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ITERATIVE METHODS AND STABILITY OF STEADY-STATE SOLUTIONS

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Abstract. This paper is devoted to the new approach in the stability analysis of steady state solutions. Two important nonlinear optics problems are used as model problems. Stability properties of classical and splitting difference schemes are investigated. Some numerical results are given.

Key words: difference schemes, stability, iterative methods, splitting methods, nonlinear optics.

Introduction. There has been a great deal of recent interest in the stability analysis of numerical methods used for the solution of partial differential equations. We can divide the analysis into the following procedures. First, it is necessary to prove the existence and uniqueness of a solution. Second, we must determine the stability characteristics of this solution. Various types of stability can be investigated, e.g., linear stability, asymptotical stability, nonlinear stability in a neighbourhood of the steady-state solution. Such an analysis can be made analytically only in exceptional cases. Hence there arises a need to develop numerical methods for solving problems of mathematical physics, which enable us not only to find a solution but also to get some information about its stability.

The method we analyse in this paper is the continuation of the ideas proposed by Čiegis (1992a). We consider a general approach for solving the stability problem. As a rule, iterative methods are used to solve nonlinear stationary boundary-value problems. Each iterative method leads to a new time dependent problem with additional terms depending on a fictitious time variable. We note that such a transition is not unique. Therefore, it is necessary to compare these various methods. The aim of the present investigation is to develop iterative methods, which lead to a stable solution for a set of parameters, as wide as possible. Note that we analyse the introduction of fictitious time as a pure mathematical operation with a very specific goal to solve the given stationary problem, and to investigate the stability of its solution at the same time. In applications stationary mathematical models are obtained by removing the temporal variable t. Therefore, we can use this evolution model as one of iterative methods and define this type of stability as 1a physical stability. As shown below, various methods of introduction of fictitious time lead to not coinciding regions of parameters for which the solution is stable and the physically correct one may be not the best.

1. The problem class. As the first example we consider a linear boundary value problem

$$\Delta u = f(x), \qquad x \in G, \tag{1.1a}$$

$$u(x) = 0, \qquad x \in \partial G, \bar{G} = G \cup \partial G, \tag{1.1b}$$

where

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

and the domain G is in R^n with boundary ∂G and sides parallel to the coordinate axes. Then introducing a fictitious time variable, we obtain a parabolic boundary value problem

$$\frac{\partial u}{\partial t} = \Delta u - f, \qquad x \in G \times (0, T], \tag{1.2a}$$

$$u(x, 0) = u_0(x), \qquad x \in \bar{G}, \tag{1.2b}$$

$$u(x, t) = 0, \qquad x \in \partial G \times (0, T]. \tag{1.2c}$$

By using the Fourier method we get that the solution of (1.2) converges to the steady-state solution of (1.1) for any initial function $u_0(x)$. Therefore, the steady-state solution is asymptotically stable.

Analogous results are valid for the discrete approximation of (1.1). Let $\bar{\omega}_h$ be a uniform space grid covering $G \cup \partial G$

$$\bar{\omega}_h = \{(x_{1i_1}, \cdots, x_{ni_n}) : \quad 0 \leqslant x_{kj} \leqslant 1, \quad x_{kj} = jh\}.$$

Then using finite difference method for (1.1) results in the following system of equations

$$\sum_{i=1}^{n} y_{\bar{x}_i x_i} = f(x), \qquad x \in \omega_h,$$
$$y(x) = 0, \qquad x \in \partial\omega_h.$$

There we use notations introduced by Samarskij (1983)

$$y = y(x, t_j), \qquad \hat{y} = y(x, t_{j+1}), \qquad y_t = (\hat{y} - y)/\tau,$$
$$y_{\bar{x}} = (y_i - y_{i-1})/h, \qquad y_x = (y_{i+1} - y_i)/h.$$

It is sufficient to consider an implicit difference scheme

$$y_t = \Lambda\hat{y} - f(x), \qquad x \in \omega_h \times \omega_\tau,$$
$$y(x) = 0, \qquad x \in \partial\omega_h \times \omega_\tau,$$

in order to prove the asymptotical stability of the steady-state difference solution.

