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Prioritized Dual Hesitant Fuzzy Aggregation Operators Based on *t*-Norms and *t*-Conorms with Their Applications in Decision Making

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Abstract. The main contributions of this paper are shown as: (1) we define dual hesitant fuzzy t-norms and t-conorms; (2) based on dual hesitant fuzzy t-norms and t-conorms, we introduce a family of prioritized dual hesitant fuzzy operators to aggregate dual hesitant fuzzy information of alternatives with regard to the prioritized attributes; (3) we propose a method to handle the dual hesitant fuzzy multi-attribute decision making (MADM) problems with prioritized attributes; (4) we show that compared to other relevant studies, the developed prioritized aggregation operators take full advantage of the given decision information, avoid the loss of original information, and thus yield better final decision results.

Key words: multi-attribute decision making, prioritization relationship, dual hesitant fuzzy t-norm, dual hesitant fuzzy t-conorm, prioritized dual hesitant fuzzy aggregation operators.

1. Introduction

In decision making, uncertainty is ubiquitous since objective things are uncertain and complex, and the managing and modelling of uncertain information are crucial for the acquisition of desirable solutions (Xu and Zhao, 2016). To date, a large number of tools have been proposed to model people's imprecise decision information from different angles (Atanassov, 1986; Torra, 2010; Zhu *et al.*, 2012; Zhou *et al.*, 2013; Pang *et al.*, 2016; Guan *et al.*, 2017; Gou *et al.*, 2017; Liao *et al.*, 2017; Zhang, 2017; Zhao *et al.*, 2018). Among them, to fully reflect the characteristics of affirmation and negation of human's cognitive performance, Atanassov (1986) introduced intuitionistic fuzzy sets (IFSs), which assign to each element a membership degree and a non-membership degree. In order to portray the human's hesitancy in the decision-making process, Torra

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(2010) introduced hesitant fuzzy sets (HFSs), which allocate to each element several different membership degrees. Afterwards, motivated by the ideas of IFSs and HFSs, Zhu *et al*. (2012) developed dual hesitant fuzzy sets (DHFSs) to manipulate the hesitation both for the membership degree and the non-membership degree by two functions from which two sets of membership and non-membership degrees can be respectively derived for each element. Obviously, DHFSs are able to express the human's hesitancy in more detail and more comprehensively than IFSs and HFSs, and cover IFSs and HFSs as special cases. DHFSs are very suitable for group deciding making as they can well handle the situations where a decision organization consisting of several experts is not sure about a value, but hesitant among several possible values when it provides the membership degree or the non-membership degree of an element.

At present, the studies on theory and application of DHFSs have attracted great attention, and a lot of research results have been achieved (Farhadinia, 2014; Ju *et al*., 2014b; Li, 2014; Singh, 2014; Zhu and Xu, 2014; Zhao *et al*., 2016; Ren *et al*., 2017). Since the aggregation operators are very important for the application of DHFSs in decision making, many scholars have studied the aggregation operators of dual hesitant fuzzy information. Wang *et al*. (2014) and Yu *et al*. (2016) developed some fundamental dual hesitant fuzzy aggregation operators in terms of arithmetic and geometric operations. Afterwards, Yu (2014) and Wang and Li (2014) extended these aggregation operators (Wang *et al*., 2014), and presented a family of generalized dual hesitant fuzzy aggregation operators. Based on Hamacher t-norm and t-conorm, Einstein t-conorm and t-norm, Ju *et al*. (2014b) and Zhao *et al*. (2017) developed a series of new dual hesitant fuzzy aggregation operators, respectively. For fusing the correlative dual hesitant fuzzy information, Ju *et al*. (2014a) developed some dual hesitant fuzzy Choquet aggregation operators; Wang *et al*. (2016) proposed a wide range of dual hesitant fuzzy power aggregation operators based on Archimedean t-conorm and t-norm; Zhang (2015) proposed the dual hesitant fuzzy Hamacher correlated geometric operator. In addition, considering that there may exist some interactions between the membership functions and the non-membership functions of different DHFSs, Xu *et al*. (2015) presented a class of dual hesitant fuzzy interaction operators. Analysing the above-mentioned dual hesitant fuzzy aggregation operators, we can find that these operators are all defined under the assumption that all attributes are at the same priority level. However, in many practical multi-attribute decision making (MADM) problems, the attributes are in different priority levels. For instance, a consumer who is unfamiliar with cars is selecting a car from five candidate cars. Due to his/her ignorance of cars, he/she invites three friends who are proficient in cars to evaluate these cars by some indices among which safety level and comfort level are important. Generally, the safety attribute has a higher priority than comfort, which is to say, it is not allowed to compensate a loss in safety by a benefit in comfort. When evaluating the safety level of a car, one of his/her friends thinks that its satisfaction degree is 0.5, while the others believe that it should be 0.7, and they can't persuade each other. Meanwhile, one of his/her friends deems the dissatisfaction degree of safety as 0.1, another views it as 0.2, and the other considers it as 0.3, and they all strongly insist on their own opinions. Then the above problem is a dual hesitant fuzzy MADM problem with prioritized attributes, and the evaluation for safety level of the car can be represented by the dual hesitant fuzzy element (DHFE) $\{\{0.5, 0.7\}, \{0.1, 0.2, 0.3\}\}$.

The MADM problem with prioritized attributes is an interesting and hot research topic and has received great attention from scholars. For example, Yager (2008, 2009) first investigated the MADM problems with prioritized attributes, and proposed the prioritized averaging (PA) operator and prioritized ordered weighted average (POWA) operator, in which the weight of an attribute with lower priority is dependent upon the satisfactions of the higher priority attributes. After that, Yu and Xu (2013) generalized the PA operator (Yager, 2008) to the intuitionistic fuzzy environment and developed several prioritized intuitionistic fuzzy aggregation (PIFA) operators in light of some basic intuitionistic fuzzy operations. Based on the POWA operator (Yager, 2009), Xu *et al.* (2011) proposed the intuitionistic fuzzy POWA operator. Yu (2012) formulated the generalized intuitionistic fuzzy prioritized weighted geometric operator based on Archimedean t-conorm and t-norm. Wei (2012) extended the prioritized operators (Yager, 2008) to the hesitant fuzzy environment and developed some hesitant fuzzy prioritized operators. Yu *et al.* (2013) generalized Wei (2012)'s prioritized operators and presented some generalized hesitant fuzzy prioritized operators. Zhou and Li (2014) developed two generalized hesitant fuzzy prioritized Einstein operators with the help of Einstein operations. By means of the softmax function, Torres *et al.* (2014) developed the hesitant fuzzy prioritized softmax average and hesitant fuzzy prioritized softmax geometric operators. Recently, based on a correctional score function and a dice similarity measure of DHFEs, Ren and Wei (2017) developed a method to solve the decision making problems in which there exists a prioritization relationship over attributes and the attribute values take the form of DHFEs.

It is worth pointing out that when we apply the PIFA operators (Yu and Xu, 2013) to solve the intuitionistic fuzzy MADM problems with prioritized attributes, the PIFA operators may not capture the prioritization relationship over attributes since there may exist compensation between attributes (for details, please refer to Section 5.1). Furthermore, in the process of generating weights for attributes (Yu and Xu, 2013; Xu *et al.*, 2011; Yu, 2012; Wei, 2012; Yu *et al.*, 2013; Zhou and Li, 2014; Torres *et al.*, 2014; Ren and Wei, 2017), the concept of score function is used to transform the original decision information into single real numbers, which leads to distortion and loss of information, and further results in inaccurate decision results (for details, please refer to Section 5.1). Consequently, it is very necessary to develop new prioritized aggregation operators. Since both IFSs and HFSs are special cases of DHFSs, we just need to investigate the aggregation of dual hesitant fuzzy information in the MADM problems with prioritized attributes. Therefore, the main objective of this paper is to develop a family of prioritized dual hesitant fuzzy aggregation operators based on t-norms and t-conorms and explore their applications in the dual hesitant fuzzy MADM problems with prioritized attributes.

To do so, we organize the paper as follows: Section 2 reviews some basic knowledge that will be used in our work. In Section 3, we define dual hesitant fuzzy t-norm and t-conorm and survey a special class of DHF t-norms and t-conorms. In Section 4, based on dual hesitant fuzzy t-norms and t-conorms, we first develop the prioritized dual hesitant fuzzy weighted triangular operator, which preserves both the monotonicity and boundary conditions, and from which a family of prioritized dual hesitant fuzzy aggregation operators can be derived. Then, based upon the developed prioritized dual hesitant fuzzy aggregation operators, we propose a method to handle the dual hesitant fuzzy MADM problems with prioritized attributes and demonstrate the feasibility and applicability through a car selection example. In Section 5, two comparisons are implemented to illustrate the validity and superiority of the developed prioritized aggregation operators and the proposed decision making method. Section 6 ends the paper with some conclusions.

2. Preliminaries

In this section, we review some preliminary knowledge that will be used thereinafter.

Definition 1 (See Batyrshin and Kaynak, 1999; Klement *et al.*, 2000). A t-norm T is a mapping from $[0, 1]^2$ to $[0, 1]$ possessing the properties of commutativity and associativity, which is increasing in both arguments and refers to 1 as identity, i.e. $T(1, a) = a$ for $a \in [0, 1]$.

A t-conorm S from $[0, 1]^2$ to $[0, 1]$ has the similar properties with the t-norm T. The only difference is that the t-conorm S regards 0 as identity, i.e. $S(0, a) = a$ for $a \in [0, 1]$ (Alsina *et al.*, 1983; Nguyen and Walker, 2005).

Some typical examples of t-norms and t-conorms are shown as follows (Kolesárová and Komorníková, 1999; Kolesárová *et al.*, 2007) ($\forall a, b \in [0, 1]$):

(i) $T_M(a, b) = \min(a, b) = a \wedge b$; $T_P(a, b) = a \cdot b$; $T_L(a, b) = \max(a + b - 1, 0)$;

(ii) $S_M(a, b) = \max(a, b) = a \vee b$; $S_S(a, b) = a + b - ab$; $S_L(a, b) = \min(a + b, 1)$.

