INFORMATICA, 1994, Vol. 5, No. 1–2, 98–109

CHARACTERISTICS OF LINEAR PERIODICALLY TIME–VARYING SYSTEMS

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Abstract. The input–output relationship of the periodically time–varying (PTV) systems, impulse response of the PTV state–space system, and the transfer function of the PTV system are presented. A coefficient sensitivity is investigated by using a virtual PTV state–space system in which periodically time–varying coefficients are stochastically varied.

Key words: periodically time–varying systems, impulse response, transfer function, coefficient sensitivity.

1. Introduction. PTV systems are systems whose coefficients vary periodically with time. PTV systems have been used in applications such as transmultiplexing, bandwidth compression, and frequency warping (Critchley, 1988; Leonard, 1986; Bonnerot, 1978; Meyer, 1976). Periodically time–varying systems have been investigated for a number of problems, however, design techniques for such systems are usually limited to those systems without a feedback. Infinite impulse response of PTV systems have been designed using approximations based on the time–domain response (Critchley, 1988), but such method have error characteristics which are not easily related to design criteria and no guarantee of stability.

Having reviewed PTV system representations, this paper investigates the input–output relationship of the PTV system. It is shown that the PTV system may be analyzed as a parallel bank of the frozen–time systems. Impulse response and transfer function of the PTV system are derived. The coefficient sensitivity of the PTV system is defined and a measure of statistical coefficient sensitivity is analyzed.

2. PTV system representation. The coefficients of PTV system vary periodically with time. PTV systems are characterized by Green's function (Claasen, 1982; Prater, 1992) which reduce to the system impulse response in the time–invariant case. The input–output relationship of a time–varying system is given by (Zadeh, 1961)

$$y(n) = \sum_{k=-\infty}^{\infty} h(n,k)x(k),$$

where $x(k)$ is the input sequence, $y(k)$ is the output sequence, and $h(n,k)$ is the response of the system at time n to an impulse at time k. $h(n,k)$ is the Green's function of the system. Two other forms of the Green's function are

$$r(m,k) = h(m+k,k)$$

and

$$p(n,m) = h(n,n-m).$$

The corresponding input–output relationships are

$$y(n) = \sum_{k=-\infty}^{\infty} r(n-k,k)x(k)$$

and

$$y(n) = \sum_{k=-\infty}^{\infty} p(n,n-k)x(k),$$

where $p(n,k)$ is the response at time n to the impulse at time $n-k$. For PTV system with periodicity L, Green's functions are characterized by

$$\begin{aligned} h(n,k) &= h(n+L,k+L), \quad \forall n,k, \\ r(m,k) &= r(m,k+L), \quad \forall m,k, \\ p(n,m) &= p(n+L,m), \quad \forall n,m. \end{aligned}$$

$h(n,k)$, $r(m,k)$ and $p(n,m)$ can describe the time–domain characteristics of PTV systems. However $h(n,k)$ is not used, because it has the periodicity for two variables, and it is difficult to use z-transform. On the other hand, $r(m,k)$ or $p(n,m)$ have the periodicity for one variable k or n and thus z-transform may be used to analyze PTV systems.

Since r is periodic in k, it can be expanded in a Fourier series

$$r(m,k) = \sum_{l=0}^{L-1} q_l(m) e^{-j(2\pi/L)lk},$$

where

$$q_l(m) = \frac{1}{L} \sum_{k=0}^{L-1} r(m,k) e^{j(2\pi/L)lk}.$$

Taking the discrete time Fourier transform of q_l, we obtain

$$Q^l(\omega_y) = \sum_{i=-\infty}^{\infty} q_l(i) e^{-j\omega_y i}.$$

This yields an input–output relationship called a bifrequency map (Loeffler, 1984)

$$Y(\omega_y) = \frac{1}{L} \sum_{l=0}^{L-1} Q^l(\omega_y) X(\omega_x),$$

where

$$\omega_x = \omega_y - \frac{2\pi}{L} l.$$

This equation describes a parallel bank of time invariant systems with modulated inputs.

3. Input–output relationship of PTV system. The difference equation form of a time–varying filter is

$$y(n) = \sum_{j=0}^{M-1} \alpha_j(n)x(n-j) - \sum_{l=1}^{N} \beta_l(n)y(n-l), \quad n = 0,1,2,\ldots,$$

where $\alpha_j(n)$ is the values of the jth feedforward coefficient and $\beta_l(n)$ is the value of the lth feedback coefficient in the system at time n. For the PTV system $\alpha_j(n) = \alpha_j(n+L)$ and $\beta_l(n) = \beta_l(n+L)$.

