by a digital signal. In the sequel, this digital signal can be processed and filtered very efficiently, for example, on digital computers. The sampling theory is one of the theoretical foundations of the conversion from analog to digital signals. There is a long list of impressive research results in this area starting with the classical work of Shannon. Note that the well known Shannon sampling theory is mainly for one variable signals. In this paper, we concern with bandlimited signals of several variables, whose restriction to Euclidean space $\mathbb{R}^n$ has finite p-energy. We present sampling series, where signals are sampled at Nyquist rate. These series involve digital samples of signals and also samples of their partial derivatives. It is important that our reconstruction is stable in the sense that sampling series converge absolutely and uniformly on the whole $\mathbb{R}^n$. Therefore, having a stable reconstruction process, it is possible to bound the approximation error, which is made by using only of the partial sum with finitely many samples.

Key words: Shanon's sampling formula, multidimensional sampling series, sampling with partial derivatives, bandlimited signal, truncation error, Benstein's spaces.

1. Introduction

In simplistic terms, a signal might be discrete, such as letters or digits sequences, or it might be analogue, i.e. continuous functions, reading such as a temperature or a pressure. It is a remarkable fact that under additional assumptions, analogue signals and discrete signals are equivalent: an analogue signal f can be recovered exactly from its samples $\{f(x_j)\}_{j=-\omega}^{\omega}$, i.e. from a digital signal. This is the essence of the sampling theorems. These theorems are fundamental, in particular, in information theory and communication, particularly since the advent of modern digital computers.

An analogue signal is, in the classical sense, a function such that it is continuous with respect to a real variable (for example, time) and has finite energy, i.e. it is a square-integrable on $\mathbb{R}$. The following classical Shannon sampling theorem (see, e.g. Higgins, 1996; p. 51) states that if $f \in L^2(\mathbb{R})$ is bandlimited to $[-\sigma, \sigma]$, $\sigma > 0$, i.e. if the Fourier transform of f is supported on $[-\sigma, \sigma]$, then

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\sigma}\right) \frac{\sin(\sigma x - \pi k)}{\sigma x - \pi k} \tag{1.1}$$

and this series converge in $L^2(\mathbb{R})$-norm and uniformly on $\mathbb{R}$. This theorem was the starting point for many further developments in sampling theory. Series (1.1) is called a cardinal series of f. The sequence $\{\pi k/\sigma\}_{k=-\infty}^{\infty}$ is called a sampling sequence and the corresponding values $f(\pi k/\sigma)$ are called sample values with Nyquist sampling rate of σ/π samples per unit time. The first problem is to find conditions under which f can be reconstructed completely by (1.1). In most applications, the assumption that f is bandlimited, or equivalently that the spectrum of f is supported to certain bounded area in $\mathbb{R}$, is completely justified. More precisely, if a signal f is bandlimited to $[-\sigma, \sigma]$, $0 < \sigma < \infty$, or in other words, the quantity σ is the maximal frequency in the spectrum of f, then (1.1) perfectly recovers f by samples, taken every π/σ seconds.

There are several ways by which (1.1) may be generalized (see, e.g. Zayed and Schmeisser, 2014). For example, in 1955, Fogel (Jagerman and Fogel, 1956) was motivated by a problem in aircraft instrument communications (a pilot in the case of pointer-on-scale displays may estimate the pointer position and also the rate information concerning the acceleration of the pointer, corresponding to first- and second-time derivatives) and studied a sampling expansion which involved sample values of a function and its derivatives.

Next, one possible application of sampling theory is in sensor networks. In this case, a large number of sensors is used to monitor some quantity, e.g. temperature. This quantity varies continuously in space and hence, can be viewed as a signal in multidimensional space.

In this paper, we concern with the sampling representation for bandlimited signals f of several variables such that f has finite p-energy for certain $1 \leqslant p < \infty$ (see Nashed and Sun, 2010 and Nguyen and Unser, 2017).

Note that the reconstruction (1.1) is called stable, if it converges uniformly on $\mathbb{R}$. Such a stability is important not only from the theoretical point of view but, in particular, for applications. If (1.1) is stable, then it is possible to bound the approximation error, which is made by using in (1.1) only finitely many samples. Finally, we note that our sampling series have such a stable reconstruction properties. Namely, they converge absolutely and uniformly on the whole $\mathbb{R}^n$.

2. Definitions, Notation and the Main Result

Let $\mathbb{Z}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$ be the usual n-dimensional integer lattice, real Euclidean space and complex Euclidean space, respectively. Next, $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, denotes the Lebesgue space of complex-values measurable functions on $\mathbb{R}^n$ with finite p-energy, i.e. such that

$|f|^p$ is integrable on $\mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$, we define the Fourier transform of f as usual by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-i\langle x,t\rangle} f(t)\,dt,$$

$x \in \mathbb{R}^n$, where $\langle x, t\rangle = \sum_{k=1}^n x_k t_k$ is the scalar product on $\mathbb{R}^n$. If $f \in L^p(\mathbb{R}^n)$ and $1 < p \leqslant \infty$, then we understand $\hat{f}$ in a distributional sense of tempered distributions $S'(\mathbb{R}^n)$. We recall that the Schwartz space of test functions $S(\mathbb{R}^n)$ consists of the complex-valued infinitely differentiable functions φ on $\mathbb{R}^n$ satisfying

$$\sup_{x\in\mathbb{R}^n;\ |\alpha|\leqslant m} \Big(\big(1 + \|x\|\big)^k \big|D^\alpha \varphi(x)\big|\Big) < \infty$$

for all nonnegative integers m and k, also for any nonnegative multi-index α, where D^α is the partial derivative of order α and $\|x\|$ denotes the standard Euclidean norm on $\mathbb{R}^n$. The dual space $S'(\mathbb{R}^n)$ of $S(\mathbb{R}^n)$ is called the space of tempered distributions.

Given a closed subset Ω of $\mathbb{R}^n$, a function $\omega : \mathbb{R}^n \to \mathbb{C}$ is called bandlimited to Ω if $\widehat{\omega}$ vanishes outside Ω. For $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^n$ with $\sigma_m > 0$, $m = 1, \ldots, n$, we define the spaces of bandlimited signals by

$$B^p_{Q^n_\sigma} = \big\{f \in L^p(\mathbb{R}^n)\text{: supp}\,\hat{f} \subset Q^n_\sigma\big\},$$

where

$$Q^n_\sigma = \big\{x \in \mathbb{R}^n\text{: } |x_k| \leqslant \sigma_k,\ k = 1, \ldots, n\big\}.$$

Hence, any $f \in B^p_{Q^n_\sigma}$ is bandlimited to hyperrectangle or n-orthotope Q^n_σ, i.e. $\hat{f}$ vanishes outside Q^n_σ. If we equip $B^p_{Q^n_\sigma}$ with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}^n} \big|f(x)\big|^p\,dx\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

$$\text{or} \quad \|f\|_\infty = \operatorname*{ess\,supp}_{x\in\mathbb{R}^n} \big|f(x)\big| \quad \text{for } p = \infty,$$

then $B^p_{Q^n_\sigma}$ is a Banach space and it is called the Bernstein space. By the Paley-Wiener-Schwartz theorem (see Hörmander, 1990; p. 181), any $f \in B^p_{Q^n_\sigma}$, $1 \leqslant p \leqslant \infty$, is infinitely differentiable on $\mathbb{R}^n$ and has an (unique) extension onto the complex space $\mathbb{C}^n$ to an entire function.

Remark 1. Recall that the identity theorem for analytic functions shows that the extension of $f \in B^p_{Q^n_\sigma}$ from $\mathbb{R}^n$ to $\mathbb{C}^n$ is unique. Therefore, we can identify further each $f \in B^p_{Q^n_\sigma}$ with bandlimited function defined on $\mathbb{R}^n$ and in other cases consider the same f as entire function defined on the whole $\mathbb{C}^n$. Note that terms “bandlimited signal on $\mathbb{R}^n$”and “bandlimited function on $\mathbb{R}^n$” are equivalent.

Below we shall use the technique of complex variable functions on $\mathbb{C}^n$. In particular, we shall need certain facts in entire functions. Therefore, we use below only function theory terms, i.e. "bandlimited function" term.

Next, to simplify the notation, we shall write B_σ^p in the case of functions of one variable B_σ^p instead of $B^p_{Q^1_\sigma}$.

If we define the sinc -function to be

$$\operatorname{sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0, \end{cases}$$

$z \in \mathbb{C}$, then (1.1) can be rewritten as

$$f(x) = \sum_{k \in \mathbb{Z}} f\left(\frac{\pi k}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma}{\pi} x - k\right). \tag{2.1}$$

This expression holds for each $f \in B_\sigma^p$, $1 \leqslant p < \infty$. Next, (2.1) converges absolutely and uniformly on compact subsets of $\mathbb{C}$. The Nyquist sampling rate of σ/π samples per unit time in (2.1) is exact, i.e. $f \in B_\sigma^p$ cannot be recovered from its samples taken at a lower rate. This means that, for any $\varepsilon > 1$, there exist two $f_j \in B_\sigma^p$, $j = 1, 2$ such that $f_1 \not\equiv f_2$, but f_1 coincides with f_2 on every point of the sampling sequence $\{(\varepsilon \pi k/\sigma\}_{k \in \mathbb{Z}}$.

