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ON THE STABILITY IN STOCHASTIC PROGRAMMING – GENERALIZED SIMPLE RECOURSE PROBLEMS

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Abstract. Stochastic programming problems with simple recourse belong to problems depending on a random element only through the corresponding probability measure. Consequently, this probability measure can be treated as a parameter of the problem.

In this paper the stability with respect to the above mentioned parameter will be studied for generalized simple recourse problems.

Key words: stochastic programming, generalized simple recourse problem, stability, Kolmogorov metric, empirical estimates.

1. Introduction. Let (Ω, S, P) be a probability space,

$\xi = \xi(\omega) = [\xi_1(\omega), \dots, \xi_l(\omega)]$ be an l-dimensional random vector defined on (Ω, S, P),

$F(z)$ denote the distribution function of the random vector $\xi(\omega)$,

$f_i(x),\ i = 1, \dots, l$ be real-valued, continuous functions defined on E_n,

$f(x) = [f_1(x), \dots, f_l(x)],\ x \in E_n$,

$g_i(x, z_i),\ i = 1, \dots, l$ be real-valued, continuous functions defined on $E_n \times E_1$,

$X \subset E_n$ be a nonempty set,

(E_n denotes an n-dimensional Euclidean space).

A specific, simple optimization problem with random element can be introduced as the problem. Find

$$\max\left\{\sum_{i=1}^{l} g_i(x,\xi_i(\omega))\,\middle|\, x\in X\colon\ f(x)\leqslant \xi(\omega)\ \text{componentwise}\right\}. \quad (1)$$

If the solution x has to be found without knowing the realization of the random vector $\xi(\omega)$, then evidently, it is necessary first to determine the decision rule. It means to assign to the original stochastic optimization problem (1) some deterministic equivalent. Problems with penalty function, two-stage stochastic programming problems and chance constrained programming problems are well-known types of the deterministic equivalents.

In this paper we shall consider a special case of two-stage stochastic programming problem. In particular, we shall deal with a generalized simple recourse problem. This problem, corresponding to (1), can be introduced as the following problem. Find

$$\max_{x\in X} E_F\left\{\sum_{i=1}^{l} g_i(x,\xi_i(\omega)) + \max_{\substack{y_i\in\mathcal{K}_i(x,\xi_i(\omega)),\\ i=1,2,\dots,l}} \sum_{i=1}^{l}\left[h_i^+(y_i^+)+h_i^-(y_i^-)\right]\right\} = \varphi(F), \quad (2)$$

where $h_i^+(y_i^+)$, $h_i^-(y_i^-)$, $i=1,2,\dots,l$ are real-valued, continuous functions defined on E_1, $y_i=(y_i^+,y_i^-)\in E_2$, $\mathcal{K}_i(x,z_i)$, $i=1,2,\dots,l$ are mappings of $X\times E_1$ into the space of nonempty subsets of E_2 determined by

$$\mathcal{K}_i(x,z_i)=\Big\{\,y_i\in E_2\colon\ y_i=(y_i^+,y_i^-),\ f_i(x)+y_i^+-y_i^-=z_i,\ y_i^+,\ y_i^-\in E_1^+,\ i=1,2,\dots,l\Big\}, \quad (3)$$

$E_n^+ = \{x \in E_n\colon\ x = (x_1, \dots, x_n),\ x_i \geqslant 0,\ i = 1, 2, \dots, n\}$, E_F denotes the operator of the mathematical expectation considered with respect to the probability measure $P_F(\cdot)$ given by the distribution function $F(\cdot)$.

Evidently, the distribution function $F(\cdot)$ can be considered as a parameter of the problem (2). The aim of this paper will be to study the stability of the problem (2) with respect to this parameter. The importance of this problem is well-known from the theory and practice of stochastic programming.

REMARKS. 1. The problem (1) includes also linear and some quadratic models with random elements in the objective function and on the right-hand side of the constraints.

2. If there exists a function $\bar{g}(x)$ defined on E_n such that $g_1(x, z_1) = \bar{g}(x)$, $g_i(x, z_i) = 0$, $i = 2, \dots, l$, $z_i \in E_1$, $i = 1, 2, \dots, l$, $x \in E_n$, then the problem (1) is an optimization problem with a random element only on the right-hand side of the constraints.

3. Evidently, the deterministic equivalent (2) is only rather generalized, well-known stochastic programming problem with simple recourse (see for example [4]).

