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Models for Multiple Attribute Decision Making with Interval-Valued Pythagorean Fuzzy Muirhead Mean Operators and Their Application to Green Suppliers Selection

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Abstract. In this paper, we extend MM operator and dual MM (DMM) operator to process the interval-valued Pythagorean fuzzy numbers (IVPFNs) and then to solve the MADM problems. Firstly, we develop some interval-valued Pythagorean fuzzy Muirhead mean operators by extending MM and DMM operators to IVPFNs. Then, we prove some properties and discuss some special cases with respect to the parameter vector. Moreover, we present some new methods to deal with MADM problems with the IVPFNs based on the proposed MM and DMM operators. Finally, we verify the validity and reliability of our methods by using an application example for green supplier selections, and analyse the advantages of our methods by comparing it with other existing methods.

Key words: multiple attribute decision making (MADM), Muirhead mean (MM) operator, dual Muirhead mean (DMM) operator, interval-valued Pythagorean fuzzy numbers (IVPFNs), interval-valued Pythagorean Fuzzy Muirhead mean (IVPFMM) operator, interval-valued Pythagorean Fuzzy dual Muirhead mean (IVPFDMM) operator, green suppliers selection.

1. Introduction

Atanassov (1986) defined the intuitionistic fuzzy set (IFS) based on the fuzzy set (Zadeh, 1965) such that its sum is not greater than one. After it was defined, researchers have applied these theories in different disciplines (Xu, 2007; Xu and Yager, 2006; Li *et al.*, 2018a; Garg and Arora, 2018; Ngan *et al.*, 2018; Li and Chen, 2018; Liu *et al.*, 2018; Arya and Yadav, 2018; Baccour, 2018; Kahraman *et al.*, 2018; Jafarian *et al.*, 2018; Xia, 2018; Hao and Chen, 2018; Xian *et al.*, 2018; Deng *et al.*, 2018a) and found that they are more productive to handle the uncertainties during the analysis. Although the above theories have been successfully defined, in some cases, they are unable to handle the situation by IFS. For instance, if a decision maker (DM) takes the membership degrees of any element as 0.8 and 0.5, then clearly their sum is not less than one. Hence, under such types of cases, IFS has some sort of deficiencies. In order to solve it, Pythagorean fuzzy set (PFS)

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(Yager, 2013, 2014), an extension of IFSs, has appeared as a good tool for describing the indeterminacy in uncertain MADM. For this set, the condition of the sum of the degrees is replaced with their sum of squares, which should be less than one and hence the PFS is more general than the IFS. Further, it is clear that $0.8^2 + 0.5^2 \leqslant 1$ and hence PFS stand, for such cases. After its appearance, Zhang and Xu (2014) presented the PFS TOPSIS for MADM. Zhang (2016a) presented PFS similarity measure for solving MADM. Peng and Yang (2015) developed some fundamental properties of the PFNs. Reformat and Yager (2014) used the PFSs in solving the recommender system. Zeng *et al.* (2014) developed a hybrid method for Pythagorean fuzzy MADM. Garg (2016a, 2017b) proposed some generalized PFS aggregation operators based on Einstein Operations. Zhang (2016b) extended the PFS to the interval-valued PFSs (IVPFSs). Garg (2016b) presented some aggregation operators with IVPFNs. Also, a new accuracy function has been presented to rank the IVPFNs. However, in terms of the information measure theory, a novel accuracy function (Garg, 2016b), correlation coefficient (Garg, 2016c), improved accuracy function (Garg, 2017a) were introduced. Li *et al.* (2018b) defined the Hamy Mean Operators with PFNs. Li *et al.* (2018c) extended the methods of Li *et al.* (2018b) to IVPFNs. Wei and Lu (2018a) defined the power aggregation operators with PFNs. Gao *et al.* (2018b) developed some novel interaction aggregation operators with PFNs. Wei and Lu (2018b) presented Maclaurin Symmetric Mean Operators with PFNs. Wei and Wei (2018a) defined the similarity measures of PFSs. Gao (2018) introduced the Hamacher prioritized operators with PFNs. Wei *et al.* (2018a) proposed some Pythagorean hesitant fuzzy Hamacher operators. Wei and Lu proposed some dual hesitant Pythagorean fuzzy Hamacher aggregation operators. Lu *et al.* (2017) defined some hesitant Pythagorean fuzzy Hamacher operators. Some MADM models with Pythagorean 2-tuple linguistic information are defined in Wei *et al.* (2017a), Huang and Wei (2018), Tang and Wei (2018). Some MADM methods with 2-tuple linguistic Pythagorean fuzzy information are proposed in Deng *et al.* (2018b), Wang *et al.* (2018a). Wang *et al.* (2018a) proposed some Heronian mean operators with q-Rung Orthopair Fuzzy information.

In some real MADM problems, there exist interrelationships among the attributes. Bonferroni mean (BM) operators (Bonferroni, 1950; Liu *et al.*, 2017; Wang *et al.*, 2018b; Wei, 2017a, 2017b; Jiang and Wei, 2017; Wei *et al.*, 2013) and the Heronian mean (HM) (Yu, 2012; Liu *et al.*, 2013, 2014; Yu *et al.*, 2015; Chu and Liu, 2015) operators provided a tool to consider interrelationships of aggregated arguments; however, they can only consider interrelationships between two attributes and cannot process interrelationships among three or more than three attributes. Muirhead mean (MM) (Muirhead, 1902) is a well-known aggregation operator which can consider interrelationships among any number of arguments assigned by a variable vector, and some existing operators, such as arithmetic and geometric operators (not considering the interrelationships), both BM operator and Maclaurin symmetric mean (Maclaurin, 1729) are the special cases of MM operator. Thus, the MM can offer a robust and flexible mechanism to deal with the information fusion problem and make it more adequate to cope with MADM. However, the original MM can only deal with the numeric arguments. Qin and Liu (2016) extended the MM operator to process the 2-tuple linguistic information, and proposed some 2-tuple linguistic MM operators and apply the proposed operators to solve the MADM problems.

Because IVPFNs can easily describe the fuzzy information, and the MM operator and the dual MM (DMM) operator can capture interrelationships among any number of arguments assigned by a variable vector, it is necessary to expand the MM and the DMM operators to deal with the IVPFNs. The purpose of this paper is to propose some IVPF MM operators by extending the MM and the DMM operators to IVPFNs, then to study some properties of these operators, and apply them to cope with the IVPFN MADM.

In order to achieve this purpose, the rest of this paper is set out as follows. Section 2 introduces the basic definitions and theory of IVPFSs. In Section 3, we propose the some MM and DMM operators with IVPFNs, and study some good properties of these proposed operators. In Section 4, we propose two MADM methods for IVPFNs with the PFWMM operator and PFWDMM operator. In Section 5, an illustrative example for green supplier selections is given to verify the validity of the proposed methods. In Section 6, we give some conclusions of this study.

2. Basic Concepts

In this section, we introduce some fundamental concept of IVPFSs and MM, which will be used in the next section. These concepts base on a fixed set X.

2.1. *Pythagorean Fuzzy Set (PFS)*

Definition 1. (See Yager, 2013, 2014.) A PFS P is defined as

$$P = \{\langle x, (\mu_p(x), \nu_p(x))\rangle | x \in X\}, \tag{1}$$

where the function, $\mu_p : X \to [0, 1]$ defines the degree of membership and the function $\nu_p : X \to [0, 1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively, and, for every $x \in X$, the condition $(\mu_p)^2 + (\nu_p)^2 \leqslant 1$ holds.

Definition 2. (See Ren *et al.*, 2016.) The $p = (\mu, \nu)$ is called a Pythagorean fuzzy number (PFN) and defines the score and accuracy functions as $S(p) = \mu^2 - \nu^2$ and $H(p) = \mu^2 + \nu^2$. In order to compare two or more PFNs p_1 and p_2, a comparison law is defined as

(1) if $S(p_1) < S(p_2)$, then $p_1 < p_2$;
(2) if $S(p_1) = S(p_2)$, then
  (a) if $H(p_1) = H(p_2)$, then $p_1 = p_2$;
  (b) if $H(p_1) < H(p_2)$, then $p_1 < p_2$.

2.2. *Interval Valued Pythagorean Fuzzy Set (IVPFS)*

Lang *et al.* (2015) extended the PFS to the IVPFSs which are defined as follows over the fixed set X.

Definition 3. (See Lang *et al.*, 2015.) An IVPFS $\tilde{p}$ is defined as

$$\tilde{p} = \{\langle x, (\tilde{\mu}_{\tilde{p}}(x), \tilde{\nu}_{\tilde{p}}(x))\rangle | x \in X\}, \tag{2}$$

where $\tilde{\mu}_{\tilde{p}}(x) = [\mu^L_{\tilde{p}}(x), \mu^R_{\tilde{p}}(x)]$, $\tilde{\nu}_{\tilde{p}}(x) = [\nu^L_{\tilde{p}}(x), \nu^R_{\tilde{p}}(x)]$ are the interval numbers of $[0, 1]$ with the condition $0 \leqslant (\mu^R_{\tilde{P}}(x))^2 + (\nu^R_{\tilde{P}}(x))^2 \leqslant 1$, $\forall x \in X$. The pair $\tilde{p} = ([\mu^L_{\tilde{p}}, \mu^R_{\tilde{p}}], [\nu^L_{\tilde{p}}, \nu^R_{\tilde{p}}])$ is called an IVPF number (IVPFN), where $\mu_{\tilde{p}}, \nu_{\tilde{p}} \in [0, 1]$ and $(\mu^R_{\tilde{p}})^2 + (\nu^R_{\tilde{p}})^2 \leqslant 1$.