On the other hand, if we know sample values of $f \in B_\sigma^p$ only the sampling sequence $\{2\pi k/\sigma\}_k$ with the sampling rate $\sigma/(2\pi)$, then reconstruction of f is also possible if we know, in addition, its derivative values $\{f'(2\pi k/\sigma)\}_k$. More precisely, if $f \in B_\sigma^p$, $1 \leqslant p < \infty$, then

$$f(z) = \sum_{k \in \mathbb{Z}} \left(f\left(\frac{2\pi k}{\sigma}\right) + f'\left(\frac{2\pi k}{\sigma}\right)\left(z - \frac{2\pi k}{\sigma}\right)\right) \operatorname{sinc}^2\left(\frac{\sigma}{2\pi} z - k\right) \tag{2.2}$$

(see Jagerman and Fogel, 1956; p. 145 or Butzer *et al.*, 2011; p. 442).

The following n-dimensional sampling theorem is a standard extension of (2.1) in $B^p_{Q^n_\sigma}$

$$f(z) = \sum_{u \in \mathbb{Z}^n} \left(f\left(\frac{\pi}{\sigma} u\right) \prod_{m=1}^{n} \operatorname{sinc}\left(\frac{\sigma_m}{\pi} z_m - u_m\right)\right),$$

$z \in \mathbb{C}^n$ (Gosselin, 1977; p. 172) (see also Jerri, 2017 for references). Here and below we are taking

$$\left(\tau \frac{a}{b}\right) = \left(\tau \frac{a_1}{b_1}, \dots, \tau \frac{a_n}{b_n}\right)$$

for any $\tau \in \mathbb{C}$ and all $a, b \in \mathbb{C}^n$ such that $b_1 \neq 0, \dots, b_n \neq 0$. These series converge absolutely and uniformly on compact subsets of $\mathbb{C}^n$.

The aim of this paper is to prove a multidimensional version of (2.2). In the case of two variables, in Fang and Li (2006) (see also a tutorial review Jerri, 2017; p. 40) a version of (2.2) involving only the following sample values sequences was given

$$\left\{f\left(2\pi\frac{u}{\sigma}\right)\right\}_{u\in\mathbb{Z}^2}, \qquad \left\{\frac{\partial f}{\partial x_1}\left(2\pi\frac{u}{\sigma}\right)\right\}_{u\in\mathbb{Z}^2} \quad \text{and} \quad \left\{\frac{\partial f}{\partial x_2}\left(2\pi\frac{u}{\sigma}\right)\right\}_{u\in\mathbb{Z}^2}. \tag{2.3}$$

Namely, in Fang and Li (2006) the following representation was given:

$$\begin{aligned} f(z) = \sum_{u\in\mathbb{Z}^2}\bigg[& f\left(2\pi\frac{u}{\sigma}\right)+\left(z_1-\frac{2\pi u_1}{\sigma_1}\right)\cdot\frac{\partial f}{\partial x_1}\left(2\pi\frac{u}{\sigma}\right) \\ & +\left(z_2-\frac{2\pi u_2}{\sigma_2}\right)\cdot\frac{\partial f}{\partial x_2}\left(2\pi\frac{u}{\sigma}\right)\bigg] \\ & \times \operatorname{sinc}^2\left(\frac{\sigma_1}{2\pi}z_1-u_1\right)\operatorname{sinc}^2\left(\frac{\sigma_2}{2\pi}z_2-u_2\right) \end{aligned} \tag{2.4}$$

for all $f \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$. We say that such a sampling theorem fails in general. Indeed, let us take any $\chi \in S(\mathbb{R}^2)$, $\chi \not\equiv 0$, such that $\operatorname{supp}\chi \subset \{x \in \mathbb{R}^2\colon |x_k| \leqslant \sigma_k/2,\ k = 1, 2\}$ and define

$$\Lambda(x) = \sin\left(\frac{\sigma_1}{2}x_1\right)\sin\left(\frac{\sigma_2}{2}x_2\right)\cdot\widehat{\chi}(x),$$

$x \in \mathbb{R}^2$. Since $S(\mathbb{R}^2)$ is invariant under the Fourier transform, it follows that $\widehat{\chi} \in S(\mathbb{R}^2)$, and consequently Λ is in $B^p_{Q^2_\sigma}$ for all $1 \leqslant p \leqslant \infty$. On the other hand, if $f = \Lambda$, then all sequences in (2.3) are null sequences. Hence, (2.4) generates null function, but not Λ. Even more, this example with our function Λ shows that the same is still true if we add to (2.4) an arbitrary number of sample values sequences

$$\left\{\frac{\partial^j\Lambda}{\partial x_k^j}\left(2\pi\frac{u}{\sigma}\right)\right\}_{u\in\mathbb{Z}^2},$$

$k = 1, 2$, $j = 2, 3, \ldots$. Therefore, any multidimensional version of (2.2) must necessarily contain also mixed partial derivatives of f.

Our basic idea based on observation that any sampling series are also an interpolation formula. For example, the m-th coefficient in (2.1) equals to the value of the sum, i.e. the value of f at the point with the number m, i.e. at $\pi m/\sigma$.

We now state the main result of this paper.

Theorem 1. *If* $f \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$, *then*

$$\begin{aligned} f(z) = \sum_{u\in\mathbb{Z}^2}\bigg[& f\left(2\pi\frac{u}{\sigma}\right)+\left(z_1-\frac{2\pi u_1}{\sigma_1}\right)\cdot\frac{\partial f}{\partial x_1}\left(2\pi\frac{u}{\sigma}\right) \\ & +\left(z_2-\frac{2\pi u_2}{\sigma_2}\right)\cdot\frac{\partial f}{\partial x_2}\left(2\pi\frac{u}{\sigma}\right) \end{aligned}$$

$$+\left(z_1-\frac{2\pi u_1}{\sigma_1}\right)\left(z_2-\frac{2\pi u_2}{\sigma_2}\right)\cdot\frac{\partial^2 f}{\partial x_1\partial x_2}\left(2\pi\frac{u}{\sigma}\right)\Bigg]$$
$$\times\ \mathrm{sinc}^2\left(\frac{\sigma_1}{2\pi}z_1-u_1\right)\mathrm{sinc}^2\left(\frac{\sigma_2}{2\pi}z_2-u_2\right). \tag{2.5}$$

The series (2.5) *converge absolutely and uniformly on* $\mathbb{R}^2$ *and on compact subsets of* $\mathbb{C}^2$.

3. Preliminaries and Proofs

Set $Q_\pi^2=\{x\in\mathbb{R}^2\colon |x_1|,|x_2|\leqslant\pi\}$. We may assume without loss of generality below that $\sigma_1=\sigma_2=\pi$, since the operator

$$A(f)(z)=f\left(\frac{\pi}{\sigma_1}z_1,\frac{\pi}{\sigma_2}z_2\right)$$

is an isometric isomorphism between $B^p_{Q^2_\sigma}$ and $B^p_{Q^2_\pi}$.

Let $1\leqslant p<q\leqslant\infty$. Then there exists $0<M(p;q)<\infty$ such that

$$\|f\|_{B^q_{Q^2_\pi}}\leqslant M(p;q)\|f\|_{B^p_{Q^2_\pi}} \tag{3.1}$$

for all $f\in B^q_{Q^2_\pi}$ (see Triebel, 1983; pp. 21–22). Therefore,

$$B^1_{Q^2_\pi}\subset B^p_{Q^2_\pi}\subset B^q_{Q^2_\pi}\subset B^\infty_{Q^2_\pi}. \tag{3.2}$$

If $f\in B^p_{Q^2_\pi}$ with $1\leqslant p<\infty$, then

$$\lim_{x\in\mathbb{R}^n;\,x\to\infty}f(x)=0 \tag{3.3}$$

(see Nikol'skii, 1975; p. 118). If $f\in B^\infty_{Q^2_\pi}$, then (see, e.g. Nikol'skii, 1975; p. 117; Jagerman and Fogel, 1956; p. 181 or Hörmander, 1990; p. 181).

$$\left|f(z)\right|\leqslant\sup_{x\in\mathbb{R}^2}\left|f(x)\right|e^{\pi(|y_1|+|y_2|)} \tag{3.4}$$

for all $z=x+iy$, $x,y\in\mathbb{R}^2$. Note that (3.2) shows that (3.4) also holds true for any $f\in B^p_{Q^2_\pi}$, $1\leqslant p<\infty$.

In the sequel, we shall use several times the following function in $B^\infty_{Q^2_{\pi/2}}$:

$$s_\pi(z)=\sin\left(\frac{\pi z_1}{2}\right)\sin\left(\frac{\pi z_2}{2}\right).$$

Let us start by proving a simple auxiliary statement for functions of one variable. For completeness, we also give its proof.

