

Multi-Attribute Decision Making with Interval-Valued Hesitant Fuzzy Information, a Novel Synthetic Grey Relational Degree Method

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Received: October 2017; accepted: May 2018

Abstract. Quantitative and qualitative fuzzy information measures have been proposed to solve multi-attribute decision making (MADM) problems with interval-valued hesitant fuzzy information from different points. We analyse the existing fuzzy information measures of the interval-valued hesitant fuzzy sets (IVHFSs) in detail and classify them into two categories. One is based on the closeness of the data, such as the distance, and the other is based on the linear relationship or variation tendency, such as the correlation coefficient. These two kinds of information measures are actually partial measures which pay attention to only one factor of the data. Therefore, we construct a novel synthetic grey relational degree by considering both the closeness and the variation tendency factors of the data to improve the existing information measures and enhance the grey relational analysis (GRA) theory for IVHFSs. However, the notion of the synthetic grey relational degree is not only restricted to the IVHFSs but can be extended to other sets. Furthermore, we employ two practical MADM examples about emergency management evaluation and pattern recognition to validate and compare the proposed synthetic grey relational degree with other information measures, which demonstrate its superiorities in discrimination and accuracy.

Key words: multi-attribute decision making (MADM), interval-valued hesitant fuzzy sets (IVHFSs), synthetic grey relational degree, information measures.

1. Introduction

Multi-attribute decision making (MADM) is pervasive around us and as the aggregating information tends to be uncertain and vague, the fuzzy MADM is more and more popular (Yu, 2017; Rostamzadeh *et al.*, 2017). Due to the superiority in expressing the imprecise and vague information, the hesitant fuzzy sets (HFSs) are regarded as one of the most efficient tool to deal with fuzzy MADM problems (Mu *et al.*, 2015). Torra (2010) originally introduced the hesitant fuzzy set (HFS), and Chen *et al.* (2013a, 2013b) extended the HFS to interval-valued hesitant fuzzy set (IVHFS) by using the interval to represent the membership. Since the IVHFS is more general than the HFS, we devote ourselves to this set and intend to investigate the information measures of it to solve MADM problems in this paper.

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Chen *et al.* (2013a, 2013b) first introduced interval-valued hesitant fuzzy preference relations to describe uncertain evaluation information in group decision making (GDM) processes. They also presented some aggregation operators and defined the correlation coefficients for IVHFSs. Chen and Xu (2014) further investigated the properties, operational laws and relationships of fundamental operations on IVHFS. Recently, Verma (2017) also proposed four new operations on IVHFS and studied their properties and relations in details. Besides, Yang *et al.* (2017) proposed a new comparative law based on the possibility degree to compare interval-valued hesitant fuzzy elements (IVHFEs). Following Chen *et al.*'s work, many researchers contributed to the IVHFS and applied it to various decision making problems. To our knowledge of the existing analyses of IVHFS in decision making, we summarize them to three categories. The first is based on the information measures (Chen *et al.*, 2013a; Wei *et al.*, 2014a, 2014b; Farhadinia, 2013; Jin *et al.*, 2016b; Meng *et al.*, 2016; Peng *et al.*, 2017; Liu *et al.*, 2018), the second is based on the aggregation operators (Wei *et al.*, 2013; Zhang *et al.*, 2014; Meng and Chen, 2014; He *et al.*, 2016; Jin *et al.*, 2016a) and the third is based on the preference, outranking or consensus relational models (Gitinavard *et al.*, 2017; Zhang, 2016; Asan *et al.*, 2018). Among which, the information measures take important occupations in the MADM. Some primary and classical decision making methods as EDAS (Evaluation based on Distance from Average Solution), TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making), TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), VIKOR (Visekriterijumska Optimizacija I Kompromisno Resenje) and MOORA (Multi-Objective Optimization by Ratio Analysis) are all established on the basis of information measures. Therefore, in this paper, we mainly focus on this point and aim at improving the existing information measures for IVHFS. To date, such information measures as distance, similarity, entropy, cross-entropy and correlation coefficients for IVHFS have been proposed and applied in various MADM fields. Wei *et al.* (2014a, 2014b) proposed a variety of distance, similarity and correlation coefficients for IVHFSs. Farhadinia (2013) discussed the distance, similarity and entropy measure for IVHFSs and the transformation techniques between each other. Besides, Farhadinia (2015) also introduced the division and subtraction formulas for IVHFSs. Jin *et al.* (2016b) defined the entropy, similarity measures and cross-entropy for IVHFSs based on continuous ordered weighted averaging operator. Meng *et al.* (2016) defined several new correlation coefficients which do not need to consider the lengths of IVHFEs and the arrangement of their possible interval values. Peng *et al.* (2017) exploited some (weighted) distance measures for IVHFSs based on the COWA operator and used relative ratio to make the decision. Liu *et al.* (2018) developed the distance and similarity measures for IVHFSs and transferred distance to similarity by set-theoretic approach. Gitinavard *et al.* (2016) introduced a novel multi-criteria weighting and ranking model for IVHFS and applied it to location and supplier selection problems. Zhang and Xu (2014) extended the TODIM to the IVHFS domain based on the defined measured functions and compared it with the TOPSIS (Zhang and Xu, 2013) to make the decision. Further, Fernández *et al.* (2015) introduced finite interval-valued hesitant fuzzy sets, defined a new order, entropy between them considering the fuzziness, lack of knowledge and hesitance and applied it in the business selection. Al-

though numerous information measures have been defined for IVHFSs, in a further analysis of these measures, we classify them into two types. One is based on the closeness and the other is based on the linear relations or the variation tendency of IVHFSs. Because the distance, similarity and entropy can be transferred to each other (Farhadinia, 2013; Jin *et al.*, 2016b), they are all equivalent in a sense and based on the closeness. Instead, correlation coefficient is based on the linear relations or the variation tendency. Therefore, we claim that these existing information measures are all one side of a coin in the real measures. They are not the real and actual measures between data.