It can be easily validated that for any t-norm T, it holds that $T(0, a) = 0$ and $T(a, b) \leqslant T_M(a, b)$; for any t-conorm S, we have $S(1, a) = 1$ and $S_M(a, b) \leqslant S(a, b)$, where $a, b \in [0, 1]$.

Atanassov (1986) first proposed the notion of intuitionistic fuzzy sets as follows:

Definition 2 (See Atanassov, 1986). Let X be a fixed set, then an intuitionistic fuzzy set (IFS) I on X is defined as:

$$I = \{\langle x, u_I(x), v_I(x)\rangle | x \in X\},$$

where the functions $u_I : X \to [0, 1]$ and $v_I : X \to [0, 1]$ ascertain the membership degree and non-membership degree of the element $x \in X$ to the set I, respectively, with the condition: $u_I(x) + v_I(x) \leqslant 1$. For convenience, Xu (2007) called $\alpha = (u_\alpha, v_\alpha)$ an intuitionistic fuzzy number (IFN).

Some operations on IFNs are presented as follows:

Definition 3 (See Xu, 2007). Let $\alpha_i = (u_{\alpha_i}, v_{\alpha_i})$, $i = 1, 2$ be two IFNs, then

(i) $\alpha_1 \wedge \alpha_2 = (u_{\alpha_1} \wedge u_{\alpha_2}, v_{\alpha_1} \vee v_{\alpha_2})$;

(ii) $\alpha_1 \vee \alpha_2 = (u_{\alpha_1} \vee u_{\alpha_2}, v_{\alpha_1} \wedge v_{\alpha_2})$;
(iii) $\alpha_1 \oplus \alpha_2 = (u_{\alpha_1} + u_{\alpha_2} - u_{\alpha_1} u_{\alpha_2}, v_{\alpha_1} v_{\alpha_2})$;
(iv) $\alpha_1 \otimes \alpha_2 = (u_{\alpha_1} u_{\alpha_2}, v_{\alpha_1} + v_{\alpha_2} - v_{\alpha_1} v_{\alpha_2})$.

However, there exist some cases in which the difficulty of determining the membership degree of an element is not because we have a margin of error (IFSs), but because we have several possible values. In order to cope with such cases, Torra (2010) put forward the notion of hesitant fuzzy sets.

Definition 4 (See Torra, 2010). Let X be a fixed set, then a hesitant fuzzy set (HFS) on X is in terms of a function which applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu (2011) mathematically interpreted the HFS as:

$$E = \{\langle x, h_E(x)\rangle \mid x \in X\},$$

where $h_E(x)$ is a set of several values in $[0, 1]$, representing the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xia and Xu (2011) called $h = h_E(x)$ a hesitant fuzzy element (HFE).

Assumption. As pointed out in Bedregal *et al.* (2014), although the membership of an HFS could be any subset of [0, 1], practical works always assume explicitly or implicitly that the membership of an HFS is a finite and non-empty subset. Bedregal *et al.* (2014) defined the HFSs whose memberships are finite and non-empty subsets of [0, 1] as the typical HFSs, and the elements in the typical HFSs as the typical HFEs. From now on, the paper will only pay attention to the typical HFEs. For notational convenience, we still adopt the HFEs to denote the typical HFEs.

Meanwhile, Xia and Xu (2011) developed the following method to compare HFEs.

Definition 5 (See Xia and Xu, 2011). For an HFE h, $q(h) = (1/\#h)\sum_{\gamma_i \in h} \gamma_i$ is defined as the score function of h, where $\#h$ denotes the number of elements in h. Then for two HFEs h_1 and h_2, if $q(h_1) > q(h_2)$, then h_1 is superior to h_2, denoted by $h_1 \succ h_2$; if $q(h_1) = q(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 \sim h_2$.

Recently, inspired by the ideas of IFSs and HFSs, Zhu *et al.* (2012) introduced dual hesitant fuzzy sets, which allow the membership and non-membership degrees of an element to a given set to be several different values in [0, 1], respectively.

Definition 6 (See Zhu *et al.*, 2012). Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) A on X is given in the following form:

$$A = \{\langle x, h_A(x), g_A(x)\rangle \mid x \in X\},$$

where $h_A(x)$ and $g_A(x)$ are two sets of several values in [0, 1], indicating the possible membership and non-membership degrees of the element $x \in X$ to the set A, respectively, with the condition: $\max_{\gamma \in h_A(x)}\{\gamma\} + \max_{\eta \in g_A(x)}\{\eta\} \leqslant 1$ for all $x \in X$.

Zhu *et al.* (2012) called the component $d(x) = (h_A(x), g_A(x))$ a dual hesitant fuzzy element (DHFE), denoted by $d = (h, g)$. Apparently, for the DHFE $d = (h, g)$, if $\#h = \#g = 1$ ($\#h$ and $\#g$ denote the numbers of elements in h and g, respectively), then d is simplified to an IFN; if $h \neq \phi$ and $g = \phi$ (ϕ denotes the null set), then d is simplified to an HFE. In addition, when $h \neq \phi$ and $g \neq \phi$, the complement of d is given by $d^c = (g, h)$ (Zhu *et al.*, 2012). Since DHFEs are the basic units of a DHFS, for the purpose of simplicity, we thereinafter will only focus on the DHFEs.

For convenience, we give the following two definitions:

Definition 7. Let $d = (h, g)$ be a DHFE, then h and g are respectively called the membership degree set and non-membership degree set of d, and the numbers of elements in h and g are respectively denoted by $\#h$ and $\#g$.

Definition 8. For any subset v of [0, 1], a permutation is defined by the mapping $\sigma_v :$ $\{1, 2, \dots, \#v\} \rightarrow v$ such that for any $t = 1, 2, \dots, \#v - 1$, we have $\sigma_v(t) < \sigma_v(t + 1)$, i.e. $\sigma_v(t)$ is the tth smallest element in v, where $\#v$ denotes the number of elements in v.

Farhadinia (2014) proposed a linear order to implement the comparison of two DHFEs under the assumption that the DHFEs have the same numbers of membership degrees and non-membership degrees, respectively.

Definition 9 (See Farhadinia, 2014). Let $d_i = (h_i, g_i)$, $i = 1, 2$ be two DHFEs, then a linear order $\preceq_L$ is defined as:

$$d_1 \preceq_L d_2 \text{ iff } d_1^+ < d_2^+ \text{ or } (d_1^+ = d_2^+ \text{ and } d_1^- \leqslant d_2^-),$$

where $d_i^+ = Score(h_i) + Score(g_i)$ and $d_i^- = |Score(h_i) - Score(g_i)|$ for $i = 1, 2$. Here, the score of a set v as a subset of [0, 1] is given by

$$Score(v) = \frac{2}{\#v(\#v + 1)} \sum_{t=1}^{\#v} t\sigma_v(t). \tag{1}$$

Below we show the above comparison method through an example:

Example 1. Let $d_1 = \{\{0.1, 0.2\}, \{0.5, 0.8\}\}$ and $d_2 = \{\{0.1, 0.4\}, \{0.4, 0.6\}\}$ be two DHFEs. Clearly, d_1 should be inferior to d_2 from intuition. By Eq. (1), we get $Score(h_1) = 0.1667$, $Score(g_1) = 0.7$, and $Score(h_2) = 0.3$, $Score(g_2) = 0.5333$. Then we get $d_2^+ = 0.8333 < 0.8667 = d_1^+$, which indicates that $d_2 \preceq_L d_1$. It is obvious that the comparative result is counterintuitive.

For comparing two DHFEs effectively, motivated by the comparison method for IFNs (Xu and Yager, 2006), we adjust the comparison principle in Definition 9 as:

DEFINITION 10. Let $d_i = (h_i, g_i)$, $i = 1, 2$ be two DHFEs, $d_i^\circ = Score(h_i) - Score(g_i)$ be the score function of d_i, and $d_i^* = Score(h_i) + Score(g_i)$ be the accuracy function of d_i, then we define

(i) if $d_1^\circ < d_2^\circ$, then d_1 is inferior to d_2, denoted by $d_1 \prec d_2$;
(ii) if $d_1^\circ = d_2^\circ$, then
  (1) if $d_1^* = d_2^*$, then d_1 is indifferent to d_2, denoted by $d_1 \sim d_2$;
  (2) if $d_1^* < d_2^*$, then d_1 is inferior to d_2, denoted by $d_1 \prec d_2$.

For the DHFEs d_1 and d_2 in Example 1, we have $d_1^\circ = -0.5333 < -0.2333 = d_2^\circ$. Thus, d_1 is inferior to d_2, which is in line with human's intuition.

3. Dual Hesitant Fuzzy t-Norm and t-Conorm

The aim of this section is to define dual hesitant fuzzy t-norms and t-conorms and study a special class of dual hesitant fuzzy t-norms and t-conorms in detail, which are very helpful for the development of prioritized dual hesitant fuzzy aggregation operators.

From Definition 6, we know that for two different DHFEs, the numbers of membership degrees (or non-membership degrees) may be different. Hence, a normalization process should be considered. Inspired by the normalization method for HFEs (Xu and Xia, 2011), below we normalize them by adding membership degrees (or non-membership degrees):

DEFINITION 11. Assume that $d = (h, g)$ is a DHFE, then $\overline{h} = (1 - \tau)\sigma_h(1) + \tau\sigma_h(\#h)$ and $\overline{g} = \tau\sigma_g(1) + (1 - \tau)\sigma_g(\#g)$ are called the added membership and non-membership degrees, respectively, where τ $(0 \leqslant \tau \leqslant 1)$ is a parameter provided by the decision maker according to his/her risk preference.