Lemma 1. *Suppose that $F \in B_\pi^p$, $1 \leqslant p < \infty$. If*

$$F(2n) = F'(2n) = 0 \tag{3.5}$$

for all $n \in \mathbb{Z}$, then $F \equiv 0$.

Proof. Define

$$G(z) = \frac{F(z)}{\sin^2(\frac{\pi z}{2})}. \tag{3.6}$$

Then (3.5) implies that G is a well-defined entire function on $\mathbb{C}$. Now, for each positive integer m, set

$$D_m = \bigl\{z \in \mathbb{C}\colon |\Re z| \leqslant 1 + 2m,\ |\Im z| \leqslant 2\bigr\}.$$

It can easily be checked that

$$\min_{z\in\partial D_m} \left|\sin\frac{\pi z}{2}\right| \geqslant 1 \tag{3.7}$$

for each $m = 1, 2, \ldots$. On the other hand, according to (3.2) and (3.4), we have that there is a finite number $M_1 > 0$ such that

$$\bigl|F(z)\bigr| \leqslant M_1 e^{\pi|\Im z|} \tag{3.8}$$

for all $z \in \mathbb{C}$. Therefore, by the maximum principle for analytic functions, we see that $|G|$ is bounded on D_m. Moreover, (3.7) and (3.8) imply that $|G|$ is bounded by the same constant on each D_m. Thus, $|G|$ is bounded on

$$D = \bigcup_{m=1}^{\infty} D_m = \bigl\{z \in \mathbb{C}\colon |\Im z| \leqslant 2\bigr\}. \tag{3.9}$$

Now take any $z \in \mathbb{C} \setminus D$. Then it is easy to see that

$$\left|\sin\frac{\pi z}{2}\right| \geqslant \frac{1}{2} e^{\frac{\pi}{2}|\Im z|}.$$

Combining this with (3.8), we conclude that $|G|$ is also bounded on $\mathbb{C} \setminus D$. By Liouville's theorem, G is a constant. Therefore, (3.6) implies that there exists $c \in \mathbb{C}$ such that $F(z) = c\sin^2(\pi z/2)$ on $\mathbb{C}$. Finally, using (3.3), since $F \in B_\pi^p$, $1 \leqslant p < \infty$, we see that $c = 0$. Thus, $F \equiv 0$. □

Lemma 2. *Let $F \in B_{Q_\pi^2}^p$, $1 \leqslant p \leqslant \infty$. Assume that*

$$F(2k, \lambda) = 0 \quad \textit{and} \quad F(\lambda, 2k) = 0 \tag{3.10}$$

for all $\lambda \in \mathbb{C}$ *and each* $k \in \mathbb{Z}^2$. *Then there exists an entire function* $G : \mathbb{C}^2 \to \mathbb{C}$ *such that*

$$F(z) = s_\pi(z)G(z), \tag{3.11}$$

$z \in \mathbb{C}^2$.

Proof. First we claim that there is an entire function $H : \mathbb{C}^2 \to \mathbb{C}$ such that

$$F(z) = z_1 z_2 H(z) \tag{3.12}$$

for all $z \in \mathbb{C}^2$. To that end, we expand F in a series of homogeneous polynomials

$$F(z) = \sum_{m=0}^{\infty} p_m(z), \qquad p_m(z) = \sum_{|n|=m} c_n z_1^{n_1} z_2^{n_2}, \tag{3.13}$$

where $n \in \mathbb{Z}^2$ is a non-negative multi-index and $|n| = n_1 + n_2$. Note that (3.13) converges uniformly on compact subsets of $\mathbb{C}^2$ (see, e.g. Shabat, 1992; p. 36). In particular, the first condition in (3.10) shows that

$$F(0, z_2) = 0 \tag{3.14}$$

for all $z_2 \in \mathbb{C}$. Using this and the identity theorem for entire function $\widetilde{F}(\lambda) := F(0, \lambda)$ on $\mathbb{C}$, we see that (3.14) is equivalent to the condition $c_n = 0$ in (3.13) for each $n = (n_1, n_2)$ such that $n_1 = 0$. Hence in (3.13) we have that $p_0 = 0$ and $p_m = z_1 q_m$, $m = 1, 2, \ldots$, where q_m is a homogeneous polynomial of degree $m - 1$ or $q_m \equiv 0$. Next, using the second condition in (3.10), we obtain in the same manner that $p_1 \equiv 0$ and

$$p_m(z) = z_1 q_m(z) = z_1 z_2 r_m(z),$$

$m = 2, 3, \ldots$, where r_m is a homogeneous polynomial of degree $m - 2$ or $r_m \equiv 0$. The series $\sum_2^\infty r_m$ converge uniformly on compact subsets of $\mathbb{C}^2$, since $\sum_0^\infty p_m$ has this property. Therefore, $\sum_2^\infty r_m$ defines on $\mathbb{C}^2$ an entire function, say H. Therefore, (3.13) implies our claim (3.12).

We denote by $N(s_\pi)$ the zero set of s_π. Clearly,

$$N(s_\pi) = \big\{(\lambda, k)\colon \lambda \in \mathbb{C},\ k \in \mathbb{Z}\big\} \cup \big\{(k, \lambda)\colon \lambda \in \mathbb{C},\ k \in \mathbb{Z}\big\}. \tag{3.15}$$

Hence, the function

$$G(z) = \frac{F(z)}{s_\pi(z)} \tag{3.16}$$

is well-defined on $\mathbb{C}^2 \setminus N(s_\pi)$. We claim that G can be extended to an entire function on $\mathbb{C}^2$. Our proof is based on the Riemann removable singularity theorem (see, e.g. Shabat, 1992; p. 175) for G and the analytic set $N(s_\pi)$. Since $N(s_\pi)$ is an analytic set of the codimension codim $(N(s_\pi)) = 1$ (see Scheidemann, 2005; p. 72), it follows that it remains to prove that G is locally bounded on $N(s_\pi)$, i.e. for every $z \in N(s_\pi)$ there is an open neighbourhood U_z of z such that G is bounded on $(\mathbb{C}^2 \setminus N(s_\pi)) \cap U_z$.

Fix $\lambda \in N(s_\pi)$. By (3.15), we see that the proof of the fact that G is bounded on $(\mathbb{C}^2 \setminus N(s_\pi)) \cap U_z$ can be divided into two following cases:

a) $\lambda_1 \in 2\mathbb{Z}$ but $\lambda_2 \notin 2\mathbb{Z}$ or $\lambda_1 \notin 2\mathbb{Z}$ but $\lambda_2 \in 2\mathbb{Z}$;
b) $\lambda_1, \lambda_2 \in 2\mathbb{Z}$.

Note that if $\omega \in \mathbb{Z}^2$, then

$$s_\pi(z + 2\omega) = (-1)^{\omega_1+\omega_2} s_\pi(z)$$

for all $z \in \mathbb{C}^2$. Next, $s_\pi(z_1, z_2) = s_\pi(z_2, z_1)$. Moreover, the functions $F_\omega(z) = F(z + 2\omega)$ and $\widetilde{F}(z) = F(z_2, z_1)$ also satisfy (3.10). Therefore, we may assume without loss of generality that we have the only following two cases:

a1) $\lambda_1 = 0$ and $\lambda_2 \notin 2\mathbb{Z}$ or $\lambda_1 \notin 2\mathbb{Z}$ and $\lambda_2 = 0$;
b1) $\lambda_1 = \lambda_2 = 0$.

Suppose that $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 = 0$ and $\lambda_2 \notin 2\mathbb{Z}$. Substituting (3.12) into (3.16), we get

$$G(z) = \frac{z_1 z_2 H(z)}{s_\pi(z)} = \frac{H_0(z)}{H_1(z) H_2(z)}, \tag{3.17}$$

where

$$H_0(z) = z_2 H(z), \qquad H_1(z) = \frac{\sin \frac{\pi z_1}{2}}{z_1} \quad \text{and} \quad H_2(z) = \sin \frac{\pi z_2}{2}. \tag{3.18}$$

Of course, H_0, H_1 and H_2 are entire functions on $\mathbb{C}^2$ such that

$$H_1(\lambda) = H_1(0, \lambda_2) = \frac{2}{\pi} \neq 0 \quad \text{and} \quad H_2(\lambda) = H_2(0, \lambda_2) \neq 0,$$

since in this case $\lambda_2 \notin 2\mathbb{Z}$. Therefore, there is an $\varepsilon > 0$ and a neighbourhood $U_\lambda \subset \mathbb{C}^2$ of λ such that

$$|H_1(z)| > \varepsilon \quad \text{and} \quad |H_2(z)| > \varepsilon \tag{3.19}$$

for all $z \in U_\lambda$. Using this and (3.17), we see that G is bounded on $(\mathbb{C}^2 \setminus N(s_\pi)) \cap U_z$.

In the second part of a1), i.e. if $\lambda_1 \notin 2\mathbb{Z}$ and $\lambda_2 = 0$, then we are proving (3.19) in a similar way.