4. In general, it may happen that some symbols in (2) are not reasonable. However, this situation cannot appear under the assumptions considered in this paper.

The inner problem in (2) means the following problem. Find

$$\max_{\substack{y_i \in \mathcal{K}_i(x, z_i),\\ i=1,2,\dots,l}} \sum_{i=1}^{l} \left[h_i^+(y_i^+) + h_i^-(y_i^-)\right], \tag{4}$$

$$x \in E_n, \quad z_i \in E_1, \quad i = 1, 2, \dots, l,$$

that corresponds to the possibility to correct the total effect after the realization of the random element $\xi(\omega)$ ($z = (z_1, \dots, z_l)$) by a new decision problem. Namely, the solution

$y = (y_1, \dots, y_l)$ of the inner problem in (2) can depend on the random element realization while the solution $x \in X$ of the outer problem in (2) cannot depend on this one.

It is easy to see that the inner problem (4) is equivalent to l separated optimization problems, in particular to these problems. Find

$$\max_{y_i \in \mathcal{K}_i(x, z_i)} \left[h_i^+(y_i^+) + h_i^-(y_i^-)\right] = \psi_i(x, z_i), \qquad i = 1, 2, \dots, l. \tag{5}$$

Each of these problems depends only on one-dimensional random element. More pricesely, it depends only on one component of the vector $z \in E_l$, that corresponds also to just one component of the random vector $\xi(\omega)$.

The outer problem then can be rewritten as the following problem. Find

$$\max_X E_F \sum_{i=1}^{l} \varphi_i(x, \xi_i(\omega)), \tag{6}$$

where $\varphi_i(x, z_i) = g_i(x, z_i) + \psi_i(x, z_i)$, $i = 1, \dots, l$.

2. Some auxiliary assertions and definitions. In this section we shall try to present some auxiliary assertions. First, we shall deal with the behaviour of the optimal value and the optimal solution of the inner problem. It means, first, we shall deal with the problem of the type (5). To this end we shall study the parametric problem. Find

$$\max_{y \in \mathcal{K}(x,z)} \left[h^+(y^+) + h^-(y^-)\right], \tag{7}$$

$$\mathcal{K}(x, z) = \left\{y = (y^+, y^-) : f(x) + y^+ - y^- = z,\ y^+,\ y^- \geqslant 0\right\}, \qquad x \in E_n,\ z \in E_1.$$

Evidently, in this case, $l = 1$, $\xi(\omega) = \xi_1(\omega)$, $f(x) = f_1(x)$, $z \in E_1$, $h^+(y^+)$, $h^-(y^-)$ are real-valued functions defined on

E_1. Moreover, $f(x)$ and z can be considered as parameters of the problem (7).

Lemma 1. *Let $h^+(\cdot)$, $h^-(\cdot)$ be differentiable functions on E_1. If one of the following assumptions is fulfilled*

1)
$$\sup_{y^+ \in E_1^+} [h^+(y^+)]' \leqslant \inf_{y^- \in E_1^+} \left| [h^-(y^-)]' \right|,$$
$$[h^+(y^+)]' \geqslant 0, \quad [h^-(y^-)]' \leqslant 0$$
$$\text{for } \; y^-, \, y^+ \in E_1^+,$$

2)
$$\sup_{y^- \in E_1^+} [h^-(y^-)]' \leqslant \inf_{y^+ \in E_1^+} \left| [h^+(y^+)]' \right|,$$
$$[h^-(y^-)]' \geqslant 0, \quad [h^+(y^+)]' \leqslant 0$$
$$\text{for } \; y^-, \, y^+ \in E_1^+,$$

3)
$$[h^+(y^+)]' \leqslant 0, \; [h^-(y^-)]' \leqslant 0 \quad \text{for} \quad y^-, \, y^+ \in E_1^+,$$

and if $h^+(0) = h^-(0) = 0$,

then the optimal solution (y^+_{opt}, y^-_{opt}) of the problem (7) is determined by the relations

$$\begin{array}{lll} y^+_{opt} = z - f(x), & y^-_{opt} = 0 & \text{if} \;\; f(x) \leqslant z, \\ y^+_{opt} = 0, & y^-_{opt} = f(x) - z & \text{if} \;\; f(x) \geqslant z. \end{array} \tag{8}$$

($[\cdot]'$ denotes the derivative).