As mentioned above, the existing information measures pay attention to either the closeness or the variation tendency of IVHFSs. None of them includes both sides. For this reason, the motivation of this paper is to develop a novel information measure of IVHFSs which considers both the closeness and the variation tendency factors to improve the existing ones. We attempt to explore this information measure by the grey relational analysis (GRA) of the IVHFSs. Comparing with other information measures, the GRA of the IVHFSs is relatively weak. Therefore, another purpose of this paper is to enhance the GRA in the IVHFSs field. Actually, the traditional GRA of the fuzzy sets takes an important occupation in the fuzzy measure fields. It can measure the closeness of two fuzzy sets just like the distance, similarity and entropy measures. Many researchers focused on the GRA of fuzzy sets and proposed several approaches to solve decision making problems. Turskis and Zavadskas (2010) used the additive ratio assessment method with grey numbers to multiple criteria analysis. Wei (2011a, 2011b, 2011c) established a series of GRA methods to investigate the multiple attribute decision-making problems with intuitionistic fuzzy information, 2-tuple linguistic information and the dynamic hybrid multiple attribute decision information. Kong *et al.* (2011) presented a new algorithm based on GRA to discuss fuzzy soft set decision problems. Kuo and Liang (2011) combined the concepts of VIKOR and GRA to present an effective fuzzy MCDM method. Zhang *et al.* (2011, 2013) and Guo (2013) also developed the GRA method for solving MCDM problems with interval-valued triangular fuzzy numbers, intuitionistic trapezoidal fuzzy number and hybrid multiple attribute information respectively. Tang (2015) and Li *et al.* (2015) proposed a novel fuzzy soft set approach in decision making based on GRA and Dempster-Shafer theory of evidence respectively. Liou *et al.* (2016) combined the DEMATEL (DEcision-MAking Trial and Evaluation Laboratory), ANP (Analytical Network Process) and COPRAS-G (COMplex Proportional ASsessment of alternatives with Grey relations) techniques together to make the decision with interval grey numbers. As to the HFSs domain, Li and Wei (2014) established an optimization model based on GRA to get the weight vector of the HFSs criteria. Zang *et al.* (2017) proposed a grey relational projection method based on the distance measure between the interval-valued dual hesitant fuzzy elements. Although there are so many GRA methods for various types of fuzzy sets, none of them is special for IVHFSs. Furthermore, the existing GRA in the HFSs domain can not be directly transferred for IVHFSs. Even if they can be transferred through some techniques, the transferred GRA for IVHFSs from the existing methods is also one side of a coin just like the aforementioned distance, similarity and entropy measures. That is to say, the existing GRA methods for fuzzy sets only pay attention to the

closeness between the fuzzy sets and neglect their variation tendency and relations. Obviously, these kinds of information measures are unreasonable. They can reflect only one aspect of the real measures. Sun *et al.* (2018) and Guan *et al.* (2018) presented a synthetic grey relational degree considering both sides by defining the slope grey relational degree, however, the slope grey relational degree can not be used for IVHFSs directly. Furthermore, the combination of synthetic grey relational degree is simple and can not reflect the influence of the whole index space of the grey theory. Nevertheless, what luck is that we can draw lessons from Sun *et al.*'s notions and construct a novel synthetic grey relational degree as information measure for IVHFSs which takes both the closeness and the variation tendency factors into account. Consequently, in this paper, we commit ourselves to construct a novel synthetic grey relational degree for IVHFSs which can achieve the aforementioned two goals: (1) develop a novel information measure of IVHFSs which considers both the closeness and the variation tendency factors; (2) enhance the GRA in the IVHFSs field.

As debated above, the main contribution of this paper is the novel synthetic grey relational degree for IVHFSs. It consists of two aspects: the grey relational degree accounting for the closeness and the variation tendency. As to the grey relational degree describing the closeness, we can extend the traditional grey relational degree from HFSs to IVHFSs. We call it the closeness grey relational degree in this paper. And for the grey relational degree expressing the variation tendency, we do not transfer the slope grey relational degree of HFSs in Sun *et al.* (2018) to IVHFSs. Instead, we define a novel variation rate grey relational degree. We use the variation rate of the mean value of the interval membership to represent the variation tendency. We define two different variation rates of the mean value and use them to construct the variation rate grey relational degree. Based on the closeness and the variation rate grey relational degree, we further develop the novel synthetic grey relational degree which can reflect the influence of the whole index space better than (Sun *et al.*, 2018).

The rest of the paper is as follows: Section 2 briefly reviews the concepts of IVHFSs and GRA theory. In Section 3, we extend the traditional grey relational degree from HFSs to IVHFSs and define the closeness grey relational degree for IVHFSs. We also propose the novel variation rate grey relational degree for IVHFSs in this section. Furthermore, we construct the synthetic grey relational degree with the help of the former two. In Section 4, we use the synthetic grey relational degree in MADM based on TOPSIS. In Section 5, a practical MADM problem is used to validate the synthetic grey relational degree. We also compare it with the similarity and correlation coefficient through a pattern recognition example. Finally, the paper ends with some concluding remarks and future challenges in Section 6.

2. Preliminaries

In this section, we recall the IVHFSs and the GRA theory.

2.1. Interval-Valued Hesitant Fuzzy Sets

In many real problems, due to insufficiency in available information, it may quantify the attribute with an interval number within $[0, 1]$ instead of a crisp number. Thus, Chen *et al.* (2013a, 2013b) introduced the concept of IVHFSs, which permits the membership degrees of an element to a given set to have a few different interval values.

DEFINITION 1. Suppose that $X = \{x_1, x_2, \dots, x_n\}$ is a reference set, an IVHFS \tilde{A} on X is defined as

$$\tilde{A} = \{ \{x, \tilde{h}_{\tilde{A}}(x)\} | x \in X \} \tag{1}$$

where $\tilde{h}_{\tilde{A}}(x)$ is a set of some different values in $[0, 1]$, denotes all possible interval-valued membership degrees of the element and represents the possible membership degrees of the element $x \in X$ to the set \tilde{A} . For convenience, they call $\tilde{h}_{\tilde{A}}(x)$ an interval-valued hesitant fuzzy element (IVHFE), which is a basic unit of IVHFS.

$$\tilde{h}_{\tilde{A}}(x) = \{ \tilde{\gamma} | \tilde{\gamma} \in \tilde{h}_{\tilde{A}}(x) \} \tag{2}$$

where $\tilde{\gamma}$ is an interval number, $\tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$, $\tilde{\gamma}^L$ and $\tilde{\gamma}^U$ represent the lower and upper limits of $\tilde{\gamma}$, respectively.

2.2. GRA Theory

GRA theory was originally introduced by Deng (1989). It has been widely applied in some uncertain problems as decision making, pattern recognition and alike, particularly under the discrete data and fuzzy information.

DEFINITION 2. For reference set $X_0 = (x_0(j), j = 1, 2, \dots, k)$ and $X_i = (x_i(j), j = 1, 2, \dots, k)$, the grey relational coefficient is defined by

$$r(x_0(j), x_i(j)) = \frac{\min_i \min_j |x_0(j) - x_i(j)| + \rho \cdot \max_i \max_j |x_0(j) - x_i(j)|}{|x_0(j) - x_i(j)| + \rho \cdot \max_i \max_j |x_0(j) - x_i(j)|} \tag{3}$$

where $\rho \in [0, 1]$, represents the resolution coefficient which is given by the decision makers, generally we let $\rho = 0.5$.

The grey relational degree is defined as:

$$\gamma(X_0, X_i) = \frac{1}{k} \cdot \sum_{j=1}^k r(x_0(j), x_i(j)) \tag{4}$$

If we take the weight into consideration and let the weight vector of X_i be $w = (w_1, w_2, \dots, w_k)^T$, $\sum_{j=1}^k w_j = 1$, $j = 1, 2, \dots, k$, then the grey relational degree is extended to the weighted grey relational degree:

$$\gamma(X_0, X_i) = \sum_{j=1}^k w_j \cdot r(x_0(j), x_i(j)). \quad (5)$$

3. GRA for IVHFSs

In this section, we firstly extend the traditional grey relational degree to the IVHFSs domain and form a closeness grey relational degree for IVHFSs. Subsequently, we propose the variation rate grey relational degree and further construct the synthetic grey relational degree.