Definition 4. (See Garg, 2016b.) For three IVPFNs $\tilde{p}_1 = ([\mu^L_{\tilde{p}_1}, \mu^R_{\tilde{p}_1}], [\nu^L_{\tilde{p}_1}, \nu^R_{\tilde{p}_1}])$, $\tilde{p}_2 = ([\mu^L_{\tilde{p}_2}, \mu^R_{\tilde{p}_2}], [\nu^L_{\tilde{p}_2}, \nu^R_{\tilde{p}_2}])$, and $\tilde{p} = ([\mu^L_{\tilde{p}}, \mu^R_{\tilde{p}}], [\nu^L_{\tilde{p}}, \nu^R_{\tilde{p}}])$, the basic operational laws are defined as follows:

$$(1)\ \ \tilde{p}_1 \oplus \tilde{p}_2 = \Big(\Big[\sqrt{(\mu^L_{\tilde{p}_1})^2 + (\mu^L_{\tilde{p}_2})^2 - (\mu^L_{\tilde{p}_1})^2(\mu^L_{\tilde{p}_2})^2}, \sqrt{(\mu^R_{\tilde{p}_1})^2 + (\mu^R_{\tilde{p}_2})^2 - (\mu^R_{\tilde{p}_1})^2(\mu^R_{\tilde{p}_2})^2}\Big], \big[\nu^L_{\tilde{p}_1}\nu^L_{\tilde{p}_2}, \mu^R_{\tilde{p}_1}\mu^R_{\tilde{p}_1}\big]\Big);$$

$$(2)\ \ \tilde{p}_1 \otimes \tilde{p}_2 = \Big(\Big[\nu^L_{\tilde{p}_1}\mu^L_{\tilde{p}_2}, \mu^R_{\tilde{p}_1}\mu^R_{\tilde{p}_1}\Big], \Big[\sqrt{(\nu^L_{\tilde{p}_1})^2 + (\nu^L_{\tilde{p}_2})^2 - (\nu^L_{\tilde{p}_1})^2(\nu^L_{\tilde{p}_2})^2}, \sqrt{(\nu^R_{\tilde{p}_1})^2 + (\nu^R_{\tilde{p}_2})^2 - (\nu^R_{\tilde{p}_1})^2(\nu^R_{\tilde{p}_2})^2}\Big]\Big);$$

$$(3)\ \ \pi\tilde{p} = \Big(\Big[\sqrt{1 - (1 - (\mu^L_{\tilde{p}})^2)^\pi}, \sqrt{1 - (1 - (\mu^R_{\tilde{p}})^2)^\pi}\Big], \big[(\nu^L_{\tilde{p}})^\pi, (\nu^R_{\tilde{p}})^\pi\big]\Big), \pi > 0;$$

$$(4)\ \ (\tilde{p})^\pi = \Big(\big[(\mu^L_{\tilde{p}})^\pi, (\mu^R_{\tilde{p}})^\pi\big], \Big[\sqrt{1 - (1 - (\nu^L_{\tilde{p}})^2)^\pi}, \sqrt{1 - (1 - (\nu^R_{\tilde{p}})^2)^\pi}\Big]\Big), \pi > 0;$$

$$(5)\ \ (\tilde{p})^c = \big(\big[\nu^L_{\tilde{p}}, \nu^R_{\tilde{p}}\big], \big[\mu^L_{\tilde{p}}, \mu^R_{\tilde{p}}\big]\big).$$

Based on the Definition 4, Garg (2016b) derived the following properties easily.

Theorem 1. *Let $\tilde{p}_1 = ([\mu^L_{\tilde{p}_1}, \mu^R_{\tilde{p}_1}], [\nu^L_{\tilde{p}_1}, \nu^R_{\tilde{p}_1}])$, and $\tilde{p}_2 = ([\mu^L_{\tilde{p}_2}, \mu^R_{\tilde{p}_2}], [\nu^L_{\tilde{p}_2}, \nu^R_{\tilde{p}_2}])$ be two IVPFNs, $\pi, \pi_1, \pi_2 > 0$ be three real numbers, then*

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_1$;
(2) $\tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1$;
(3) $\pi(\tilde{p}_1 \oplus \tilde{p}_2) = \pi\tilde{p}_1 \oplus \pi\tilde{p}_2$;
(4) $(\tilde{p}_1 \otimes \tilde{p}_2)^\pi = (\tilde{p}_1)^\pi \otimes (\tilde{p}_2)^\pi$;
(5) $\pi_1\tilde{p}_1 \oplus \pi_2\tilde{p}_1 = (\pi_1 + \pi_2)\tilde{p}_1$;
(6) $\tilde{p}_1^{\pi_1} \otimes \tilde{p}_1^{\pi_2} = \tilde{p}_1^{(\pi_1+\pi_2)}$;
(7) $(\tilde{p}_1^{\pi_1})^{\pi_2} = (\tilde{p}_1)^{\pi_1\pi_2}$.

Definition 5. For an IVPFN $\tilde{p} = ([\mu^L_{\tilde{p}}, \mu^R_{\tilde{p}}], [\nu^L_{\tilde{p}}, \nu^R_{\tilde{p}}])$, the score and accuracy functions of it are defined as $S(\tilde{p}) = \frac{1}{4}[(1 + (\mu^L_{\tilde{p}})^2 - (\nu^L_{\tilde{p}})^2) + (1 + (\mu^R_{\tilde{p}})^2 - (\nu^R_{\tilde{p}})^2)]$, and $H(\tilde{p}) = \frac{(\mu^L_{\tilde{p}})^2 + (\mu^R_{\tilde{p}})^2 + (\nu^L_{\tilde{p}})^2 + (\nu^R_{\tilde{p}})^2}{2}$, respectively. Further, in order to compare two different IVPFNs $\tilde{p}_1$ and $\tilde{p}_2$, an order relation is defined as

(1) if $S(\tilde{p}_1) < S(\tilde{p}_2)$, then $\tilde{p}_1 < \tilde{p}_2$,
(2) if $S(\tilde{p}_1) = S(\tilde{p}_2)$, then
  (i) if $H(\tilde{p}_1) = H(\tilde{p}_2)$, then $\tilde{p}_1 = \tilde{p}_2$,
  (ii) if $H(\tilde{p}_1) < H(\tilde{p}_2)$, then $\tilde{p}_1 < \tilde{p}_2$.

2.3. *Muirhead Mean (MM)*

Muirhead (1902) proposed the MM operator.

Definition 6. (See Muirhead, 1902.) Let α_j $(j = 1, 2, \dots, n)$ be a group of crisp numbers and $[\pi] = (\pi_1, \pi_2, \dots, \pi_n) \in R$, then the Muirhead mean (MM) operator is defined as

$$MM^{\pi}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{1}{n!} \sum_{\varphi \in \phi_n} \prod_{j=1}^{n} \alpha_{\varphi(j)}^{\pi_j} \right)^{\frac{1}{\sum_{j=1}^{n} \pi_j}}, \tag{3}$$

where $\varphi(j)$ $(j = 1, 2, \dots, n)$ is any permutation of $(1, 2, \dots, n)$ and ϕ_n is set of all permutations of $(1, 2, \dots, n)$.

By assigning some special vectors to π, we can obtain some special cases of the MM:
(1) If $\pi = (1, 0, \dots, 0)$, the MM is reduced to

$$MM^{(1,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{n} \sum_{j=1}^{n} \alpha_j, \tag{4}$$

which is the arithmetic averaging operator.
(2) If $\pi = (1, 1, 0, \dots, 0)$, the MM is reduced to

$$MM^{(1,1,0,\dots,0)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{n(n+1)} \sum_{i,j=1; i \neq j}^{n} \alpha_i \alpha_j, \tag{5}$$

which is the BM operator.
(3) If $\pi = (\overbrace{1, 1, \dots, 1}^{k}, \overbrace{0, 0, \dots, 0}^{n-k})$, the MM is reduced to

$$MM^{(\overbrace{1, 1, \dots, 1}^{k}, \overbrace{0, 0, \dots, 0}^{n-k})}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{\sum_{1 \leqslant i_1 \leqslant \dots \leqslant i_i \leqslant n} \prod_{j=1}^{n} \alpha_{i_j}}{C_n^k} \right)^{1/k}, \tag{6}$$

which is the Maclaurin symmetric mean (MSM) (Maclaurin, 1729) operator.
(4) If $\pi = (1/n, 1/n, \dots, 1/n)$, the MM is reduced to

$$MM^{(1/n,1/n,\dots,1/n)}(\alpha_1, \alpha_2, \dots, \alpha_n) = \prod_{j=1}^{n} (\alpha_j)^{1/n}, \tag{7}$$

which is the arithmetic averaging operator.

3. Interval Valued Pythagorean Fuzzy Muirhead Mean (IVPFMM) Operators

In this section, we shall develop some Muirhead mean operators with IVPFNs.

3.1. *IVPFMM Operator*

The MM operator has usually been utilized in situations with interaction relationships. Next, we extend MM operator to IVPFS. From Definitions 4 and 6, we can obtain:

Definition 7. Let $\tilde{p}_j = ([\mu_j^L, \mu_j^R], [\nu_j^L, \nu_j^R])$ $(j = 12, \dots, n)$ be a set of IVPFNs and $[\pi] = (\pi_1, \pi_2, \dots, \pi_n) \in R$ be a vector of parameters, then the interval valued Pythagorean Fuzzy Muirhead mean (IVPFMM) operator is defined as

$$IVPMM^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\tilde{p}_{\varphi(j)}^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}, \tag{8}$$

where $\varphi(j)$ $(j = 1, 2, \dots, n)$ is any permutation of $(1, 2, \dots, n)$ and ϕ_n is the collection of all permutations of $(1, 2, \dots, n)$.

Based on the operations of the IVPFN described, we can get the Theorem 2.