Proof. Let $x \in E_n$, $z \in E_1$ be arbitrary given. Let us, first, consider the case 1). It follows from this assumption that there exist q^+, $q^- \in E_1$, $q^- \leqslant 0$ such that

$$[h^+(y^+)]' \leqslant q^+ \leqslant |q^-| \leqslant \left| [h^-(y^-)]' \right|, \; [h^-(y^-)]' \leqslant 0 \quad \text{for every} \quad y^+, \, y^- \in E_1^+. \tag{9}$$

It is easy to see that one of the two cases has to be valid

$$\text{a) } f(x) \leqslant z,$$
$$\text{b) } f(x) > z.$$

We shall consider, first, the case a). It is easy to see that the solution given by (8) is optimal for the problem (7) (in this case) if and only if

$$h^+\big(z - f(x) + K\big) + h^-(K) \leqslant h^+\big(z - f(x)\big) \quad \text{for an arbitrary } K \in E_1^+.$$

Since it follows from the relation (9) that

$$\begin{aligned} h^-(K) &+ h^+\big(z - f(x) + K\big) - h^+\big(z - f(x)\big) \\ &\leqslant q^- K + q^+ \Big[\big(z - f(x) + K\big) - \big(z - f(x)\big)\Big] \\ &\leqslant K(q^+ + q^-) \leqslant 0 \quad \text{for every} \quad K \in E_1^+, \end{aligned}$$

we have verified the assertion of Lemma 1 in the case a).

It remains to consider the case b). In this case the solution given by the relation (8) will be optimal if and only if the inequality

$$h^+(K) + h^-\big(f(x) - z + K\big) \leqslant h^-\big(f(x) - z\big)$$

holds for every $K \in E_1^+$, $x \in X$, $z \in E_1$. However since it follows from the relation (9) that

$$\begin{aligned} h^+(K) &+ h^-\big(f(x) - z + K\big) - h^-\big(f(x) - z\big) \\ &\leqslant q^+ K + q^- \Big[\big(f(x) - z + K\big) - \big(f(x) - z\big)\Big] \\ &= K(q^+ + q^-) \leqslant 0 \end{aligned}$$

for every $K \in E_1^+$, we have finished the proof of Lemma 1 under the Assumption 1. Since the situation under the Assumption 2) is quite similar, we omit it.

At last we shall consider the Assumption 3). Evidently, utilizing the technique of the proof in the previous cases we can see that the assertion of Lemma 1 is valid in this case too.

REMARK. If $h^+(y^+) = q^+y^+$, $h^-(y^-) = g^-y^-$ for some q^+, $q^- \in E_1$, then we obtain that the relation (8) is valid if $q^+ + q^- \leqslant 0$. This is a well-known assertion from the theory of stochastic linear programming (see for example [4]).

Lemma 2. *Let the assumptions of Lemma 1 be fulfilled. If $h^+(y^+)$, $h^-(y^-)$ are Lipschitz functions with Lipschitz constants L^+, L^-, then for every $x \in E_n$*

$$\max_{\mathcal{K}(x,z)} [h^+(y^+) + h^-(y^-)]$$

is a Lipschitz function of $z \in E_1$ with a Lipschitz constant not greater than $\max(L^+, L^-)$.

If, moreover, $f(x)$ is a Lipschitz function on E_n with the Lipschitz constant L', then

$$\max_{\mathcal{K}(x,z)} [h^+(y^+) + h^-(u^-)]$$

is a Lipschitz function of $x \in E_n$ with a Lipschitz constant independent of $z \in E_1$, and not greater than $L' \max(L^+, L^-)$.

Proof. First, since according to Lemma 1, we have

$$\max_{\mathcal{K}(x,z)} [h^+(y^+) + h^-(u^-)] = \begin{cases} h^+\big(z - f(x)\big) & \text{if} \quad z \geqslant f(x), \\ h^-\big(f(x) - z\big) & \text{if} \quad z \leqslant f(x), \end{cases}$$

we can see that the first assertion holds. Further, as evidently, we get under the additional assumptions that

$$\begin{aligned} \big|h^+\big(z - f(x^1)\big) - h^+\big(z - f(x^2)\big)\big| &\leqslant L^+ \big|f(x^1) - f(x^2)\big| \\ &\leqslant L^+ L' \|x^1 - x^2\|, \end{aligned}$$

and simultaneously

$$\begin{aligned}\left|h^-(z-f(x^1)) - h^-(z-f(x^2))\right| &\leqslant L^-\left|f(x^1)-f(x^2)\right| \\ &\leqslant L^-L'\|x^1-x^2\|\end{aligned}$$

for an arbitrary $x^1,\ x^2 \in E_n,\ z \in E_1$, we can see that the second assertion is valid too ($\|\cdot\|$ denotes the Euklidean norm in E_n).