3.1. Closeness Grey Relational Degree for IVHFSs

DEFINITION 3. For two IVHFSs on the fixed set $X = \{x_1, x_2, \dots, x_n\}$, $\tilde{A} = \{(x, \tilde{h}_{\tilde{A}}(x_i)) | x_i \in X, i = 1, 2, \dots, n\}$ and $\tilde{B}_j = \{(x_i, \tilde{h}_{\tilde{B}_j}(x_i)) | x_i \in X, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$ with $\tilde{h}_{\tilde{A}}(x_i) = \{\tilde{\gamma}_{\tilde{A}i1}, \tilde{\gamma}_{\tilde{A}i2}, \dots, \tilde{\gamma}_{\tilde{A}il_{\tilde{A}i}}\}$, $\tilde{h}_{\tilde{B}_j}(x_i) = \{\tilde{\gamma}_{\tilde{B}_ji1}, \tilde{\gamma}_{\tilde{B}_ji2}, \dots, \tilde{\gamma}_{\tilde{B}_jil_{\tilde{B}_ji}}\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, then we extend the traditional grey relational coefficient to be the traditional grey relational coefficient between IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{\tilde{B}_j}(x_i)$ as:

$$r(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) = \frac{\min_j \min_i \{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))\} + \rho \cdot \max_j \max_i \{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))\}}{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) + \rho \cdot \max_j \max_i \{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))\}} \quad (6)$$

where $d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))$ is the distance between IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{\tilde{B}_j}(x_i)$, which can be calculated according to:

$$d_{hne}(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) = \left[\left(\frac{1}{2l_{Ai}} \sum_{k=1}^{l_{Ai}} (|\tilde{\gamma}_{\tilde{A}ik}^L - \tilde{\gamma}_{\tilde{B}_jik}^L|^p + |\tilde{\gamma}_{\tilde{A}ik}^U - \tilde{\gamma}_{\tilde{B}_jik}^U|^p) \right) \right]^{1/p}. \quad (7)$$

For more distance between IVHFEs, please refer to Wei *et al.* (2014a, 2014b), Farhadinia (2013), Jin *et al.* (2016b), Peng *et al.* (2017), Liu *et al.* (2018). The traditional grey relational coefficient between IVHFEs describes the closeness of the IVHFSs data, so we also call it the closeness grey relational coefficient in this paper.

REMARK 1. In this paper, we assume the number of the membership in each IVHFE to be compared with is equal. For the moment, we do not discuss the unequal case. Actually, if the number of the membership in each IVHFE is different, we have to extend the shorter

one until both of them have the same length when we compare them. We can extend them according to the optimistic or the pessimistic methods and some other rules.

Based on the closeness grey relational coefficient between IVHFEs, the closeness grey relational degree between IVHFSs \tilde{A} and \tilde{B}_j is defined as:

$$\gamma(\tilde{A}, \tilde{B}_j) = \frac{1}{n} \cdot \sum_{i=1}^n r(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \tag{8}$$

If we take the weight into consideration and let the weight vector of X be $w = (w_1, w_2, \dots, w_n)^T$, $\sum_{i=1}^n w_i = 1$, $i = 1, 2, \dots, n$, then we extend the IVHFSs closeness grey relational degree to the weighted IVHFSs closeness grey relational degree as:

$$\gamma_w(\tilde{A}, \tilde{B}_j) = \sum_{i=1}^n w_i \cdot r(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \tag{9}$$

3.2. Variation Rate Grey Relational Degree for IVHFSs

In this section, we define the variation rate grey relational degree which can represent the variation tendency of IVHFSs. We use the variation rate of the mean value in the interval membership to represent this variation tendency. We define two different variation rates of the mean value and use them to construct the variation rate grey relational degrees.

For IVHFE $\tilde{h}(x) = \{\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_k, \dots, \tilde{\gamma}_l\}$ with interval membership $\tilde{\gamma}_k = [\tilde{\gamma}_k^L, \tilde{\gamma}_k^U]$, $k = 1, 2, \dots, l$, the mean value of the interval membership in IVHFE can be represented by

$$m(\tilde{h}(x)) = \left\{ m(\tilde{\gamma}_1), m(\tilde{\gamma}_2), \dots, m(\tilde{\gamma}_k), \dots, m(\tilde{\gamma}_l) \mid m(\tilde{\gamma}_k) = \frac{\tilde{\gamma}_k^L + \tilde{\gamma}_k^U}{2}, k = 1, 2, \dots, l \right\}. \tag{10}$$

With the help of the mean value sequence, we define two different variation rates of the mean value to represent the variation tendency of IVHFSs: the global variation rate and the local variation rate.

The global variation rate of the mean value is described as:

$$\tilde{h}'_{glo}(x) = \{m'_{glo}(\tilde{\gamma}_k), k = 1, 2, \dots, l - 1\} \tag{11}$$

where

$$m'_{glo}(\tilde{\gamma}_k) = \frac{m(\tilde{\gamma}_{k+1}) - m(\tilde{\gamma}_k)}{\bar{m}(\tilde{\gamma}_k)}, \quad k = 1, 2, \dots, l - 1 \tag{12}$$

where $\bar{m}(\tilde{\gamma}_k)$ is the mean of mean value of the interval membership.

$$\bar{m}(\tilde{\gamma}_k) = \sum_{k=1}^l m(\tilde{\gamma}_k). \quad (13)$$

The local variation rate of the mean value is described as:

$$\tilde{h}'_{l_o}(x) = \{m'_{l_o}(\tilde{\gamma}_k), k = 1, 2, \dots, l - 1\} \quad (14)$$

where

$$m'_{l_o}(\tilde{\gamma}_k) = \frac{m(\tilde{\gamma}_{k+1}) - m(\tilde{\gamma}_k)}{m(\tilde{\gamma}_k)}, \quad k = 1, 2, \dots, l - 1. \quad (15)$$

With the help of these two different variation rates, we defined the variation rate grey relational degree as follows.

DEFINITION 4. For two IVHFSs on the fixed set $X = \{x_1, x_2, \dots, x_n\}$, $\tilde{A} = \{(x, \tilde{h}_{\tilde{A}}(x_i)) | x_i \in X, i = 1, 2, \dots, n\}$ and $\tilde{B}_j = \{(x_i, \tilde{h}_{\tilde{B}_j}(x_i)) | x_i \in X, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$ with $\tilde{h}_{\tilde{A}}(x_i) = \{\tilde{\gamma}_{\tilde{A}i1}, \tilde{\gamma}_{\tilde{A}i2}, \dots, \tilde{\gamma}_{\tilde{A}il_{\tilde{A}i}}\}$, $\tilde{h}_{\tilde{B}_j}(x_i) = \{\tilde{\gamma}_{\tilde{B}_ji1}, \tilde{\gamma}_{\tilde{B}_ji2}, \dots, \tilde{\gamma}_{\tilde{B}_jil_{\tilde{B}_ji}}\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, the variation rate grey relational coefficient between the IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{\tilde{B}_j}(x_i)$ is defined as:

$$r_v(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) = \frac{1}{l_{A_i} - 1} \sum_{k=1}^{l_{A_i} - 1} \varepsilon_v[\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i)]_k \quad (16)$$

where l_{A_i} is the number of membership in $\tilde{h}_{\tilde{A}}(x_i)$,

$$\begin{aligned} & \varepsilon_v[\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i)]_k \\ &= \frac{1 + |m'(\tilde{\gamma}_{\tilde{A}ik})|}{1 + |m'(\tilde{\gamma}_{\tilde{A}ik})| + |m'(\tilde{\gamma}_{\tilde{A}ik}) - m'(\tilde{\gamma}_{\tilde{B}_jik})|}, \quad k = 1, 2, \dots, l_{A_i} - 1 \end{aligned} \quad (17)$$

where $m'(\tilde{\gamma}_{\tilde{A}ik})$ and $m'(\tilde{\gamma}_{\tilde{B}_jik})$ are the variation rate of IVHFSs, which can be obtained in two ways: the global variation rate (12) and the local variation rate (15).