Suppose now that $\lambda_1 = \lambda_2 = 0$. In this case it is enough to change H_0, H_1 and H_2 in (3.18) by the following functions:

$$H_0(z) = H(z), \qquad H_1(z) = \frac{\sin \frac{\pi z_1}{2}}{z_1} \quad \text{and} \quad H_1(z) = \frac{\sin \frac{\pi z_2}{2}}{z_2}$$

and repeat the above proof. This finished the proof that G is locally bounded on $N(s_\pi)$. Thus, G is entire and (3.16) implies (3.11). Lemma 2 is proved. □

Theorem 2. *Let $f_1, f_2 \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$. Assume that*

$$f_1\left(2\pi\frac{u}{\sigma}\right) = f_2\left(2\pi\frac{u}{\sigma}\right), \qquad \frac{\partial f_1}{\partial x_j}\left(2\pi\frac{u}{\sigma}\right) = \frac{\partial f_2}{\partial x_j}\left(2\pi\frac{u}{\sigma}\right) \tag{3.20}$$

and

$$\frac{\partial^2 f_1}{\partial x_1 \partial x_2}\left(2\pi\frac{u}{\sigma}\right) = \frac{\partial^2 f_2}{\partial x_1 \partial x_2}\left(2\pi\frac{u}{\sigma}\right) \tag{3.21}$$

for $j = 1, 2$ and all $u \in \mathbb{Z}^2$. Then $f_1 \equiv f_2$.

Proof. Set $f = f_1 - f_2$. We must prove that $f \equiv 0$. In our case $\sigma_1 = \sigma_2 = \pi$. Therefore (3.20) and (3.21) are equivalent to

$$f(2u) = 0, \qquad \frac{\partial f(2u)}{\partial x_1} = \frac{\partial f(2u)}{\partial x_2} = 0 \tag{3.22}$$

and

$$\frac{\partial^2 f(2u)}{\partial x_1 \partial x_2} = 0, \tag{3.23}$$

$u \in \mathbb{Z}^2$, respectively. First, we claim that

$$f(2k, \lambda) = f(\lambda, 2k) = 0 \tag{3.24}$$

for all $\lambda \in \mathbb{C}$ and each $k \in \mathbb{Z}$. Indeed, fix $k \in \mathbb{Z}$ and let us define $F_1(\lambda) := f(\lambda, 2k)$ and $F_2(\lambda) := f(2k, \lambda)$ for $\lambda \in \mathbb{C}$. Then (3.22) implies that

$$F_1(2m) = F_2(2m) = 0, \qquad F_1'(2m) = \frac{\partial f(2m, 2k)}{\partial x_1} = 0 \quad \text{and}$$
$$F_2'(2m) = \frac{\partial f(2m, 2k)}{\partial x_2} = 0$$

for all $m \in \mathbb{Z}$. Since $F_1, F_2 \in B^p_\pi$, Lemma 1 yields our claim (3.24).

Now, using Lemma 2, we see that there is an entire function g on $\mathbb{C}^2$ such that

$$f(z) = s_\pi(z) g(z). \tag{3.25}$$

Now we claim that

$$g(2k, \lambda) = g(\lambda, 2k) = 0 \tag{3.26}$$

for each $k \in \mathbb{Z}$ and all $\lambda \in \mathbb{C}$. To that end, fix $k \in \mathbb{Z}$ and set

$$F_1(\lambda) = \frac{\partial f(z)}{\partial x_1}(2k, \lambda) \quad \text{and} \quad F_2(\lambda) = \frac{\partial f(z)}{\partial x_2}(\lambda, 2k),$$

$\lambda \in \mathbb{C}$. From (3.22) we have that

$$F_1(2m) = F_2(2m) = 0 \tag{3.27}$$

for all $m \in \mathbb{Z}$. Since

$$F_1'(\lambda) = \frac{\partial^2 f(z)}{\partial x_1 \partial x_2}(2k, \lambda) \quad \text{and} \quad F_2'(\lambda) = \frac{\partial^2 f(z)}{\partial x_2 \partial x_1}(\lambda, 2k),$$

it follows from (3.23) that

$$F_1'(2m) = F_2'(2m) = 0, \tag{3.28}$$

$m \in \mathbb{Z}$. Next, according to Bernstein's inequality (Nikol'skii, 1975; p. 116), if $f \in B^p_{Q^2_\pi}$, then all partial derivatives of f also are elements of $B^p_{Q^2_\pi}$. Hence, $F_1, F_2 \in B^p_\pi$. Therefore, using (3.27) and (3.28), we see that

$$F_1 = F_2 \equiv 0 \tag{3.29}$$

by Lemma 1. On the other hand, from (3.25) we get

$$F_1(\lambda) = (-1)^k \frac{\pi}{2} \sin\left(\frac{\pi}{2}\lambda\right) g(2k, \lambda) \quad \text{and} \quad F_2(\lambda) = (-1)^k \frac{\pi}{2} \sin\left(\frac{\pi}{2}\lambda\right) g(\lambda, 2k).$$

Together with (3.29), this implies that

$$g(2k, \lambda) = g(\lambda, 2k) = 0$$

for all $\lambda \in \mathbb{C}$ such that $\lambda \neq 2\pi n$, $n \in \mathbb{Z}$. Since $h_1(\lambda) := g(2k, \lambda)$ and $h_2(\lambda) := g(\lambda, 2k)$, $\lambda \in \mathbb{C}$, are entire functions on $\mathbb{C}$, we obtain that $h_1 = h_2 \equiv 0$, which yields our claim (3.26).

Now Lemma 2 shows that then there is an entire function h on $\mathbb{C}^2$ such that $g(z) = s_\pi(z)h(z)$. Then (3.25) implies that

$$h(z) = \frac{f(z)}{s^2_\pi(z)}.$$

Combining (3.4), (3.6) and (3.9) with the definition of s_π, we see that $|h|$ is bounded on the whole $\mathbb{C}^2$. Therefore, by Liouville's theorem, we obtain that $h \equiv c_h$, $c_h \in \mathbb{C}$. Hence, $f(z) = c_h s^2_\pi(z)$. Since $f \in B^p_{Q^2_\pi}$ with $1 \leqslant p < \infty$, it follows from (3.3) that $c_h = 0$, i.e. $f \equiv 0$. This proves Theorem 2. $\square$

Set

$$B_{Q^2_\sigma} = \{ f \in C_0(\mathbb{R}^2) \colon \operatorname{supp} \hat{f} \subset Q^2_\sigma \},$$

where $C_0(\mathbb{R}^2)$ is the usual space of continuous functions on $\mathbb{R}^2$ that vanish at infinity. $B_{Q^2_\sigma}$ is a Banach space with respect to the sup-norm inherited from $C_0(\mathbb{R}^2)$. Next, (3.3) implies that

$$B^p_{Q^2_\sigma} \subset B_{Q^2_\sigma} \tag{3.30}$$

for all $1 \leqslant p < \infty$.

Let us look again at the proof of Theorem 2. Then we can see that together with (3.20) and (3.21), we used, in addition, only the following two properties of f_j, $j = 1, 2$: a) $\hat{f}_j$ is supported in Q^2_π; b) f_j satisfies (3.3). Hence, the following corollary is true:

Corollary 1. *Let $f_1, f_2 \in B_{Q^2_\pi}$. If f_1 and f_2 satisfy* (3.20) *and* (3.21)*, then $f_1 \equiv f_2$.*

Next, for each $1 < r < \infty$ and any $w > 0$, the following estimate is true (see Splettstösser, 1982; p. 811):

$$\sum_{k \in \mathbb{Z}} \left| \operatorname{sinc}(wt - k) \right|^r \leqslant \frac{r}{r-1} \tag{3.31}$$

for all $t \in \mathbb{R}$. In particular, this implies that the series in (3.31) converge on the whole $\mathbb{R}$ and its sum is a bounded function on $\mathbb{R}$.

Proof of Theorem 1. To begin, we recall the following Nikol'skii's inequality (Nikol'skii, 1975; p. 123): for any $1 \leqslant p < \infty$ and each $0 < \varrho < \infty$, there exists a finite positive number $c(\sigma; p; \varrho)$ such that

$$\left(\sum_{u \in \mathbb{Z}^2} \left| f(\varrho u) \right|^p \right)^{1/p} \leqslant c(\sigma; p; \varrho) \| f \|_{B^p_{Q^2_\sigma}}$$

for all $f \in B^p_{Q^2_\sigma}$. A similar inequality holds also for all partial derivatives of f, since by Bernstein's inequality (Nikol'skii, 1975; p. 116), any partial derivative of f also is an element of $B^p_{Q^2_\sigma}$.