Next we investigate two important nonlinear mathematical models. The first model arises when we treat counterpropagating waves in a Brillouin-active medium (see Narum *et al.*, 1988)

$$\frac{du(z)}{dz} = i\gamma|v(z)|^2 u(z), \qquad u(0) = 1, \tag{1.3a}$$
$$\frac{dv(z)}{dz} = -i\gamma|u(z)|^2 v(z), \qquad v(1) = B, \tag{1.3b}$$

where $u(z), v(z)$ are the amplitudes of the forward and backward travelling waves.

The second mathematical model is more complicated. It describes optical phase conjugation in stimulated Brillouin scattering (SBS) with pump depletion. Under steady-state conditions, the complex pump and SBS amplitudes $u(z,r), v(z,r)$ satisfy the equations (see Lehmberg, 1982; Volkova, 1988)

$$\frac{\partial u(z)}{\partial z} + i\mu Au = -\gamma(z)|v(z)|^2 u(z), \qquad u(0,r) = u_0(r), \qquad (1.4a)$$

$$\frac{\partial v(z)}{\partial z} - i\mu Av = -\gamma(z)|u(z)|^2 v(z), \qquad v(L,r) = v_L(r), \qquad (1.4b)$$

$$u(z,R) = v(z,R) = 0, \qquad \partial u(z,0)/\partial r = \partial v(z,0)/\partial r = 0, \qquad (1.4c)$$

$$Au = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big), \qquad \mu = 0.5k,$$

where, k is the magnitude of the propagation vectors (assumed equal for the two vectors), r is the transverse coordinate, and $\gamma(z) \geqslant \gamma_0 > 0$ is the coupling coefficient. Numerical solving of (1.4) is a nontrivial problem, because the initial conditions for the two coupled functions $u(z,r)$, $v(z,r)$ are defined on opposite sides of the region $0 \leqslant z \leqslant L$. The problem (1.4) is solved numerically by Buzelis *et al.* (1990), Čiegis, Kairytė and Norvaišas (1990), Lehmberg (1982, 1983), Volkova (1988), (see also the references cited in these papers). Iterative methods for the problem (1.4) are investigated by Čiegis (1990). As to the stability of the solution of (1.4) this problem remains open.

We also consider the simplified one dimensional problem $(\mu = 0)$

$$\frac{du}{dz} = -\gamma(z)|v(z)|^2 u(z), \qquad u(0) = 1, \qquad (1.5a)$$

$$\frac{dv}{dz} = -\gamma(z)|u(z)|^2 v(z), \qquad v(1) = B. \qquad (1.5b)$$

2. Stability analysis of the problem (1.3). Equations (1.3) yield the following simple steady-state solution (designated by the superscript zero):

$$\overset{o}{u}(z) = \exp(i\gamma|B|^2 z), \qquad \overset{o}{v}(z) = B\exp(i\gamma(1-z)). \qquad (2.1)$$

In order to determine the stability characteristics of the steady-state solution (2.1) a nonstationary time - dependent model can be used (see Narum *et al.*, 1988)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = i\gamma\Omega^2 \rho v, \qquad u(0,t) = 1, \tag{2.2a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} = i\gamma\Omega^2 \rho^* u, \qquad v(1,t) = B, \tag{2.2b}$$

$$\frac{\partial^2 \rho}{\partial t^2} + \Gamma\frac{\partial \rho}{\partial t} + \Omega^2 \rho = uv^*, \tag{2.2c}$$

$$u(z,0) = u_0(z), \quad v(z,0) = v_0(z), \tag{2.2d}$$

$$\rho(z,0) = \rho_0(z), \quad \rho'(z,0) = 0,$$

where the additional equation for the variation of the density ρ from its mean value ρ_0 is introduced. We note that the problem (2.2) defines the stability of the steady-state solution (2.1) connected with some physical model of the process in investigation.

A fictitious time can be introduced into (1.3) similarly to the method (1.2)

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = i\gamma |v|^2 u, \qquad u(0,t) = 1, \tag{2.3a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} = i\gamma |u|^2 v, \qquad v(1,t) = B. \tag{2.3b}$$