Thus, for two DHFEs with different numbers of membership degrees (or non-membership degrees), we may employ the parameter τ to normalize them such that they have the same number of membership degrees (or non-membership degrees). For example, for two DHFEs $d_1 = \{\{0.1, 0.2\}, \{0.4, 0.5, 0.6\}\}$ and $d_2 = \{\{0.2, 0.3, 0.4, 0.6\}, \{0.3, 0.4\}\}$, it is obvious that $\#h_1 < \#h_2$ and $\#g_2 < \#g_1$. In Definition 11, let $\tau = 1$, then in this case, $\overline{h_1} = 0 \times \sigma_{h_1}(1) + 1 \times \sigma_{h_1}(\#h_1) = 0.2$ and $\overline{g_2} = 1 \times \sigma_{g_2}(1) + 0 \times \sigma_{g_2}(\#g_2) = 0.3$. Thus, we can extend d_1 by adding 0.2 into its membership degree set h_1 and extend d_2 by adding 0.3 into its non-membership degree set g_2. As a result, d_1 becomes $\{\{0.1, 0.2, 0.2, 0.2\}, \{0.4, 0.5, 0.6\}\}$ and d_2 becomes $\{\{0.2, 0.3, 0.4, 0.6\}, \{0.3, 0.3, 0.4\}\}$. In general, the parameter τ is determined by the decision maker used to reflect his/her risk attitude. Clearly, if $\tau = 1$, then the largest membership degree and the smallest non-membership degree are added, respectively, which means that the decision maker is optimistic. For other cases, for example, $\tau = 0$ indicates that the decision maker is pessimistic, and $\tau = 0.5$ indicates that the decision maker is indifferent. In this paper, we only consider the case in which the decision makers are optimists, and other cases can be studied in a similar way.

For facilitating the definitions of dual hesitant fuzzy t-norms and t-conorms, a partial order "$\leqslant_D$" on the set of all DHFEs D is defined as follows:

Definition 12. Let $d_i = (h_i, g_i)$, $i = 1, 2$ be two DHFEs with $\#h_1 = \#h_2 = k$ and $\#g_1 = \#g_2 = l$, then a partial order between d_1 and d_2 is defined as:

$$d_1 \leqslant_D d_2 \text{ iff } \sigma_{h_1}(t) \leqslant \sigma_{h_2}(t) \quad \text{for any } t = 1, 2, \dots, k \quad \text{and}$$
$$\sigma_{g_1}(t) \geqslant \sigma_{g_2}(t) \quad \text{for any } t = 1, 2, \dots, l.$$

Note 1. For two DHFEs with different numbers of membership degrees (or non-membership degrees), the partial order "$\leqslant_D$" is defined between their normalization forms (see Definition 11).

Example 2. Consider two DHFEs $d_1 = \{\{0.5, 0.6, 0.8\}, \{0.1, 0.2\}\}$ and $d_2 = \{\{0.4, 0.6\}, \{0.2, 0.3, 0.4\}\}$. Through normalization, d_1 and d_2 become $d_1 = \{\{0.5, 0.6, 0.8\}, \{0.1, 0.1, 0.2\}\}$ and $d_2 = \{\{0.4, 0.6, 0.6\}, \{0.2, 0.3, 0.4\}\}$, respectively. Thus, by Definition 12, we get $d_2 \leqslant_D d_1$.

For two DHFEs $d_i = (h_i, g_i)$, $i = 1, 2$ with $\#h_1 = \#h_2 = k$ and $\#g_1 = \#g_2 = l$, we define the meet-operation as $d_1 \wedge d_2 = \{\{\sigma_{h_1}(t) \wedge \sigma_{h_2}(t) | t = 1, 2, \dots, k\}, \{\sigma_{g_1}(t) \vee \sigma_{g_2}(t) | t = 1, 2, \dots, l\}\}$ and the joint-operation as $d_1 \vee d_2 = \{\{\sigma_{h_1}(t) \vee \sigma_{h_2}(t) | t = 1, 2, \dots, k\}, \{\sigma_{g_1}(t) \wedge \sigma_{g_2}(t) | t = 1, 2, \dots, l\}\}$. Then for any DHFE d, it can be easily deduced that $d \wedge 0_D = 0_D$ and $d \vee 1_D = 1_D$, where $0_D = \{\{0\}, \{1\}\}$ and $1_D = \{\{1\}, \{0\}\}$. Therefore, the set D with the partial order "$\leqslant_D$" is a bounded lattice, denoted as $(D, \leqslant_D)$, whose smallest and largest elements are $0_D = \{\{0\}, \{1\}\}$ and $1_D = \{\{1\}, \{0\}\}$, respectively.

Now with the help of the bounded lattice $(D, \leqslant_D)$, we can straightforwardly extend the definitions of t-norms and t-conorms to the dual hesitant fuzzy setting.

Definition 13. A dual hesitant fuzzy t-norm (DHF t-norm) is a mapping $DT : D^2 \to D$ having the following properties:

(i) Commutativity: $DT(d_1, d_2) = DT(d_2, d_1)$;
(ii) Associativity: $DT(d_1, DT(d_2, d_3)) = DT(DT(d_1, d_2), d_3)$;
(iii) Monotonicity: if $d_1 \leqslant_D d_3$ and $d_2 \leqslant_D d_4$, then $DT(d_1, d_2) \leqslant_D DT(d_3, d_4)$;
(iv) 1_D as identity: $DT(1_D, d_1) = d_1$.

A dual hesitant fuzzy t-conorm (DHF t-conorm) is a mapping $DS : D^2 \to D$ possessing the following properties:

(i) Commutativity: $DS(d_1, d_2) = DS(d_2, d_1)$;
(ii) Associativity: $DS(d_1, DS(d_2, d_3)) = DS(DS(d_1, d_2), d_3)$;

(iii) Monotonicity: if $d_1 \leqslant_D d_3$ and $d_2 \leqslant_D d_4$, then $DS(d_1, d_2) \leqslant_D DS(d_3, d_4)$;
(iv) 0_D as identity: $DS(0_D, d_1) = d_1$,

where $d_1, d_2, d_3, d_4 \in D$.

The associativity property of DHF t-norms and t-conorms permits us to extend them to any number of dual hesitant fuzzy arguments, i.e. given n DHFEs d_i, $i = 1, 2, \ldots, n$, we have $DT(d_1, \ldots, d_n) = DT(DT(d_1, \ldots, d_{n-1}), d_n)$ and $DS(d_1, \ldots, d_n) = DS(DS(d_1, \ldots, d_{n-1}), d_n)$. Besides, we get the notable properties: $DT(d_1, \ldots, d_n, d_{n+1}) \leqslant_D DT(d_1, \ldots, d_n)$ and $DS(d_1, \ldots, d_n) \leqslant_D DS(d_1, \ldots, d_n, d_{n+1})$ for $n \geqslant 2$.

Let $d_i = (h_i, g_i)$, $i = 1, 2$ be two DHFEs with $\#h_1 = \#h_2 = k$ and $\#g_1 = \#g_2 = l$, then some typical examples of DHF t-norms are listed as follows:

1) Dual hesitant fuzzy minimum (DHFM) t-norm:

$$\begin{aligned} DT_{T_M,S_M}(d_1, d_2) &= d_1 \wedge d_2 \\ &= \big\{\big\{\sigma_{h_1}(t) \wedge \sigma_{h_2}(t) \big| t = 1, 2, \ldots, k\big\}, \big\{\sigma_{g_1}(t) \vee \sigma_{g_2}(t) \big| t = 1, 2, \ldots, l\big\}\big\}. \end{aligned} \tag{2}$$

2) Dual hesitant fuzzy product (DHFP) t-norm:

$$\begin{aligned} DT_{T_P,S_S}(d_1, d_2) &= d_1 \otimes d_2 \\ &= \big\{\big\{\sigma_{h_1}(t)\sigma_{h_2}(t) \big| t = 1, 2, \ldots, k\big\}, \\ &\quad \big\{\sigma_{g_1}(t) + \sigma_{g_2}(t) - \sigma_{g_1}(t)\sigma_{g_2}(t) \big| t = 1, 2, \ldots, l\big\}\big\}. \end{aligned} \tag{3}$$

3) Dual hesitant fuzzy Lukasiewicz (DHFL) t-norm:

$$\begin{aligned} DT_{T_L,S_L}(d_1, d_2) &= \big\{\big\{(\sigma_{h_1}(t) + \sigma_{h_2}(t) - 1) \vee 0 \big| t = 1, 2, \ldots, k\big\}, \\ &\quad \big\{(\sigma_{g_1}(t) + \sigma_{g_2}(t)) \wedge 1 \big| t = 1, 2, \ldots, l\big\}\big\}. \end{aligned} \tag{4}$$

Three typical examples of DHF t-conorms are presented as follows:

1) Dual hesitant fuzzy maximum (DHFM) t-conorm:

$$\begin{aligned} DS_{S_M,T_M}(d_1, d_2) &= d_1 \vee d_2 \\ &= \big\{\big\{\sigma_{h_1}(t) \vee \sigma_{h_2}(t) \big| t = 1, 2, \ldots, k\big\}, \\ &\quad \big\{\sigma_{g_1}(t) \wedge \sigma_{g_2}(t) \big| t = 1, 2, \ldots, l\big\}\big\}. \end{aligned} \tag{5}$$

2) Dual hesitant fuzzy probabilistic sum (DHFPS) t-conorm:

$$\begin{aligned} DS_{S_S,T_P}(d_1, d_2) &= d_1 \oplus d_2 \\ &= \big\{\big\{\sigma_{h_1}(t) + \sigma_{h_2}(t) - \sigma_{h_1}(t)\sigma_{h_2}(t) \big| t = 1, 2, \ldots, k\big\}, \\ &\quad \big\{\sigma_{g_1}(t)\sigma_{g_2}(t) \big| t = 1, 2, \ldots, l\big\}\big\}. \end{aligned} \tag{6}$$

3) Dual hesitant fuzzy Lukasiewicz (DHFL) t-conorm:

$$\begin{aligned} DS_{S_L,T_L}(d_1,d_2) = \Big\{&\big\{(\sigma_{h_1}(t)+\sigma_{h_2}(t))\wedge 1\big|t=1,2,\dots,k\big\},\\ &\big\{(\sigma_{g_1}(t)+\sigma_{g_2}(t)-1)\vee 0\big|t=1,2,\dots,l\big\}\Big\}. \end{aligned} \tag{7}$$

Note 2. The above-mentioned dual hesitant fuzzy t-norms and t-conorms are all defined on two DHFEs with the same number of membership degrees and the same number of non-membership degrees. For two DHFEs with different numbers of membership degrees (or non-membership degrees), we may define dual hesitant fuzzy t-norms and t-conorms on their normalization forms (see Definition 11).