Suppose again that $\sigma_1 = \sigma_2 = \pi$ and let l^p_2 be the usual space of sequences $\{c_u \in \mathbb{C} : \sum_{u \in \mathbb{Z}^2} |c_u|^p < \infty\}$. First, we claim that the series (2.5) converge absolutely and uniformly on $\mathbb{R}^2$. Indeed, if $z = x \in \mathbb{R}^2$, then (2.5) can be divided into four series of the following type

$$\begin{aligned} I_{\nu;\infty}(x) = \sum_{m \in \mathbb{Z}^2} \biggl[& \alpha_m \left(\frac{x_1}{2} - m_1 \right)^{\nu_1} \left(\frac{x_2}{2} - m_2 \right)^{\nu_2} \\ & \times \operatorname{sinc}^2 \left(\frac{x_1}{2} - m_1 \right) \operatorname{sinc}^2 \left(\frac{x_2}{2} - m_2 \right) \biggr] \end{aligned} \tag{3.32}$$

with certain $\alpha \in l_2^p$ and some $\nu = (\nu_1, \nu_2)$, where $\nu_1, \nu_2 \in \{0, 1\}$. Therefore, it suffices to show that for any $\varepsilon > 0$ there is a positive integer N such that

$$\begin{aligned}\left|I_{\nu;\,N}(x)\right| = \Bigg| \sum_{\substack{m\in\mathbb{Z}^2\\ |m_1|,|m_2|\geqslant N}} & \left[\alpha_m\left(\frac{x_1}{2}-m_1\right)^{\nu_1}\left(\frac{x_2}{2}-m_2\right)^{\nu_2}\operatorname{sinc}^2\left(\frac{x_1}{2}-m_1\right)\right.\\ & \left.\times\operatorname{sinc}^2\left(\frac{x_2}{2}-m_2\right)\right]\Bigg| < \varepsilon \qquad (3.33)\end{aligned}$$

for all $x \in \mathbb{R}^2$. Take any $p_1 \geqslant p$ such that $1 < p_1 < \infty$. Then Hölder's inequality implies that

$$\begin{aligned}\left|I_{\nu;\,N}(x)\right| \leqslant \Bigg(\sum_{\substack{m\in\mathbb{Z}^2\\ |m_1|,|m_2|\geqslant N}} |\alpha_m|^{p_1}\Bigg)^{1/p_1} & \Bigg(\sum_{\substack{m\in\mathbb{Z}^2\\ |m_1|,|m_2|\geqslant N}} \left|\frac{x_1}{2}-m_1\right|^{q_1\nu_1}\left|\frac{x_2}{2}-m_2\right|^{q_1\nu_2}\\ & \times\left|\operatorname{sinc}\left(\frac{x_1}{2}-m_1\right)\right|^{2q_1}\left|\operatorname{sinc}\left(\frac{x_2}{2}-m_2\right)\right|^{2q_1}\Bigg)^{1/q_1} \qquad (3.34)\end{aligned}$$

with $1 < q_1 < \infty$ such that $1/p_1 + 1/q_1 = 1$. Since $\alpha \in l_2^p \subset l_2^{p_1}$, it follows that, for any $\varepsilon_1 > 0$, there exists a positive integer N_1 such that

$$\Bigg(\sum_{\substack{m\in\mathbb{Z}^2\\ |m_1|,|m_2|\geqslant N_1}} |\alpha_m|^{p_1}\Bigg)^{1/p_1} < \varepsilon_1. \qquad (3.35)$$

On the other hand, it is clear that

$$\left|\frac{x_j}{2}-m_j\right|^{q_1\nu_j}\left|\operatorname{sinc}\left(\frac{x_j}{2}-m_j\right)\right|^{2q_1} \leqslant \left|\operatorname{sinc}\left(\frac{x_j}{2}-m_j\right)\right|^{(2-\nu_j)q_1} \leqslant 1$$

for $j = 1, 2$, $m_j \in \mathbb{Z}$ and all $x_j \in \mathbb{R}$. Hence, by (3.31), we see that the second series in (3.34) converge and

$$\begin{aligned}&\Bigg(\sum_{\substack{m\in\mathbb{Z}^2\\ |m_1|,|m_2|\geqslant N}} \left|\frac{x_1}{2}-m_1\right|^{q_1\nu_1}\left|\frac{x_2}{2}-m_2\right|^{q_1\nu_2}\left|\operatorname{sinc}\left(\frac{x_1}{2}-m_1\right)\right|^{2q_1}\left|\operatorname{sinc}\left(\frac{x_2}{2}-m_2\right)\right|^{2q_1}\Bigg)^{1/q_1}\\ &\quad\leqslant \Bigg(\Bigg(\sum_{m_1\in\mathbb{Z},\,|m_1|\geqslant N}\left|\operatorname{sinc}\left(\frac{x_1}{2}-m_1\right)\right|^{(2-\nu_1)q_1}\Bigg)^{1/q_1}\Bigg)\\ &\quad\leqslant \left(\frac{(2-\nu_1)q_1}{(2-\nu_1)q_1-1}\right)^{1/q_1} \qquad (3.36)\end{aligned}$$

for all $x \in \mathbb{R}^2$ and each $N \geqslant 0$. Combining this with (3.33), (3.34) and (3.35), we get that (3.32) converges absolutely and uniformly on $\mathbb{R}^2$.

Next, for any compact subset K of $\mathbb{C}^2$, there is a strip $E_\tau = \{z \in \mathbb{C}^2 : |\Im z_1|, |\Im z_2| \leqslant \tau\}$, $0 < \tau < \infty$, such that $K \subset E_\tau$. If $f \in B^p_{Q^2_\pi}$, then (3.2) shows that $f \in B^\infty_{Q^2_\pi}$. Therefore, using (3.4), we obtain that (2.5) also converges absolutely and uniformly on K.

Let us denote by F the sum of (2.5). Any sum of partial finite subseries of (2.5) belongs to $B^p_{Q^2_\pi}$ for any $1 \leqslant p < \infty$. By (3.30), it is also an element of $B_{Q^2_\pi}$. Since $B_{Q^2_\pi}$ is a Banach space and (2.5) converges absolutely and uniformly on $\mathbb{R}^2$, i.e. in $B_{Q^2_\pi}$-norm, it follows that $F \in B_{Q^2_\pi}$. Using the fact, that $f \in B^p_{Q^2_\pi} \subset B_{Q^2_\pi}$ for each $1 \leqslant p < \infty$, it is easy to check that the functions $f_1 := f$ and $f_2 = F$ satisfy the conditions of Corollary 1. Hence, $f \equiv F$ and this proves our theorem. □

4. The Truncation Error

The sampling formula (2.5) requires us to know values of a signal f at infinitely many points $\{2\pi u/\sigma\}_{u\in\mathbb{Z}^2}$. In practice, only finitely many samples are available. For $N \in \mathbb{Z}^2$ with certain positive integers N_1 and N_2, let us define the partial sum of (2.5) by

$$\begin{aligned} f_{N-1}(z) = \sum_{\substack{u\in\mathbb{Z}^2\\ |u_1|\leqslant N_1-1,\, |u_2|\leqslant N_2-1}} &\left[f\left(2\pi \frac{u}{\sigma}\right) + \left(z_1 - \frac{2\pi u_1}{\sigma_1}\right)\frac{\partial f}{\partial z_1}\left(2\pi\frac{u}{\sigma}\right)\right.\\ &+ \left(z_2 - \frac{2\pi u_2}{\sigma_2}\right)\frac{\partial f}{\partial z_2}\left(2\pi\frac{u}{\sigma}\right)\\ &\left.+ \left(z_1 - \frac{2\pi u_1}{\sigma_1}\right)\left(z_2 - \frac{2\pi u_2}{\sigma_2}\right)\frac{\partial^2 f}{\partial z_1 \partial z_2}\left(2\pi\frac{u}{\sigma}\right)\right]\\ &\times \operatorname{sinc}^2\left(\frac{\sigma_1}{2\pi}z_1 - u_1\right)\operatorname{sinc}^2\left(\frac{\sigma_2}{2\pi}z_2 - u_2\right). \end{aligned} \tag{4.1}$$

Then the truncation error of f is defined by

$$E_{f;\,N}(x) = f(x) - f_{N-1}(x), \tag{4.2}$$

$x \in \mathbb{R}^2$. The best known $E_N(x)$ estimates are local, i.e. these estimates are valid only on certain compacts subsets of $\mathbb{R}^2$. We shall indicate some uniform bounds of $E_{f;N}(x)$, i.e. the bounds of

$$e_{N;f} := \sup_{x\in\mathbb{R}^2} \left|E_{f;\,N}(x)\right|.$$

Such an estimate is simpler in the case if we apply to functions $f \in B^p_{Q^2_\sigma}$ certain additional conditions of constructiveness relating to the decay of f at infinity in (3.3) (see, e.g. Lin, 2019 and Wang *et al.*, 2018). Note that the spectral function, i.e. the Fourier transform $\hat{f}$, of many important signals f are smooth enough. Hence, these signals f have rapid decay in the time domain for large time. In light of this, it is natural to study the truncation error for functions $f \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$ that satisfy the following simple decay condition

$$\left|f(x)\right| \leqslant \frac{c_f}{|x_1||x_2|}, \tag{4.3}$$

for all $x \in \mathbb{R}^2$ such that , $|x_1| \geqslant N_1 \geqslant 1$, $|x_2| \geqslant N_2 \geqslant 1$. Here c_f is a positive number that depends on f. Note that the function

$$f(x) = \operatorname{sinc}\left(\frac{\sigma_1}{\pi}x_1\right)\operatorname{sinc}\left(\frac{\sigma_2}{\pi}x_2\right)$$

satisfies (4.3) with $c_f = 1/(\sigma_1\sigma_2)$.