This model gives another stability definition of the solution (2.1). First we investigate the model (2.3). In order to determine a linear stability of the steady-state solution, we perturb the amplitudes of the forward and backward waves so that

$$u(z,t) = \overset{o}{u}(z)\big(1 + u_1(z)\exp(\lambda t) + u_2(z)\exp(\lambda^* t)\big), \tag{2.4a}$$

$$v(z,t) = \overset{o}{v}(z)\big(1 + v_1(z)\exp(\lambda t) + v_2(z)\exp(\lambda^* t)\big), \tag{2.4b}$$

where $u_j(z), v_j(z)$ represent small perturbations to the steady-state solution. Inserting these expressions into Eqs. (2.3) we derive linearized equations for the perturbation amplitudes

$$\frac{d\dot{u}_1(z)}{dz} = -\lambda u_1(z) + i\gamma |B|^2 \Big(v_1(z) + v_2^*(z)\Big), \tag{2.5a}$$

$$\frac{du_2^*(z)}{dz} = -\lambda u_2^*(z) - i\gamma |B|^2 \Big(v_1(z) + v_2^*(z)\Big), \tag{2.5b}$$

$$\frac{dv_1(z)}{dz} = \lambda v_1(z) - i\gamma \Big(u_1(z) + u_2^*(z)\Big), \tag{2.5c}$$

$$\frac{dv_2(z)}{dz} = \lambda v_2(z) + i\gamma \Big(u_1(z) + u_2^*(z)\Big). \tag{2.5d}$$

$$u_1(0) = 0, \qquad u_2(0) = 0, \qquad v_1(1) = 0, \qquad v_2(1) = 0. \tag{2.5e}$$

If for $Re\lambda > 0$ a nontrivial solution of (2.5) exists, then the steady state solution (2.1) will be temporally unstable to the growth of these perturbations (linear instability). By adding (2.5a,b) we obtain the equation

$$\frac{d(u_1(z) + u_2^*(z))}{dz} = -\lambda(u_1(z) + u_2^*(z)), \qquad u_1(0) + u_2^*(0) = 0,$$

the particular solution of which is equal to $u_1(z) + u_2^*(z) = 0$. Then it follows from (2.5c,d) and boundary conditions (2.5e) that $v_1(t) = 0, v_2(t) = 0$. Finally we obtain from (2.5a,b,c) that $u_1(z) = 0, u_2(z) = 0$, i.e. only a trivial solution $u_j(z) = 0, v_j(z) = 0$ of the problem (2.5) with $Re\lambda > 0$ exists. Hence we have proved a linear stability of the steady-state solution (2.1), if the stability is defined by the nonstationary problem (2.3).

Next we investigate an asymptotical stability of the same solution. After simple calculations we get that the following conservation properties hold for the solution of (2.3)

$$\frac{\partial |u|^2}{\partial t} + \frac{\partial |u|^2}{\partial z} = 0, \qquad \frac{\partial |v|^2}{\partial t} - \frac{\partial |v|^2}{\partial z} = 0. \tag{2.6}$$

It follows from the maximum principle and the boundary conditions (2.3) that the solution of (2.3) is globally bounded. Moreover, the method of characteristics yields us the equalities

$$|u(z,t)|^2 = |A|^2, \qquad |v(z,t)|^2 = |B|^2, \qquad 0 \leqslant z \leqslant 1, \quad t > 1,$$

hence, the system (2.3) becomes decoupled for $t \geqslant 1$. By solving these simple problems we get

$$u(z,t) = \overset{o}{u}(z), \qquad v(z,t) = \overset{o}{v}(z), \qquad 0 \leqslant z \leqslant 1, \quad t \geqslant 2,$$

and for any initial perturbations the solution of the nonstationary problem (2.3) converges to the exact steady-state solution (2.1). We have proved the asymptotical stability of this solution when its stability is defined by the time - dependent model (2.3).

Similar results are valid for the solution of the difference scheme

$$y_z = i\gamma \frac{|w_{j+1}|^2 + |w_j|^2}{2} \frac{|y_{j+1}|^2 + |y_j|^2}{y_{j+1}^* + y_j^*}, \qquad y_0 = 1, \qquad (2.7a)$$

$$w_z = -i\gamma \frac{|y_{j+1}|^2 + |y_j|^2}{2} \frac{|w_{j+1}|^2 + |w_j|^2}{w_{j+1}^* + w_j^*}, \qquad w_N = \dot{B}. \qquad (2.7b)$$