The next assertion follows immediately from Lemma 2.

Lemma 3. *Let the assumptions of Lemma* 1 *be fulfilled. If $h^+(y^+)$, $h^-(y^-)$ are Lipschitz functions and if a finite $E_F\xi(\omega)$ exists, then there exists also the finite*

$$E_F \max_{\mathcal{K}(x,\xi(\omega))} \left[h^+(y^+) + h^-(y^-)\right]$$

for every $x \in E_n$.

Proof. The assertion of Lemma 3 follows immediately from the assertion of Lemma 2.

The class of strongly concave (convex) functions is rather important for concave (convex) programming problems. We shall remember here the corresponding definition.

DEFINITION. Let $h(x)$ be a real-valued function defined on a convex, nonempty set $\mathcal{K} \subset E_n$. $h(x)$ is a strongly concave function with a parameter $\rho > 0$ if

$$\begin{aligned}h(\lambda x^1 + (1-\lambda)x^2) \geqslant \lambda h(x^1) &+ (1-\lambda)h(x^2) \\ &+ \lambda(1-\lambda)\rho\|x^1-x^2\|^2\end{aligned}$$

for every $x^1,\ x^2 \in \mathcal{K},\ \lambda \in \langle 0,1\rangle$.

The next assertion has been proved in [7].

Lemma 4. *Let $\mathcal{K} \subset E_n$ be a nonempty, convex, compact set. Let, moreover, $h(x)$ be a strongly concave with a parameter $\rho > 0$, continuous function on $\mathcal{K}$. If $x^0 \in \mathcal{K}$ is defined by the relation*

$$x^0 = \arg \max_{x \in \mathcal{K}} h(x),$$

then

$$\|x - x^0\|^2 \leqslant \frac{2}{\rho}\left[h(x^0) - h(x)\right]$$

for every $x \in \mathcal{K}$.

The next assertion will deal with a sum of concave and strongly concave function.

Lemma 5. *Let $K \subset E_n$ be a nonempty, convex, compact set. Let, moreover, $h_1(x)$, $h_2(x)$ be concave, continuous functions on $\mathcal{K}$. If, moreover, $h_1(x)$ is a strongly concave with a parameter $\varrho > 0$, continuous function on $\mathcal{K}$, then*

$$h_1(x) + h_2(x)$$

is a strongly concave with a parameter $\varrho > 0$, continuous function on $\mathcal{K}$.

Proof. The proof of Lemma 5 follows immediately from the definition of concave and strongly concave functions.

REMARK. The assumptions under which a quadratic function is a strongly concave (convex) one with a parameter $\varrho > 0$ are introduced in [10], for example.

Evidently, it follows from Lemma 1 and Lemma 2 that

$$\psi(x, z) = \max_{\mathcal{K}(x,z)} \left[h^+(y^+) + h^-(y^-)\right] \tag{10}$$

is a continuous and Lipschitz function under relatively general assumptions. Moreover, it is easy to see that for every

$x \in E'_n$, the optimal value $\psi(x, \xi(\omega))$ depends only on one-dimensional random element. Since, consequently, $\psi_i(x, \xi_i(\omega))$ and $\varphi_i(x, \xi_i(\omega))$, $i = 1, 2, \ldots, l$ given by (5) and (6) are also functions of this type (under relatively general assumptions), it is surely reasonable to study the stability of the mathematical expectation of the function depending on one-dimensional random element.

3. Stability studies. In this section, first, we shall deal again with the parametric optimization problem (7). Consequently $\xi(\omega)$ is one-dimensional random element defined on (Ω, S, P) and $F(z)$ one-dimensional distribution function. We can define for $\delta > 0$ two distribution functions $\underline{F}_\delta(\cdot)$, $\overline{F}_\delta(\cdot)$ by

$$\begin{aligned} \underline{F}_\delta(z) &= F(z - \delta), \\ \overline{F}_\delta(z) &= F(z + \delta). \end{aligned} \tag{11}$$

If $\kappa(z)$ is a real-valued, continuous function defined on E_1, then we shall prove the following auxiliary assertion.