To date, we have introduced the definitions of DHF t-norms and t-conorms, which are consistent with those of traditional t-norms and t-conorms (similar work can be found in Deschrijver *et al.*, 2004; Dimuro *et al.*, 2011). In what follows, we research a special class of DHF t-norms and t-conorms, which are constructed by traditional t-norms and t-conorms and will play a crucial role in the development of prioritized dual hesitant fuzzy aggregation operators.

Theorem 1. *Let $T, S : [0,1]^2 \to [0,1]$ be a t-norm and a t-conorm, respectively, such that $T(x,y)+S(1-x,1-y)\leqslant 1$ for $x,y\in[0,1]$. Then the mapping DT defined by*

$$\begin{aligned} DT_{T,S}(d_1,d_2) = \Big\{&\big\{T(\sigma_{h_1}(t),\sigma_{h_2}(t))\big|t=1,2,\dots,k\big\},\\ &\big\{S(\sigma_{g_1}(t),\sigma_{g_2}(t))\big|t=1,2,\dots,l\big\}\Big\} \end{aligned} \tag{8}$$

is a DHF t-norm, and the mapping DS defined by

$$\begin{aligned} DS_{S,T}(d_1,d_2) = \Big\{&\big\{S(\sigma_{h_1}(t),\sigma_{h_2}(t))\big|t=1,2,\dots,k\big\},\\ &\big\{T(\sigma_{g_1}(t),\sigma_{g_2}(t))\big|t=1,2,\dots,l\big\}\Big\} \end{aligned} \tag{9}$$

is a DHF t-conorm, where $d_i=(h_i,g_i)\in D$, $i=1,2$ with $\#h_1=\#h_2=k$ and $\#g_1=\#g_2=l$.

Theorem 1 can be easily proven by the definitions of t-norms and t-conorms. Obviously, the DHF t-norms and t-conorms respectively presented by Eqs. (2)–(4) and Eqs. (5)–(6) can be formed through Theorem 1.

Property 1. *Let $DT_{T,S}$ and $DS_{S,T}$ be a DHF t-norm and t-conorm constructed through Theorem 1, respectively, then for any two DHFEs d_i, $i=1,2$, we have*

(i) *$DT_{T,S}(0_D,d_1)=0_D$ and $DS_{S,T}(1_D,d_1)=1_D$;*
(ii) *if for $i=1,2$, $d_i=(h_i,g_i)$ with $h_i\neq\phi$ and $g_i\neq\phi$, then $(DT_{T,S}(d_1,d_2))^c = DS_{S,T}(d_1^c,d_2^c)$ and $(DS_{S,T}(d_1,d_2))^c=DT_{T,S}(d_1^c,d_2^c)$;*

(iii) $DT_{T,S}(d_1, d_2) \leqslant_D DT_{T_M,S_M}(d_1, d_2)$ *and* $DS_{S_M,T_M}(d_1, d_2) \leqslant_D DS_{S,T}(d_1, d_2)$, *i.e.* DT_{T_M,S_M} *is the largest DHF t-norm among all* $DT_{T,S}$*, and* DS_{S_M,T_M} *is the smallest DHF t-conorm among all* $DS_{S,T}$.

Proof. (1) Since for any t-norm T and t-conorm S, it holds that $T(0, a) = 0$ and $S(1, a) = 1$ for $a \in [0, 1]$. Then

$$DT_{T,S}(0_D, d_1) = \big\{\big\{T(0, \sigma_{h_1}(t))\big|t = 1, 2, \dots, \#h_1\big\}, \big\{S(1, \sigma_{g_1}(t))\big|t = 1, 2, \dots, \#g_1\big\}\big\} = 0_D$$

and

$$DS_{S,T}(1_D, d_1) = \big\{\big\{S(1, \sigma_{h_1}(t))\big|t = 1, 2, \dots, \#h_1\big\}, \big\{T(0, \sigma_{g_1}(t))\big|t = 1, 2, \dots, \#g_1\big\}\big\} = 1_D$$

$$\begin{aligned}(2)\ (DT_{T,S}(d_1, d_2))^c &= \big(\big\{\big\{T(\sigma_{h_1}(t), \sigma_{h_2}(t))\big|t = 1, 2, \dots, k\big\}, \big\{S(\sigma_{g_1}(t), \sigma_{g_2}(t))\big|t = 1, 2, \dots, l\}\big\}\big)^c \\ &= \big\{\big\{S(\sigma_{g_1}(t), \sigma_{g_2}(t))\big|t = 1, 2, \dots, l\big\}, \big\{T(\sigma_{h_1}(t), \sigma_{h_2}(t))\big|t = 1, 2, \dots, k\big\}\big\} = DS_{S,\,T}\big(d_1^c, d_2^c\big).\end{aligned}$$

Similarly, it can be proven that $(DS_{S,\,T}(d_1, d_2))^c = DT_{T,\,S}(d_1^c, d_2^c)$.

(3) Since for any t-norm T and t-conorm S, it holds that $T(a, b) \leqslant T_M(a, b)$ and $S_M(a, b) \leqslant S(a, b)$ for $a, b \in [0, 1]$. Then, $T(\sigma_{h_1}(t), \sigma_{h_2}(t)) \leqslant \sigma_{h_1}(t) \wedge \sigma_{h_2}(t)$ for $t = 1, 2, \dots, k$, and $S(\sigma_{g_1}(t), \sigma_{g_2}(t)) \geqslant \sigma_{g_1}(t) \vee \sigma_{g_2}(t)$ for $t = 1, 2, \dots, l$. Therefore, $DT_{T,\,S}(d_1, d_2) \leqslant_D DT_{T_M,\,S_M}(d_1, d_2)$. In a similar way, we prove $DS_{S_M,\,T_M}(d_1, d_2) \leqslant_D DS_{S,\,T}(d_1, d_2)$. □

4. Prioritized Dual Hesitant Fuzzy Multi-Attribute Decision Making

In this section, we focus on the resolution of dual hesitant fuzzy MADM problems in which there exists a prioritization relationship over attributes. We first describe this kind of problems as follows: suppose that there are several attributes c_j $(j = 1, 2, \dots, m)$ according to which a decision organization containing several experts assesses several feasible alternatives x_i $(i = 1, 2, \dots, n)$ by using the DHFEs $d_j(x_i)$. Assume that there exists the prioritization relationship over the attributes: $c_1 \succ c_2 \succ \cdots \succ c_m$, i.e. the attribute c_r has a higher priority than the attribute c_j if $r < j$. The goal of the decision maker is to get the ranking of all feasible alternatives.

A simple method to solve the aforesaid dual hesitant fuzzy MADM problems with prioritized attributes is to conduct an aggregation for the dual hesitant fuzzy evaluation values of each alternative under all attributes to obtain a collective value, and then to compare the collective values of all alternatives to obtain a ranking of alternatives. However, from the introduction, we note that no studies have been done on the aggregation of dual hesitant fuzzy information in the MADM problems with prioritized attributes. Therefore, below we attempt to develop prioritized dual hesitant fuzzy aggregation operators based on the DHF t-norms and t-conorms defined in the former section.

4.1. *The Prioritized Dual Hesitant Fuzzy Weighted Triangular Operator*

In this subsection, we pay attention to the aggregation of dual hesitant fuzzy information in the MADM problems where there exists a prioritization relationship over attributes. Before that, we first introduce the following definition:

Definition 14. A dual hesitant fuzzy aggregation function (DHFAF) of dimension n is a mapping $DF : D^n \to D$ with the following conditions:

(i) $DF(0_D, 0_D, \dots, 0_D) = 0_D$ and $DF(1_D, 1_D, \dots, 1_D) = 1_D$;
(ii) $DF(d_1, d_2, \dots, d_n) \leqslant_D DF(d_1^{'}, d_2^{'}, \dots, d_n^{'})$ for $d_i \leqslant_D d_i^{'}$, $i = 1, 2, \dots, n$.

The conditions (i) and (ii) can be interpreted as follows: the condition (i) indicates that when the aggregated dual hesitant fuzzy arguments are all the smallest element 0_D of the lattice $(D, \leqslant_D)$, their aggregation result should be 0_D; analogously, when the aggregated dual hesitant fuzzy arguments are all the largest element 1_D of the lattice $(D, \leqslant_D)$, the obtained aggregation result should be 1_D; the condition (ii) says that the aggregation result should not decrease with the individual dual hesitant fuzzy argument increasing regarding the partial order $\leqslant_D$, that is, an aggregation function on D should be non-decreasing in relation to the aggregated dual hesitant fuzzy arguments.

