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# AUTOREGRESSIVE MODELS OF 2D RANDOM FIELDS

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**Abstract.** The problem of mathematical modelling and simulating of two-dimensional (2D) random fields, using space autoregressive models, is analyzed. Algorithms for the estimation of parameters of models, procedures for finding correlation coefficients and for synthesis of the realizations of given parameter fields are presented.

**Key words:** random fields, space autoregression, autoregressive models, covariation and correlation coefficients.

**1. Introduction.** Solving the problems of modelling and controlling of the objects, having parameters, distributed in space, and random fields, space autoregressive models are widely used. Autoregressive models in two dimensions generating Markov fields as defined by Woods, 1972. Correspondence of autoregressive models to finite difference approximation of partial differential equations, covariance of models and their application in image processing as considered by Jain, 1977. Autoregressive models were used in the research of fields and images (Vittich, Sergeev and Soifer, 1982), in procedures of recognition of stochastic textures (Therrien *et al.*, 1986). The attention to using autoregressive models is not decreasing. This is proved by articles in the magazine *Pattern Recognition* (Mhidra *et al.*, 1993; Oe, 1993), in other publications.

The author of this article and the collaborators were analyzing questions of modelling random fields and images. Some theorical aspects of this work were published earlier (Balkevičienė, Valteris, 1989a, b; Valteris, 1989). Application of autoregressive models for 2D random field analysis is summarized in this article.

**2. Statement of the problem.** We present a two-dimensional field in a plane surface by his values $x(i,j)$ at the points of the grid of a finite rectangle, with coordinates $(i,j)$.

Let's say, that field $x(i,j)$ is created by a two-dimensional linear filter through which comes white Gaussian noise $w(i,j)$, having zero average and finite dispersion $\sigma_w^2$.

The forming of two-dimensional field $x(i,j)$ by using linear filter we describe by space autoregression equation:

$$x(i,\, j) = \sum_{k,\, l \in D} a(k,\, l)\, x(i-k,\, j-l) + w(i,\, j), \qquad (1)$$

in which $D$ – non-zero definition area of autoregression coefficients $a(k,l)$.

The problem of forming autoregressive model Eq. (1) consists of the selection of $D$ area, of finding coefficients $a(k,l)$ and noise dispersion $\sigma_w^2$.

**3. Estimation of parameters.** For more detailed analysis of Eq. (1), we will part three available configurations of definition area $D$ of coefficients $a(k,l)$ and the models corresponding them:

- one quadrant causal model (CM1), whose $D$ area is formed by values $(k,l):\ k=\overline{0,K},\ l=\overline{0,K},\ (k,l)\neq(0,0)$;
- asymmetrical half-plane (two quadrant) causal model (CM2): $k=\overline{0,K},\ l=\overline{-K,K},\ k=0,\ l>0$;
- non-causal model (NCM): $k=\overline{-K,K},\ l=\overline{-K,K},\ (k,l)\neq(0,0)$, where $K$ – model's rank.

Multiplied (1) by $x(i-k',j-l')$, where $k',\ l'\in G$, and averaged for all $i$, $j$ values, we get a set of equations:

$$\begin{cases} \underset{k',\, l' \in G}{r(k',\, l')} = \sum\limits_{k,\, l \in D} a(k,\, l)\, r(k'-k,\, l'-l), \\ r(0,\, 0) = \sum\limits_{k,\, l \in D} a(k,\, l)\, r(k,\, l) + \sigma_w^2. \end{cases} \tag{2}$$

In this set:

$$r(k',\, l') = \underset{i,\, j \in N}{\mathbf{E}} \left[x(i,\, j)\, x(i-k',\, j-l')\right]$$

– field covariation coefficients in area $G = (k' > 0\, \forall\, l')\, \cup (k' = 0,\ l' > 0)$, valid for Eq. (2), $r(0,0) = \sigma_x^2$ – field dispersion.

For NCM with square area $D$, according to Eq. (1), the value of field at point $(i,j)$ is described by dependency:

$$x(i,\, j) = \sum_{\substack{k=-K \\ (k,\, l) \neq (0,\, 0)}}^{K} \sum_{l=-K}^{K} a(k,\, l)\, x(i-k,\, j-l) + w(i,\, j). \tag{3}$$

Because of two-dimensional covariation functions symmetry with respect of the origin of the coordinates, we can assume NCM coefficients $a(k,l)$ symmetry: $a(k,l) = a(-k,-l)$, $a(m,n) = a(-m,-n)$. Then in Eq. (3), we can pass from summing in all definition plane to half-plane, as it was done with CM2.

Multiplied (3) by $x(i-k',j-l')$ and averaged for all $(i,j) \in N$, we get Eq. 4.