Theorem 2. *Let $\tilde{p}_j = ([\mu_j^L, \mu_j^R], [\nu_j^L, \nu_j^R])$ $(j = 12, \dots, n)$ be a group of IVPFNs, then the corresponding aggregated value by utilizing IVPFMM operator is also an IVPFN, and*

$$\begin{aligned}
&\mathrm{IVPFMM}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\tilde{p}_{\varphi(j)}^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\
&= \left(\left[\begin{array}{c}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{L}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\ \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{R}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{array}\right],\right.\\
&\left.\left[\begin{array}{c}\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon_{\varphi(j)}^{L}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}},\\ \sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon_{\varphi(j)}^{R}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{array}\right]\right).
\end{aligned} \tag{9}$$

Proof.

$$(\tilde{p}^{R}_{\varphi(j)})^{\pi_j} = \left(\begin{array}{c} \left[\left(\mu^{L}_{\varphi(j)}\right)^{\pi_j}, \left(\mu^{R}_{\varphi(j)}\right)^{\pi_j}\right], \\ \left[\begin{array}{c} \sqrt{1-\left(1-\left(v^{L}_{\varphi(j)}\right)^{2}\right)^{\pi_j}}, \\ \sqrt{1-\left(1-\left(v^{R}_{\varphi(j)}\right)^{2}\right)^{\pi_j}} \end{array} \right] \end{array} \right), \tag{10}$$

$$\prod_{j=1}^{n} \tilde{p}^{\pi_j}_{\varphi(j)} = \left(\begin{array}{c} \left[\prod_{j=1}^{n}\left(\mu^{L}_{\varphi(j)}\right)^{\pi_j}, \prod_{j=1}^{n}\left(\mu^{R}_{\varphi(j)}\right)^{\pi_j}\right], \\ \left[\begin{array}{c} \sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{L}_{\varphi(j)}\right)^{2}\right)^{\pi_j}}, \\ \sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{R}_{\varphi(j)}\right)^{2}\right)^{\pi_j}} \end{array} \right] \end{array} \right). \tag{11}$$

Thereafter,

$$\sum_{\varphi\in\phi_n}\prod_{j=1}^{n} \tilde{p}^{\pi_j}_{\varphi(j)} = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^{L}_{\varphi(j)}\right)^{2\pi_j}\right)}, \\ \sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^{R}_{\varphi(j)}\right)^{2\pi_j}\right)} \end{array} \right], \\ \left[\begin{array}{c} \prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{L}_{\varphi(j)}\right)^{2}\right)^{\pi_j}}, \\ \prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{R}_{\varphi(j)}\right)^{2}\right)^{\pi_j}} \end{array} \right] \end{array} \right), \tag{12}$$

$$\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n} \tilde{p}^{\pi_j}_{\varphi(j)} = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{1-\left(1-\left(1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^{L}_{\varphi(j)}\right)^{2\pi_j}\right)\right)\right)^{\frac{1}{n!}}}, \\ \sqrt{1-\left(1-\left(1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^{R}_{\varphi(j)}\right)^{2\pi_j}\right)\right)\right)^{\frac{1}{n!}}} \end{array} \right], \\ \left[\begin{array}{c} \left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{L}_{\varphi(j)}\right)^{2}\right)^{\pi_j}}\right)^{\frac{1}{n!}}, \\ \left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v^{R}_{\varphi(j)}\right)^{2}\right)^{\pi_j}}\right)^{\frac{1}{n!}} \end{array} \right] \end{array} \right). \tag{13}$$

Therefore,

$$
\begin{aligned}
&\left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\tilde{p}_{\varphi(j)}^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\
&=\left(\left[\begin{array}{l}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{L}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\ \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{R}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{array}\right],\right.\\
&\qquad\left.\left[\begin{array}{l}\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{L}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}},\\ \sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{array}\right]\right)
\end{aligned}
\tag{14}
$$

and then, we can know:

$$
0\leqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{R}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\leqslant 1,
\tag{15}
$$

$$
0\leqslant\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\leqslant 1.
\tag{16}
$$

We can obtain $(\mu_{\phi(j)}^{R})^2+(v_{\phi(j)}^{R})^2\leqslant 1$ from the definition of IVPFS, so

$$
\begin{aligned}
&\left(\left(\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu_{\varphi(j)}^{R}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\right)^{2}\right.\\
&\quad+\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\right)^{2}\\
&\leqslant\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}
\end{aligned}
$$

$$+1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon_{\varphi(j)}^{R}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}=1. \tag{17}$$

We complete the proof. □

Example 1. Let $x_1=([0.2,0.5],[0.4,0.6])$, $x_2=([0.4,0.6],[0.2,0.3])$, and $x_3=([0.6,0.8],[0.1,0.2])$ be three IVPFNs, and $\pi=(0.2,0.5,0.3)$, then we have

$$\begin{aligned}
&\mathrm{IVPFMM}^{[\pi]}(x_1,x_2,x_3)\\
&=\left(\left[\begin{array}{l}\left(\sqrt{1-\left(\begin{array}{l}(1-0.2^{0.4}\times 0.4^{1}\times 0.6^{0.6})\times(1-0.2^{0.4}\times 0.6^{1}\times 0.4^{0.6})\times\\(1-0.4^{0.4}\times 0.2^{1}\times 0.6^{0.6})\times(1-0.4^{0.4}\times 0.6^{1}\times 0.2^{0.6})\times\\(1-0.6^{0.4}\times 0.4^{1}\times 0.2^{0.6})\times(1-0.6^{0.4}\times 0.2^{1}\times 0.4^{0.6})\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}},\\
\left(\sqrt{1-\left(\begin{array}{l}(1-0.5^{0.4}\times 0.6^{1}\times 0.8^{0.6})\times(1-0.5^{0.4}\times 0.8^{1}\times 0.6^{0.6})\times\\(1-0.6^{0.4}\times 0.8^{1}\times 0.5^{0.6})\times(1-0.6^{0.4}\times 0.5^{1}\times 0.8^{0.6})\times\\(1-0.8^{0.4}\times 0.5^{1}\times 0.6^{0.6})\times(1-0.8^{0.4}\times 0.6^{1}\times 0.5^{0.6})\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}\end{array}\right],\right.\\
&\left.\left[\begin{array}{l}\sqrt{1-\left(1-\left(\begin{array}{l}(1-0.84^{0.2}\times 0.96^{0.5}\times 0.99^{0.3})\times(1-0.84^{0.2}\times 0.99^{0.5}\times 0.96^{0.3})\times\\(1-0.96^{0.2}\times 0.84^{0.5}\times 0.99^{0.3})\times(1-0.96^{0.2}\times 0.99^{0.5}\times 0.84^{0.3})\times\\(1-0.99^{0.2}\times 0.96^{0.5}\times 0.84^{0.3})\times(1-0.99^{0.2}\times 0.84^{0.5}\times 0.96^{0.3})\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}},\\
\sqrt{1-\left(1-\left(\begin{array}{l}(1-0.64^{0.2}\times 0.91^{0.5}\times 0.96^{0.3})\times(1-0.64^{0.2}\times 0.96^{0.5}\times 0.91^{0.3})\times\\(1-0.91^{0.2}\times 0.64^{0.5}\times 0.96^{0.3})\times(1-0.91^{0.2}\times 0.96^{0.5}\times 0.64^{0.3})\times\\(1-0.96^{0.2}\times 0.64^{0.5}\times 0.91^{0.3})\times(1-0.96^{0.2}\times 0.91^{0.5}\times 0.64^{0.3})\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}\end{array}\right]\right)\\
&=\big([0.3694,0.6241],[0.2648,0.4136]\big).
\end{aligned}$$

In the following, we give some properties of IVPFMM operator.

Property 1 (*Idempotency*). *Let* $\tilde{p}_j=([\mu_{\tilde{p}_j}^{L},\mu_{\tilde{p}_j}^{R}],[\nu_{\tilde{p}_j}^{L},\nu_{\tilde{p}_j}^{R}])=\tilde{p}=([\mu_{\tilde{p}}^{L},\mu_{\tilde{p}}^{R}],[\nu_{\tilde{p}}^{L},\nu_{\tilde{p}}^{R}])$, *(* $j=1,2,\dots,n$ *), then*

$$IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)=\tilde{p}. \tag{18}$$

Proof.

$$\begin{aligned}
IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)&=\left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\tilde{p}_{\varphi(j)}^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\
&=\left(\frac{1}{n!}n!\tilde{p}^{\sum_{j=1}^{n}\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}=\tilde{p}. \qquad \square
\end{aligned} \tag{19}$$

Property 2 (*Monotonicity*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [\nu^L_{\tilde{p}_j}, \nu^R_{\tilde{p}_j}])$ *and* $\tilde{q}_j = ([\mu^L_{\tilde{q}_j}, \mu^R_{\tilde{q}_j}], [\nu^L_{\tilde{q}_j}, \nu^R_{\tilde{q}_j}])$, $(j = 1, 2, \ldots, n)$ *be two sets of IVPFNs, if* $(\mu^L_{\tilde{p}_j})^2 + (\mu^R_{\tilde{p}_j})^2 \leqslant (\mu^L_{\tilde{q}_j})^2 + (\mu^R_{\tilde{q}_j})^2$, *and* $(\nu^L_{\tilde{p}_j})^2 + (\nu^R_{\tilde{p}_j})^2 \geqslant (\nu^L_{\tilde{q}_j})^2 + (\nu^R_{\tilde{q}_j})^2$ *then*

$$IVPFMM^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leqslant IVPFMM^{[\pi]}(\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_n). \tag{20}$$

Proof.

$$\prod_{j=1}^{n} \left(\mu^R_{\varphi(\tilde{p}_j)}\right)^{2\pi_j} \leqslant \prod_{j=1}^{n} \left(\mu^R_{\varphi(\tilde{q}_j)}\right)^{2\pi_j}, \tag{21}$$

$$\left(\sum_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^R_{\varphi(\tilde{p}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}} \geqslant \left(\sum_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^R_{\varphi(\tilde{q}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}. \tag{22}$$

Therefore,

$$\begin{aligned}&\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^R_{\varphi(\tilde{p}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\&\quad\leqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^R_{\varphi(\tilde{q}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}.\end{aligned} \tag{23}$$

Similarly, we also can obtain

$$\begin{aligned}&\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon^R_{\varphi(\tilde{p}_j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\\&\quad\geqslant\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon^R_{\varphi(\tilde{q}_j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{aligned} \tag{24}$$

and

$$\begin{aligned}&\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^L_{\varphi(\tilde{p}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\&\quad\leqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^L_{\varphi(\tilde{q}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\end{aligned} \tag{25}$$

$$\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^L_{\varphi(\tilde{p}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}$$

$$\geqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(\mu^L_{\varphi(\tilde{q}_j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}, \tag{26}$$

then, the proof is completed. □

Then

(1) If $(\mu^L_{\tilde{p}_j})^2+(\mu^R_{\tilde{p}_j})^2<(\mu^L_{\tilde{q}_j})^2+(\mu^R_{\tilde{q}_j})^2$, and $(\nu^L_{\tilde{p}_j})^2+(\nu^R_{\tilde{p}_j})^2>(\nu^L_{\tilde{q}_j})^2+(\nu^R_{\tilde{q}_j})^2$, then

$$IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)<IVPFMM^{[\pi]}(\tilde{q}_1,\tilde{q}_2,\dots,\tilde{q}_n).$$

(2) If $(\mu^L_{\tilde{p}_j})^2+(\mu^R_{\tilde{p}_j})^2<(\mu^L_{\tilde{q}_j})^2+(\mu^R_{\tilde{q}_j})^2$, and $(\nu^L_{\tilde{p}_j})^2+(\nu^R_{\tilde{p}_j})^2=(\nu^L_{\tilde{q}_j})^2+(\nu^R_{\tilde{q}_j})^2$, then

$$IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)<IVPFMM^{[\pi]}(\tilde{q}_1,\tilde{q}_2,\dots,\tilde{q}_n).$$