The time - dependent problem (2.3) is replaced by the difference scheme (the iterative method)

$$\frac{\hat{y} - y(-1)}{h} = i\gamma \frac{|\hat{w}(-1)|^2 + |w|^2}{2} \frac{|\hat{y}|^2 + |y(-1)|^2}{\hat{y}^* + y^*(-1)}, \qquad \hat{y}_0 = 1, \quad (2.8a)$$

$$\frac{\hat{w}(-1) - w}{h} = -i\gamma \frac{|\hat{y}|^2 + |y(-1)|^2}{2} \frac{|\hat{w}(-1)|^2 + |w|^2}{\hat{w}^*(-1) + w^*}, \quad \hat{w}_N = B. (2.8b)$$

The following discrete conservation laws hold for (2.7),(2.8)

$$|y_{j+1}|^2 = |y_j|^2 = 1, \quad |w_j|^2 = |w_{j+1}|^2 = B, \; j = 0, 1, \cdots, N-1, (2.9a)$$

$$|\hat{y}|^2 = |y(-1)|^2, \qquad |\hat{w}(-1)|^2 = |w|^2. \qquad (2.9b)$$

We obtain from (2.9) that the steady-state solution of (2.7) is asymptotically stable.

Next we consider the problem (2.2), which also defines some iterative method. As stated above the stability defined according to this model is called a physical stability. Linearized equations for perturbation amplitudes are given by

$$\frac{du_1}{dz} = -\lambda u_1 - i\gamma|B|^2 u_1 + i\gamma|B|^2 v_1 + i\gamma_1|B|^2(u_1 + v_2^*), \quad (2.10a)$$

$$\frac{du_2^*}{dz} = -\lambda u_2^* + i\gamma|B|^2 u_2^* - i\gamma|B|^2 v_2^* - i\gamma_1|B|^2(u_2^* + v_1), \quad (2.10b)$$

$$\frac{dv_1}{dz} = \lambda v_1 + i\gamma v_1 - i\gamma u_1 - i\gamma_1(u_2^* + v_1), \qquad (2.10c)$$

$$\frac{dv_2^*}{dz} = \lambda v_2^* - i\gamma v_2^* + i\gamma u_2^* + i\gamma_1(u_1 + v_2^*), \qquad (2.10d)$$

$$\gamma_1 = \gamma\Omega^2/(\lambda^2 + \Gamma\lambda + \Omega^2).$$

The investigation of particular solutions of (2.10) shows that there exist nonstable modes with $Re\lambda \geqslant 0$. The instability threshold for the system (2.2) is defined as the lowest intensity that yields a solution to the system of linearized equations with $Re\lambda = 0$ for any value of $Im\lambda$ (see Narum *et al.*, 1988). We see that the steady-state solution is not unconditionally stable, if the time-dependent problem (2.2) is used to define the stability. A numerical investigation of the whole dynamic system (2.2) confirms the results of linear analysis: for some value of $\gamma > \gamma_0$ the solution becomes unstable and evolves from a stable state to a chaotic state as the input intensities are increased following the period-doubling route (see also Narum *et al.*, 1988). These examples confirm our statement that the stability of the steady-state solution depends on the iterative method (nonstationary problem) used to find this solution.

3. Stability analysis of the problem (1.5). It is easy to prove that the problem (1.5) has a unique solution for any parameters γ, B (see Čiegis and Norvaišas, 1989). Taking into account the results of Sect.2 we investigate the stability of this steady-state solution by using a nonstationary problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = -\gamma_1 |v(z,t)|^2 u(z,t), \qquad u(0,t) = 1, \tag{3.1a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} = \gamma_1 |u(z,t)|^2 v(z,t), \qquad v(1,t) = B. \tag{3.1b}$$

We note that the asymptotical stability is most important in applications, therefore we restrict ourselves only to its analysis (for the linear stability analysis see Čiegis, 1992a). After simple calculations we prove the following conservation property of the problem (3.1)

$$\frac{\partial(|u|^2 + |v|^2)}{\partial t} + \frac{\partial(|u|^2 - |v|^2)}{\partial z} = 0.$$