It can be easily seen that DHF t-norms and t-conorms depicted in Theorem 1 are binary DHFAFs. In the following, we develop another DHFAF for aggregating the dual hesitant fuzzy information in the MADM problems with prioritized attributes. The key to developing such a function is to determine the weights of attributes, by which the prioritization relationship over attributes can be reflected. For the alternative x_i $(i = 1, 2, \dots, n)$, based on the DHF t-norm $DT_{T,S}$ (constructed through Theorem 1), we define the priority weight of the attribute c_j as:

$$w_j(x_i) = \begin{cases} 1_D, & j = 1, \\ DT_{T,\,S}(d_0(x_i), d_1(x_i), \dots, d_{j-1}(x_i)), & j = 2, 3, \dots, m, \end{cases} \tag{10}$$

where $d_0(x_i) = 1_D$. Then, based on the DHF t-norm DT_{T_1,S_1} and DHF t-conorm DS_{S_2,T_2}(constructed through Theorem 1), we define the prioritized dual hesitant fuzzy weighted triangular (PDHFWT) operator as:

$$\begin{aligned} F(x_i) &= \mathrm{PDHFWT}_{w(x_i)}\big(d_1(x_i), \dots, d_m(x_i)\big) \\ &= DS_{S_2,T_2}\big(DT_{T_1,S_1}\big(w_1(x_i), d_1(x_i)\big), \dots, DT_{T_1,S_1}\big(w_m(x_i), d_m(x_i)\big)\big). \end{aligned} \tag{11}$$

From Eqs. (10) and (11), we can easily observe that: (i) the weights $w_j(x_i)$ $(i = 1, 2, \dots, n,\ j = 1, 2, \dots, m)$ are all DHFEs; (ii) for $i = 1, 2, \dots, n$, the weights $w_j(x_i)$ are non-increasing in j, i.e., $w_j(x_i) \leqslant_D w_r(x_i)$ if $r < j$; (iii) for $i = 1, 2, \dots, n$, if $d_1(x_i) =$

0_D, then $w_1(x_i) = 1_D$ and $w_j(x_i) = 0_D$ for $j > 1$, and in this case, $F(x_i) = 0_D$, that is, for any alternative x_i, 0_D satisfaction to the first prioritized attribute c_1 always results in the overall aggregation result $F(x_i) = 0_D$; furthermore, if $d_j(x_i) = 0_D$, then $w_r(x_i) = 0_D$ for $r > j$, and $F(x_i) = DS_{S_2,T_2}(DT_{T_1,S_1}(w_1(x_i), d_1(x_i)), \ldots, DT_{T_1,S_1}(w_{j-1}(x_i), d_{j-1}(x_i)))$, which illustrates that 0_D satisfaction to some attribute c_j leads to no contributions made by the attributes with lower priority than c_j to the overall aggregation result, that is, the lack of satisfaction to the higher prioritized attributes cannot be compensated by satisfaction to the lower prioritized attributes (i.e., the PDHFWT operator has the property of non-compensation); (iv) for different alternatives x_i and $x_k (k \neq i)$, the weights $w_j(x_i)$ and $w_j(x_k)$ are usually different, that is, the yielded weights rely on the alternatives x_i $(i = 1, 2, \ldots, n)$.

All the above say that the developed PDHFWT operator can well capture the prioritization relationship over attributes. Furthermore, we get:

Theorem 2. *The PDHFWT operator is a DHFAF.*

Theorem 2 is straightforward.

In Eqs. (10) and (11), when different DHF t-norms and t-conorms are employed, different forms of prioritized dual hesitant fuzzy aggregation operators can be obtained. A specific one is shown as follows: assume that for the alternatives x_i $(i = 1, 2, \ldots, n)$, $d_j(x_i) = (h_{ij}, g_{ij})$ with $\#h_{ij} = k_i$ and $\#g_{ij} = l_i$ for $j = 1, 2, \ldots, m$. Let $DT_{T,S} = DT_{T_1,S_1} = DT_{T_P,S_S}$, shown as Eq. (3), and $DS_{S_2,T_2} = DS_{S_S,T_P}$, shown as Eq. (6). Then by Eq. (10), we obtain

$$\begin{aligned} w_j(x_i) &= DT_{T_P,\,S_S}(d_0(x_i), d_1(x_i), \ldots, d_{j-1}(x_i)) \\ &= \left\{ \left\{ \prod_{r=1}^{j-1} \sigma_{h_{ir}}(t) \middle| t = 1, 2, \ldots, k_i \right\}, \right. \\ &\qquad \left. \left\{ 1 - \prod_{r=1}^{j-1} \big(1 - \sigma_{g_{ir}}(t)\big) \middle| t = 1, 2, \ldots, l_i \right\} \right\} \end{aligned} \tag{12}$$

for $j = 2, 3, \ldots, m$. Moreover, by Eq. (11), we acquire

$$\begin{aligned} F(x_i) &= DS_{S_S,T_P}\big(DT_{T_P,S_S}(w_1(x_i), d_1(x_i)), \ldots, DT_{T_P,\,S_S}\big(w_m(x_i), d_m(x_i)\big)\big) \\ &= \left\{ \left\{ 1 - \prod_{j=1}^{m} \left(1 - \prod_{r=1}^{j} \sigma_{h_{ir}}(t) \right) \middle| t = 1, 2, \ldots, k_i \right\}, \right. \\ &\qquad \left. \left\{ \prod_{j=1}^{m} \left(1 - \prod_{r=1}^{j} (1 - \sigma_{g_{ir}}(t)) \right) \middle| t = 1, 2, \ldots, l_i \right\} \right\}. \end{aligned} \tag{13}$$

We illustrate the developed PDHFWT operator by the following example:

EXAMPLE 3. Consider the prioritized attributes $c_1 \succ c_2 \succ c_3$, and assume that for the alternative x the evaluation values under the attributes c_j $(j = 1, 2, 3)$ provided by a decision organization are DHFEs:

$$d_1(x) = \{\{0.4, 0.6\}, \{0.1, 0.3, 0.4\}\}, \qquad d_2(x) = \{\{0.5\}, \{0.2, 0.5\}\},$$
$$d_3(x) = \{\{0.5, 0.7\}, \{0.2\}\}.$$

Firstly, by Eq. (12), the priority weights $w_j(x)$ of the attributes c_j $(j = 1, 2, 3)$ can be calculated:

$$w_1(x) = \{\{1\}, \{0\}\}, \qquad w_2(x) = \{\{0.4, 0.6\}, \{0.1, 0.3, 0.4\}\},$$
$$w_3(x) = \{\{0.2, 0.3\}, \{0.28, 0.44, 0.7\}\}.$$

Evidently, $w_3(x) \leqslant_D w_2(x) \leqslant_D w_1(x)$, which accords with the prioritization relationship over the three attributes.

Then, by Eq. (13), we obtain the overall attribute value of the alternative x, shown as:

$$F(x) = \{\{0.568, 0.7788\}, \{0.0119, 0.0729, 0.2128\}\}.$$

Since the DHFS is a generalization of the IFS and the HFS, below we apply the developed PDHFWT operator shown in Eqs. (12) and (13) to the intuitionistic fuzzy and hesitant fuzzy environments.

In the intuitionistic fuzzy context, by Eq. (12), the priority weight of the attribute c_j regarding the alternative x_i is given as:

$$w_j(x_i) = \left(\prod_{r=1}^{j-1} u_{ir}, 1 - \prod_{r=1}^{j-1}(1 - v_{ir})\right), \quad \text{for } j = 2, 3, \ldots, m, \tag{14}$$

where $w_1(x_i) = (1, 0)$, and $\alpha_j(x_i) = (u_{ij}, v_{ij})$ is an IFN representing the attribute value of the alternative x_i under the attribute c_j. Then, by Eq. (13), we get the following prioritized intuitionistic fuzzy weighted triangular (PIFWT) operator:

$$\begin{aligned}\alpha(x_i) &= \text{PIFWT}_{w(x_i)}\big(\alpha_1(x_i), \alpha_2(x_i), \ldots, \alpha_m(x_i)\big) \\ &= \left(1 - \prod_{j=1}^{m}\left(1 - \prod_{r=1}^{j} u_{ir}\right), \prod_{j=1}^{m}\left(1 - \prod_{r=1}^{j}(1 - v_{ir})\right)\right).\end{aligned} \tag{15}$$

In the hesitant fuzzy context, by Eq. (12), the priority weight of the attribute c_j with respect to the alternative x_i is given as:

$$w_j(x_i) = \left\{\prod_{r=1}^{j-1} \sigma_{h_{ir}}(t) \,\middle|\, t = 1, 2, \ldots, k_i\right\}, \quad \text{for } j = 2, 3, \ldots, m, \tag{16}$$

Table 1
Dual hesitant fuzzy attribute values.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	$\{\{0.5, 0.7\}, \{0.1, 0.2, 0.3\}\}$	$\{\{0.3, 0.5, 0.6\}, \{0.2, 0.4\}\}$	$\{\{0.1, 0.2\}, \{0.5, 0.6\}\}$	$\{\{0.8, 0.9\}, \{0.1\}\}$	$\{\{0.6, 0.8\}, \{0.1, 0.2\}\}$	$\{\{0.1, 0.3\}, \{0.5, 0.7\}\}$
x_2	$\{\{0.4\}, \{0.2, 0.3\}\}$	$\{\{0.6, 0.7\}, \{0.1\}\}$	$\{\{0.6, 0.7, 0.8\}, \{0.1, 0.2\}\}$	$\{\{0.5, 0.7\}, \{0.1\}\}$	$\{\{0.5, 0.6, 0.7\}, \{0.1, 0.2\}\}$	$\{\{0.2\}, \{0.6, 0.7\}\}$
x_3	$\{\{0.3, 0.4\}, \{0.1, 0.2\}\}$	$\{\{0.6, 0.8\}, \{0.1\}\}$	$\{\{0.4, 0.6\}, \{0.4\}\}$	$\{\{0.6, 0.7\}, \{0.2, 0.3\}\}$	$\{\{0.8, 0.9\}, \{0.1\}\}$	$\{\{0.4, 0.6\}, \{0.3, 0.4\}\}$
x_4	$\{\{0.6, 0.7\}, \{0.1, 0.2\}\}$	$\{\{0.6, 0.8\}, \{0.1\}\}$	$\{\{0.2, 0.3\}, \{0.4, 0.6\}\}$	$\{\{0.7, 0.8\}, \{0.2\}\}$	$\{\{0.4, 0.5, 0.6\}, \{0.4\}\}$	$\{\{0.1, 0.2\}, \{0.4, 0.5\}\}$
x_5	$\{\{0.4, 0.5\}, \{0.3\}\}$	$\{\{0.4, 0.6\}, \{0.1, 0.2, 0.3\}\}$	$\{\{0.5, 0.6\}, \{0.2, 0.4\}\}$	$\{\{0.8, 0.9\}, \{0.1\}\}$	$\{\{0.4\}, \{0.5, 0.6\}\}$	$\{\{0.6, 0.7\}, \{0.3\}\}$

where $w_1(x_i) = \{1\}, h_j(x_i) = \{\sigma_{h_{ij}}(t) | t = 1, 2, \ldots, k_i\}$ is a HFE representing the possible attribute values of the alternative x_i under the attribute c_j. Then, by Eq. (13), we obtain the following prioritized hesitant fuzzy weighted triangular (PHFWT) operator:

$$\begin{aligned} h(x_i) &= \mathrm{PHFWT}_{w(x_i)}\big(h_1(x_i), h_2(x_i), \ldots, h_m(x_i)\big) \\ &= \left\{1 - \prod_{j=1}^{m}\left(1 - \prod_{r=1}^{j}\sigma_{h_{ir}}(t)\right) \middle| t = 1, 2, \ldots, k_i\right\}. \end{aligned} \tag{17}$$