(3) If $(\mu^L_{\tilde{p}_j})^2+(\mu^R_{\tilde{p}_j})^2=(\mu^L_{\tilde{q}_j})^2+(\mu^R_{\tilde{q}_j})^2$, and $(\nu^L_{\tilde{p}_j})^2+(\nu^R_{\tilde{p}_j})^2>(\nu^L_{\tilde{q}_j})^2+(\nu^R_{\tilde{q}_j})^2$, then

$$IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)<IVPFMM^{[\pi]}(\tilde{q}_1,\tilde{q}_2,\dots,\tilde{q}_n).$$

(4) If $(\mu^L_{\tilde{p}_j})^2+(\mu^R_{\tilde{p}_j})^2=(\mu^L_{\tilde{q}_j})^2+(\mu^R_{\tilde{q}_j})^2$, and $(\nu^L_{\tilde{p}_j})^2+(\nu^R_{\tilde{p}_j})^2>(\nu^L_{\tilde{q}_j})^2+(\nu^R_{\tilde{q}_j})^2$, then

$$IVPFMM^{[\pi]}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)=IVPFMM^{[\pi]}(\tilde{q}_1,\tilde{q}_2,\dots,\tilde{q}_n).$$

Property 3 (*Boundedness*). *Let* $\tilde{p}_j=([\mu^L_{\tilde{p}_j},\mu^R_{\tilde{p}_j}],[\nu^L_{\tilde{p}_j},\nu^R_{\tilde{p}_j}])$, $(j=1,2,\dots,n)$ *be a set of IVPFNs, if* $\tilde{p}^+_j=([\max_j(\mu^L_{\tilde{p}_j}),\max_j(\mu^R_{\tilde{p}_j})],[\min_j(\nu^L_{\tilde{p}_j}),\min_j(\nu^R_{\tilde{p}_j})])$, *and* $\tilde{p}^+=([\min_j(\mu^L_{\tilde{p}_j}),\min_j(\mu^R_{\tilde{p}_j})],[\max_j(\nu^L_{\tilde{p}_j}),\max_j(\nu^R_{\tilde{p}_j})])$, *According to the process of property of Monotonicity and Idempotency, it is easy to get that*

$$\tilde{p}^-_j\leqslant IVPFMM^{\pi}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_n)\leqslant\tilde{p}^+_j. \tag{27}$$

3.2. *IVPFWMM Operator*

In Section 3.1, it can be seen that the IVPFMM operator doesn't consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes play an important role in the process of aggregation. To overcome the limitation of IVPFMM, we shall develop the interval valued Pythagorean fuzzy weighted MM (IVPFWMM) operator as follows.

Definition 8. Let $\tilde{p}_j=([\mu^L_{\tilde{p}_j},\mu^R_{\tilde{p}_j}],[\nu^L_{\tilde{p}_j},\nu^R_{\tilde{p}_j}])$, $(j=1,2,\dots,n)$ be a group of IVPFNs with weight $w=(w_1,w_1,\dots,w_n)^T$, $\sum_{j=1}^{n}w_j=1$ and $[\pi]=(\pi_1,\pi_2,\dots,\pi_n)\in R$, then the interval valued Pythagorean fuzzy weighted Muirhead mean (IVPFWMM) operator is given as

$$IVPFMM_{w}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n} nw_{\varphi(j)}\tilde{p}_{\varphi(j)}^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}. \tag{28}$$

Theorem 3 can be derived by the operations of the IVPFN.

Theorem 3. *Let* $\tilde{p}_j = ([\mu_{\tilde{p}_j}^L, \mu_{\tilde{p}_j}^R], [\nu_{\tilde{p}_j}^L, \nu_{\tilde{p}_j}^R])$, $(j = 1, 2, \ldots, n)$ *be a collection of IVPFNs, then the corresponding aggregated value of IVPFWMM operator is also an IVPFN, and*

$$\begin{aligned}
&\mathrm{IVPFWMM}_{w}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \cdots \tilde{p}_n) = \left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\left(nw_{\varphi(j)}\tilde{p}_{\varphi(j)}\right)^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\
&= \left(\begin{bmatrix}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\
\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{bmatrix},\right.\\
&\left.\begin{bmatrix}\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}},\\
\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{bmatrix}\right).
\end{aligned} \tag{29}$$

Proof.

$$nw_{\varphi(j)}\tilde{p}_{\varphi(j)} = \left(\begin{bmatrix}\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\vartheta(j)}}},\\ \sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\vartheta(j)}}}\end{bmatrix}, \atop \left[\left(v_{\varphi(j)}^{L}\right)^{nw_{\vartheta(j)}}, \left(v_{\varphi(j)}^{R}\right)^{nw_{\vartheta(j)}}\right]\right), \tag{30}$$

$$\begin{aligned}
&\left(nw_{\varphi(j)}\tilde{p}_{\varphi(j)}\right)^{\pi_j}\\
&= \left(\begin{matrix}\left[\left(\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}}\right)^{\pi_j}, \left(\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}}\right)^{\pi_j}\right],\\
\left[\sqrt{1-\left(1-\left(v_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}, \sqrt{1-\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\right]\end{matrix}\right).
\end{aligned} \tag{31}$$

Thereafter,

$$\prod_{j=1}^{n}\left(nw_{\varphi(j)}\tilde{p}_{\varphi(j)}\right)^{\pi_j}$$

$$=\left(\begin{bmatrix}\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}}\right)^{\pi_j},\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}}\right)^{\pi_j}\end{bmatrix},\begin{bmatrix}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}},\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\end{bmatrix}\right), \tag{32}$$

$$\sum_{\phi\in\phi_n}\prod_{j=1}^{n}\left(nw_{\varphi(j)}\tilde{p}_{\varphi(j)}\right)^{\pi_j}$$

$$=\left(\begin{bmatrix}\sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)},\\ \sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)}\end{bmatrix},\begin{bmatrix}\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}},\\ \prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\end{bmatrix}\right). \tag{33}$$

Then,

$$\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\left(nw_{\varphi(j)}\tilde{p}_{\varphi(j)}\right)^{\pi_j}$$

$$=\left(\left[\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\phi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}},\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\phi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right],\left[\left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\phi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\right)^{\frac{1}{n!}},\left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(v_{\phi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\right)^{\frac{1}{n!}}\right]\right). \tag{34}$$

Therefore,

$$\left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}(nw_{\varphi(j)}\tilde{p}_{\varphi(j)})^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}$$

$$=\left(\left[\begin{array}{l}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\ \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{array}\right],\right.$$

$$\left.\left[\begin{array}{l}\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\right),\\ \left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\right)\end{array}\right]\right) \tag{35}$$

and we can get the followed easily:

$$0\leqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\leqslant 1, \tag{36}$$

$$0\leqslant\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\leqslant 1. \tag{37}$$

Therefore,

$$\left(\left(\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\right)^2\right.$$

$$+\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(v_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\right)^2$$

$$\leqslant \left(\left(\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\right)^2$$

$$+\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\right)^2$$

$$= 1. \tag{38}$$

We complete the proof. □

Example 2. Let $x_1 = ([0.2, 0.5], [0.4, 0.6])$, $x_2 = ([0.4, 0.6], [0.2, 0.3])$, and $x_3 = ([0.6, 0.8], [0.1, 0.2])$ be three IVPFNs, and $\pi = (0.2, 0.5, 0.3)$, $W = (0.2, 0.5, 0.3)$, then we have

$$\text{IVPFWMM}_W^{[\pi]}(x_1, x_2, x_3)$$

$$= \left(\left[\left(\sqrt{1-\begin{pmatrix}(1-0.0361^{0.2}\times 0.1888^{0.5}\times 0.3308^{0.3})\times\\(1-0.0361^{0.2}\times 0.3308^{0.5}\times 0.1888^{0.3})\times\\(1-0.1888^{0.2}\times 0.0361^{0.5}\times 0.3308^{0.3})\times\\(1-0.1888^{0.2}\times 0.3308^{0.5}\times 0.0361^{0.3})\times\\(1-0.3308^{0.2}\times 0.1888^{0.5}\times 0.0361^{0.3})\times\\(1-0.3308^{0.2}\times 0.0361^{0.5}\times 0.1888^{0.3})\end{pmatrix}^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}, \left(\sqrt{1-\begin{pmatrix}(1-0.2281^{0.2}\times 0.4146^{0.5}\times 0.6013^{0.3})\times\\(1-0.2281^{0.2}\times 0.6013^{0.5}\times 0.4146^{0.3})\times\\(1-0.4146^{0.2}\times 0.6013^{0.5}\times 0.2281^{0.3})\times\\(1-0.4146^{0.2}\times 0.2281^{0.5}\times 0.6013^{0.3})\times\\(1-0.6013^{0.2}\times 0.4146^{0.5}\times 0.2281^{0.3})\times\\(1-0.6013^{0.2}\times 0.2281^{0.5}\times 0.4146^{0.3})\end{pmatrix}^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}\right],\right.$$

$$\left.\left[\sqrt{1-\left(1-\begin{pmatrix}(1-0.1922^{0.2}\times 0.0210^{0.5}\times 0.0158^{0.3})\times\\(1-0.1922^{0.2}\times 0.0158^{0.5}\times 0.0210^{0.3})\times\\(1-0.0210^{0.2}\times 0.1922^{0.5}\times 0.0158^{0.3})\times\\(1-0.0210^{0.2}\times 0.0158^{0.5}\times 0.1922^{0.3})\times\\(1-0.0158^{0.2}\times 0.0210^{0.5}\times 0.1922^{0.3})\times\\(1-0.0158^{0.2}\times 0.1922^{0.5}\times 0.0210^{0.3})\times\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}, \sqrt{1-\left(1-\begin{pmatrix}(1-0.3987^{0.2}\times 0.0556^{0.5}\times 0.0552^{0.3})\times\\(1-0.3987^{0.2}\times 0.0552^{0.5}\times 0.0556^{0.3})\times\\(1-0.0556^{0.2}\times 0.3987^{0.5}\times 0.0552^{0.3})\times\\(1-0.0556^{0.2}\times 0.0552^{0.5}\times 0.3987^{0.3})\times\\(1-0.0552^{0.2}\times 0.0556^{0.5}\times 0.3987^{0.3})\times\\(1-0.0552^{0.2}\times 0.3987^{0.5}\times 0.0556^{0.3})\times\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}\right]\right)$$

$$= \big([0.3683, 0.6229], [0.2773, 0.4252]\big).$$

The IVPFWMM operator has the following property.