Let $n = mL+i$, $m = 0,1,2,\ldots$, $i = 0,1,\ldots,L-1$. Then

$$y(mL+i) = \sum_{j=0}^{M-1} \alpha_j(i)x(mL+i-j) - \sum_{l=1}^{N} \beta_l(i)y(mL+i-j).$$

The z-transform of the output $y(n)$ is

$$\begin{aligned} Y(z) &= \sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{i=0}^{L-1} \sum_{m=0}^{\infty} y(mL+i)z^{-mL}z^{-i} \\ &= \sum_{i=0}^{L-1} Y(z^L,i)z^{-i}, \end{aligned} \tag{2}$$

where

$$Y(z^L,i) = \sum_{m=0}^{\infty} y(mL+i)z^{-mL}. \tag{3}$$

Using equations (1) and (3), we get the z-transform of $y(mL+i)$

$$
\begin{aligned}
Y(z^L, i) &= \sum_{m=0}^{\infty} \Big[\sum_{j=0}^{M-1} \alpha_j(i) x(mL + i - j) \\
&\qquad - \sum_{l=1}^{N} \beta_l(i) y(mL + i - l) \Big] z^{-mL} \\
&= \sum_{j=0}^{M-1} \alpha_j(i) z^{-j} \sum_{m=0}^{\infty} x(mL + i) z^{-mL} \\
&\qquad - \sum_{l=1}^{N} \beta_l(i) z^{-l} \sum_{m=0}^{\infty} y(mL + i) z^{-mL} \\
&= \sum_{j=0}^{M-1} \alpha_j(i) z^{-j} X(z^L, i) - \sum_{l=1}^{N} \beta_l(i) z^{-l} Y(z^L, i), \quad (4)
\end{aligned}
$$

where

$$
X(z^L, i) = \sum_{m=0}^{\infty} x(mL + i) z^{-mL}.
$$

Using the equation (4), we derive that

$$
Y(z^L, i) + \sum_{l=1}^{N} \beta_l(i) z^{-l} Y(z^L, i) = \sum_{j=0}^{M-1} \alpha_j(i) z^{-j} X(z^L, i).
$$

Thus

$$
\begin{aligned}
Y(z^L, i) &= \frac{\sum_{j=0}^{M-1} \alpha_j(i) z^{-j}}{1 + \sum_{l=1}^{N} \beta_l(i) z^{-l}} \cdot X(z^L, i) \\
&= K(z, i) X(z^L, i). \qquad (5)
\end{aligned}
$$

Inserting (5) into (2), we have the input–output relationship of PTV system

$$
Y(z) = \sum_{i=0}^{L-1} \frac{\sum_{j=0}^{M-1} \alpha_j(i) z^{-j}}{1 + \sum_{l=1}^{N} \beta_l(i) z^{-l}} z^{-i} X(z^L, i). \qquad (6)
$$

This equation describes the PTV system as a parallel bank of the frozen–time systems $K(z, i)$.

4. Impulse response of PTV state–space system. Consider a state–space system $(A(n), b(n), c(n), d(n))$ described by the state–space equations

$$\begin{aligned} \nu(n+1) &= A(n)\nu(n) + b(n)x(n), \\ y(n) &= c(n)\nu(n) + d(n)x(n), \end{aligned} \tag{7}$$

where $x(n)$ is the scalar input, $y(n)$ is the scalar output, $\nu(n)$ is an N-dimensional state vector, $A(n)$, $b(n)$, $c(n)$ and $d(n)$ are $N \times N$, $N \times 1$, $1 \times N$, and 1×1 real matrices with period L, i.e.,

$$\begin{aligned} A(n+L) &= A(n), \\ b(n+L) &= b(n), \\ c(n+L) &= c(n), \\ d(n+L) &= d(n). \end{aligned}$$

Using the state–space equations (7), we obtain a general impulse response of PTV system

$$h(n,k) = \begin{cases} 0, & n < k, \\ d(k), & n = k, \\ c(n)b(k), & n = k+1, \\ c(n)\prod_{i=n-1}^{k+1} A(i)b(k), & n > k+1. \end{cases} \tag{8}$$