Based on the variation rate grey relational coefficient between the IVHFEs, the variation rate grey relational degree between the IVHFSs \tilde{A} and \tilde{B}_j is defined as:

$$\gamma_v(\tilde{A}, \tilde{B}_j) = \frac{1}{n} \cdot \sum_{i=1}^n r_v(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \quad (18)$$

If we take the weight into consideration and let the weight vector of X be $w = (w_1, w_2, \dots, w_n)^T$, $\sum_{i=1}^n w_i = 1$, $i = 1, 2, \dots, n$, then we extend the variation rate grey

relational degree between the IVHFSs to the weighted IVHFSs variation rate grey relational degree as:

$$\gamma_{vw}(\tilde{A}, \tilde{B}_j) = \sum_{i=1}^n w_i \cdot r_v(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \tag{19}$$

3.3. Synthetic Grey Relational Degree for IVHFSs

Based on the closeness and the variation rate grey relational degree, we further construct the novel synthetic grey relational degree which takes into consideration both the closeness and the variation tendency factors.

DEFINITION 5. For two IVHFSs on the fixed set $X = \{x_1, x_2, \dots, x_n\}$, $\tilde{A} = \{(x, \tilde{h}_{\tilde{A}}(x_i)) | x_i \in X, i = 1, 2, \dots, n\}$ and $\tilde{B}_j = \{(x_i, \tilde{h}_{\tilde{B}_j}(x_i)) | x_i \in X, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$ with $\tilde{h}_{\tilde{A}}(x_i) = \{\tilde{\gamma}_{\tilde{A}i1}, \tilde{\gamma}_{\tilde{A}i2}, \dots, \tilde{\gamma}_{\tilde{A}il_{\tilde{A}i}}\}$, $\tilde{h}_{\tilde{B}_j}(x_i) = \{\tilde{\gamma}_{\tilde{B}_ji1}, \tilde{\gamma}_{\tilde{B}_ji2}, \dots, \tilde{\gamma}_{\tilde{B}_jil_{\tilde{B}_ji}}\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, the synthetic grey relational coefficient between the IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{\tilde{B}_j}(x_i)$ is defined as:

$$r_s(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) = \frac{1 + \xi \cdot \max_j \max_i \{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))\} + \zeta \cdot \max_j \max_i \{d(\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i))\}}{1 + \lambda_1 \cdot d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) + \lambda_2 \cdot d(\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i)) + \xi \cdot \max_j \max_i \{d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))\} + \zeta \cdot \max_j \max_i \{d(\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i))\}} \tag{20}$$

where $\lambda_1, \lambda_2 > 0$, which indicate the importance of the closeness and the variation rate of the IVHFSs, respectively, $\lambda_1 + \lambda_2 = 1$. ξ and ζ denote the resolution coefficient of the closeness and the variation rate, $\xi, \zeta \in [0, 1]$. $\tilde{h}'_{\tilde{A}}(x_i)$ and $\tilde{h}'_{\tilde{B}_j}(x_i)$ are the variation rates of the mean value in the interval membership, which can be gotten in two ways: equations (11) and equations (14). $d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))$ is the distance between IVHFEs $\tilde{h}_{\tilde{A}}(x_i)$ and $\tilde{h}_{\tilde{B}_j}(x_i)$ and $d(\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i))$ is the distance between the variation rate of IVHFEs $\tilde{h}'_{\tilde{A}}(x_i)$ and $\tilde{h}'_{\tilde{B}_j}(x_i)$. $d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i))$ can be calculated by equations (7) and $d(\tilde{h}'_{\tilde{A}}(x_i), \tilde{h}'_{\tilde{B}_j}(x_i))$ can be calculated by:

$$d(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)) = \left[\left(\frac{1}{l_{\tilde{A}i} - 1} \sum_{k=1}^{l_{\tilde{A}i} - 1} |m'(\tilde{\gamma}_{\tilde{A}ik}) - m'(\tilde{\gamma}_{\tilde{B}_j ik})|^p \right)^{1/p} \right]. \tag{21}$$

Based on IVHFEs synthetic grey relational coefficient, the IVHFSs synthetic grey relational degree is defined as:

$$\gamma_s(\tilde{A}, \tilde{B}_j) = \frac{1}{n} \cdot \sum_{i=1}^n r_s(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \tag{22}$$

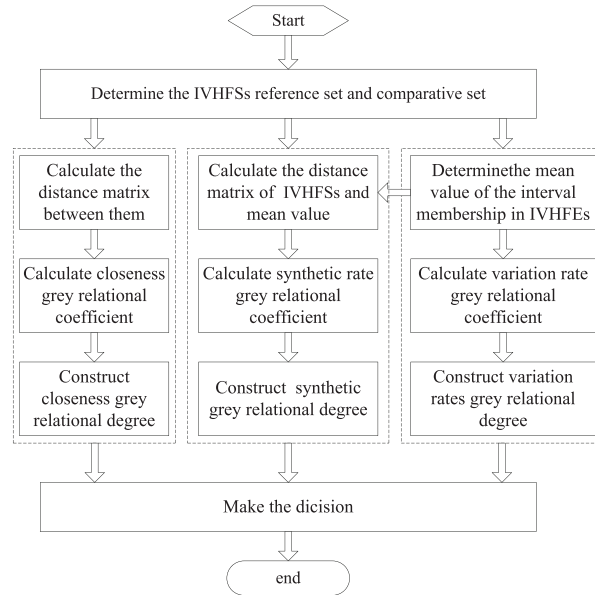


Fig. 1. The process of the construction of the IVHFSs grey relational degree.

If we take the weight into consideration and let the weight vector of X be $w = (w_1, w_2, \dots, w_n)^T$, $\sum_{i=1}^n w_i = 1$, $i = 1, 2, \dots, n$, then we extend the IVHFSs synthetic grey relational degree to the weighted IVHFSs synthetic grey relational degree as:

$$\gamma_{sw}(\tilde{A}, \tilde{B}_j) = \sum_{i=1}^n w_i \cdot r_s(\tilde{h}_{\tilde{A}}(x_i), \tilde{h}_{\tilde{B}_j}(x_i)). \quad (23)$$

Actually, we can use either the global variation rate of the mean value or the local variation rate of the mean value in constructing the synthetic grey relational degree, which are called global synthetic grey relational degree and local synthetic grey relational degree, respectively.

The IVHFSs synthetic grey relational degree takes the considerations of both the closeness and the variation tendency factors of IVHFSs together, which can better distinguish two IVHFSs than the existing fuzzy information measures.