Property 4 (*Monotonicity*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [\nu^L_{\tilde{p}_j}, \nu^R_{\tilde{p}_j}])$ *and* $\tilde{q}_j = ([\mu^L_{\tilde{q}_j}, \mu^R_{\tilde{q}_j}], [\nu^L_{\tilde{q}_j}, \nu^R_{\tilde{q}_j}])$ $(j = 1, 2, \dots, n)$ *be two sets of IVPFNs, if* $(\mu^L_{\tilde{p}_j})^2 + (\mu^R_{\tilde{p}_j})^2 \leqslant (\mu^L_{\tilde{q}_j})^2 + (\mu^R_{\tilde{q}_j})^2$, *and* $(\nu^L_{\tilde{p}_j})^2 + (\nu^R_{\tilde{p}_j})^2 \geqslant (\nu^L_{\tilde{q}_j})^2 + (\nu^R_{\tilde{q}_j})^2$, *then*

$$IVPFWMM^{[\pi]}_W(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leqslant IVPFWMM^{[\pi]}_W(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n). \tag{39}$$

Property 5 (*Boundedness*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [\nu^L_{\tilde{p}_j}, \nu^R_{\tilde{p}_j}])$, $(j = 1, 2, \dots, n)$ *be a set of IVPFNs with weights* $W = (w_1, w_2, \dots, W_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^n = 1$, *if* $\tilde{p}_j^+ = ([\max_j(\mu^L_{\tilde{p}_j}), \max_j(\mu^R_{\tilde{p}_j})], [\min_j(\nu^L_{\tilde{p}_j}), \min_j(\nu^R_{\tilde{p}_j})])$, *and* $\tilde{p}^- = ([\min_j(\mu^L_{\tilde{p}_j}), \min_j(\mu^R_{\tilde{p}_j})], [\max_j(\nu^L_{\tilde{p}_j}), \max_j(\nu^R_{\tilde{p}_j})])$, *because of Property 4, then*

$$\begin{aligned} IVPFWMM^{\pi}_W(\tilde{p}_1^-, \tilde{p}_2^-, \dots, \tilde{p}_n^-) &\leqslant P2TLWMM^{\pi}_W(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \\ &\leqslant P2TLWMM^{\pi}_W(\tilde{p}_1^+, \tilde{p}_2^+, \dots, \tilde{p}_n^+). \end{aligned} \tag{40}$$

3.3. *IVPFDMM Operator*

Qin and Liu (2016) proposed the dual Muirhead mean (DMM) operator.

Definition 9. (See Qin and Liu, 2016.) Let α_j $(j = 1, 2, \dots, n)$ be a set of nonnegative real numbers, and $[\pi] = (\pi_1, \pi_2, \dots, \pi_n) \in R$ be a vector of parameters. If

$$DMM^{[\pi]}(\alpha_1, \alpha_2, \dots, \alpha_n) = \frac{1}{\sum_{j=1}^n \pi_j} \left(\sum_{j=1}^n \pi_j \alpha_{\varphi(j)} \right)^{\frac{1}{n!}}, \tag{41}$$

where φ_j $(j = 1, 2, \dots, n)$ is any a permutation of $(1, 2, \dots, n)$ and ϕ_n is a set of all permutations of $(1, 2, \dots, n)$.

In the following, we proposed the interval valued Pythagorean fuzzy dual MM (IVPFDMM) operator for IVPFNs.

Definition 10. Let $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [\nu^L_{\tilde{p}_j}, \nu^R_{\tilde{p}_j}])$, $(j = 1, 2, \dots, n)$ be a set of IVPFNs and there exists parameter vector $[\pi] = (\pi_1, \pi_2, \dots, \pi_n) \in R$, then

$$\text{IVPFDMM}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \frac{1}{\sum_{j=1}^n \pi_j} \left(\prod_{\varphi \in \phi_n} \sum_{j=1}^n \pi_j \tilde{p}_{\varphi(j)} \right)^{\frac{1}{n!}}. \tag{42}$$

Theorem 4 is derived by operations of the IVPFN.

Theorem 4. *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [\nu^L_{\tilde{p}_j}, \nu^R_{\tilde{p}_j}])$*,* $(j = 1, 2, \ldots, n)$ *be a collection of IVPFNs, then the corresponding aggregated value of IVPFDMM operator is also an IVPFN, and*

$$\begin{aligned}
&\mathrm{IVPFDMM}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \frac{1}{\sum_{j=1}^{n}\pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j \tilde{p}_{\varphi(j)}\right)^{\frac{1}{n!}} \\
&= \left(\begin{array}{c}
\left[\begin{array}{c}
\sqrt{1-\left(1-\left(\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}, \\
\sqrt{1-\left(1-\left(\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}
\end{array}\right], \\
\left[\begin{array}{c}
\left(\sqrt{1-\left(\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(\upsilon^L_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}, \\
\left(\sqrt{1-\left(\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(\upsilon^R_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}
\end{array}\right]
\end{array}\right). \qquad (43)
\end{aligned}$$

Proof.

$$\pi_j \tilde{p}_{\varphi(j)} = \left(\begin{array}{c}
\left[\sqrt{1-\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}}, \sqrt{1-\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}}\right], \\
\left[\left(\upsilon^L_{\varphi(j)}\right)^{\pi_j}, \left(\upsilon^R_{\varphi(j)}\right)^{\pi_j}\right]
\end{array}\right), \qquad (44)$$

$$\sum_{j=1}^{n}\pi_j \tilde{p}_{\varphi(j)} = \left(\begin{array}{c}
\left[\sqrt{1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}}, \sqrt{1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}}\right], \\
\left[\prod\limits_{j=1}^{n}\left(\upsilon^L_{\varphi(j)}\right)^{\pi_j}, \prod\limits_{j=1}^{n}\left(\upsilon^R_{\varphi(j)}\right)^{\pi_j}\right]
\end{array}\right). \qquad (45)$$

Therefore,

$$\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j \tilde{p}_{\sigma(j)} = \left(\begin{array}{c}
\left[\begin{array}{c}
\prod\limits_{\varphi\in\phi_n}\sqrt{1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}}, \\
\prod\limits_{\varphi\in\phi_n}\sqrt{1-\prod\limits_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}}
\end{array}\right], \\
\left[\begin{array}{c}
\sqrt{1-\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(\upsilon^L_{\varphi(j)}\right)^{2\pi_j}\right)}, \\
\sqrt{1-\prod\limits_{\varphi\in\phi_n}\left(1-\prod\limits_{j=1}^{n}\left(\upsilon^R_{\varphi(j)}\right)^{2\pi_j}\right)}
\end{array}\right]
\end{array}\right), \qquad (46)$$

$$\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j\tilde{p}_{\varphi(j)}\right)^{\frac{1}{n!}}=\left(\left[\begin{array}{l}\left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}}\right)^{\frac{1}{n!}},\\ \left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}}\right)^{\frac{1}{n!}}\end{array}\right],\right.$$
$$\left.\left[\begin{array}{l}\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^L_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}},\\ \sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^R_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\end{array}\right]\right), \tag{47}$$

then, we can get

$$\frac{1}{\sum_{j=1}^{n}\pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j\tilde{p}_{\varphi(j)}\right)^{\frac{1}{n!}}$$
$$=\left(\left[\begin{array}{l}\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^L_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}},\\ \sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{array}\right],\right.$$
$$\left.\left[\begin{array}{l}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^L_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\ \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^R_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{array}\right]\right). \tag{48}$$

From the aggregation result above, we prove the result of IVPFDMM aggregation is also an IVPFN in the following, then

$$0\leqslant\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\leqslant 1, \tag{49}$$

$$0\leqslant\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^R_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\leqslant 1. \tag{50}$$

And, we can prove

$$
\begin{aligned}
&\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\right)^2\\
&\quad+\left(\left(\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(v^R_{\varphi(j)}\right)^{2\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\right)\right)^2\\
&\quad\leqslant 1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\\
&\quad+\left(1-\left(m\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu^R_{\varphi(j)}\right)^2\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}=1.
\end{aligned}
\tag{51}
$$

So, we proved that the aggregation result of IVPFDMM is also an IVPFN. □

EXAMPLE 3. Let $x_1 = ([0.2, 0.5], [0.4, 0.6])$, $x_2 = ([0.4, 0.6], [0.2, 0.3])$, and $x_3 = ([0.6, 0.8], [0.1, 0.2])$ be three IVPFNs, and $[\pi] = (0.2, 0.5, 0.3)$, then we have

$$
\begin{aligned}
&\mathrm{IVPFDMM}^{[\pi]}(x_1, x_2, x_3)\\
&=\left(\left[\begin{array}{c}
\sqrt{1-\left(1-\left(\begin{array}{c}(1-0.96^{0.2}\times 0.84^{0.5}\times 0.64^{0.3})\times(1-0.96^{0.2}\times 0.64^{0.5}\times 0.84^{0.3})\times\\(1-0.84^{0.2}\times 0.96^{0.5}\times 0.64^{0.3})\times(1-0.84^{0.2}\times 0.64^{0.5}\times 0.96^{0.3})\times\\(1-0.64^{0.2}\times 0.84^{0.5}\times 0.96^{0.3})\times(1-0.64^{0.2}\times 0.96^{0.5}\times 0.84^{0.3})\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}},\\
\sqrt{1-\left(1-\left(\begin{array}{c}(1-0.75^{0.2}\times 0.64^{0.5}\times 0.36^{0.3})\times(1-0.75^{0.2}\times 0.36^{0.5}\times 0.64^{0.3})\times\\(1-0.64^{0.2}\times 0.75^{0.5}\times 0.36^{0.3})\times(1-0.64^{0.2}\times 0.36^{0.5}\times 0.75^{0.3})\times\\(1-0.36^{0.2}\times 0.75^{0.5}\times 0.64^{0.3})\times(1-0.36^{0.2}\times 0.64^{0.5}\times 0.75^{0.3})\end{array}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}
\end{array}\right]\right.\\
&\left.\left[\begin{array}{c}
\left(\sqrt{1-\left(\begin{array}{c}(1-0.4^{0.4}\times 0.2^{1}\times 0.1^{0.6})\times(1-0.4^{0.4}\times 0.1^{1}\times 0.2^{0.6})\times\\(1-0.2^{0.4}\times 0.4^{1}\times 0.1^{0.6})\times(1-0.2^{0.4}\times 0.1^{1}\times 0.4^{0.6})\times\\(1-0.1^{0.4}\times 0.2^{1}\times 0.4^{0.6})\times(1-0.1^{0.4}\times 0.4^{1}\times 0.2^{0.6})\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}},\\
\left(\sqrt{1-\left(\begin{array}{c}(1-0.6^{0.4}\times 0.3^{1}\times 0.2^{0.6})\times(1-0.6^{0.4}\times 0.2^{1}\times 0.3^{0.6})\times\\(1-0.3^{0.4}\times 0.6^{1}\times 0.2^{0.6})\times(1-0.3^{0.4}\times 0.2^{1}\times 0.6^{0.6})\times\\(1-0.2^{0.4}\times 0.6^{1}\times 0.3^{0.6})\times(1-0.2^{0.4}\times 0.3^{1}\times 0.6^{0.6})\end{array}\right)^{\frac{1}{3!}}}\right)^{\frac{1}{0.2+0.5+0.3}}
\end{array}\right]\right)\\
&=\big([0.4403, 0.6624], [0.2047, 0.3356]\big).
\end{aligned}
$$