To create autoregressive model Eq. (1), the sets of (2) or (4) must be solved in $a(k,l)$ and $\sigma_w^2$ respect, and for this must be known the statistical characteristics of the modeling field: average, dispersion and covariation coefficients. As they are not known beforehand, they must be determined from the data of observation.

$$
\begin{cases}
r(k',l') = \displaystyle\sum_{\substack{k=-K \\ (k,\ l)\neq(0,\ 0)}}^{K} \sum_{l=-K}^{K} a(k,\ l) \Big[ r(k'-k,\ l'-l) \\
\qquad + r(k'+k,\ l'+l) \\
\qquad - \displaystyle\sum_{\substack{m=-K \\ (m,\ n)\neq(0,\ 0)}}^{K} \sum_{n=-K}^{K} a(m,\ n) \\
\qquad \times r(k'+k-m,\ l'+l-n) \Big], \\
r(0,\ 0) = \displaystyle\sum_{\substack{k=-K \\ (k,\ l)\neq(0,\ 0)}}^{K} \sum_{l=-K}^{K} a(k,\ l) \Big[ 2\, r(k,\ l) \\
\qquad - \displaystyle\sum_{\substack{m=-K \\ (m,\ n)\neq(0,\ 0)}}^{K} \sum_{n=-K}^{K} a(m,\ n) \\
\qquad \times r(k-m,\ l-n) \Big] + \sigma_w^2..
\end{cases}
\tag{4}
$$

**4. Adequation models.** After choosing a model and after estimating its parameters, its adequacy for modeling field must be checked. One of the methods of checking adequation is a comparison of correlation or covariation functions of the modelling field and the model. For this, correlation coefficients matrix of the model must be known.

The connection between covariation coefficients and autoregressive model parameters is shown in Eq. (2) or (4), which were used for estimating models parameters. They can't be used directly for inverse problem, because the quantity of covariation coefficients in them is larger then a quantity of autoregression coefficients $a(k,l)$.

Dividing Eq. (2) first expression by $r(0,0)$, we get:

$$\underset{k',\, l' \in G}{\varrho(k',\, l')} = \sum_{k,\, l \in D} a(k,l)\, \varrho(k'-k,\, l'-l), \tag{5}$$

where $\varrho(.)$ – correlation coefficients.

On the base of (5), we will create iteration procedure for finding correlation coefficients:

$$\overset{(n)}{\varrho}(k',\, l') = \sum_{k,\, l \in D} a(k,l)\, \overset{(n-1)}{\varrho}(k'-k,\, l'-l), \tag{6}$$

where $\overset{(n)}{\varrho}(.)$, $\overset{(n-1)}{\varrho}(.)$ – $\varrho(.)$ value at $n$ and $(n-1)$ iterations.

We define the starting and limiting conditions for the procedure: $\overset{(0)}{\varrho}(k',l') = 1$, if $k',\ l' \in G$, $\varrho(k',l') = 0$, if $k',\ l' \notin G$.

The iteration sequence Eq. (6) converges into solution Eq. (5), if the condition Eq. (7) is fulfilled:

$$A = \sum_{k,\, l \in D} \left| a(k,\, l) \right| < 1. \tag{7}$$

Condition Eq. (7) is valid for non-causal models too. In this case the iteration procedure is

$$\begin{aligned} \overset{(n)}{\varrho}(k',l') = \sum_{k,l \in D} a(k,l) \Big[ & \overset{(n-1)}{\varrho}(k'-k, l'-l) \\ & + \varrho(k'+k,\, l'+l) \\ & - \sum_{m,n \in D} a(m,n)\, \overset{(n-1)}{\varrho}(k'+k-m, l'+l-n) \Big]. \end{aligned} \tag{8}$$

The procedures Eq. (6) and (8) are ceased after fulfilling the condition:

$$\sum_{k',\, l' \in G} \left( \frac{\overset{(n)}{\varrho}(k',\, l') - \overset{(n-1)}{\varrho}(k',\, l')}{\overset{(n-1)}{\varrho}(k',\, l')} \right)^2 \leqslant \varepsilon, \tag{9}$$

where $\varepsilon$ – chosen small number.

The models adequacy to the modeling field is determined by comparing correlation coefficients $\varrho(k', l')$ of the real field with $\widehat{\varrho}(k', l')$ of the model.

Let's say, we have a random field with known correlation function $a(k', l')$. The best autoregressive model must be chosen for this field. Using the comparison of the correlational functions of the field and the model, the model are sorted, beginning with the model with a minimal number of autoregressive coefficients $a(k, l)$. While sorting the mean square error of models correlational function equivalence to real is checked. The best approach is chosen from the minimum condition:

$$\mathop{\mathbf{E}}_{k', l' \in G} \left[\varrho(k'; l') - \widehat{\varrho}(k', l')\right]^2 = \min.$$

It is defined from the analysis of the model's correlation identity, that CM1 and CM2 fully ensures correlational identity only for some fields with correlational functions, distinctive according to arguments. The first order models CM1 and CM2 can't guarantee correlational identity for fields with correlational functions, not distinctive according to the arguments, though they are giving smaller estimation errors. Acceptable precision is got only for higher order models (Valteris, 1989).