The nonlinearity of the problem (3.1) is to complicated to investigate it analytically. We use the numerical experiment method to analyse the stability of the steady state solution. Our aim is not only to analyse this stability in the case of the model 1D problem (1.5), but also to propose and to test a general technique which can be used for the 2D problem (1.4). The difference scheme for the nonlinear problem (3.1) is defined by

$$y_t + \hat{y}_{\bar{z}} = -\gamma\left|\frac{w_j + w_{j-1}}{2}\right|^2\frac{\hat{y}_j + \hat{y}_{j-1}}{2}, \qquad \hat{y}_0 = 1, \tag{3.2a}$$

$$w_t - \hat{w}_z = \gamma\left|\frac{\hat{y}_{j+1} + \hat{y}_j}{2}\right|^2\frac{\hat{w}_{j+1} + \hat{w}_j}{2}, \qquad \hat{w}_N = B, \tag{3.2b}$$

$$y(z_j, 0) = u_0(z_j), \qquad w(z_j, 0) = v_0(z_j). \tag{3.2c}$$

The convergence of the difference scheme (3.2) solution to the solution of (3.1) can be obtained following similar results of Čiegis (1992b). The steady state solution satisfies the finite difference scheme

$$\overset{o}{y}_z = -\gamma\left|\frac{\overset{o}{w}_j + \overset{o}{w}_{j+1}}{2}\right|^2\frac{\overset{o}{y}_j + \overset{o}{y}_{j+1}}{2}, \qquad \overset{o}{y}_0 = 1, \tag{3.3a}$$

$$\overset{o}{w}_z = -\gamma\left|\frac{\overset{o}{y}_j + \overset{o}{y}_{j+1}}{2}\right|^2\frac{\overset{o}{w}_j + \overset{o}{w}_{j+1}}{2}, \qquad \overset{o}{w}_N = B. \tag{3.3b}$$

We use iterative methods proposed by Čiegis and Norvaišas (1989) to find the solution of (3.3).

In numerical experiments the parameters (γ, B) were selected so that to investigate the region of these parameters important for applications. The case when backscatter grows from a small noise wave, introduced at $z = L$, was also considered:

$$1 \leqslant \gamma \leqslant 60, \qquad B = b_0(1 - i), \qquad b_0 = 10^{-3}, 10^{-2}, 10^{-1}, 1.$$

The main result of the numerical experiment is that in all cases the solution of (3.2) converges to the steady-state solution $\overset{o}{y}(z), \overset{o}{w}(z)$ and for any ε the following estimates hold

$$||y(z,t) - \overset{o}{y}(z)||_c \leqslant \varepsilon, \qquad ||w(z,t) - \overset{o}{w}(z)||_c \leqslant \varepsilon, \quad \text{for } \ t \geqslant T(\varepsilon).$$

Hence, the steady-state solution is asymptotically stable if the stability is defined by the time-dependent problem (3.1).

Similarly to the analysis of Sect.2, we can use one more time dependent model

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = ig\sigma v, \qquad u(0,t) = 1, \tag{3.4a}$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} = ig\sigma^* u, \qquad v(1,t) = B, \tag{3.4b}$$

$$\frac{\partial \sigma}{\partial t} + (a_1 + ia_2)\sigma = iguv^*, \qquad \gamma = g^2/a, \tag{3.4c}$$

where g is defined by $g^2 = \gamma(a_1 + ia_2)$. Mathematical model (3.4) gives one more stability definition of the steady state solution. By using the numerical experiment method we have determined such sets of parameters (γ, B, a_1, a_2) for which the instability of the steady state solution arises (see also Chu, Kanefsky and Falk, 1992; Chow and Bers, 1993). Recall that our goal is to develop iterative methods (time dependent problems) for which the steady-state solution remains stable as long as possible. Therefore the following values of the parameters $a_1 = 0$, $a_2 > 0$ must be recomended for stability computations.

4. Stability of the problem (1.4) solution. The $2D$ problem is more complicated. No results are known about the existence and uniqueness of its solution (obviously excluding the case when a classical fixed point theorem of Schauder may be used). One more interesting detail about the problem (1.4) must be noted. In many papers in which the results of numerical simulation are reported the convergence of iterative methods is obtained only for some finite value of the coupling coefficient $\gamma \leqslant \gamma^*$ (see Lehmberg, 1982,1983; Volkova, 1988; Moyer, Valley and Cimolino, 1988). Two hypothesis can be proposed. The first conjecture is that this fact is connected with the internal property of the problem(1.4), i.e. with the stability of its solution. The second conjecture is that the convergence of iterative methods depends on the difference scheme properties. Our goal is to investigate these conjectures. Having in mind the results obtained in Sect. 2, 3 we propose the following time dependent problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} + i\mu A u = -g|v|^2 u, \qquad u(0,r,t) = u_0(r), \qquad (4.1a)$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} + i\mu A u = g|u|^2 v, \qquad v(L,r,t) = v_L(r). \qquad (4.1b)$$