Lemma 6. *Let $\delta > 0$ be arbitrary, $\underline{F}_\delta(z), \overline{F}_\delta(z)$ fulfil the relation* (11). *Let, further, $\kappa(z)$ be a Lipschitz function defined on E_1 with Lipschitz constant L. If there exists a finite $E_F \kappa(\xi(\omega))$ and if $G(z)$ is an arbitrary one-dimensional distribution function such that*

$$G(z) \in \langle \underline{F}_\delta(z), \overline{F}_\delta(z) \rangle \quad \text{for} \quad z \in E_1, \tag{12}$$

then

$$\left| E_F \kappa(\xi(\omega)) - E_G \kappa(\xi(\omega)) \right| \leqslant \delta L.$$

REMARK. It is evident that there exists an inaccuracy in the form of the assertion of Lemma 6. The exact form should be $|E_F \kappa(\xi(\omega)) - E_G \kappa(\xi^G(\omega))| \leqslant \delta L$, where $\xi^G(\omega)$ is a random value defined on (Ω, S, P) with the corresponding distribution function $G(z)$.

Proof. If $d_N > 0$, $d_N \in E_1$, $N = 1, 2, \ldots$ is a sequence for which $d_N \downarrow 0$, $(N \to \infty)$, then we can define the points $z_i(N) \in E_1$, $i = \ldots, -1, 0, 1, 2, \ldots$ by

$$z_0(N) = 0, \; z_{i+1}(N) = z_i(N) + d_N, \quad N = 1, 2, \ldots . \tag{13}$$

Evidently, it follows from the assumption of Lemma 6 that there exists a finite value of the sum

$$\sum_{i=-\infty}^{+\infty} \kappa\big(z_i(N)\big)\Big[F\big(z_{i+1}(N)\big) - F\big(z_i(N)\big)\Big]. \tag{14}$$

If $G(z)$ is a continuous function, then there have to exist points $z'_i(N)$, $i = \ldots, -1, 0, 1, \ldots$ such that

$$G\big(z'_i(N)\big) = F\big(z_i(N)\big), \quad i = \ldots, -1, 0, 1, \ldots, \; N = 1, 2, \ldots .$$

According to the assumptions it is easy to see that the assertion of Lemma 6 is valid in this case.

It reminds to consider the case of an arbitrary $G(z)$. Evidently then the points $z'_i(N)$, $i = \ldots, -1, 0, 1, 2, \ldots$, $N = 1, 2, \ldots$ have to be chosen more carefully. We define these points by the relation

$$z'_i(N) = \sup\Big\{z \in E_1\colon \; G(z) \leqslant F\big(z_i(N)\big)\Big\}, \tag{17}$$

$i = \ldots -1, 0, 1, 2, \ldots, \; N = 1, 2, \ldots .$
If the points $\bar{z}'_{i_j}(N)$, $j = 1, 2, \ldots, r_i$, $i = \ldots -1, 0, 1, 2, \ldots$, $N = 1, 2, \ldots$ fulfil the relations

$$\bar{z}'_{i_1}(N) = z'_i(N), \; \bar{z}'_{i_{r_i}} = z'_{i+1}(N), \; \big|\bar{z}'_{i_{j+1}}(N) - \bar{z}'_{i_j}(N)\big| \leqslant d_N,$$

then, evidently, it holds that

$$\begin{aligned} &\big|z'_i(N) - z_i(N)\big| \leqslant \delta, \\ &\bar{z}'_{i_j}(N) \in \big\langle z_i(N) - \delta - d_N, z_i(N) + \delta + d_N\big\rangle, \\ &\Big|\kappa\big(\bar{z}'_{i_j}(N)\big) - \kappa\big(z_i(N)\big)\Big| \leqslant L(\delta + d_N), \end{aligned} \tag{18}$$

for $j = 1,2,\ldots,r_i$, $i = \ldots,-1,0,1,2,\ldots$, $N = 1,2,\ldots$. If, further, $p_i(N)$, $p'_{i_j}(N)$, $q'_{i_j}(N)$ are defined by the system