4.2. *A Method to Solve Dual Hesitant Fuzzy MADM Problems with Prioritized Attributes*

In the previous section, we have developed the PDHFWT operator by using the DHF t-norms and t-conorms, which manages to capture the prioritization relationship over attributes, and proved that it is essentially a DHFAF, which indicates that the PDHFWT operator can be applied to aggregate the dual hesitant fuzzy information in MADM problems. Based on the PDHFWT operator, we here provide a method to handle the dual hesitant fuzzy MADM problems with prioritized attributes, whose procedures are presented as follows:

Step 1. For the alternatives x_i $(i = 1, 2, \ldots, n)$, we calculate the priority weights $w_j(x_i)$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ of the attributes c_j $(j = 1, 2, \ldots, m)$ by Eq. (12).

Step 2. By Eq. (13), we calculate the overall attribute values $F(x_i)$ $(i = 1, 2, \ldots, n)$ of alternatives x_i $(i = 1, 2, \ldots, n)$.

Step 3. Rank $F(x_i)$, $i = 1, 2, \ldots, n$, according to Definition 10, and then the corresponding ranking orders of alternatives x_i $(i = 1, 2, \ldots, n)$ can be obtained.

Example 4 (See Yu and Xu, 2013). Assume that a consumer wants to buy a new car from five cars $x_1, x_2, \ldots, x_5$. Since he/she has poor understanding about cars, he/she invites three friends who are proficient in cars to assess these cars by considering six attributes: 1) safety level c_1; 2) price c_2; 3) fuel consumption c_3; 4) comfort level c_4; 5) maximum gas mileage c_5; 6) fashion c_6. In the consumer's opinion, there exists the prioritization relationship among these attributes: $c_1 \succ c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_6$. Suppose that the assessment values $d_j(x_i)$ of cars x_i $(i = 1, 2, \ldots, 5)$ with regard to the attributes c_j $(j = 1, 2, \ldots, 6)$ provided by the three friends can be represented by DHFEs, see Table 1.

Obviously, the above problem is a dual hesitant fuzzy MADM problem with prioritized attributes. In the following, we adopt the proposed method to solve it.

Table 2
Dual hesitant fuzzy priority weights of first three attributes.

	w_1	w_2	w_3
x_1	{{1}, {0}}	{{0.5, 0.7}, {0.1, 0.2, 0.3}}	{{0.15, 0.35, 0.42}, {0.28, 0.36, 0.58}}
x_2	{{1}, {0}}	{{0.4}, {0.2, 0.3}}	{{0.24, 0.28}, {0.28, 0.37}}
x_3	{{1}, {0}}	{{0.3, 0.4}, {0.1, 0.2}}	{{0.18, 0.32}, {0.19, 0.28}}
x_4	{{1}, {0}}	{{0.6, 0.7}, {0.1, 0.2}}	{{0.36, 0.56}, {0.19, 0.28}}
x_5	{{1}, {0}}	{{0.4, 0.5}, {0.3}}	{{0.16, 0.3}, {0.37, 0.44, 0.51}}

Table 3
Dual hesitant fuzzy priority weights of last three attributes.

	w_4	w_5	w_6
x_1	{{0.015, 0.07, 0.084}, {0.64, 0.68, 0.832}}	{{0.012, 0.063, 0.0756}, {0.676, 0.712, 0.8488}}	{{0.0072, 0.0504, 0.0605}, {0.7084, 0.7408, 0.8791}}
x_2	{{0.144, 0.196, 0.224}, {0.424, 0.559}}	{{0.072, 0.1372, 0.1568}, {0.4816, 0.6031}}	{{0.036, 0.0823, 0.1098}, {0.5334, 0.6825}}
x_3	{{0.072, 0.192}, {0.514, 0.568}}	{{0.0432, 0.1344}, {0.6112, 0.6976}}	{{0.0346, 0.1209}, {0.6501, 0.7278}}
x_4	{{0.072, 0.168}, {0.514, 0.712}}	{{0.0504, 0.1344}, {0.6112, 0.7696}}	{{0.0202, 0.0672, 0.0806}, {0.7667, 0.8618}}
x_5	{{0.08, 0.18}, {0.496, 0.552, 0.706}}	{{0.064, 0.162}, {0.5464, 0.5968, 0.7354}}	{{0.0256, 0.0648}, {0.7732, 0.7984, 0.8942}}

Firstly, by Eq. (12), we calculate the priority weights $w_j(x_i)$ of the attributes c_j $(j = 1, 2, \dots, 6)$ for the cars x_i $(i = 1, 2, \dots, 5)$, see Tables 2 and 3. Then, by Eq. (13), the overall attribute values of the cars x_i $(i = 1, 2, \dots, 5)$ are calculated, shown as:

$$F(x_1) = \big\{\{0.5897, 0.8411, 0.8641\}, \{0.0073, 0.0225, 0.1041\}\big\},$$
$$F(x_2) = \big\{\{0.6533, 0.7295, 0.7539\}, \{0.0049, 0.0231\}\big\},$$
$$F(x_3) = \big\{\{0.5148, 0.7674\}, \{0.0029, 0.0135\}\big\},$$
$$F(x_4) = \big\{\{0.7794, 0.9125, 0.914\}, \{0.0039, 0.0246\}\big\},$$
$$F(x_5) = \big\{\{0.5836, 0.7853\}, \{0.0196, 0.0298, 0.0658\}\big\}.$$

Finally, according to Definition 10, we obtain the scores of $F(x_i)$ $(i = 1, 2, \dots, 5)$, listed as:

$$d_1^\circ = 0.7499, \quad d_2^\circ = 0.715, \quad d_3^\circ = 0.7171, \quad d_4^\circ = 0.8768, \quad d_5^\circ = 0.4438,$$

from which we get the ranking orders of the five optional cars: $x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$, where "$\succ$" means "prefer to". Therefore, the consumer should buy the fourth car x_4.

It is not hard to find that in the above procedures the priority weights of attributes are indeed induced by the DHFP t-norm, shown as Eq. (3), and the overall attribute value of each car is indeed derived from the DHFP t-norm and DHFPS t-conorm, shown as Eqs. (3) and (6), respectively. Now we consider other DHF t-norms and t-conorms. Firstly, if we adopt the DHFM t-norm and DHFL t-norm, shown as Eqs. (2) and (4), respectively, to induce the priority weights of attributes, then by means of our proposed method, the score of the obtained overall attribute value of each car x_i can be computed, see Table 4.

Moreover, if we use the DHFM t-norm, DHFP t-norm, and DHFL t-norm, shown as Eqs. (2)–(4), respectively, to generate the priority weights of attributes, and the DHFL

Table 4
Scores and rankings of cars with different DHF t-norms inducing priority weights.

DHF t-norms	Scores of cars					Ranking orders
	x_1	x_2	x_3	x_4	x_5	
DHFM t-norm	0.7901	0.8289	0.8041	0.8915	0.8606	$x_4 \succ x_5 \succ x_2 \succ x_3 \succ x_1$
DHFP t-norm	0.7499	0.715	0.7171	0.8768	0.4438	$x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$
DHFL t-norm	0.6351	0.5581	0.5181	0.7995	0.5169	$x_4 \succ x_1 \succ x_2 \succ x_3 \succ x_5$

Table 5
Scores and ranking orders of cars with different DHF t-norms inducing priority weights.

DHF t-norms	Scores of cars					Ranking orders
	x_1	x_2	x_3	x_4	x_5	
DHFM t-norm	0.75	0.6667	0.7	0.9	0.8	$x_4 \succ x_5 \succ x_1 \succ x_3 \succ x_2$
DHFP t-norm	0.8	0.4933	0.4672	0.9	0.54	$x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$
DHFL t-norm	0.8	0.46	0.45	0.9	0.5	$x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$

t-norm and DHFL t-conorm, shown as Eqs. (4) and (7), to aggregate the attribute values, then by our proposed method, the score of the obtained overall attribute value of each car x_i can be computed, see Table 5.

From Tables 4 and 5, it can be clearly seen that the car x_4 is always the best choice no matter which DHF t-norm is applied to generate the priority weights of attributes and whichever DHF t-norm and t-conorm is adopted to aggregate the attribute values of cars. However, the ranking orders of cars are different in different cases. On one hand, the ranking orders of cars are different when different DHF t-norms are used to induce the priority weights of attributes. On the other hand, different ranking orders of cars are generated when different DHF t-norms and t-conorms are used to aggregate the attribute values.

All of these illustrate that different ranking orders of alternatives will be yielded when different DHF t-norms are used to induce the priority weights of attributes or different DHF t-norms and t-cornoms are adopted to aggregate the attribute values of alternatives. Then two important questions arise: which DHF t-norm performs better in generating priority weights, and which DHF t-norm and DHF t-conorm conduct better in aggregating attribute values? Since the solving process is complicated, we will resolve these questions in our future work.

5. Comparative Analyses and Discussions

In this section, two comparisons are conducted to illustrate the validity and superiority of the developed prioritized aggregation operators and the proposed decision making method.