The process of the construction of the IVHFSs grey relational degree is shown in Fig. 1.

4. The MADM Methodology with IVHFSs Information Based on the Grey Relational Degree

In this section, we investigate the MADM problems with IVHFSs information based on the synthetic grey relational degree and the TOPSIS method.

Suppose an interval-valued hesitant fuzzy MADM problem, that there are m alternatives \tilde{A}_i ($i = 1, 2, \dots, m$) to be evaluated, each alternative has n interval-valued hesitant

fuzzy attributes C_j ($j = 1, 2, \dots, n$), denote $h_{\tilde{A}_i}(C_j) = \{\gamma_{\tilde{A}_i 1}, \gamma_{\tilde{A}_i 2}, \dots, \gamma_{\tilde{A}_i k}, \dots, \gamma_{\tilde{A}_i l_{ij}}\}$ represent the interval-valued hesitant fuzzy information of the alternatives A_i on the attribute C_j , l_{ij} is the number of the membership values in $h_{\tilde{A}_i}(C_j)$, let $w = (w_1, w_2, \dots, w_j, \dots, w_n)^T$ be the relative weight vector of the attribute, satisfying the normalization conditions: $0 \leq w_j \leq 1$ and $\sum_{j=1}^n w_j = 1$. Then all the interval-valued hesitant fuzzy information can be concisely expressed in matrix format as:

$$\tilde{A} = \begin{bmatrix} h_{\tilde{A}_1}(C_1) & h_{\tilde{A}_1}(C_2) & \cdots & h_{\tilde{A}_1}(C_n) \\ h_{\tilde{A}_2}(C_1) & \ddots & \cdots & h_{\tilde{A}_2}(C_n) \\ \vdots & \vdots & h_{\tilde{A}_i}(C_j) & \vdots \\ h_{\tilde{A}_m}(C_1) & h_{\tilde{A}_m}(C_2) & \cdots & h_{\tilde{A}_m}(C_n) \end{bmatrix}_{m \times n} \quad (24)$$

According to the process of TOPSIS, we express the steps of MADM with IVHFSs information based on the synthetic grey relational degree as follows:

Step 1: Determine the positive ideal solution (PIS) and the negative ideal solution (NIS) of each attribute in the normalized interval-valued hesitant fuzzy decision matrix to form the positive and the negative IVHFSs:

$$\tilde{A}^+ = \{ \{C_j, h_{\tilde{A}}^+(C_j)\} | C_j \in C, j = 1, 2, \dots, n \}, \quad (25)$$

$$\tilde{A}^- = \{ \{C_j, h_{\tilde{A}}^-(C_j)\} | C_j \in C, j = 1, 2, \dots, n \} \quad (26)$$

where $h_{\tilde{A}}^+(C_j)$ and $h_{\tilde{A}}^-(C_j)$ are the positive and the negative IVHFEs:

$$h_{\tilde{A}}^+(C_j) = \{ \tilde{\gamma}_1^+, \tilde{\gamma}_2^+, \dots, \tilde{\gamma}_k^+, \dots, \tilde{\gamma}_{l_j^+}^+ \}, \quad (27)$$

$$h_{\tilde{A}}^-(C_j) = \{ \tilde{\gamma}_1^-, \tilde{\gamma}_2^-, \dots, \tilde{\gamma}_k^-, \dots, \tilde{\gamma}_{l_j^-}^- \} \quad (28)$$

where

$$\gamma_k^+ = \left(\max_{1 \leq i \leq m} \{ \gamma_{\tilde{A}_i k} \} \text{ if } \gamma_{\tilde{A}_i k} \in \Omega_b, \min_{1 \leq i \leq m} \{ \gamma_{\tilde{A}_i k} \} \text{ if } \gamma_{\tilde{A}_i k} \in \Omega_c \right), \quad (29)$$

$$\gamma_k^- = \left(\min_{1 \leq i \leq m} \{ \gamma_{\tilde{A}_i k} \} \text{ if } \gamma_{\tilde{A}_i k} \in \Omega_b, \max_{1 \leq i \leq m} \{ \gamma_{\tilde{A}_i k} \} \text{ if } \gamma_{\tilde{A}_i k} \in \Omega_c \right) \quad (30)$$

where Ω_b and Ω_c are related to benefit attribute and cost attribute, l_j^+ and l_j^- are the number of the membership values in the positive and the negative IVHFEs, respectively, $l_j^+ = l_j^-$. We can use the comparative law in Chen and Xu (2014) to calculate the maximum and the minimum value in equations (29) and (30).

Step 2: Calculate the IVHFSs positive and negative synthetic grey relational degrees between each alternative and the PIS and the NIS according to the process of the construction

of the IVHFSs synthetic grey relational degree.

$$\gamma_{sw}^+(\tilde{A}_i, \tilde{A}^+) = \sum_{j=1}^n w_j \cdot r_s(h_{\tilde{A}_i}(C_j), h_{\tilde{A}^+}(C_j)), \quad (31)$$

$$\gamma_{sw}^-(\tilde{A}_i, \tilde{A}^-) = \sum_{j=1}^n w_j \cdot r_s(h_{\tilde{A}_i}(C_j), h_{\tilde{A}^-}(C_j)). \quad (32)$$

Step 3: Construct the relative closeness of the alternative \tilde{A}_i ($i = 1, 2, \dots, m$) with respect to the ideal solution based on the calculated positive and negative IVHFSs synthetic grey relational degrees which is defined as:

$$\eta_{si} = \frac{\gamma_{sw}^+(\tilde{A}_i, \tilde{A}^+)}{\gamma_{sw}^+(\tilde{A}_i, \tilde{A}^+) + \gamma_{sw}^-(\tilde{A}_i, \tilde{A}^-)}, \quad i = 1, 2, \dots, m. \quad (33)$$

Step 4: Rank the alternatives according to the decreasing order of their relative closeness. That is, the best alternative is the one with the greatest relative closeness to the ideal solution.

5. MADM Applications

In this section, we employ the proposed grey relational degree to deal with MADM problems with IVHFSs information. We use example 1 about emergency management evaluation to validate the proposed grey relational degree and example 2 about pattern recognition to compare the proposed grey relational degree with other information measures. We also make a sensitive analysis of some parameters in the synthetic grey relational degree in this section.

5.1. Apply the Proposed Grey Relational Degree to Emergency Management Evaluation Example

In this subsection, an MADM example about emergency management evaluation problems with interval-valued hesitant fuzzy information is used to validate the proposed grey relational degree. The interval-valued hesitant fuzzy data are extracted from Jin *et al.* (2016b).

EXAMPLE 1. Suppose that there are four alternatives A_i ($i = 1, 2, 3, 4$) to be evaluated by evaluators, each alternative has these six attributes C_i ($i = 1, 2, \dots, 6$). To determine the attribute weight is not the key point in this paper, so to simplify we let the weight be $w = (0.1074, 0.1205, 0.2101, 0.1428, 0.2474, 0.1718)^T$, which is the same in Jin *et al.* (2016b). The evaluated values are expressed by interval-valued hesitant fuzzy information, which is shown in Table 1.

Table 1
The interval-valued hesitant fuzzy attributes' information.