Non-causal models have no above mentioned durations, so they must be firstly used, in spite of their non-linearity in parameters $a(k, l)$ respect.

**5. Synthesis of realizations.** Exploring random fields, realizations with described characteristics are needed. For this may be used synthesized realizations of given parameter fields. Field values $x(i, j)$ generation, when the models rank $K$ and its parameters – autoregression coefficients $a(k, l)$ and field dispersion $\sigma_x^2$ – are known, is called random field synthesis.

For the synthesis of a field, described by Eq. (1) equation, we use iteration procedure:

$$\overset{(n)}{x}(i, j) = \sum_{k, l \in D} a(k,l) \overset{(n-1)}{x}(i-k, j-l) + w(i, j), \quad (10)$$

which is converging at Eq. (1) solution, if the condition of stability EQ. (7) and the starting condition $\overset{(0)}{x}(i,j) = w(i,j)$ are fulfilled.

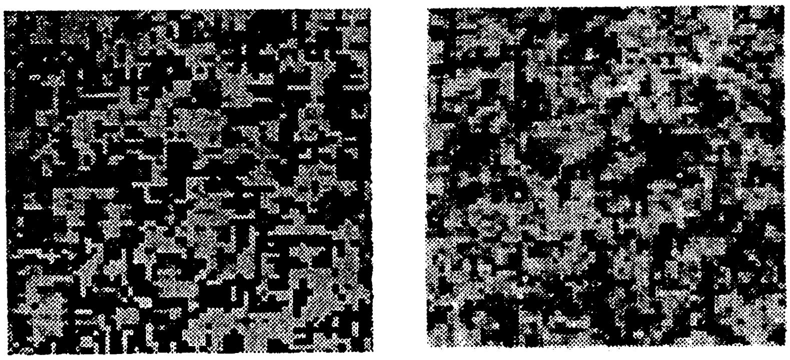
The procedure of synthesis (10) is ceased, when

$$\sum_{i j \in N} \left( \frac{\overset{(n)}{x}(i, j) - \overset{(n-1)}{x}(i, j)}{\overset{(n-1)}{x}(i, j)} \right)^2 \leqslant \varepsilon, \quad (11)$$

For obtaining synthesized fields optical image, the field meanings must be positive. For this, gotten values $x(i,j)$ are added together with some fields brightness mean value $m_x$. It is convenient to use the rule of square deviations, defining that $m_x = 3\,\sigma_x$. Fig. 1 shows the examples of the synthesized fields.

While using discrete measurement means for controlling fields, because of the limited number of measured channels, observations are only partial, only a limited number of values is measured. Wanting to rebuild the whole image of the field, the methods of rebuilding or interpolation must be used. The rebuilding of unobserved values of the field in this article is offered to be changed into statistical interpolation using the algorithm of autoregressive field synthesis Eq. (10). It is enough to find the estimations of fields correlation coefficients matrix elements for this, using obtained observations, and to build the autoregressive model of the field.

To fill unobserved values of the field, we will use iteration



*a* *b*

**Fig. 1.** Images of synthesized fields of CM1 (a) and NCM (b) of first order: a – $a(0,1) = a(1,0) = 0.8$, $a(1,1) = -0.64$; b – $a(0,1) = a(0,-1) = a(1,0) = a(-1,0) = 0.20$, $\sigma_w = 30$.

procedure, analogous to Eq. (10):

$$\overset{(n)}{x}(i,\,j) = \begin{cases} \overset{(n-1)}{x}(i,\,j) \quad - \quad \text{for observing } x(i,\,j), \\ \sum\limits_{k,\,l \in D} a(k,l)\, \overset{(n-1)}{x}(i-k,\,j-l) \\ \qquad + w(i,\,j) - \big[1 - \sum\limits_{k,\,l \in D} a(k,l)\big]\, m_x \\ \qquad\qquad - \text{for the rest of } x(i,\,j). \end{cases} \tag{12}$$

Condition Eq. (11) suits for ceasing the procedure. The entered component $\left[1 - \sum a(k,l)\right] m_x$ in Eq. (12), guarantees the holding of average $m_x$, if the $x(i,j)$ are not-centered.

The essence of the suggested algorithms for fields values statistical interpolation is, that field around the observing values is filled with values, statistically connected with them, keeping given statistical characteristics of the field. It is especially actual while generating realizations, near to real fields.

**6. Conclusions.** The methods of 2D random fields autoregressive models parameters estimation, of definition of correlation functions and of synthesis of realization of fields with given characteristics, described in this article, allows the creation of models, adequate to real fields. They successfully might be used in solving problems of modelling objects with parameters, distributed in space, of analysis of random textures, of controlling the quality of tape production.

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