INFORMATICA, 1994, Vol. 5, No. 1–2, 167–174

OPTIMIZATION PROBLEMS IN SIMPLE COMPETITIVE MODEL

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Abstract. We consider here the optimization problems of simple competitive model. There are two servers providing the some service. Each server fix the price and the rate of service. The rate of service defines the customer losses waiting in line for the service. The customer go to the server with lesser total service cost. The total cost includes the service price plus waiting losses. A customer goes away, if the total cost exceeds some critical level. The flow of customers and the service time both are stochastic. There is no known analytical solution for this model. We get the results by Monte Carlo simulation. We get the analytical solution of the simplyfied model.

We use the model as an illustration to show the possibilities and limitations of optimization theory and numerical techniques in the competitive models.

We consider optimization in two different mathematical frameworks: the fixed point and the Lagrange multipliers. We consider two different economic and social objectives, too: the equilibrium and the social cost minimization.

We use the model teaching Operations Research. The simple model may help to design more realistic models describing the processes of competition.

Key words: optimization, competition, stochastic, Monte Carlo, equilibrium, Lagrange multipliers, queuing theory.

1. Competitive model. We consider two servers providing the some service:

$$u_i = u_i(x_1, x_2, y_1, y_2) = a_i y_i - x_i, \quad i = 1, 2, \tag{1}$$

where u_i is the profit, y_i is the service price, x_i is the service rate[1] , a_i is the rate of customers, i is the server index.

A customer goes away, if

$$\min_i c_i > c. \tag{2}$$

A customer goes to the server i, if

$$c_i < c_j \tag{3}$$

and

$$\min_i c_i \leqslant c. \tag{4}$$

Here the service cost

$$c_i = y_i + \gamma_i, \quad i = 1, 2. \tag{5}$$

where c is the critical cost and γ_i is the average waiting time at the server i.

We fix the total rate of consumers

$$a = \sum_{i=1}^{3} a_i, \tag{6}$$

where a_3 is the rate of lost customers.

Conditions (3) and (2) separate the flow of incoming customers into three flows and so makes the problem very difficult for analytical solution. The separated flow is not simple even in the simplest Poisson case, see Gnedenko and Kovalenko (1987). Thus we need Monte Carlo simulation, to define the average rates of customers a_i, $i = 1, 2, 3$ by expressions (3) (2), and the average profits u_i, $i = 1, 2$, by expression (1).

[1] Average number of customers per unit time.

2. Equilibrium and fixed point. We fix the "initial contract" – the initial values $x_1^n, x_2^n, y_1^n, y_2^n$, $n = 0$. We get the "next contract" $n+1$ by maximizing the profits of each server, under the assumption that the patner will honour the previous contract.

$$\begin{aligned}(x_1^{n+1}, y_1^{n+1}) &= \max_{x_1, y_1} u_1(x_1, x_2^n, y_1, y_2^n),\\ (x_2^{n+1}, y_2^{n+1}) &= \max_{x_2, y_2} u_2(x_1^n, x_2, y_1^n, y_2).\end{aligned} \tag{7}$$

The conditions (7) transforms the vector z^n into the vector z^{n+1}. We denote this transformation by T

$$z^{n+1} = T(z^n), \quad n = 0, 1, 2, \ldots \tag{8}$$

Here vector $z = (x_i, y_i,\ i = 1, 2) \in B \subset R^4$. We get the equilibrium at the fixed point z^n, where

$$z^n = T(z^n). \tag{9}$$

The fixed point z^n exists, if the feasible set B and the profit functions (1) are all convex, see Michael, (1976). We may obtain the equilibrium directly by (7), if the transformation T is contracting, see Neuman and Morgenstern, (1953). If not, then we may optimize the square deviation

$$\min_{z \in B} \|z - T(z)\|^2. \tag{10}$$

We reach the equilibrium, if the minimum (10) is zero. The equilibrium does not exists, if the minimum (10) is positive. We may minimize (10) by the usual stochastic approximation techniques, see Ermoljev and Wets (1980), if the square deviation (10) is unimodal. If not, then we may use the Bayesian techniques of global stochastic optimization, see Mockus (1989).

We can simplify the model by ignoring the queuing theory and expressing the average waiting time as

$$\gamma_i = a_i/x_i. \tag{11}$$

We get expression (11) by deleting the second factor in the Poission case

$$\gamma_i = \frac{a_i}{x_i}\frac{1}{x_i - a_i}. \tag{12}$$

In the steady–state

$$a_i/x_i + y_i = q, \quad i = 1, 2. \tag{13}$$

Here q is the steady–state factor. From (2)

$$q \leqslant c.$$

From (13)

$$a_i = (q - y_i)x_i \tag{15}$$

and

$$u_i = (q - y_i)x_i y_i - x_i. \tag{16}$$

We maximize profit assuming that $x_i \leqslant a$.

$$\max_{x_i y_i q} x_i\big((q - y_i)y_i - 1\big); \quad x_i \leqslant a, \quad q \leqslant c. \tag{17}$$

We get the optimal values

$$x_i = a, \quad y_i = q/2, \quad i = 1, 2, \quad q = c. \tag{18}$$

From (17) and (18) the maximal profit

$$u_i = a\left((c/2)^2 - 1\right). \tag{19}$$

Table 1. The simulation results of the first server

Parameters						
k	y_1	y_2	u_1	$a_1 + a_2$	γ_1	γ_2
1	104	10	55.07	1.06	3027	3086
2	66	69	32.25	0.85	787	756
3	64	66	30.86	0.89	810	78
4	62	64	31.06	0.94	243	23
5	61	63	30.27	0.97	205	19
6	60	60	29.84	1.00	0	4
7	61	61	30.52	0.98	2	2
8	61	61	29.88	0.99	0	2

We get the positive profit equilibrium, if $c > 2$.