We approximate the problem (4.1) by the difference scheme

$$y_t + \hat{y}_{\bar{z}} + i\mu\Lambda\hat{y} = -g|\bar{w}|^2\hat{y}, \qquad y(0,r_k,t_j) = u_0(r_k), \qquad (4.2a)$$

$$w_{t,i-1} - \hat{w}_z + i\mu\Lambda\hat{w} = g|\hat{y}|^2\hat{w}, \qquad \hat{w}(L,r_k,t_j) = v_L(r_k), \qquad (4.2b)$$

where the following notations are used

$$\bar{y} = 0.5(y_i + y_{i-1}), \qquad \hat{\bar{y}} = 0.5(\hat{y}_i + \hat{y}_{i-1}).$$

The convergence of the solution of (4.2) to the solution of (4.1) is investigated by Čiegis (1989).

As is noted above, we have no existence and uniqueness results for the problem (1.4). In order to get a boundary condition (4.1b) for which some steady-state solution really exists (perhaps not unique), we solve an auxiliary initial value problem

$$y_z + i\mu\Lambda\bar{y} = -g|\bar{w}|^2\bar{y}, \qquad y(0, r_k) = u_0(r_k), \qquad (4.3a)$$

$$w_z - i\mu\Lambda\bar{w} = -g|\bar{y}|^2\bar{w}, \qquad w(0, r_k) = v_0(r_k), \qquad (4.3b)$$

and define the boundary condition as $v_L(r_k) = w(L, r_k)$.

REMARK 1. Such a choice enables us not only to get the boundary condition $v_L(r_k)$, but also to obtain the exact steady-state solution. Hence, we can investigate the asymptotical stability in a neighbourhood of the solution.

In our numerical simulations we take

$$u_0(r) = d_0L_0(0, r, f) + d_1L_1(0, r, f) + \cdots d_mL_m(0, r, f),$$

where $L_i(z, r, f)$ are the Laguerre functions. For example in the most simple case, when $m = 0$

$$u_0(r) = \exp(-0.5r^2(1 + i1/f)).$$

The initial condition for the backscatter is given by a random representation

$$v_0(x) = c_0L_0(0, r, f) + c_1L_1(0, r, f) + \cdots + c_pL_p(0, r, f). \qquad (4.4)$$

The numerical experiments proves the following results.

C1. *Convergence of the iterative method* (4.2). In all cases for sufficiently small value of τ the iterative method (4.3) converges to some steady state solution.

C2. *Stability of the solution of* (4.3). The stability of the solution of (4.3) depends on the number of modes presented in (4.4) (we fixed the parameter $m = 0$ in all computations). If $p = 0$, then the steady state solution remains stable for any values of the parameter g. For example the simulations give the correlation function value $H = 0.9992$ and the total reflectivity $R_0 = 0.95$ for $g = 0.07$, where we defined

$$H = \frac{|\int_0^R r u(r,0) v(r,0) dr|^2}{||u(0)||^2 \, ||v(0)||^2}, \quad R_0 = \frac{||v(0)||^2 - ||v(L)||^2}{||u(0)||^2}.$$

A situation becomes more complicated when we use $p = 5$ in (4.4) and select c_i such that for the exact solution $H_0 = 0.396$. The exact steady state solution remains stable till $g = 0.005$ (or $R_0 \approx 0.85$). After this for $g > 0.005$ it becomes unstable and (4.2) converges to the another solution with $H > H_0$. This new solution remains stable and the conjugation charakteristic H is improved monotonically with increased values of the coupling coefficient g, e.g., the simulations give $R_0 = 0.958$, $H = 0.872$ for $g = 0.02$. More detailed results of a computational experiment are given by Buzelis *et al.*, 1990.