5.1. *Comparative Studies of Prioritized Aggregation Operators*

From the introduction, we note that no studies have been done on the aggregation of dual hesitant fuzzy information in the MADM problems with prioritized attributes. Since IFSs and HFSs are two special cases of DHFSs, below we compare the developed prioritized aggregation operators with those in the intuitionistic fuzzy and hesitant fuzzy environments shown in Yu and Xu (2013), Wei (2012).

(1) A comparison with prioritized intuitionistic fuzzy aggregation operators

Based upon the basic operations of IFNs shown in Definition 3, Yu and Xu (2013) developed the following two kinds of prioritized intuitionistic fuzzy aggregation (PIFA) operators to manage the intuitionistic fuzzy MADM problems with prioritized attributes:

$$\alpha(x_i) = \bigwedge_{j=1}^{m} \big(w_j(x_i) \vee \alpha_j(x_i)\big), \tag{18}$$

$$\alpha(x_i) = \bigotimes_{j=1}^{m} \big(w_j(x_i) \oplus \alpha_j(x_i)\big), \tag{19}$$

where $\alpha_j(x_i)$ is an IFN representing the attribute value of the alternative x_i under the attribute c_j $(j = 1, 2, \dots, m)$ and $w_j(x_i)$ is the priority weight of the attribute c_j regarding the alternative x_i defined by

$$w_j(x_i) = \alpha_0(x_i) \otimes \alpha_1(x_i) \otimes \cdots \otimes \alpha_{j-1}(x_i), \tag{20}$$

where $\alpha_0(x_i) = (1, 0)$.

Assume that there are two prioritized attributes $c_1 \succ c_2$, $\alpha_1(x_i) = (0, 1)$ and $\alpha_2(x_i) = (0.3, 0.5)$. Then, by Eq. (20), we get $w_1(x_i) = (1, 0)$ and $w_2(x_i) = (1, 0) \otimes (0, 1) = (0, 1)$. Furthermore, by Eqs. (18) and (19), we respectively obtain

$$\alpha(x_i) = \big[(1, 0) \vee (0, 1)\big] \wedge \big[(0, 1) \vee (0.3, 0.5)\big] = (0.3, 0.5)$$

and

$$\alpha(x_i) = \big[(1, 0) \oplus (0, 1)\big] \otimes \big[(0, 1) \oplus (0.3, 0.5)\big] = (0.3, 0.5).$$

From the above, we notice that for the alternative x_i, when the attribute value under the first prioritized attribute is (0, 1), neither of the aggregation results derived from Eqs. (18) and (19) is (0, 1), which indicates that there exists compensation between the two attributes. Thus, the PIFA operators shown in Eqs. (18) and (19) do not capture the prioritization relationship between the two attributes and they are not suitable to handle the above intuitionistic fuzzy MADM problems with the prioritized attributes. However, by using our prioritized aggregation operator defined as Eqs. (14) and (15), we get

$$w_1(x_i) = (1, 0), \qquad w_2(x_i) = (0, 1)$$

and

$$\alpha(x_i) = (0, 1),$$

which are rational. Therefore, it can be concluded that our developed prioritized aggregation operators are also applicable for solving the intuitionistic fuzzy MADM problems with prioritized attributes and are more effective than the PIFA operators shown in Yu and Xu (2013).

(2) A comparison with the hesitant fuzzy prioritized weighted average operator

In order to solve the hesitant fuzzy MADM problems with prioritized attributes, Wei (2012) developed the following hesitant fuzzy prioritized weighted average (HFPWA) operator:

$$h(x_i) = \bigoplus_{j=1}^{m} \left(\frac{T_j(x_i) h_{ij}}{\sum_{j=1}^{m} T_j(x_i)} \right) = \bigcup_{\gamma_{i1} \in h_{i1}, \dots, \gamma_{im} \in h_{im}} \left\{ 1 - \prod_{j=1}^{m} (1 - \gamma_{ij})^{\frac{T_j(x_i)}{\sum_{j=1}^{m} T_j(x_i)}} \right\}, \quad (21)$$

where h_{ij} is an HFE representing the possible attribute values of the alternative x_i under the attribute c_j $(j = 1, 2, \dots, m)$ and $T_j(x_i)$ is the priority weight of the attribute c_j regarding the alternative x_i defined by

$$T_j(x_i) = \begin{cases} 1, & j = 1, \\ \prod_{r=1}^{j-1} q(h_{ir}), & j \geqslant 2, \end{cases} \quad (22)$$

where $q(h_{ij})$ is the score of h_{ij} derived from Definition 5.

Below we make a detailed comparison of our developed prioritized aggregation operators with the HFPWA operator through the following example concerning the introduction of overseas outstanding teachers.

Example 5 (See Wei, 2012). To strengthen academic education and promote the building of a teaching body, the school of management in a Chinese university wants to introduce overseas outstanding teachers. This introduction has received great attention from the school. A panel of decision makers constituted by the university president, the dean of management school and the human resource officer will take the whole responsibility for this introduction. They came up with a strict evaluation for five candidates x_i $(i = 1, 2, \dots, 5)$ from the following four aspects: morality c_1, research capability c_2, teaching skill c_3 and educational background c_4. This introduction will be in strict accordance with the principle of combining ability with political integrity. There is a prioritization relationship over the attributes: $c_1 \succ c_2 \succ c_3 \succ c_4$. Each candidate x_i is evaluated using the HFE $h_j(x_i)$ by the panel of decision makers under the attribute c_j, see Table 6.

Below we utilize the proposed method to deal with the above hesitant fuzzy MADM problem with prioritized attributes.

Firstly, by Eq. (16), we calculate the priority weights $w_j(x_i)$ $(i = 1, 2, \dots, 5,\ j = 1, 2, \dots, 4)$ of the attributes c_j $(j = 1, 2, \dots, 4)$ for candidates x_i $(i = 1, 2, \dots, 5)$, see Table 7.

Table 6
Hesitant fuzzy assessment values.

	c_1	c_2	c_3	c_4
x_1	$\{0.4, 0.5, 0.7\}$	$\{0.5, 0.8\}$	$\{0.6, 0.7, 0.9\}$	$\{0.5, 0.6\}$
x_2	$\{0.6, 0.7, 0.8\}$	$\{0.5, 0.6\}$	$\{0.4, 0.6, 0.7\}$	$\{0.4, 0.5\}$
x_3	$\{0.6, 0.8\}$	$\{0.2, 0.3, 0.5\}$	$\{0.4, 0.6\}$	$\{0.5, 0.7\}$
x_4	$\{0.5, 0.6, 0.7\}$	$\{0.4, 0.5\}$	$\{0.8, 0.9\}$	$\{0.3, 0.4, 0.5\}$
x_5	$\{0.6, 0.7\}$	$\{0.5, 0.7\}$	$\{0.7, 0.8\}$	$\{0.2, 0.3, 0.4\}$

Table 7
Hesitant fuzzy priority weights.

	w_1	w_2	w_3	w_4
x_1	$\{1\}$	$\{0.4, 0.5, 0.7\}$	$\{0.2, 0.4, 0.56\}$	$\{0.12, 0.28, 0.504\}$
x_2	$\{1\}$	$\{0.6, 0.7, 0.8\}$	$\{0.3, 0.42, 0.48\}$	$\{0.12, 0.252, 0.336\}$
x_3	$\{1\}$	$\{0.6, 0.8\}$	$\{0.12, 0.24, 0.4\}$	$\{0.048, 0.144, 0.24\}$
x_4	$\{1\}$	$\{0.5, 0.6, 0.7\}$	$\{0.2, 0.3, 0.35\}$	$\{0.16, 0.27, 0.315\}$
x_5	$\{1\}$	$\{0.6, 0.7\}$	$\{0.3, 0.49\}$	$\{0.21, 0.392\}$

Moreover, by Eq. (17), we compute the overall attribute values of the candidates x_i $(i = 1, 2, \dots, 5)$, shown as:

$$h(x_1) = \{0.6029, 0.8203, 0.9543\}, \qquad h(x_2) = \{0.7654, 0.8862, 0.9425\},$$
$$h(x_3) = \{0.6729, 0.8830, 0.9241\}, \qquad h(x_4) = \{0.6801, 0.8177, 0.8875\},$$
$$h(x_5) = \{0.7881, 0.9179, 0.9216\}.$$

Finally, according to Definition 5, the scores of $h(x_i)$ $(i = 1, 2, \dots, 5)$ are obtained:

$$q(h(x_1)) = 0.7925, \qquad q(h(x_2)) = 0.8647, \qquad q(h(x_3)) = 0.8267,$$
$$q(h(x_4)) = 0.7951, \qquad q(h(x_5)) = 0.8759.$$

Consequently, the ranking of the five candidates is $x_5 \succ x_2 \succ x_3 \succ x_4 \succ x_1$.

In order to obtain the ranking of the five candidates, Wei (2012) adopted the HFPWA operator defined as Eqs. (21) and (22) to compute the overall attribute values of candidates, and then rank the candidates according to the scores of the overall attribute values defined in Definition 5. Here we omit the detailed procedure of Wei's method and only list the obtained ranking of candidates: $x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_3$.