Those results may help considering more complicated model (1).

Now we describe some results of Monte Carlo simulation. We assume that the rate of lost customers depends on the average service price.

$$a_3 = \alpha a \frac{(y_1 + y_2)/2 - y^0}{y^0}. \tag{20}$$

It is a different assumption comparing with (2). The Table 1 shows the simulation results for the first server. The Table 2 shows that for the second server. The initial price $y^0 = 10$, the customer rate $a = 2$, the factor $\alpha = 0.1$, the service rate $x_i \leqslant a, i = 1, 2$, and the initial service rate $x_i = 0.5, i = 1, 2$.

Table 1 and Table 2 show the possibility of direct Monte Carlo simulation of transformation (8).

Table 2. The simulation results of the second server

Parameters						
k	y_1	y_2	u_1	$a_1 + a_2$	γ_1	γ_2
1	69	10	48.95	1.41	3315	3264
2	66	104	32.60	0.50	0	25
3	64	66	31.49	0.90	264	278
4	63	64	30.84	0.93	152	163
5	60	62	30.61	0.98	21	19
6	61	61	30.55	0.98	23	27
7	66	60	30.20	0.94	36	28
8	61	61	30.30	0.98	2	4

3. Social model. We minimize social service cost

$$\sum_i(-a_iy_i + x_i + a_iy_i + a_i\gamma_i) = \sum_i(x_i + a_i\gamma_i). \tag{21}$$

From here, in the simplified case (11), we get the optimal service rates

$$x_i = a_i, \qquad i = 1,2. \tag{22}$$

Here the prices y are eliminated, because we did not limit the supply of service rates.

4. Lagrange multipliers. We consider now the Lagrangian model. We limit the total service rate of both servers by b

$$\sum_i x_i \leqslant b. \tag{23}$$

We fix the customer rates a_i, $i = 1,2$ and minimize the service losses

$$\min_x \sum_i a_i\gamma_i. \tag{24}$$

We can solve the problem (24) (23) by Lagrange multipliers, assuming convexity of γ_i.

$$\max_{y\geqslant 0}\min_{x\geqslant 0}\Big(\sum_i a_i\gamma_i + y\big(\sum_i x_i - b\big)\Big). \tag{25}$$

Here the Lagrange multiplier y means the "price" which the servers pay to the supplier of service recourses x_i. At first we fix y and minimize (25) by x. Thus we define the optimal rate $x_i = x_i(y)$ as a function of price y. Then we maximize (25) by $y \geqslant 0$ to get the equilibrium (max–min) price $y = y_0$. Now each server can define the optimal service rate $x_i = x_i(y_0)$ by minimizing the social service cost

$$\min_{x_i\geqslant 0}(y_0 x_i + a_i\gamma_i), \quad i = 1, 2. \tag{26}$$

As we may see this model is not exactly competitive, because the customer rate is fixed for each server. We define equilibrium of the supplier and the servers. Thus we assume competition not between servers, as in (1) but between the supplier and the servers. The servers are of "non-profit" type. They minimize the social service cost including the customer waiting losses γ_i plus the price $y_0 x_i$ which the server pays to get the recourse x_i.

In the simplified case (11)

$$\max_{y\geqslant 0}\min_{x\geqslant 0}\Big(\sum_i a_i^2/x_i + y\big(\sum_i x_i - b\big)\Big). \tag{27}$$

First we fix y and get the optimal $x_i = x_i(y)$ as a function of y

$$x_i(y) = a_i/\sqrt{y}. \tag{28}$$

Now we maximize by y

$$\max_{y\geqslant 0}\big(\sqrt{y}\sum_i a_i + y(1/\sqrt{y}\sum_i a_i - b)\big) = \max_{y\geqslant 0}(2a\sqrt{y} - by) \tag{29}$$

and get the optimal price

$$y = y_0 = (a/b)^2. \tag{30}$$

From here and (28)

$$x_i = a_i/ab. \tag{31}$$

All the solutions of simplified models are illustrative and may be used considering more complicated models correctly representing the stochastic service and processes.

REFERENCES

Mockus, J. (1989). *Bayesian Approach to Global Optimization.* Kluwer Academic Publishers, Dordrecht–Boston–London.

Neuman, J., and O. Morgenstern (1953). *Theory of Games and Economic Behavior.* Princeton University Press, Princeton.

Gnedenko, B.V., and I.N.Kovalenko (1987). *Vvedenie v Teoriju Massovovo Obsluzivanija.* Nauka, Moskva. (in Russian).

Ermoljev, Ju., and R. Wets (1980). *Numerical Techniques for Stochastic Optimization.* Springer–Verlag, Berlin.

Michael, J. (1976). *The Computation of Fixed Points and Applications.* Springer–Verlag, Berlin.

Received March, 1994

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