Next we investigate our second conjecture. For this purpose we approximate the problem (1.4) by the following splitting scheme (see Volkova, 1988; Lehmberg, 1982)

$$\frac{y^0 - y_k}{h} + i\mu\Lambda \frac{y^0 + y_k}{2} = 0, \tag{4.5a}$$

$$\frac{w^0 - w_k}{h} - i\mu\Lambda \frac{w^0 + w_k}{2} = 0, \tag{4.5b}$$

$$\frac{y_{k+1} - y^0}{h} = -g\left|\frac{w_{k+1} + w^0}{2}\right|^2 \frac{y_{k+1} + y^0}{2}, \tag{4.5c}$$

$$\frac{w_{k+1} - w^0}{h} = -g\left|\frac{y_{k+1} + y^0}{2}\right|^2 \frac{w_{k+1} + w^0}{2}. \tag{4.5d}$$

Analogycally to (4.3) we can state initial value problem for (4.5)

$$y(0, r_k) = u_0(r_k), \qquad w(0, r_k) = v_0(r_k). \tag{4.5e}$$

The obtained solution $w(L, r_k)$ is used as a boundary condition for the problem (4.5) with two point conditions ($v_L(r_k) = w(L, r_k)$)

$$y(0, r_k) = u_0(r_k), \qquad v_N(r_k) = v_L(r_k). \tag{4.5f}$$

The covergence of the initial value problem solution is investigated by many authors (see Sanz-Serna, 1984; Čiegis, 1989; Ivanauskas, 1989). The following error estimates are valid

$$||y(z) - u(z)|| \leqslant C(h + h_0^2), \qquad ||w(z) - v(z)|| \leqslant C(h + h_0^2),$$

where h_0 is the mesh size of the discrete grid w_r. In order to solve nonlocal problem (4.5a-d, f) we use the iterative method similar to (4.2)

$$\frac{\hat{y}^0 - y^0}{\tau} + \frac{\hat{y}^* - \hat{y}_k}{h} + i\mu\Lambda\frac{\hat{y}^0 + \hat{y}_k}{2} = 0, \tag{4.6a}$$

$$\frac{\hat{y}_{k+1} - y_{k+1}}{\tau} + \frac{\hat{y}_{k+1} - \hat{y}^0}{h} = -g\Big|\frac{w_{k+1} + w^0}{2}\Big|^2 \frac{\hat{y}_{k+1} + \hat{y}^0}{2}, \tag{4.6b}$$

$$\frac{\hat{w}^0 - w^0}{\tau} - \frac{\hat{w}_{k+1} - \hat{w}^0}{h} = g\Big|\frac{\hat{y}_{k+1} + \hat{y}^0}{2}\Big|^2 \frac{\hat{w}_{k+1} + \hat{w}^0}{2}, \tag{4.6c}$$

$$\frac{\hat{w}_k - w_k}{\tau} - \frac{\hat{w}^0 - \hat{w}_k}{h} + i\mu\Lambda\frac{\hat{w}^0 + \hat{w}_k}{2} = 0. \tag{4.6d}$$

Some results of numerical simulations are presented in Tables 1, 2. Boundary conditions for (1.4) were given by formulas

$$\begin{aligned}
&u_0(r) = \exp(-0.5r^2(1 + i/f)),\\
&v_L(r) = c_0(L_0(L, r, f) + c_1L_1(L, r, f) + \cdots + c_4L_4(L, r, f)),\\
&c_0 = 0.025,\ c_1 = 0.1 + i0.8,\ c_2 = -0.3 + i0.6,\\
&c_3 = 1 + i0.1,\ c_4 = 0.5 - i0.7.
\end{aligned}$$

We solved (1.4) for the following values of parameters

$$\begin{aligned}
&L = 0.0231, \quad f = 0.0169, \quad \mu = 0.5, \quad R = 3.18,\\
&h = L/(N-1), \quad h_0 = R/(M - 0.5), \quad N = 100, \quad M = 100.
\end{aligned}$$

The well-known Talanov transformation was implemented in our algorithm in order to solve numerically the propagation of focused laser beams accurately (see Volkova, 1988.) First we used the difference scheme (4.2). The values of H and R_0 are presented in Table 1 for various values g. We see that the steady state solution is stable for all g.

Table 1. Results for the difference scheme (4.2)

g	3.462	5.193	8.654	13.847	25.963
R_0	0.042	0.219	0.524	0.718	0.863
H	0.953	0.957	0.953	0.961	0.977

Next we solved (1.4) using the splitting difference scheme (4.5) and the iterative method (4.6). The convergence was obtained only for $g \leqslant g_0$, $N = 50$, $M = 50$ (see Table 2). Such stability property of the steady state solution (4.5) is due to the discretization properties of the splitting method and is not connected with the internal stability of the differential problem (1.4). A more detailed analysis of this interesting phenomena will be given in a separate paper.