Theorem 3. *Let $f \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$, and let N_1 and N_2 be two positive integers. Assume that f satisfies* (4.3)*. Then, for any $1 < \omega < \infty$, we have that*

$$\begin{aligned} e_{f;N} \leqslant{}& \frac{c_f\sigma_1\sigma_2}{4\pi^2}\Bigg[\left(\frac{4\omega^2}{(\omega+1)^2}\right)^{1-1/\omega} + 2\left(\frac{2\omega^2}{\omega+1}\right)^{1-1/\omega}\left(2+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}\right) \\ &+ 4\omega^{2(1-1/\omega)}\left(1+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}+\frac{3}{4\pi^2 N_1 N_2}\right)\Bigg]\frac{(2(\omega-1)^2)^{1/\omega}}{(N_1N_2)^{1-1/\omega}}. \end{aligned} \tag{4.4}$$

The estimate (4.4) can be substantially simplified. Indeed, the following corollary holds:

Corollary 2. *Under the assumptions of Theorem* 3*, we get*

$$e_{f;N} \leqslant \frac{3c_f\sigma_1\sigma_2}{\pi^2}\omega^{2(1-1/\omega)}\left(2(\omega-1)^2\right)^{1/\omega}\frac{1}{(N_1N_2)^{1-1/\omega}} \tag{4.5}$$

for all $x \in \mathbb{R}^2$.

Of course, (4.5) is less exact than (4.4).

Proof of Theorem 3. For $\nu = (\nu_1, \nu_2)$ with $\nu_1, \nu_2 \in \{0; 1\}$, set

$$\alpha_\nu(u) = \frac{\partial^{\nu_1+\nu_2} f}{\partial^{\nu_1}x_1\partial^{\nu_2}x_2}\left(2\pi\,\frac{u}{\sigma}\right),$$

$u \in \mathbb{Z}^2$. The truncation error $E_{N;f}(x)$ can be divided into four series of the type

$$I_\nu(x) = \sum_{\substack{u\in\mathbb{Z}^2 \\ |u_1|\geqslant N_1,\,|u_2|\geqslant N_2}} \left[\alpha_\nu(u)\left(x_1 - \frac{2\pi}{\sigma_1}u_1\right)^{\nu_1}\left(x_2 - \frac{2\pi}{\sigma_2}u_2\right)^{\nu_2}\operatorname{sinc}^2\left(\frac{\sigma_1}{2\pi}x_1 - u_1\right)\right.$$
$$\left.\times \operatorname{sinc}^2\left(\frac{\sigma_2}{2\pi}x_2 - u_2\right)\right]. \tag{4.6}$$

Next, for any ω such that $1 < \omega < \infty$, Hölder's inequality implies that

$$\left|I_\nu(x)\right| \leqslant \Bigg(\sum_{\substack{u\in\mathbb{Z}^2 \\ |u_1|\geqslant N_1,|u_2|\geqslant N_2}} \left|\alpha_\nu(u)\right|^\omega\Bigg)^{1/\omega}\Bigg(\sum_{\substack{u\in\mathbb{Z}^2 \\ |u_1|\geqslant N_1,\,|u_2|\geqslant N_2}} \left|x_1 - \frac{2\pi}{\sigma_1}u_1\right|^{s\nu_1}\left|x_2 - \frac{2\pi}{\sigma_2}u_2\right|^{s\nu_2}$$
$$\times \left|\operatorname{sinc}\left(\frac{\sigma_1}{2\pi}x_1 - u_1\right)\right|^{2s}\left|\operatorname{sinc}\left(\frac{\sigma_2}{2\pi}x_2 - u_2\right)\right|^{2s}\Bigg)^{1/s} \tag{4.7}$$

with $1 < s < \infty$ such that $1/\omega + 1/s = 1$.

We claim that

$$\Bigg(\sum_{\substack{u\in\mathbb{Z}^2 \\ |u_1|\geqslant N_1,\,|u_2|\geqslant N_2}} \left|x_1 - \frac{2\pi}{\sigma_1}u_1\right|^{s\nu_1}\left|x_2 - \frac{2\pi}{\sigma_2}u_2\right|^{s\nu_2}\left|\operatorname{sinc}\left(\frac{\sigma_1}{2\pi}x_1 - u_1\right)\right|^{2s}\Bigg|$$
$$\times \operatorname{sinc}\left(\frac{\sigma_2}{2\pi}x_2 - u_2\right)\Bigg|^{2s}\Bigg)^{1/s} \leqslant \frac{2^{\nu_1+\nu_2}}{\sigma_1^{\nu_1}\sigma_2^{\nu_2}}\left(s^2\frac{(2-\nu_1)(2-\nu_2)}{[(2-\nu_1)s-1)][(2-\nu_2)s-1]}\right)^{1/s} \tag{4.8}$$

for all $x \in \mathbb{R}^2$. Indeed,

$$\left|x_j - \frac{2\pi}{\sigma_j}u_j\right|^{s\nu_j}\left|\operatorname{sinc}\left(\frac{\sigma_j}{2\pi}x_j - u_j\right)\right|^{2s} \leqslant \left(\frac{2}{\sigma_j}\right)^{s\nu_j}\left|\operatorname{sinc}\left(\frac{\sigma_j}{2\pi}x_j - u_j\right)\right|^{(2-\nu_j)s},$$

$j = 1, 2$, $x \in \mathbb{R}^2$. Therefore, by a similar argument as is used in the proof of (3.36), we obtain (4.8).

Now we estimate the first series in (4.6). Here we have four cases. Start with $\nu_1 = \nu_2 = 0$. Then

$$\alpha_{0,0}(u) = f\left(\frac{2\pi}{\sigma}u\right),$$

$u \in \mathbb{Z}^2$. Hence, using (4.3), we get

$$\sum_{\substack{u\in\mathbb{Z}^2\\ u_1\geqslant N_1,\, u_2\geqslant N_2}} \left|\alpha_{0,0}(u)\right|^{\omega} \leqslant c_f^{\omega} \sum_{\substack{u\in\mathbb{Z}^2\\ u_1\geqslant N_1,\, u_2\geqslant N_2}} \frac{1}{|2\pi u_1/\sigma_1|^{\omega}|2\pi u_2/\sigma_2|^{\omega}}. \tag{4.9}$$

Since

$$\sum_{\substack{k\in\mathbb{Z}\\ k\geqslant N>0}} \frac{1}{|2\pi k/\sigma|^{\omega}} \leqslant \left(\frac{\sigma}{2\pi}\right)^{\omega} \int_N^{\infty} \frac{1}{t^{\omega}}\,dt = \left(\frac{\sigma}{2\pi}\right)^{\omega} \frac{\omega-1}{N^{\omega-1}},$$

it follows from (4.9) that

$$\sum_{\substack{u\in\mathbb{Z}^2\\ u_1\geqslant N_1,\, u_2\geqslant N_2}} \left|\alpha_{0,0}(u)\right|^{\omega} \leqslant \left(\frac{c_f\sigma_1\sigma_2}{4\pi^2}\right)^{\omega} \frac{(\omega-1)^2}{(N_1N_2)^{\omega-1}}. \tag{4.10}$$

Thus,

$$\left(\sum_{\substack{u\in\mathbb{Z}^2\\ |u_1|\geqslant N_1,\, |u_2|\geqslant N_2}} \left|\alpha_{0,0}(u)\right|^{\omega}\right)^{1/\omega} \leqslant \frac{c_f\sigma_1\sigma_2}{4\pi^2}\frac{(2(\omega-1)^2)^{1/\omega}}{(N_1N_2)^{1-1/\omega}}. \tag{4.11}$$

Next, assume that $\nu_1 = 1$ and $\nu_2 = 0$. Then

$$\alpha_{1,0}(u) = \frac{\partial f}{\partial x_1}\left(2\pi\frac{u}{\sigma}\right), \quad u \in \mathbb{Z}^2.$$

We shall estimate these $\alpha_{1,0}$. For this purpose let us define the function

$$F(x) := x_1x_2 f(x), \tag{4.12}$$

$x \in \mathbb{R}^2$. According to (4.3), it is in $B^{\infty}_{Q^2_\sigma}$ and

$$\|F\|_{\infty} \leqslant c_f.$$

Given any nonnegative multi-index $\nu \in \mathbb{Z}^2$ with $\nu_1, \nu_2 \in \{0; 1\}$, we get by Bernstein's inequality in $B^{\infty}_{Q^2_\sigma}$ (see Nikol'skii, 1975; p. 116) that

$$\left|\frac{\partial^{\nu_1+\nu_2}F(x)}{\partial^{\nu_1}x_1\partial^{\nu_2}x_2}\right| \leqslant \sigma_1^{\nu_1}\sigma_2^{\nu_2}c_f \tag{4.13}$$

for all $x \in \mathbb{R}^2$. Now $\nu_1 = 1$ and $\nu_2 = 0$. Hence, (4.13) implies that

$$\left| x_2 f(x) + x_1 x_2 \frac{\partial f}{\partial x_1}(x) \right| = \left| \frac{\partial F}{\partial x_1}(x) \right| \leqslant c_f \sigma_1.$$