It can be easily seen that the ranking of the five candidates derived from our method is slightly different from that yielded by Wei's method. The main reason may be that in the two methods the priority weights of attributes are generated in different ways. In our method, we indeed use a t-norm to induce the priority weights of attributes. In this process, we make full use of the given hesitant fuzzy decision information and avoid the loss of original information; while in Wei's method, we first need to compute the scores of

Table 8
Hesitant fuzzy attribute values.

	c_1	c_2	c_3
x_1	$\{0.2, 0.3, 0.5, 0.8\}$	$\{0.2, 0.3, 0.4\}$	$\{0.4, 0.6\}$
x_2	$\{0.3, 0.45, 0.6\}$	$\{0.2, 0.4\}$	$\{0.3, 0.5, 0.7\}$

Table 9
Real-valued priority weights.

	w_1	w_2	w_3
x_1	1	0.45	0.135
x_2	1	0.45	0.135

Table 10
Hesitant fuzzy priority weights.

	w_1	w_2	w_3
x_1	$\{1\}$	$\{0.2, 0.3, 0.5, 0.8\}$	$\{0.04, 0.09, 0.2, 0.32\}$
x_2	$\{1\}$	$\{0.3, 0.45, 0.6\}$	$\{0.06, 0.18, 0.24\}$

hesitant fuzzy attribute values, which are real numbers, and then calculate the priority weights of attributes by the product of the scores, which are also real numbers. In this course, the given hesitant fuzzy decision information is not fully utilized and just transformed into single real numbers, which leads to the distortion and loss of information and thus results in inaccurate decision results. We shall further illustrate this by the following example:

Suppose that an MADM problem is composed of two alternatives x_1 and x_2, and three attributes c_1, c_2 and c_3 over which there exists the prioritization relationship $c_1 \succ c_2 \succ c_3$. The attribute value of each alternative under each attribute expressed by an HFE is shown in Table 8.

In the following, we shall investigate the priority weights of attributes generated by Wei's method and our method, respectively. At first, we use Wei's method, i.e. Eq. (22), to calculate the priority weights of attributes. For convenience, we denote the hesitant fuzzy attribute values of the alternatives x_i $(i = 1, 2)$ under the attributes c_j $(j = 1, 2, 3)$ as h_{ij} $(i = 1, 2,\ j = 1, 2, 3)$. Then, according to Definition 5, we calculate the scores of all hesitant fuzzy attribute values, shown as:

$$q(h_{11}) = q(h_{21}) = 0.45, \qquad q(h_{12}) = q(h_{22}) = 0.3, \qquad q(h_{13}) = q(h_{23}) = 0.5.$$

Moreover, by Eq. (22), we compute the priority weights of attributes, see Table 9.

Again, we use the developed method, i.e. Eq. (16), to compute the priority weights of attributes and list the results in Table 10.

From Tables 9 and 10, it can be easily observed that by Wei's method, for the alternatives x_1 and x_2, the generated priority weight of each attribute is the same although their assessment information under the attribute is different, which is unreasonable. The reason

Table 11
Real-valued priority weights.

	w_1	w_2	w_3	w_4	w_5	w_6
x_1	1	0.748	0.4543	0.1008	0.0902	0.0742
x_2	1	0.6104	0.5255	0.4322	0.3663	0.2913
x_3	1	0.675	0.5886	0.3269	0.2361	0.2112
x_4	1	0.81	0.7063	0.2373	0.1873	0.1041
x_5	1	0.5984	0.4239	0.2741	0.2453	0.1033

is that in the process of generating priority weights for attributes, the score function is used to transform the given hesitant fuzzy attribute values into single real numbers, which leads to distortion and loss of information. Nevertheless, our method takes full advantage of the given hesitant fuzzy decision information, and for the alternatives x_1 and x_2, the generated priority weights of each attribute are different, except for those of the first prioritized attribute, that is, the generated priority weights are dependent upon the alternatives, which capture the characteristic of the MADM problems with prioritized attributes.

5.2. *Comparative Studies of Prioritized Dual Hesitant Fuzzy MADM Methods*

As mentioned in the introduction, in order to solve the dual hesitant fuzzy MADM problems with prioritized attributes, Ren and Wei (2017) established a decision making method based on a correctional score function and a dice similarity measure of DHFEs. In this part, we make a detailed comparison with this method to illustrate the advantages of our proposed method through the car selection problem shown in Example 4.

Now we adopt Ren and Wei (2017)'s method to deal with the car selection problem described in Example 4. Firstly, by the formulae $w_1(x_i) = 1$ and $w_j(x_i) = \prod_{r=1}^{j-1} S_{D_\alpha}(d_r(x_i))$ for $j \neq 1$, we calculate the priority weights $w_j(x_i)$ $(i = 1, 2, \dots, 5,$ $j = 1, 2, \dots, 6)$ of the attributes c_j $(j = 1, 2, \dots, 6)$ for the cars x_i $(i = 1, 2, \dots, 5)$, see Table 11. Here, $S_{D_\alpha}(d_r(x_i))$ is the correctional score function of the dual hesitant fuzzy attribute value $d_r(x_i)$ (for details about the correctional score function of DHFEs, please see Ren and Wei (2017).

Then, based on the formula $PD(x^*, x_i) = \sum_{j=1}^{6} \frac{w_j(x_i)}{\sum_{j=1}^{6} w_j(x_i)} \frac{2S_{D_\alpha}(d_j^*)S_{D_\alpha}(d_j(x_i))}{S_{D_\alpha}(d_j^*)^2 + S_{D_\alpha}(d_j(x_i))^2}$, we calculate the dice similarity measure between the ideal alternative x^*, defined as $x^* = (d_1^*, d_2^*, \dots, d_6^*)$ with $d_j^* = \{\{1\}, \{0\}\}$, $j = 1, 2, \dots, 6$, and each car x_i $(i = 1, 2, \dots, 5)$, shown as:

$$PD(x^*, x_1) = 0.8264, \quad PD(x^*, x_2) = 0.9059, \quad PD(x^*, x_3) = 0.9301,$$
$$PD(x^*, x_4) = 0.8698, \quad PD(x^*, x_5) = 0.8985.$$

Therefore, the ranking of the five optional cars is $x_3 \succ x_2 \succ x_5 \succ x_4 \succ x_1$, and the consumer should buy the third car x_3.

Obviously, the ranking of the five optional cars derived from Ren and Wei (2017)'s method is markedly different from that derived from our proposed method, which is

$x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$. The reason may be that in the two methods the dual hesitant fuzzy decision information is handled in different ways. In Ren and Wei's method, all original dual hesitant fuzzy decision information is not sufficiently utilized and just translated into single real numbers by the correctional score function, which leads to distortion and loss of information. However, in our method, all original dual hesitant fuzzy decision information is first dealt with by means of a dual hesitant fuzzy t-norm to generate the priority weights of attributes, and then the single dual hesitant fuzzy attribute values are conducted a weighted aggregation by the developed PDHFWT operator to yield the overall attribute values of alternatives. In the whole process, we make the best of the given dual hesitant fuzzy decision information and avoid the loss of information. Furthermore, it is worthwhile to mention that by using our method, the obtained priority weights of attributes and overall attribute values of alternatives are all in the form of DHFEs, which can well embody the dual hesitant essence of dual hesitant fuzzy MADM problems.

From the above two comparisons, it is not hard to see that the developed prioritized aggregation operators can well circumvent the drawbacks of some existing prioritized aggregation operators, and have great superiority in manipulating such dual hesitant fuzzy MADM problems in which there is a prioritization relationship over attributes. The main advantages of the developed prioritized aggregation operators are summarized as follows:

(1) The developed prioritized aggregation operators take full advantage of the given dual hesitant fuzzy decision information and avoid the loss of original information.

(2) By the developed prioritized aggregation operators, the obtained priority weights of attributes and overall attribute values of alternatives take the form of DHFEs, which can well embody the dual hesitant essence of dual hesitant fuzzy MADM problems.

(3) The developed prioritized aggregation operators permit the decision makers to have more choices when choosing aggregation techniques since in these operators, the dual hesitant fuzzy t-norms and t-conorms are changeable.

(4) The developed prioritized aggregation operators are also applicable for solving the intuitionistic fuzzy and hesitant fuzzy MADM problems with prioritized attributes, and are more effective than relevant prioritized aggregation operators.

6. Conclusions

In this paper, we have been concerned with the aggregation of dual hesitant fuzzy information in multi-attribute decision making (MADM) problems with prioritized attributes. Firstly, we have defined dual hesitant fuzzy t-norms and t-conorms and studied a special class of dual hesitant fuzzy t-norms and t-conorms in detail, which are constructed by traditional t-norms and t-conorms. Secondly, we have developed the prioritized dual hesitant fuzzy weighted triangular operator to aggregate dual hesitant fuzzy assessment information of alternatives under the prioritized attributes, from which a family of prioritized dual hesitant fuzzy aggregation operators can be derived. Thirdly, we have applied the developed prioritized aggregation operators to solve the dual hesitant fuzzy MADM problems with prioritized attributes. Finally, by comparative analyses, we have shown that the developed prioritized aggregation operators take full advantage of the given decision information, avoid the loss of original information, and thus yield better final decision results. Moreover, the priority weights of attributes induced by a dual hesitant fuzzy t-norm take the form of dual hesitant fuzzy elements, which can well embody the dual hesitant essence of dual hesitant fuzzy MADM problems. Besides, the developed prioritized aggregation operators are also applicable for handling the intuitionistic fuzzy and hesitant fuzzy MADM problems with prioritized attributes, and are more effective than relevant prioritized aggregation operators.

As shown in Subsection 4.2, when we use different dual hesitant fuzzy t-norms to induce priority weights of attributes or different dual hesitant fuzzy t-norms and t-conorms to aggregate the attribute values of alternatives, different rankings of alternatives will be gotten. Which dual hesitant fuzzy t-norm performs better in generating priority weights, and which dual hesitant fuzzy t-norm and dual hesitant fuzzy t-conorm conducts better in aggregating attribute values? In our future work, we will try to solve these two issues. Moreover, as Zhao *et al.* (2017) and Liu *et al.* (2017) pointed out that in the decision making process different decision makers have different knowledge, experience, culture and educational backgrounds, they always use heterogeneous preference representation structures to express their preferences. In future studies, we will investigate the resolution of heterogeneous multi-attribute group decision making problems, in which the decision information is expressed as hesitant fuzzy elements, dual hesitant fuzzy elements, hesitant fuzzy linguistic term sets and probabilistic linguistic term sets. Besides, the consensus problem is a hot topic in group decision making (Capuano *et al.*, 2017; Del Moral *et al.*, 2018; Herrera-Viedma *et al.*, 2017). In our future work, we attempt to study the consensus reaching model in group decision making based on dual hesitant fuzzy preference relations by considering social influence.

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