Property 6 (*Idempotency*). *If* $\tilde{p}_j = \tilde{p} = ([\mu^L_j, \mu^R_j], [v^L_j, v^R_j])$, *then*

$$
\mathrm{IVPFDMM}^{[\pi]}(\tilde{P}_1, \tilde{P}_2, \dots \tilde{P}_n) = \tilde{p}. \tag{52}
$$

Property 7 (*Monotonicity*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}])$ *and* $\tilde{q}_j = ([\mu^L_{\tilde{q}_j}, \mu^R_{\tilde{q}_j}], [v^L_{\tilde{q}_j}, v^R_{\tilde{q}_j}])$, $(j = 1, 2, \dots, n)$ *be two sets of IVPFNs, if* $(\mu^L_{\tilde{p}_j})^2 + (\mu^R_{\tilde{p}_j})^2 \leqslant (\mu^L_{\tilde{q}_j})^2 + (\mu^R_{\tilde{q}_j})^2$ *and* $(v^L_{\tilde{p}_j})^2 + (v^R_{\tilde{p}_j})^2 \geqslant (v^L_{\tilde{q}_j})^2 + (v^R_{\tilde{q}_j})^2$, *then*

$$\mathrm{IVPFDMM}^{[\pi]}(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leqslant \mathrm{IVPFDMM}^{[\pi]}(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n). \tag{53}$$

Property 8 (*Boundedness*). *Let*

$$\tilde{p}_j = \left(\left[\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}\right], \left[v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}\right]\right) \quad (j = 1, 2, \dots, n)$$

be a set of IVPFNs. If $\tilde{p}^+ = ([\max_j(\mu^L_{\tilde{p}_j}), \max_j(\mu^R_{\tilde{p}_j})], [\min_j(v^L_{\tilde{p}_j}), \min_j(v^R_{\tilde{p}_j})])$ *and* $\tilde{p}^- = ([\min_j(\mu^L_{\tilde{p}_j}), \min_j(\mu^R_{\tilde{p}_j})], [\max_j(v^L_{\tilde{p}_j}), \max_j(v^R_{\tilde{p}_j})])$, *because of property 7 and property 8, then*

$$p^- \leqslant \mathrm{PFDMM}^{[\pi]}(p_1, p_2, \dots, p_n) \leqslant p^+. \tag{54}$$

3.4. *IVPFWDMM Operator*

In Section 3.3, it can be seen that the IVPFDMM operator doesn't consider the importance of the aggregated arguments. However, in many real practical situations, especially in multiple attribute decision making, the weights of attributes play an important role in the process of aggregation. To overcome the limitation of IVPFDMM, we shall develop the interval valued Pythagorean fuzzy weighted DMM (IVPFWDMM) operator as follows.

Definition 11. Let $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}])$ $(j = 1, 2, 3, \dots, n)$ be a group of IVPFNs with weights $W = (w_1, w_2, \dots w_n)^T$, $w_j \in [0, 1]$, $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$ and there exists parameter vector $[\pi] = (\pi_1, \pi_2, \dots, \pi_n) \in R$, then

$$\mathrm{IVPFWDMM}^{[\pi]}_w(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \frac{1}{\sum_{j=1}^n \pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^n \pi_j \tilde{p}^{n w_{\varphi(j)}}_{\varphi(j)}\right)^{\frac{1}{n!}}. \tag{55}$$

Theorem 5 can be derived by operations of the IVPFN.

Theorem 5. *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}])$ $(j = 1, 2, 3, \dots, n)$ *be a collection of IVPFNs, then the corresponding aggregated value of IVPFWDMM operator is also an IVPFN, and*

$$\mathrm{IVPFWDMM}^{[\pi]}_W = \frac{1}{\sum_{j=1}^n \pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^n \pi_j \tilde{p}^{n w_{\varphi(j)}}_{\varphi(j)}\right)^{\frac{1}{n!}}$$

$$= \left(\left[\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}, \sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}} \right], \left[\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{L}\right)^{2}\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}, \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}} \right] \right). \quad (56)$$

Proof.

$$\pi_j \tilde{p}_{\varphi(j)}^{nw_{\varphi(j)}} = \left(\left[\sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}, \sqrt{1-\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}} \right], \left[\left(\sqrt{1-\left(1-\left(v_{\varphi(j)}^{L}\right)^{2}\right)^{nw_{\varphi(j)}}}\right)^{\pi_j}, \left(\sqrt{1-\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{nw_{\varphi(j)}}}\right)^{\pi_j} \right] \right), \quad (57)$$

$$\sum_{j=1}^{n}\pi_j \tilde{p}_{\varphi(j)}^{nw_{\varphi(j)}} = \left(\left[\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}, \sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}} \right], \left[\prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(v_{\varphi(j)}^{L}\right)^{2}\right)^{nw_{\varphi(j)}}}\right)^{\pi_j}, \prod_{j=1}^{n}\left(\sqrt{1-\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{nw_{\varphi(j)}}}\right)^{\pi_j} \right] \right). \quad (58)$$

Thereafter,

$$\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j \tilde{p}_{\varphi(j)}^{nw_{\varphi(j)}} = \left(\left[\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}, \prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}} \right], \left[\sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{L}\right)^{2}\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)}, \sqrt{1-\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{R}\right)^{2}\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)} \right] \right),$$

(59)

$$\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j\tilde{p}_{\varphi(j)}^{nw_{\varphi(j)}}\right)^{\frac{1}{n!}}$$

$$=\left(\left[\begin{array}{l}\left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\right)^{\frac{1}{n!}},\\ \left(\prod_{\varphi\in\phi_n}\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}}\right)^{\frac{1}{n!}}\end{array}\right],\left[\begin{array}{l}\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}},\\ \sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\end{array}\right]\right). \quad (60)$$

Therefore,

$$\frac{1}{\sum_{j=1}^{n}\pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j\tilde{p}_{\varphi(j)}^{nw_{\varphi(j)}}\right)^{\frac{1}{n!}}$$

$$=\left(\left[\begin{array}{l}\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{L}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}},\\ \sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\end{array}\right],\left[\begin{array}{l}\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{L}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}},\\ \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^{R}\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}\end{array}\right]\right). \quad (61)$$

Then, we can obtain

$$0\leqslant\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^{R}\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}\pi_j}}}\leqslant 1, \quad (62)$$

$$0 \leqslant \left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^R\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}} \leqslant 1. \quad (63)$$

Because $(\mu_{\varphi(j)}^R)^2 + (v_{\varphi(j)}^R)^2 \leqslant 1$, therefore,

$$\left(\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^R\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}}\right)^2$$

$$+\left(\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(v_{\varphi(j)}^R\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}\right)^2$$

$$\leqslant 1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^R\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}$$

$$+\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^R\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}} = 1. \quad (64)$$

So, the aggregation result of IVPFWDMM is also IVPFN. □

Example 4. Let $x_1 = ([0.2, 0.5], [0.4, 0.6])$, $x_2 = ([0.4, 0.6], [0.2, 0.3])$, and $x_3 = ([0.6, 0.8], [0.1, 0.2])$ be three IVPFNs, and $[\pi] = (0.2, 0.5, 0.3)$, $w = (0.3, 0.4, 0.3)$, then we have

$$\mathrm{IVPFWDMM}_W^{[\pi]}(x_1, x_2, x_3)$$

$$= \left(\left[\sqrt{1-\left(1-\begin{pmatrix}(1-0.2349^{0.2}\times 0.3330^{0.5}\times 0.6314^{0.3})\times\\(1-0.2349^{0.2}\times 0.6314^{0.5}\times 0.3330^{0.3})\times\\(1-0.3330^{0.2}\times 0.2349^{0.5}\times 0.6314^{0.3})\times\\(1-0.3330^{0.2}\times 0.6314^{0.5}\times 0.2349^{0.3})\times\\(1-0.6314^{0.2}\times 0.3330^{0.5}\times 0.2349^{0.3})\times\\(1-0.6314^{0.2}\times 0.2349^{0.5}\times 0.3330^{0.3})\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}, \sqrt{1-\left(1-\begin{pmatrix}(1-0.5359^{0.2}\times 0.5417^{0.5}\times 0.8181^{0.3})\times\\(1-0.5359^{0.2}\times 0.8181^{0.5}\times 0.5417^{0.3})\times\\(1-0.5417^{0.2}\times 0.5359^{0.5}\times 0.8181^{0.3})\times\\(1-0.5417^{0.2}\times 0.8181^{0.5}\times 0.5359^{0.3})\times\\(1-0.8181^{0.2}\times 0.5359^{0.5}\times 0.5417^{0.3})\times\\(1-0.8181^{0.2}\times 0.5417^{0.5}\times 0.5359^{0.3})\end{pmatrix}^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.5+0.3}}}\right],\right.$$

$$\left[\left(\sqrt[\frac{1}{3!}]{1-\begin{pmatrix}(1-0.1452^{0.2}\times 0.0478^{0.5}\times 0.0090^{0.3})\times\\(1-0.1452^{0.2}\times 0.0090^{0.5}\times 0.0478^{0.3})\times\\(1-0.0478^{0.2}\times 0.1452^{0.5}\times 0.0090^{0.3})\times\\(1-0.0478^{0.2}\times 0.0090^{0.5}\times 0.1452^{0.3})\times\\(1-0.0090^{0.2}\times 0.0478^{0.5}\times 0.1452^{0.3})\times\\(1-0.0090^{0.2}\times 0.1452^{0.5}\times 0.0478^{0.3})\end{pmatrix}}\right)^{\frac{1}{0.2+0.5+0.3}}, \left(\sqrt[\frac{1}{3!}]{1-\begin{pmatrix}(1-0.3308^{0.2}\times 0.1070^{0.5}\times 0.0361^{0.3})\times\\(1-0.3308^{0.2}\times 0.0361^{0.5}\times 0.1070^{0.3})\times\\(1-0.1070^{0.2}\times 0.3308^{0.5}\times 0.0361^{0.3})\times\\(1-0.1070^{0.2}\times 0.0361^{0.5}\times 0.3308^{0.3})\times\\(1-0.0361^{0.2}\times 0.3308^{0.5}\times 0.1070^{0.3})\times\\(1-0.0361^{0.2}\times 0.1070^{0.5}\times 0.3308^{0.3})\end{pmatrix}}\right)^{\frac{1}{0.2+0.5+0.3}}\right]\Bigg)$$

$$= \big([0.4453, 0.6666], [0.2039, 0.3347]\big).$$

IVPFWDMM has the following properties.