Hence, using (4.3), we get that

$$\left| \frac{\partial f}{\partial x_1}(x) \right| \leqslant \frac{c_f \sigma_1}{|x_1 x_2|} + \frac{|f(x)|}{|x_1|} \leqslant \frac{c_f}{|x_1 x_2|} \left(\sigma_1 + \frac{1}{|x_1|} \right) \tag{4.14}$$

for all $x \in \mathbb{R}^2$. Therefore, we have that

$$\begin{aligned} &\sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1,\, u_2 \geqslant N_2}} \left| \alpha_{1,0}(u) \right|^\omega \\ &\quad \leqslant c_f^\omega \sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1,\, u_2 \geqslant N_2}} \left[\frac{1}{|2\pi u_1/\sigma_1|^\omega |2\pi u_2/\sigma_2|^\omega} \left(\sigma_1 + \frac{1}{2\pi u_1/\sigma_1} \right)^\omega \right]. \end{aligned} \tag{4.15}$$

Since

$$\sup_{\substack{u_1 \in \mathbb{Z} \\ u_1 \geqslant N_1}} \left(\sigma_1 + \frac{1}{2\pi u_1/\sigma_1} \right) \leqslant \left(\sigma_1 + \frac{\sigma_1}{2\pi N_1} \right),$$

it follows from (4.15) that

$$\sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1,\, u_2 \geqslant N_2}} \left| \alpha_{1,0}(u) \right|^\omega \leqslant c_f^\omega \left(\sigma_1 + \frac{\sigma_1}{2\pi N_1} \right)^\omega \sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1,\, u_2 \geqslant N_2}} \frac{1}{|2\pi u_1/\sigma_1|^\omega |2\pi u_2/\sigma_2|^\omega}.$$

Finally, with (4.9), (4.10) and (4.11) in mind, we obtain that

$$\left(\sum_{\substack{u \in \mathbb{Z}^2 \\ |u_1| \geqslant N_1,\, |u_2| \geqslant N_2}} \left| \alpha_{1,0}(u) \right|^\omega \right)^{1/\omega} \leqslant \frac{c_f \sigma_1^2 \sigma_2}{4\pi^2} \left(1 + \frac{1}{2\pi N_1} \right) \frac{(2(\omega-1)^2)^{1/\omega}}{(N_1 N_2)^{1-1/\omega}}. \tag{4.16}$$

A similar argument shows that

$$\left(\sum_{\substack{u \in \mathbb{Z}^2 \\ |u_1| \geqslant N_1,\, |u_2| \geqslant N_2}} |\alpha_{0,1}(u)|^\omega \right)^{1/\omega} \leqslant \frac{c_f \sigma_1 \sigma_2^2}{4\pi^2} \left(1 + \frac{1}{2\pi N_2} \right) \frac{(2(\omega-1)^2)^{1/\omega}}{(N_1 N_2)^{1-1/\omega}}. \tag{4.17}$$

Assume now that $\nu_1 = \nu_2 = 1$. Then

$$\alpha_{1,1}(u) = \frac{\partial^2 f}{\partial x_1 \partial x_1}\left(2\pi \frac{u}{\sigma}\right), \quad u \in \mathbb{Z}^2.$$

Now (4.13) implies that

$$\left| f(x) + x_1 \frac{\partial f}{\partial x_1}(x) + x_2 \frac{\partial f}{\partial x_2}(x) + x_1 x_2 \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \right| = \left| \frac{\partial F^2}{\partial x_1 \partial x_2}(x) \right| \leqslant c_f \sigma_1 \sigma_2$$

for all $x \in \mathbb{R}^2$. Therefore, using (4.3) and (4.14), we have that

$$\begin{aligned} \left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) \right| &\leqslant \frac{c_f \sigma_1 \sigma_2}{|x_1||x_2|} + \frac{|f(x)|}{|x_1||x_2|} + \frac{1}{|x_2|}\left| \frac{\partial f}{\partial x_1}(x) \right| + \frac{1}{|x_1|}\left| \frac{\partial f}{\partial x_2}(x) \right| \\ &\leqslant \frac{c_f}{|x_1||x_2|}\left(\sigma_1 \sigma_2 + \frac{\sigma_1}{|x_2|} + \frac{\sigma_2}{|x_1|} + \frac{3}{|x_1||x_2|} \right), \end{aligned} \tag{4.18}$$

$x \in \mathbb{R}^2$. Since

$$\begin{aligned} &\sup_{\substack{u_1 \in \mathbb{Z} \\ u_1 \geqslant N_1,\ u_2 \geqslant N_2}} \left(\sigma_1 \sigma_2 + \frac{\sigma_1}{|2\pi u_2/\sigma_2|} + \frac{\sigma_2}{|2\pi u_1/\sigma_1|} + \frac{3}{|2\pi u_1/\sigma_1||2\pi u_2/\sigma_2|} \right) \\ &\leqslant \sigma_1 \sigma_2 \left(1 + \frac{1}{2\pi N_1} + \frac{1}{2\pi N_2} + \frac{3}{4\pi^2 N_1 N_2} \right), \end{aligned}$$

we obtain from (4.18) that

$$\begin{aligned} &\sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1, u_2 \geqslant N_2}} \left| \alpha_{1,1}(u) \right|^\omega = \sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1, u_2 \geqslant N_2}} \left| \frac{\partial^2 f}{\partial x_1 \partial x_1}\left(2\pi \frac{u}{\sigma}\right) \right|^\omega \\ &\leqslant (c_f \sigma_1 \sigma_2)^\omega \left(1 + \frac{1}{2\pi N_1} + \frac{1}{2\pi N_2} + \frac{3}{4\pi^2 N_1 N_2} \right)^\omega \\ &\times \sum_{\substack{u \in \mathbb{Z}^2 \\ u_1 \geqslant N_1, u_2 \geqslant N_2}} \frac{1}{|2\pi u_1/\sigma_1|^\omega |2\pi u_2/\sigma_2|^\omega} \\ &\leqslant \left(\frac{c_f \sigma_1^2 \sigma_2^2}{4\pi^2} \right)^\omega \left(1 + \frac{1}{2\pi N_1} + \frac{1}{2\pi N_2} + \frac{3}{4\pi^2 N_1 N_2} \right)^\omega \frac{(\omega - 1)^2}{(N_1 N_2)^{\omega - 1}}. \end{aligned}$$

Therefore,

$$\Bigg(\sum_{\substack{u\in\mathbb{Z}^2\\ |u_1|\geqslant N_1,\,|u_2|\geqslant N_2}} \big|\alpha_{1,1}(u)\big|^{\omega}\Bigg)^{1/\omega} \leqslant \frac{c_f\sigma_1^2\sigma_2^2}{4\pi^2}\bigg(1+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}+\frac{3}{4\pi^2 N_1N_2}\bigg) \times \frac{(2(\omega-1)^2)^{1/\omega}}{(N_1N_2)^{1-1/\omega}}. \tag{4.19}$$

Now, combining (4.7), (4.8), (4.11), (4.16), (4.16) and (4.19) with the estimate

$$\big|E_{f;N}(x)\big| \leqslant \sum_{\nu=(\nu_1,\nu_2)\ \nu_1,\nu_2\in\{0;1\}} \big|I_\nu(x)\big|,$$

$x\in\mathbb{R}^2$, we have that

$$\begin{aligned} e_{N;f} \leqslant \sup_{x\in\mathbb{R}^2}\big|E_{f;N}(x)\big| \leqslant \frac{c_f\sigma_1\sigma_2}{4\pi^2}\Bigg[&\bigg(\frac{4s^2}{(2s-1)^2}\bigg)^{1/s}+2\bigg(\frac{2s^2}{(s-1)(2s-1)}\bigg)^{1/s}\\ &\times\bigg(2+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}\bigg)+4\bigg(\frac{s^2}{(s-1)^2}\bigg)^{1/s}\\ &\times\bigg(1+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}+\frac{3}{4\pi^2N_1N_2}\bigg)\Bigg]\frac{(2(\omega-1)^2)^{1/\omega}}{(N_1N_2)^{1-1/\omega}}. \end{aligned} \tag{4.20}$$

Finally, since

$$\frac{s}{s-1}=\omega \quad\text{and}\quad \frac{2s}{2s-1}=\frac{2\omega}{\omega+1},$$

(4.20) implies (4.4). This proves our theorem. □

Proof of Corollary 2. Using the following elementary inequality $2\omega/(2\omega+1)<\omega$ that holds for each $\omega>1$, we get from (4.4) that

$$\begin{aligned} e_{f;N} \leqslant\ & \frac{c_f\sigma_1\sigma_2}{4\pi^2}\omega^{2(1-/\omega)}\\ &\times\Bigg[1+2\bigg(2+\frac{1}{2\pi N_1}\bigg)+4\bigg(1+\frac{1}{2\pi N_1}+\frac{1}{2\pi N_2}+\frac{3}{4\pi^2N_1N_2}\bigg)\Bigg]\\ &\times\big(2(\omega-1)^2\big)^{1/\omega}\frac{1}{(N_1N_2)^{1-1/\omega}}. \end{aligned}$$

Now since $N_1, N_2 \geqslant 1$, a direct estimation gives (4.5). Corollary 2 is proved. □

5. A Numerical Analysis of the Truncation Error

In this section, we comment the estimates (4.4) and (4.5) for the uniformly truncated error $e_{N;f}$. We also give two numerical examples of $e_{N;f}$ by tables.