Attribute	Alternatives			
	A ₁	A ₂	A ₃	A ₄
Attribute 1	{[0.7, 0.9], [0.7, 0.8], [0.6, 0.8]}	{[0.5, 0.7], [0.5, 0.6], [0.4, 0.6]}	{[0.3, 0.5], [0.2, 0.4], [0.2, 0.3]}	{[0.6, 0.7], [0.5, 0.7], [0.5, 0.6]}
Attribute 2	{[0.4, 0.5], [0.2, 0.3], [0.1, 0.3]}	{[0.5, 0.7], [0.5, 0.5], [0.4, 0.5]}	{[0.8, 0.9], [0.7, 0.8], [0.6, 0.8]}	{[0.4, 0.6], [0.3, 0.6], [0.3, 0.4]}
Attribute 3	{[0.2, 0.4], [0.2, 0.3], [0.1, 0.3]}	{[0.8, 1.0], [0.7, 0.9], [0.6, 0.8]}	{[0.3, 0.4], [0.2, 0.4], [0.1, 0.4]}	{[0.3, 0.5], [0.2, 0.4], [0.2, 0.3]}
Attribute 4	{[0.5, 0.8], [0.4, 0.7], [0.4, 0.6]}	{[0.9, 1.0], [0.7, 0.9], [0.6, 0.8]}	{[0.2, 0.3], [0.1, 0.3], [0.1, 0.2]}	{[0.3, 0.5], [0.3, 0.4], [0.2, 0.4]}
Attribute 5	{[0.2, 0.5], [0.2, 0.4], [0.1, 0.4]}	{[0.8, 0.9], [0.7, 0.9], [0.7, 0.8]}	{[0.1, 0.3], [0.0, 0.2], [0.0, 0.1]}	{[0.1, 0.2], [0.0, 0.2], [0.0, 0.1]}
Attribute 6	{[0.8, 0.9], [0.7, 0.8], [0.7, 0.7]}	{[0.9, 1.0], [0.8, 1.0], [0.8, 0.9]}	{[0.6, 0.8], [0.6, 0.7], [0.5, 0.5]}	{[0.6, 0.7], [0.4, 0.6], [0.4, 0.5]}

We utilize the proposed grey relational degree to evaluate the alternatives with IVHFSs information in the following steps:

Step 1: All the attributes are of benefit type, we select each maximum IVHFE in the five alternatives IVHFSs on the four attributes to construct the interval-valued hesitant fuzzy PIS A^+ and each minimum IVHFE to construct the interval-valued hesitant fuzzy NIS A^- . The PIS and the NIS are described as follows:

$$\begin{aligned}
 A^+ &= \{ \{ [0.7, 0.9], [0.7, 0.8], [0.6, 0.8] \}, \{ [0.8, 0.9], [0.7, 0.8], [0.6, 0.8] \}, \\
 &\quad \{ [0.8, 1.0], [0.7, 0.9], [0.6, 0.8] \}, \{ [0.9, 1.0], [0.7, 0.9], [0.6, 0.8] \}, \\
 &\quad \{ [0.8, 0.9], [0.7, 0.9], [0.7, 0.8] \}, \{ [0.9, 1.0], [0.8, 1.0], [0.8, 0.9] \} \}, \\
 A^- &= \{ \{ [0.3, 0.5], [0.2, 0.4], [0.2, 0.3] \}, \{ [0.4, 0.5], [0.2, 0.3], [0.1, 0.3] \}, \\
 &\quad \{ [0.2, 0.4], [0.2, 0.3], [0.1, 0.3] \}, \{ [0.2, 0.3], [0.1, 0.3], [0.1, 0.2] \}, \\
 &\quad \{ [0.1, 0.2], [0.0, 0.2], [0.0, 0.1] \}, \{ [0.6, 0.7], [0.4, 0.6], [0.4, 0.5] \} \}.
 \end{aligned}$$

Step 2: Calculate the IVHFSs positive and negative grey relational degrees between each alternative and the PIS and the NIS, respectively. We calculate the five grey relational degrees in this paper: the traditional or closeness grey relational degree, the global variation rate grey relational degree, the local variation rate grey relational degree, the global synthetic grey relational degree and the local synthetic grey relational degree. When calculating the traditional (closeness) grey relative degree, we set the resolution coefficient to be $\rho = 0.5$. When calculating the synthetic grey relative degree, we set the importance of the closeness and variation rate of the IVHFSs to be $\lambda_1 = \lambda_2 = 0.5$, the resolution coefficient $\rho = 0.5$ and the resolution coefficient to be $\xi = \zeta = 0.5$, too. The results of the five grey relational degrees are shown in Table 2.

Step 3: Construct the relative closeness to the ideal solution based on the calculated IVHFSs five positive and negative grey relational degrees. The IVHFSs five relative closeness of the alternative A_i ($i = 1, 2, 3, 4$) are shown in Table 3.

Step 4: Rank the alternatives according to the decreasing order of the IVHFSs grey relative closeness, also shown in Table 3.

It can be clearly seen from Table 3 that all the five kinds of grey relative closeness indicate that decision result is the alternative A_2 . It is consistent with the decision result in Jin *et al.* (2016b), which illustrates the validity and accuracy of the proposed IVHFSs

Table 2
IVHFSs positive and negative grey relational degrees from the PIS and the NIS.

Methods	Relational degrees	Alternatives			
		A ₁	A ₂	A ₃	A ₄
Traditional (closeness) grey relational degree	γ_w^+	0.5456	0.9102	0.4834	0.4498
	γ_w^-	0.7027	0.4360	0.8293	0.8292
Global variation rate grey relational degree	γ_{loww}^+	0.9392	0.9930	0.8945	0.8989
	γ_{loww}^-	0.9048	0.8525	0.9211	0.9397
Local variation rate grey relational degree	γ_{loww}^+	0.9445	0.9915	0.9075	0.9117
	γ_{loww}^-	0.9203	0.8743	0.9186	0.9363
Global synthetic grey relational degree	γ_{glosw}^+	0.8826	0.9831	0.8386	0.8411
	γ_{glosw}^-	0.9086	0.8169	0.9317	0.9463
Local synthetic grey relational degree	γ_{losw}^+	0.8822	0.9819	0.8410	0.8422
	γ_{losw}^-	0.9126	0.8233	0.9341	0.9445

Table 3
The five grey relative closeness of the 4 alternatives to the ideal solution.

Relative closeness	Alternatives				Rankings
	A ₁	A ₂	A ₃	A ₄	
Traditional (closeness) grey relative closeness	0.4371	0.6761	0.3683	0.3517	A ₂ > A ₁ > A ₃ > A ₄
Global variation rate grey relative closeness	0.5093	0.5381	0.4927	0.4889	A ₂ > A ₁ > A ₃ > A ₄
Local variation rate grey relative closeness	0.5065	0.5314	0.4970	0.4934	A ₂ > A ₁ > A ₃ > A ₄
Global synthetic grey relative closeness	0.4915	0.5439	0.4738	0.4714	A ₂ > A ₁ > A ₃ > A ₄
Local synthetic grey relative closeness	0.4927	0.5462	0.4737	0.4706	A ₂ > A ₁ > A ₃ > A ₄

grey relational degree. Though the rankings are the same, the grey relative closeness is different. For example, the closeness grey relative closeness of A₂ is 0.6761, while the global synthetic grey relative closeness is 0.5439. It is the variation rate that makes the effect. If the variation rates of the alternatives approach the variation rate of the ideal solution, then the synthetic grey relative closeness will approach the closeness grey relative closeness. At this time, the closeness factor plays the vital role. In this example, both the closeness and the variation rates indicate that A₂ is the best alternative, so all the rankings are the same. If the closeness and the variation rate factors produce different results, we can not make the decision only by the single one grey relative closeness. Under this condition, we should seek the help of the synthetic grey relative closeness. We will explain this condition in Example 2 in detail.