Property 9 (*Monotonicity*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}])$ *and* $\tilde{q}_j = ([\mu^L_{\tilde{q}_j}, \mu^R_{\tilde{q}_j}], [v^L_{\tilde{q}_j}, v^R_{\tilde{q}_j}])$, $(j = 1, 2, \dots, n)$ *be two sets of IVPFNs with weights vector being* $W = (w_1, w_2, \cdots w_n)^T$, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, *if* $(\mu^L_{\tilde{p}_j})^2 + (\mu^R_{\tilde{p}_j})^2 \leqslant (\mu^L_{\tilde{q}_j})^2 + (\mu^R_{\tilde{q}_j})^2$ *and* $(v^L_{\tilde{p}_j})^2 + (v^R_{\tilde{p}_j})^2 \geqslant (v^L_{\tilde{q}_j})^2 + (v^R_{\tilde{q}_j})^2$, *then*

$$\mathrm{IVPFWDMM}^{[\pi]}_w(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leqslant \mathrm{IVPFWDMM}^{[\pi]}_w(\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n). \tag{65}$$

Property 10 (*Boundedness*). *Let* $\tilde{p}_j = ([\mu^L_{\tilde{p}_j}, \mu^R_{\tilde{p}_j}], [v^L_{\tilde{p}_j}, v^R_{\tilde{p}_j}])$ $(j = 1, 2, \dots, n)$ *be a set of IVPFNs. If* $\tilde{p}^+ = ([\max_j(\mu^L_{\tilde{p}_j}), \max_j(\mu^R_{\tilde{p}_j})], [\min_j(v^L_{\tilde{p}_j}), \min_j(v^R_{\tilde{p}_j})])$, $\tilde{p}^- = ([\min_j(\mu^L_{\tilde{p}_j}), \min_j(\mu^R_{\tilde{p}_j})], [\max_j(v^L_{\tilde{p}_j}), \max_j(v^R_{\tilde{p}_j})])$, *because of property 10, then*

$$\begin{aligned}\mathrm{IVPFWDMM}^{[\pi]}_w\big(\tilde{p}^-, \tilde{p}^-, \dots, \tilde{p}^-\big) &\leqslant \mathrm{IVPFWDMM}^{[\pi]}_w(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n)\\ &\leqslant \mathrm{IVPFWDMM}^{[\pi]}_w\big(\tilde{p}^+, \tilde{p}^+, \dots, \tilde{p}^+\big).\end{aligned} \tag{66}$$

4. Models for MADM with IVPFNs

We shall solve the MADM with IVPFNs on the basis of IVPFWMM and IVPFWDMM operators. Let $O = \{O_1, O_2, \dots, O_m\}$ be a discrete set of alternatives, and $C = \{C_1, C_2, \dots, C_n\}$ be a set of attributes, $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weight of the attribute G_j $(j = 1, 2, \dots, n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $P = (\tilde{p}_{ij})_{m\times n} =$

$([\mu_{ij}^L, \mu_{ij}^R], [\nu_{ij}^L, \nu_{ij}^R])_{m\times n}$ is the IVPF decision matrix, $[\mu_{ij}^L, \mu_{ij}^R] \subset [0,1]$, $[\nu_{ij}^L, \nu_{ij}^R] \subset [0,1]$, $(\mu_{ij}^R)^2 + (\nu_{ij}^R)^2 \leqslant 1$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Then, we can solve the MADM with IVPFNs on the basis of IVPFWMM and IVPFWDMM operators.

Step 1. We use the IVPFNs in $\tilde{R}$, and IVPFWMM operator

$$\begin{aligned}
\tilde{p}_i &= \mathrm{IVPFWMM}_w^{[\pi]}(\tilde{p}_{i1}, \tilde{p}_{i2}, \ldots, \tilde{p}_{in}) = \left(\frac{1}{n!}\sum_{\varphi\in\phi_n}\prod_{j=1}^{n}\left(nw_{\varphi(j)}\tilde{p}_{\varphi(ij)}\right)^{\pi_j}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}} \\
&= \left(\left[\begin{array}{l}
\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(ij)}^L\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}, \\
\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\mu_{\varphi(ij)}^R\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}
\end{array}\right],\right. \\
&\qquad \left.\left[\begin{array}{l}
\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon_{\varphi(ij)}^L\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}}, \\
\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\upsilon_{\varphi(ij)}^R\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum\limits_{j=1}^n \pi_j}}}
\end{array}\right]\right)
\end{aligned} \tag{67}$$

or

$$\begin{aligned}
\tilde{p}_i &= \mathrm{IVPFWDMM}_W^{[\pi]}(\tilde{p}_{i1}, \tilde{p}_{i2}, \ldots, \tilde{p}_{in}) = \frac{1}{\sum_{j=1}^n \pi_j}\left(\prod_{\varphi\in\phi_n}\sum_{j=1}^{n}\pi_j\tilde{p}_{\varphi(ij)}^{nw_{\varphi(j)}}\right)^{\frac{1}{n!}} \\
&= \left(\left[\begin{array}{l}
\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^L\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}}, \\
\sqrt{1-\left(1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\mu_{\varphi(j)}^R\right)^{2nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}}
\end{array}\right]\right. \\
&\qquad \left.\left[\begin{array}{l}
\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\upsilon_{\varphi(ij)}^L\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}, \\
\left(\sqrt{1-\left(\prod_{\varphi\in\phi_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\upsilon_{\varphi(ij)}^R\right)^2\right)^{nw_{\varphi(j)}}\right)^{\pi_j}\right)\right)^{\frac{1}{n!}}}\right)^{\frac{1}{\sum_{j=1}^n \pi_j}}
\end{array}\right]\right)
\end{aligned} \tag{68}$$

to derive the $\tilde{p}_i$ $(i = 1, 2, \ldots, m)$ of O_i.

Table 1
IVPFN decision matrix.

	C1	C2	C3	C4
O1	([0.40, 0.50], [0.60, 0.80])	([0.30, 0.60], [0.40, 0.50])	([0.10, 0.30], [0.50, 0.60])	([0.50, 0.60], [0.50, 0.70])
O2	([0.40, 0.70], [0.20, 0.50])	([0.60, 0.70], [0.10, 0.40])	([0.60, 0.70], [0.10, 0.20])	([0.70, 0.80], [0.50, 0.60])
O3	([0.50, 0.70], [0.10, 0.50])	([0.20, 0.50], [0.50, 0.70])	([0.40, 0.50], [0.10, 0.30])	([0.30, 0.60], [0.10, 0.20])
O4	([0.40, 0.80], [0.10, 0.20])	([0.10, 0.60], [0.20, 0.30])	([0.10, 0.40], [0.30, 0.50])	([0.40, 0.60], [0.20, 0.60])
O5	([0.40, 0.60], [0.20, 0.40])	([0.10, 0.40] [0.50, 0.70])	([0.40, 0.70], [0.30, 0.50])	([0.40, 0.60], [0.50, 0.80])

Step 2. Calculate the $S(\tilde{p}_i)$ and $H(\tilde{p}_i)$ of the overall IVPFNs $\tilde{p}_i$ to rank all the alternatives O_i.

Step 3. Rank and select the best A_i $(i = 1, 2, \ldots, m)$ in accordance with $S(\tilde{p}_i)$ and $H(\tilde{p}_i)$ $(i = 1, 2, \ldots, m)$.

Step 4. End.

5. Numerical Example and Comparative Analysis

5.1. *Numerical Example*

Supplier is the "Source" of the whole supply chain, and the green supplier selection is the foundation of GSCM. The quality of suppliers will directly affect the environmental performance of enterprises. First, the green supply chain management and the traditional supply chain management were compared, then the characteristics of green supplier partnerships were analysed by various aspects. The problems of selecting green suppliers in GSCM are classical MADM problems (Lang *et al.*, 2015; Wu *et al.*, 2018; Wei *et al.*, 2018c; Wang *et al.*, 2018c; Wei, 2018a; Yue and Jia, 2013; Wang *et al.*, 2018d; Wei and Wei, 2018b; Chen and Wei, 2010; Merigó and Gil-Lafuente, 2013; Wei *et al.*, 2018d; Wei and Gao, 2018).Then, we shall give an application for selecting green suppliers in GSCM with IVPFNs. There are five potential green suppliers O_i $(i = 1, 2, 3, 4, 5)$ to be evaluated by four attributes: (1) C_1 is the price factor; (2) C_2 is the delivery factor; (3) C_3 is the environmental factor; (4) C_4 is the product quality factor. Five potential green suppliers are to be evaluated by IVPFNs under four attributes (whose weight values $W = (0.2, 0.1, 0.3, 0.4)$, $\pi = (0.2, 0.2, 0.3, 0.3)$, as shown in Table 1.

Then, in order to find the best green suppliers in GSCM, we utilize the IVPFMM, IVPFWMM, IVPFDMM and IVPFWDMM operators to solve the MADM problem with IVPFNs, which concludes the following calculating steps:

Table 2
The aggregating result of IVPFMM, IVPFWMM, IVPFDMM and IVPFWDMM operators.