The estimates (4.4) and (4.5) depend on the signal f magnitude c_f, the highest frequencies $\sigma = (\sigma_1, \sigma_2)$ of f, the numbers N_1 and N_2 of samples that are used in the partial sum (4.1), and a number ω which we can choose free. Now we will briefly comment on their action on $e_{f;N}$.

Magnitude c_f.

If $f \in B^p_{Q^2_\sigma}$, $1 \leqslant p < \infty$, satisfies (4.3), then the function $F(x) = x_1 x_2 f(x)$ is in $B^\infty_{Q^2_\sigma}$. Hence, we can choose c_f equal to the maximal amplitude of F in the domain $\{x \in \mathbb{R}^2 : |x_1| \geqslant N_1, |x_2| \geqslant N_2\}$, i.e.

$$c_f = \max\big\{|F(x)| : |x_1| \geqslant N_1, |x_2| \geqslant N_2\big\}.$$

However, instead of this choice for c_f, we can associate c_f with certain function relating to the p-energy of signal f. Indeed, if f satisfies (4.3), then we can take

$$c_f = \left(\frac{(p-1)^2}{4}\right)^{2/p} (N_1 N_2)^{1-1/p} \Sigma_{p;N}(f),$$

where $\Sigma_{p;N}(f)$ is a part of the p-energy of f supported in $\{x \in \mathbb{R}^2 \colon |x_1| \geqslant N_1,\ |x_2| \geqslant N_2\}$, i.e.

$$\Sigma_{p;N}(f) = \left(\int_{|x_1| \geqslant N_1, |x_2| \geqslant N_2} \big|f(x)\big|^p \, dx\right)^{1/p}.$$

Therefore, we shall calculate below only the quantity $e_{N;f}/c_f$ instead of the usual truncated error $e_{N;f}$.

Frequencies $\sigma = (\sigma_1, \sigma_2)$.

We recall that humans can hear a range of frequencies from 20 to 20,000 Hz, i.e. we can hear signals which have 20-20000 full cicles per second. On the other hand, in the case of smartphones (cell phones) the frequency band ranges from approximately 300 Hz to 3400 Hz. Cutting out the other frequencies reduced the amount of information that would have to be transmitted and reduced the Nyquist frequency. Consequently, our smartphones have such low quality, i.e. they are not picking up all the frequencies that make up our voices or that we can hear. For this reason, we shall calculate below the estimates of $e_{N;f}/c_f$ in the case $\sigma_1 = \sigma_2 = 3400$ Hz.

The numbers of samples N_1 and N_2.

Obviously, by getting higher values of N_1 and N_2, we get a better estimate of $e_N(f)$ and $e_N(f)/c_f$. For example, if N_1 and N_2 are such that $|e_N(f)/c_f| < 0.01$, then we can understand that the difference $|f - f_N|$ is less than 1 percent of the maximal amplitude

Table 1
Estimates of $e_{N;f}/c_f$ in (4.4) with $N_1 = N_2 = n$ and with frequencies up to $\sigma_1 = \sigma_2 = 3400$ Hz.

$\omega \setminus n$	$2 \cdot 10^4$	$5 \cdot 10^4$	10^5	$5 \cdot 10^5$	10^6	$2 \cdot 10^6$
$1,25$	10231,9	7092,18	5374,85	2823,44	2139,76	1621.64
2	328,501	131,399	65,6996	13,1399	6,56994	3.28497
4	10,6876	2,70375	0,95592	$8,5499 \cdot 10^{-2}$	$3,0229 \cdot 10^{-2}$	$10,6874 \cdot 10^{-3}$
8	3,02323	0,60824	0,18083	$1,0816 \cdot 10^{-2}$	$0,3217 \cdot 10^{-2}$	$0,9560 \cdot 10^{-3}$
12	2,72504	0,50794	0,14254	$0,7455 \cdot 10^{-2}$	$0,2092 \cdot 10^{-2}$	$0,5871 \cdot 10^{-3}$
16	3,05007	0,54722	0,14919	$0,7297 \cdot 10^{-2}$	$0,1989 \cdot 10^{-2}$	$0,5424 \cdot 10^{-3}$
20	3,60504	0,63215	0,16938	$0,7958 \cdot 10^{-2}$	$0,2132 \cdot 10^{-2}$	$0,5714 \cdot 10^{-3}$
50	11,4879	1,90666	0,49006	$2,0906 \cdot 10^{-2}$	$0,5373 \cdot 10^{-2}$	$1,3812 \cdot 10^{-3}$

c_f of f. Below we calculate the estimates of $e_N(f)/c_f$ for different values of N_1 and N_2 and the same values of other parameters σ and ω.

A parameter ω.

From the proof of Theorem 3 follows that we can choose the value of ω in $(1, \infty)$ free, since ω is only the technical parameter that was used in the proof of this theorem. However, a preliminary analysis of the action of ω on $e_{N;f}$ can be done. Indeed, let us define

$$A_{N,\omega}(f) = \bigg[\bigg(\frac{4\omega^2}{(\omega+1)^2}\bigg)^{1-1/\omega} + 2\bigg(\frac{2\omega^2}{\omega+1}\bigg)^{1-1/\omega}\bigg(2 + \frac{1}{2\pi N_1} + \frac{1}{2\pi N_2}\bigg) + 4\omega^{2(1-1/\omega)}\bigg(1 + \frac{1}{2\pi N_1} + \frac{1}{2\pi N_2} + \frac{3}{4\pi^2 N_1 N_2}\bigg)\bigg]\big(2(\omega-1)^2\big)^{1/\omega}$$

and

$$B_N(f) = \frac{1}{(N_1 N_2)^{1-1/\omega}}.$$

Then the right-hand side of (4.4) is equal to

$$\frac{c_f \sigma_1 \sigma_2}{4\pi^2} A_{N,\omega}(f) B_N(f).$$

It is obvious that for given N_1 and N_2, $B_N(f)$ can be made smaller by taking the values of ω close to infinity. On the other hand, it is easy to check that

$$\lim_{\omega \to +\infty} A_{N,\omega}(f) = \infty.$$

Therefore, the choice of optimized values of ω for the best estimates in (4.4) is a distinct and nontrivial problem. Here we give two numerical examples for estimate of $e_{N;f}$.

Those tables give us certain knowledge in the case of smartphones, where the frequency band ranges up to 3400 Hz. For example, assume that we want to know what

Table 2
Estimates of $e_{N;f}/c_f$ in (4.5) with $N_1 = N_2 = n$ and with frequencies up to $\sigma_1 = \sigma_2 = 3400$ Hz.

$\omega \setminus n$	$2 \cdot 10^4$	$5 \cdot 10^4$	10^5	$5 \cdot 10^5$	10^6	$2 \cdot 10^6$
1, 25	13856,6	9604,67	7278,96	3823,68	2897,80	1153,64
2	496,926	198,771	99,3853	19,8771	9,93853	0,99385
4	20,4711	5,17884	1,83099	$16,377 \cdot 10^{-2}$	$5,7912 \cdot 10^{-2}$	$20,471 \cdot 10^{-3}$
8	7,05162	1,41872	0,42179	$2,5229 \cdot 10^{-2}$	$0,7501 \cdot 10^{-2}$	$2,2299 \cdot 10^{-3}$
12	6,88186	1,28277	0,35996	$1,8829 \cdot 10^{-2}$	$0,5284 \cdot 10^{-2}$	$1,4827 \cdot 10^{-3}$
16	8,03339	1.44132	0.39294	$1,9220 \cdot 10^{-2}$	$0,5239 \cdot 10^{-2}$	$1,4286 \cdot 10^{-3}$
20	9,74328	1,70852	0,45778	$2,1510 \cdot 10^{-2}$	$0,5763 \cdot 10^{-2}$	$1,5442 \cdot 10^{-3}$
50	33,0651	5,48788	1,41054	$6,0174 \cdot 10^{-2}$	$1,5466 \cdot 10^{-2}$	$3,9752 \cdot 10^{-3}$

quantity $N = (N_1, N_2)$ of sample values of a signal f and its derivatives in (4.1) guarantees us that $|f - f_N|$ is less than 1 percent of the maximal amplitude c_f. In Table 1 we see that that is enough to take $N_1, N_2 \geqslant 5 \cdot 10^5$ if the parameter ω ranges from 12 to 20.

Finally, from the Table 2 we conclude that (4.5) gives us the estimate of $e_{N;f}$ such that it is several times (for example, up to three times) less that (4.4). Hence, although the estimate (4.5) is a simpler than (4.4) but (4.5) is better suited to theoretical purposes.

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