Table 2. Results for the difference scheme (4.6)

g	1.731	3.462	4.327	5.193	> 0.003
R_0	0.002	0.043	0.119	0.22	–
H	0.731	0.947	0.957	0.955	–

5. General stability analysis. In this section we present some general conclusions. Our aim is to investigate efficient iterative methods for solving a nonlinear problem

$$Ay = f. \tag{5.1}$$

In particular we consider iterative methods which are expressed in the following from

$$B(y_k)\frac{y_{k+1} - y_k}{\tau_k} + Ay_k = f. \tag{5.2}$$

In the simplest cases $B = E + \tau R$ (see, for example, (1.4), (1.6)). For $B = E$ we get the classical explicit iterative method. The error function $z_k = y - y_k$ satisfies the problem

$$B(y_k)\frac{z_{k+1} - z_k}{\tau_k} + \delta A(\widetilde{y}_k)z_k = 0, \quad \delta A(y) = (\frac{\partial a_i(y)}{\partial y_j}). \tag{5.3}$$

Next we assume that $B = E$. A sufficient condition for the convergence of the iterative method (5.2) follows from the energy equality

$$||z_{k+1}||^2 + \tau||z_t||^2 + 2\tau(\delta A z_k, z_k) = ||z_k||^2$$

which is obtained from (5.3) multiplying it by $2\tau z_k$. If $(\delta A z_k, z_k) > 0$, then we get that $||z_{k+1}|| < ||z_k||$. Hence the iterative method (5.2) is convergent and the steady-state solution of (5.1) is asymptotically stable according to our stability definition. For many problems arising in applications (e.g. the electrocontact modeling problem, which is investigated by Čiegis, Čiegis and Čiupaila, 1992) the condition $(\delta A(y)z, z) > 0$ is used to define the stability of the solution. Therefore, the iterative method (5.2) with $B = E$ enables us both to find a solution, and to prove its stability at the same time. A divergence of the iterative method shows that a steady state solution (if any) is not asymptotically stable (we deal with the global asymptotical stability there). If only stable solutions are of interest to us, then the explicit iterative method $B = E$ or its modifications $B = E + \tau R$ are recommended for solving (5.1). It follows from the analysis given above that in order to find nonstable solutions of (5.1) we must eliminate the modes v_i, $i = 1, 2, \cdots, J_0$, which correspond to eigenvalues λ_i with $Re\lambda_i \leqslant 0$. For some problems this can be done efficiently if the basic iterative method (5.2) is supplemented with the orthogonal projection step

$$y_k = P(\widetilde{y}_k), \qquad (y_k, v_i) = 0, \qquad i = 1, 2, \cdots, J_0.$$

But a more general method to overcome the instability of the steady state solution is to use the Newton method, for which $B = \delta A(y_k)$. Then the error function satisfies the equation

$$\delta A(y_k)\frac{z_{k+1} - z_k}{\tau_k} + \delta A(y_k)z_k + \frac{1}{2}z_k^T\delta^2 A(\widetilde{y}_k)z_k = 0. \tag{5.4a}$$

Multiplying this equation by $(\delta A(y^k))^{-1}$ we get

$$\frac{z_{k+1}-z_k}{\tau_k}+z_k+\frac{1}{2}z_k^T(\delta A(y_k))^{-1}\delta^2 A(\widetilde{y}_k)z_k=0. \qquad (5.4b)$$

For the classical Newton method $\tau_k = 1$, and the problem becomes even simpler

$$z_{k+1}=\frac{1}{2}z_k^T(\delta A(y_k))^{-1}\delta^2 A(\widetilde{y}_k)z_k. \qquad (5.4c)$$

It follows from (5.4) that Newton's method is convergent if a sufficiently good initial approximation of the exact steady state solution is given. Therefore, with such a choice of the operator B the convergence of (5.2) does not depend explicitly on the signs of eigenvalues $Re\lambda_k$. The divergence of the Newton method doesn't give any information about the stability of the solution either. These results show that the value of different iterative methods can be reestimated when new classes of problems are considered. For example explicit iterative methods proved to be supperior to the Newton method when a stability of the solution is analysed along with its existence.

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