5.2. Sensitivity Analysis of Some Parameters

In this subsection, we make a sensitive analysis of these parameters in the synthetic grey relational degree: the importance of the closeness and the variation rate of the IVHFSs λ_1 and λ_2 , the two resolution coefficients ξ and ζ . We use the same case in Example 1 to analyse them.

Firstly, to get the impact of λ_1 and λ_2 , we let ξ and ζ be fixed and modify the parameter λ_1 from 0.1 to 1 to see the trends of the synthetic relative closeness to the four

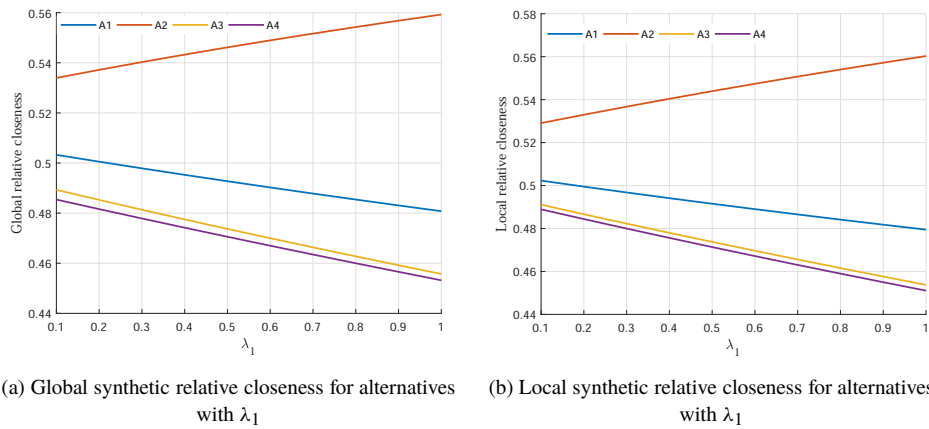


Fig. 2. The trends of the synthetic relative closeness for alternatives with λ_1 .

alternatives. Because $\lambda_1 + \lambda_2 = 1$, changing λ_1 is enough. We set the resolution coefficient to be $\xi = \zeta = 0.5$, then the changing trends of the synthetic relative closeness to the parameter λ_1 from 0.1 to 1 are shown in Fig. 2.

Figure 2(a) shows the global synthetic relative closeness and Fig. 2(b) shows the local synthetic relative closeness. We can see that the trends of the synthetic relative closeness vary with the changing of the parameter λ_1 . When the parameter λ_1 increases from 0.1 to 1, the synthetic relative closeness also increases. It is in accordance to the debate in example 1 that when the importance of the closeness takes a more important role, the result of the synthetic relative closeness will approach that of the closeness method. In this example, the closeness method and the variation rate produce the same result, so the decision results of the synthetic methods do not change with the changing of the parameter λ_1 . It is alternative A_2 all the time. However, if the results of the closeness method and the variation rate are different, the decision results of the synthetic methods will change with the changing of the parameter λ_1 . Therefore, the parameters λ_1 and λ_2 make important impacts in the decision result of the synthetic methods. Furthermore, we can see that trends of the global and the local synthetic relative closeness vary the same. It illustrates that anyone of them can be applied for decision without specific demand.

In the sequel, we modify the two resolution coefficients ξ and ζ to see the trends of the synthetic relative closeness to the four alternatives. We set the parameter to be $\lambda_1 = \lambda_2 = 0.5$ and make resolution coefficients ξ and ζ increase from 0.1 to 1 simultaneously, then the changing trends of the synthetic relative closeness of alternative A_2 with resolution coefficients ξ and ζ are shown in Fig. 3.

Figure 3(a) shows the global synthetic relative closeness and Fig. 3(b) shows the local synthetic relative closeness. According to the above figures, we can observe that although the synthetic relative closeness varies with the changing of resolution coefficients ξ and ζ , the changing range is small and it does not change the decision result. It illustrates that the synthetic relative closeness is not sensitive to the resolution coefficients ξ and ζ . The reason is that when constructing the synthetic relational degree, the numerator and the

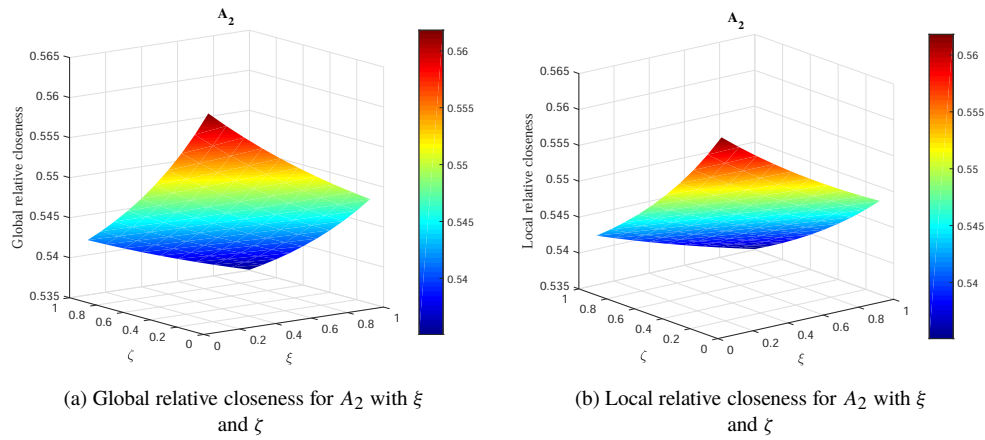


Fig. 3. The process of the construction of the IVHFSs grey relational degree.

Table 4
Interval-valued hesitant fuzzy data of the known patterns.