	O1	O2	O3	O4	O5
IVPFMM	([0.2799, 0.4827], [0.5082, 0.6742])	([0.5639, 0.7238], [0.2872, 0.4590])	([0.3316, 0.5695], [0.2742, 0.4882])	([0.2013, 0.5833], [0.2129, 0.4411])	([0.2843, 0.5639], [0.4029, 0.6448])
IVPFWMM	([0.2622, 0.4556], [0.5680, 0.7080])	([0.5266, 0.6773], [0.2972, 0.5247])	([0.3125, 0.5362], [0.4412, 0.6065])	([0.1893, 0.5542], [0.3110, 0.4671])	([0.2672, 0.5262], [0.4890, 0.6842])
IVPFDMM	([0.3627, 0.5220], [0.4951, 0.6407])	([0.5936, 0.7297], [0.1789, 0.3947])	([0.3719, 0.5874], [0.1505, 0.3820])	([0.2960, 0.6370], [0.1865, 0.3673])	([0.3529, 0.5936], [0.3508, 0.5794])
IVPFWDMM	([0.4394, 0.5960], [0.4657, 0.6007])	([0.6367, 0.7600], [0.1674, 0.3695])	([0.4396, 0.6400], [0.1426, 0.3673])	([0.3400, 0.6929], [0.1756, 0.3416])	([0.3750, 0.6280], [0.3301, 0.5410])

Table 3
The rank and score of green suppliers by using IVPFMM, IVPFWMM, IVPFDMM and IVPFWDMM operators.

	O1	O2	O3	O4	O5	Order
IVPFMM	0.3996	0.6372	0.5302	0.5352	0.4552	O2>O4>O3>O5>O1
IVPFWMM	0.3631	0.5931	0.4557	0.507	0.4102	O2>O4>O3>O5>O1
IVPFDMM	0.4371	0.6743	0.5787	0.5809	0.5046	O2>O4>O3>O5>O1
IVPFWDMM	0.4926	0.7046	0.6119	0.612	0.5333	O2>O4>O3>O5>O1

Step 1. According to Table 1, aggregate all IVPFNs p_{ij} $(j = 1, 2, \dots, n)$ by using the IVPFMM, IVPFWMM, IVPFDMM and IVPFWDMM operators to derive the overall IVPFNs $\tilde{p}_i$ $(i = 1, 2, 3, 4)$ of the alternative O_i. The results are listed in Table 2.

Step 2. According to Table 2, the score functions of the green suppliers are listed in Table 3. According to the result of green suppliers order, we can know that the best choice is supplier 4, we get the same result by different aggregation, which proved the effectiveness of result.

5.2. *Influence Analysis*

The aggregation method of extended IVPFS with MM has two advantages, one is that it can reduce the bad effects of the unduly high and low assessments on the final result, the other is that it can capture the interrelationship between IVPFNs. These aggregation operators have a parameter vector, which makes extended operator more flexible, so the different vector leads to different aggregation results, different scores and ranking results. In order to illustrate the influence of the parameter vector on the ranking result, we discuss the influence with several parameter vectors, the result you can find in Table 4.

We can see that the different parameters lead to different result and different ranking order. The more attributes we consider, the bigger the scores; the bigger the attribute value, the lower the scores. Therefore, the parameter vector can be considered as decision maker's risk preference.

Table 4
Ranking results by utilizing different parameter vector *R* in the IVPFWMM operator.

$(\pi_1, \pi_2, \pi_3, \pi_4)$	Scores					Order
	O_1	O_2	O_3	O_4	O_5	
(1,0,0,0)	0.4295	0.6829	0.5952	0.5661	0.512	$O_2>O_4>O_3>O_5>O_1$
(2,0,0,0)	0.4578	0.7032	0.608	0.5837	0.5324	$O_2>O_4>O_3>O_5>O_1$
(3,0,0,0)	0.4817	0.721	0.6172	0.5966	0.5468	$O_2>O_4>O_3>O_5>O_1$
(1,1,0,0)	0.393	0.6246	0.5469	0.5391	0.4764	$O_2>O_4>O_3>O_5>O_1$
(1,1,1,0)	0.3748	0.6051	0.5041	0.5194	0.4475	$O_2>O_4>O_3>O_5>O_1$
(1,1,1,1)	0.3624	0.5923	0.4534	0.5062	0.4084	O2>O4>O3>O5>O1

Table 5
Ranking results by utilizing different parameter vector R in the IVPFWDMM operator.

$(\pi_1, \pi_2, \pi_3, \pi_4)$	Scores					Order
	O_1	O_2	O_3	O_4	O_5	
(1,0,0,0)	0.3909	0.6347	0.5647	0.5185	0.4677	$O_2>O_3>O_4>O_5>O_1$
(2,0,0,0)	0.3744	0.5975	0.5519	0.4927	0.4405	$O_2>O_3>O_4>O_5>O_1$
(3,0,0,0)	0.362	0.5708	0.5412	0.4745	0.4172	$O_2>O_3>O_4>O_5>O_1$
(1,1,0,0)	0.4442	0.6742	0.5931	0.5678	0.5113	$O_2>O_3>O_4>O_5>O_1$
(1,1,1,0)	0.4691	0.6921	0.6054	0.5979	0.5242	$O_2>O_3>O_4>O_5>O_1$
(1,1,1,1)	0.494	0.7054	0.6124	0.613	0.5339	O2>O4>O3>O5>O1

5.3. *Comparative Analysis*

Then, we compare the proposed method with the IVPFWA and IVPFWG operator (Garg, 2016b).

Definition 12. (See Garg, 2016b.) Suppose that $\tilde{R} = (\tilde{r}_{ij})_{m\times n} = ([\mu_{ij}^L, \mu_{ij}^R], [\nu_{ij}^L, \nu_{ij}^R])_{m\times n}$ be a IVPFN matrix, $W = (w_1, w_2, \dots, w_n)$ be the weight of w_j, $0 \leqslant w_j \leqslant 1$, $\sum_{j=1}^n w_j = 1$. Then

$$\begin{aligned}\tilde{r}_i &= \text{IVPFWA}_w(\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}) \\ &= \bigoplus_{j=1}^n (w_j \tilde{r}_{ij}) \\ &= \left(\left[\sqrt{1 - \prod_{j=1}^n \left(1 - \left(\mu_{ij}^L\right)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^n \left(1 - \left(\mu_{ij}^R\right)^2\right)^{w_j}}\right],\right. \\ &\quad \left.\left[\prod_{j=1}^n \left(\nu_{ij}^L\right)^{w_j}, \prod_{j=1}^n \left(\nu_{ij}^R\right)^{w_j}\right]\right), \quad i = 1, 2, \dots, m,\end{aligned} \tag{69}$$

Table 6
The results of green suppliers by IVPFWA (IVPFWG) operators.

	IVPFWA	IVPFWG
O_1	([0.3872,0.5156],[0.5071,0.6637])	([0.2804,0.4699],[0.5149,0.6861])
O_2	([0.6188,0.7459],[0.2187,0.3995])	([0.5885,0.7384],[0.3457,0.4831])
O_3	([0.3751,0.5902],[0.1175,0.3075])	([0.3478,0.5753],[0.1926,0.3949])
O_4	([0.3208,0.6173],[0.1966,0.4255])	([0.2297,0.5627],[0.2224,0.496])
O_5	([0.3822,0.621],[0.3571,0.5968])	([0.3482,0.6034],[0.4062,0.6705])

Table 7
The score functions of the green suppliers.

	IVPFWA	IVPFWG
O_1	0.4295	0.3909
O_2	0.6829	0.6347
O_3	0.5952	0.5647
O_4	0.5661	0.5185
O_5	0.512	0.4677

Table 8
Order of the green suppliers.

	Order
IVPFWA	$O_2 > O_3 > O_4 > O_5 > O_1$
IVPFWG	$O_2 > O_3 > O_4 > O_5 > O_1$

$$\begin{aligned}\tilde{r}_i &= \text{IVPFWG}_W(\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}) = \bigotimes_{j=1}^{n} (\tilde{r}_{ij})^{w_j} \\ &= \left(\left[\prod_{j=1}^{n} \left(\mu_{ij}^L\right)^{w_j}, \prod_{j=1}^{n} \left(\mu_{ij}^R\right)^{w_j}\right], \left[\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\nu_{ij}^L\right)^2\right)^{w_j}},\right.\right. \\ &\qquad \left.\left.\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\nu_{ij}^R\right)^2\right)^{w_j}}\right]\right), \quad i = 1, 2, \dots, m. \end{aligned} \tag{70}$$

By utilizing the IVPFWA and IVPFWG operators, the results are derived in Table 6.

According to Table 6, the score values are listed in Table 7.

From the Table 7, the order is in Table 8.

From above, we get the same best green suppliers to show the effectiveness of our proposed operators. However, the IVPFWA and IVPFWG operators don't consider relationship among aggregated arguments, and thus can't eliminate the influence of unfair arguments. The IVPFWMM and IVPFWDMM operators consider the relationship among aggregated arguments.

6. Conclusion

Aggregation operators have become a hot issue and an important tool in the decision making fields in recent years. However, they still have some limitations in practical applications. For example, some aggregation operators suppose the attributes are independent of each other. However, the MM operator and the dual MM operator have a prominent characteristic that it can consider the interaction relationships among any number of attributes by a parameter vector. Motivated by the studies about the MM operator and the dual MM operator, in this paper, we proposed some new MM and DMM operators to cope with MADM with IVPFNs, including the IVPFMM operator, IVPFWMM operator, the IVPFDMM operator, and the IVPFWDMM operator. Then, the desirable properties were proved. Moreover, these proposed operators are used to deal with the MADM problems with IVPFNs. Finally, we used an illustrative example for green supplier selections in GSCM to prove the feasibility and validity of the proposed operators by comparing with the other existing methods. In subsequent studies, we shall extend the proposed operators to the different fields (De and Sana, 2014; Gao *et al.*, 2018a; Wei *et al.*, 2018b, 2018e; Chen, 2015; Wei *et al.*, 2018f; Wei, 2018b; Deli and Çağman, 2015; Wei *et al.*, 2017b; Wei and Wang, 2017) as well as to propose some new aggregation operators under the uncertain environment (Wang *et al.*, 2013; Wei *et al.*, 2017c; Wei, 2018c; Chaira, 2014; Singh, 2014; Wei, 2017c; Son, 2015).

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