$h(n, k)$ has the periodicity, i.e.,

$$h(n,k) = h(n+L, k+L).$$

From equation $p(n, m) = h(n, n-m)$ and equation (8) we get the impulse response of the PTV system

$$p(n,k) = \begin{cases} 0, & k < 0, \\ d(n-k), & k = 0, \\ c(n)b(n-k), & k = 1, \\ c(n)\prod_{i=n-1}^{n-k+1} A(i)b(n-k), & k > 1. \end{cases} \tag{9}$$

5. Transfer function of PTV system. We take the z-transform $P(n,z)$ of $p(n,k)$ as

$$P(n,z) = \sum_{k=0}^{\infty} p(n,k) z^{-n}. \tag{10}$$

We derive the expression of $P(n,z)$ in terms of the coefficients of the PTV state–space systems. It is well known that the impulse response $p(k)$ of a time–invariant state–space system is

$$p(k) = \begin{cases} 0, & k < 0, \\ d, & k = 0, \\ cA^{k-1}b, & k > 0, \end{cases}$$

and the transfer function is

$$\begin{aligned} P(z) &= \sum_{k=0}^{\infty} p(k) z^{-k} = d + cbz^{-1} + cAbz^{-2} + \dots \\ &= d + c(zI - A)^{-1} b, \end{aligned}$$

where

$$c(zI - A)^{-1} b = cbz^{-1} + cAbz^{-2} + \dots . \tag{11}$$

Evaluating $p(n,k)$ from (9) and putting $p(n,k)$ into (10) with the help of (11), we can get $P(n,z)$ in terms of the coefficients of PTV state–space systems:

$$\begin{aligned} P(0,z) =& \big[c(0)A(L-1)\cdots A(1)b(0)z^{-L} \\ &+ c(0)\overline{A}_0 A(L-1)\cdots A(1)b(0)z^{-2L} + \dots\big] z^0 + \dots \\ &+ \big[c(0)A(L-1)b(L-2)z^{-L} \\ &+ c(0)\overline{A}_0 A(L-1)b(L-2)z^{-2L} + \dots\big] z^{L-2} \\ &+ \big[c(0)b(L-1)z^{-L} + c(0)\overline{A}_0 b(L-1)z^{-2L} \\ &+ \dots\big] z^{L-1} + d(0) \\ =& P_0(0,z) + P_1(0,z) + \cdots + P_L(0,z), \end{aligned}$$

where (Kazlauskas, 1991)

$$\overline{A}_i = \left[A(i-1)A(i-2)\cdots A(i-L)\right], \quad i = 0, 1, \ldots, L-1,$$

$$\begin{aligned}
&P_0(0,z) = c(0)(z^L I - \overline{A}_0)^{-1} A(L-1)\cdots A(1)b(0)z^0,\\
&\quad\vdots\\
&P_{L-1}(0,z) = c(0)(z^L I - \overline{A}_0)^{-1} b(L-1),\\
&P_L(0,z) = d(0).
\end{aligned}$$

Also

$$P(1,z) = P_0(1,z) + P_1(1,z) + \cdots + P_L(1,z),$$

where

$$\begin{aligned}
&P_0(1,z) = c(1)(z^L I - \overline{A}_1)^{-1} A(0)\cdots A(2)b(1)z^0,\\
&\quad\vdots\\
&P_{L-1}(1,z) = c(1)(z^L I - \overline{A}_1)^{-1} b(0),\\
&P_L(1,z) = d(1).
\end{aligned}$$

And

$$\begin{aligned}
P(L-1,z) = &P_0(L-1,z) + P_1(L-1,z) + \cdots\\
&+ P_L(L-1,z),
\end{aligned}$$

where

$$\begin{aligned}
&P_0(L-1,z) = c(L-1)(z^L I - \overline{A}_{L-1})^{-1} A(L-2)\cdots\\
&\qquad\qquad A(0)b(L-1)z^0,\\
&\quad\vdots\\
&P_{L-1}(L-1,z) = c(L-1)(z^L I - \overline{A}_{L-1})^{-1} b(L-2),\\
&P_L(L-1,z) = d(L-1).
\end{aligned}$$