Alternatives	Attributes		
	Attribute 1	Attribute 2	Attribute 3
Pattern 1	{[0.05, 0.15], [0.15, 0.25], [0.25, 0.35]}	{[0.05, 0.15], [0.25, 0.35], [0.35, 0.45]}	{[0.15, 0.25], [0.35, 0.45]}
Pattern 2	{[0.25, 0.35], [0.35, 0.45], [0.45, 0.55]}	{[0.35, 0.45], [0.45, 0.55], [0.65, 0.75]}	{[0.55, 0.65], [0.65, 0.75]}
Pattern 3	{[0.10, 0.20], [0.25, 0.35], [0.40, 0.50]}	{[0.10, 0.20], [0.40, 0.50], [0.55, 0.65]}	{[0.25, 0.35], [0.55, 0.65]}
Pattern 4	{[0.15, 0.25], [0.25, 0.35], [0.65, 0.75]}	{[0.35, 0.45], [0.65, 0.75], [0.85, 0.95]}	{[0.15, 0.25], [0.45, 0.55]}
Pattern 5	{[0.05, 0.15], [0.45, 0.55], [0.55, 0.65]}	{[0.25, 0.35], [0.35, 0.45], [0.85, 0.95]}	{[0.25, 0.35], [0.55, 0.65]}
Pattern 6	{[0.06, 0.10], [0.14, 0.18], [0.22, 0.26]}	{[0.06, 0.10], [0.22, 0.26], [0.30, 0.34]}	{[0.14, 0.18], [0.30, 0.34]}
Pattern 7	{[0.09, 0.15], [0.21, 0.27], [0.33, 0.39]}	{[0.09, 0.15], [0.33, 0.39], [0.45, 0.51]}	{[0.21, 0.27], [0.45, 0.51]}

denominator all include the resolution coefficients ξ and ζ . The effect of them is reduced in the division reduction operation. Furthermore, when constructing the synthetic relative closeness, the effect of them is further reduced. Therefore, the resolution coefficients ξ and ζ make no obvious impact on the decision results. Actually, the resolution coefficients ξ and ζ can be adjusted by the decision makers' preferences.

5.3. Comparison of the Proposed Grey Relational Degree

In this section, we use a pattern recognition example to compare the proposed grey relational degree with other information measures.

EXAMPLE 2. Consider a pattern recognition problem. There are seven known patterns 1–7, which are represented by the IVHFSs. Each pattern has three attributes 1–3. The interval-valued hesitant fuzzy data of the known patterns are shown in Table 4.

Now, there is one detected unknown pattern to be recognized. The data \tilde{Q} is represented by IVHFSs, too, which is in the following:

$$\tilde{Q} = \{[0.15, 0.25], [0.35, 0.45], [0.55, 0.65]\}, \{[0.15, 0.25], [0.55, 0.65], [0.75, 0.85]\}, \{[0.35, 0.45], [0.75, 0.85]\}. \tag{34}$$

Table 5
The hesitant fuzzy measurement degree for 7 alternatives with 7 different methods.

Methods	Alternatives						
	Pattern 1	Pattern 2	Pattern 3	Pattern 4	Pattern 5	Pattern 6	Pattern 7
Similarity	0.7326	0.8791	0.8663	0.8598	0.8791	0.6777	0.7851
Correlation coefficient	0.9959	0.9732	0.9995	0.9611	0.9733	1.0000	1.0000
Traditional (closeness) grey relational degree	0.5996	0.8840	0.8431	0.8501	0.8840	0.5360	0.6761
Global variation rate grey relational degree	1.0000	0.6909	1.0000	0.7736	0.7160	1.0000	1.0000
Local variation rate grey relational degree	1.0000	0.8075	1.0000	0.8629	0.8174	1.0000	1.0000
Global synthetic grey relational degree	0.9169	0.8604	0.9566	0.8743	0.8568	0.9015	0.9321
Local synthetic grey relational degree	0.9441	0.8244	0.9712	0.8390	0.7625	0.9334	0.9546

The goal is to classify the unknown pattern in the 7 known patterns. We use the proposed synthetic grey relative degree to achieve this goal. The attribute weight of the three attributes is (0.4, 0.4, 0.2). We calculate both the global and the local synthetic grey relative degree to find the recognition results with the biggest degree. The synthetic grey relative degrees are shown in Table 5. We can see that both synthetic grey relative degrees indicate that the detected unknown pattern deserves to be known pattern 3.

In order to show the advantages of the proposed synthetic grey relative degree, we use the other 5 information measures to compare with each other: the Hamming distance in Wei *et al.* (2014b), the correlation coefficient in Chen and Xu (2014), Wei *et al.* (2014b), the traditional (closeness) grey relational degree, the global variation rate grey relational degree and the local variation rate grey relational degree proposed in this paper. The grey relative degree of these information measures are shown in Table 5, too.

From Table 5, we can see that different information measures produce different results. The similarity method and the closeness grey relational degree method regard known pattern 2 and 5 as the best recognition results, while the correlation coefficient refer as the best recognition result to be known target 6 and 7. The global and the variation rate grey relational degrees determine the best recognition results to be known pattern 1, 3, 6 and 7. These three kinds of results are completely different.

The reason is that the similarity method and the closeness grey relational degree method pay more attention to the closeness of the data, while the correlation coefficient focuses more on the linear relationship of the data and the global and the variation rate grey relational degrees emphasize the variation rate of the mean value of the data instead. When the correlation coefficients of the data are equal, the global and the variation rate grey relational degrees are equal too, but not vice versa. All these three kinds of information measures consider only one factor of the data, either the closeness or the linear relationship or the variation rate of the mean value of the data. When one of the factors is equal, they can not distinguish which one is the best and can not make the sole decision. Therefore, these information measures are only partial measures which can not reflect the real relationships of the data.

However, the recognition result of both synthetic grey relational degrees is known pattern 3 only. They can distinguish the result better than the existing information measures by considering both the closeness and the variation rate factors. It also demonstrates that the synthetic grey relational degree is superior in discrimination and accuracy than the existing information measures.

6. Conclusion

In this paper, we propose the synthetic grey relational degree of IVHFSs and use it to solve MADM problems with hesitant fuzzy information. We firstly apply the GRA theory to the IVHFSs and define the closeness grey relational degree. Since the closeness grey relational degree reflects the closeness of the data just like the distance, similarity and entropy information measures, we explore two novel variation rate grey relational degrees: the global and the local variation rate grey relational degrees. We use them to describe the variation rate of the data, which enhances the cognition of the traditional grey relational degree. Furthermore, we construct the synthetic grey relational degree with the help of the closeness and the variation rate. The synthetic grey relational degree combines both the merits of the former two grey relational degrees. It can measure not only the closeness but also the variation rate of the data, which is a novel information measure for IVHFSs. Based on the synthetic grey relational degree, we develop a MCDM process with the help of TOPSIS. We apply this notion in a real MCDM problem about the emergency management evaluation, which illustrates its validity. We also make a sensitivity analysis of the parameters in the synthetic grey relational degree. Based on the analysis, we conclude that the importance of the closeness and the variation rate of the IVHFSs λ_1 and λ_2 have obvious effect on the relative closeness while the resolution coefficients ξ and ζ have no obvious effect it. In addition, we compare the synthetic grey relational degree with 5 information measures: distance, correlation coefficient, traditional (closeness) grey relational degree, global variation rate grey relational degree and local variation rate grey relational degree, to show its advantages in discrimination and accuracy.

In the future, the notion of the construction of the synthetic grey relational degree is expected to be used in the information measures for other types of fuzzy sets. Furthermore, we will devote ourselves to other innovative information measures.

Acknowledgements. The authors are very grateful to the Editor-in-Chief Prof. G. Dzemmyda and the anonymous reviewers for their helpful comments and suggestions in improving this paper. This work is supported by the Excellent Youth Scholar of the National Defense Science and Technology Foundation of China, the Special Fund for the Taishan Scholar Project (Grant No. ts201712072) and the Natural Science Foundation of Shandong Province (Grant No. ZR2017BG014).

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