6. Coefficient sensitivity of PVT systems. Define the variations of $A(n)$, $b(n)$, $c(n)$ and $d(n)$ as $\delta A(n)$, $\delta b(n)$, $\delta c(n)$ and $\delta d(n)$. Then we get, using equations (7), the following system

$$\begin{aligned} \widehat{\nu}(n+1) &= \left[A(n)+\delta A(n)\right]\widehat{\nu}(n) + \left[b(n)+\delta b(n)\right]x(n), \\ \widehat{\nu}(n) &= \left[c(n)+\delta c(n)\right]\widehat{\nu}(n) + \left[d(n)+\delta d(n)\right]x(n), \end{aligned}$$

where the input $x(n)$ is a Gaussian signal with average 0 and variance 1. The variation $\delta a_{ij}(n)$ in the matrix $A(n)$ is assumed to be a white noise, i.e.,

$$E\left[\delta a_{ij}(n)\right] = 0$$

and

$$E\left[\delta a_{ij}(n)\delta a_{ij}(k)\right] = \sigma^2 \Delta(n-k).$$

The variations of $\delta b_i(n)$, $\delta c_i(n)$, and $\delta d_i(n)$ are the same as the variations of $\delta a_{ij}(n)$.

Subtracting (7) from (12), we have

$$\begin{aligned} e_\nu(n+1) &= A(n)e_\nu(n) + \delta A(n)\nu(n) + \delta A(n)e_\nu(n) \\ &\quad + \delta b(n)x(n), \\ e_y(n) &= c(n)e_\nu(n) + \delta c(n)\nu(n) + \delta c(n)e_\nu(n) \\ &\quad + \delta d(n)x(n), \end{aligned} \tag{13}$$

where $e_\nu(n) = \widehat{\nu}(n) - \nu(n)$ is the state error vector, and $e_y(n) = \widehat{y}(n) - y(n)$ is the output error.

The terms $\delta A(n)e_\nu(n)$ and $\delta c(n)e_\nu(n)$ are much smaller than the other terms in (13). Therefore, ignoring these terms a simplified version of equation (13) is

$$\begin{aligned} e_\nu(n+1) &= A(n)e_\nu(n) + \delta A(n)\nu(n) + \delta b(n)x(n), \\ e_y(n) &= c(n)e_\nu(n) + \delta c(n)\nu(n) + \delta d(n)x(n). \end{aligned} \tag{14}$$

From (14) we obtain

$$\begin{aligned} E\big[e_y^2(n)\big] &= E\big[c(n)e_\nu(n) + \delta c(n)\nu(n) + \delta d(n)x(n)\big] \\ &\quad \big[c(n)e_\nu(n) + \delta c(n)\nu(n) + \delta d(n)x(n)\big]^T \\ &= c(n)E\big[e_\nu(n)e_\nu^T(n)\big]c^T(n) + E\delta c(n)\nu(n)\nu^T(n) \\ &\quad \times \delta c^T(n) + E\big[\delta d^2(n)x^2(n)\big] \\ &= c(n)R_\nu(n)c^T(n) + \sigma^2 \mathrm{tr} V(n) + \sigma^2, \end{aligned} \tag{15}$$

where

$$R_\nu(n) = E\big[e_\nu(n)e_\nu^T(n)\big],$$
$$V(n) = E\big[\nu(n)\nu^T(n)\big],$$

$\mathrm{tr}[V(n)]$ is the sum of diagonal elements of the matrix $V(n)$. The measure of coefficient sensitivity is defined as a ratio of the variance of the output error $R_y(n)$ normalized by the variance σ^2 of coefficient variations

$$J(n) = \frac{E\big[e_y^2(n)\big]}{\sigma^2} = \frac{R_y(n)}{\sigma^2}. \tag{16}$$

From (15) and (16), we get

$$J(n) = \frac{1}{\sigma^2} c(n)R_\nu(n)c^T(n) + \mathrm{tr}V(n) + 1. \tag{17}$$

Equation (17) shows the degree of distortion of the input–output characteristics caused by coefficient variations, i.e., $J(n)$ is a measure of statistical coefficient sensitivity of PTV state–space systems.

Conclusions. The input–output relationship of the PTV system is analyzed. It is shown that the PTV system may be described as a parallel bank of frozen–time systems with constant coefficients. The expression of the transfer function of the PTV system in terms of the coefficients of PTV state-space systems is derived. A statistical coefficient sensitivity of PTV state-space systems is defined and investigated by using a virtual PTV state-space system. The measure of coefficient sensitivity is defined as a ratio of the variance of the output error normalized by the variance of coefficient variations.

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Received January 1994

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