$$
\begin{aligned}
p_i(N) &= F\big(z_{i+1}(N)\big) - F\big(z_i(N)\big),\\
p'_{i_j}(N) &= G\big(\bar{z}'_{i_{j+1}}(N)\big) - G_+\big(\bar{z}'_{i_j}(N)\big),\\
q'_{i_j}(N) &= G_+\big(\bar{z}'_{i_j}(N)\big) - G\big(\bar{z}'_{i_j}(N)\big),\\
&\qquad i = \ldots,-1,0,1,2,\ldots,\\
&\qquad j = 1,2,\ldots,r_i,\ N = 1,2,\ldots,\\
G_+(z') &= \lim_{z\downarrow z'} G(z),
\end{aligned}
$$

then according to the assumptions we obtain that

$$
\begin{aligned}
&\sum_{i=-\infty}^{k}\sum_{j=1}^{r_i} p'_{i_j}(N) + \sum_{i=1}^{k}\sum_{j=1}^{r_i-1} q'_{i_j} = G\big(z'_{k+1}(N)\big)\\
&\qquad \leqslant \sum_{i=-\infty}^{k} p_i(N) \leqslant F\big(z_{k+1}(N)\big),\\
&\sum_{i=-\infty}^{k+1}\sum_{j=1}^{r_i} p'_{i_j}(N) + \sum_{i=1}^{k+1}\sum_{j=1}^{r_i} q'_{i_j} \geqslant F\big(z_{k+1}(N) - \delta - d_N\big).
\end{aligned}
$$

Evidently there have to exist $\bar{p}^k_{i_j} \geqslant 0$, $\bar{q}^k_{i_j} \geqslant 0$ such that

1. $\bar{z}'_{i_j}(N) \notin \big(z_k(N) - \delta - d_N,\ z_{k+1}(N) + \delta + d_N\big)$
   $\Rightarrow \bar{p}^k_{i_j}(N) = 0,\ \bar{q}^k_{i_j}(N) = 0,$
   $j = 1,\ldots,r_i,\quad i,k = \ldots,-1,0,1,\ldots,$
2. $p_i(N) = \sum\limits_{k=-\infty}^{+\infty}\sum\limits_{j=1}^{r_k} \bar{p}^i_{k_j}(N) + \sum\limits_{k=-\infty}^{+\infty}\sum\limits_{j=1}^{r_k} \bar{q}^i_{k_j}(N)$
   $\sum\limits_{k=-\infty}^{+\infty} \bar{p}^k_{i_j}(N) = p'_{i_j}(N),\quad \sum\limits_{k=-\infty}^{+\infty} \bar{q}^k_{i_j}(N) = q'_{i_j}(N),$
   $i = \ldots,-1,0,1,\ldots,\quad j = 1,2,\ldots,r_i.$

However it follows from this, that

$$\sum_{i=-\infty}^{+\infty} \kappa(z_i(N)) \Big[F(z_{i+1}(N)) - F(z_i(N))\Big] = \sum_{i=-\infty}^{+\infty} \kappa(z_i(N)) \sum_{k=-\infty}^{+\infty} \sum_{j=1}^{r_k} [\bar{p}^i_{k_j}(N) + \bar{q}^i_{k_j}(N)]$$

and also

$$\sum_{i=-\infty}^{+\infty} \kappa(z'_i(N)) \Big[G(z'_{i+1}(N)) - G(z'_i(N))\Big] = \sum_{i=-\infty}^{+\infty} \kappa(z'_i(N)) \sum_{k=-\infty}^{+\infty} \sum_{j=1}^{r_i} [\bar{p}^{-k}_{i_j}(N) + \bar{q}^k_{i_j}(N)].$$

Utilizing the Lipschitz property of the function $\kappa(z)$ we obtain also that

$$\Big| \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \sum_{j=1}^{r_k} \kappa(z_i(N)) [\bar{p}^i_{k_j}(N) + \bar{q}^i_{k_j}(N)] - \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \sum_{j=1}^{r_k} \kappa(z'_i(N)) [\bar{p}^k_{i_j}(N) + \bar{q}^k_{i_j}(N)] \Big| \leqslant (\delta + d_N)L \quad \text{for enough large } N.$$

However now already we can see that the assertion of Lemma 6 is valid also for an arbitrary type of $G(z)$.

REMARK. Lemma 6 presents a stability interval. If we set $\kappa(z) = z$, $G(z) = F(z - \delta)$ for a $\delta > 0$, then it is $L = 1$ and $E_G\xi(\omega) = E_F\xi(\omega) + \delta$. Consequently, the interval given by (11) cannot be generally smaller.

The next auxiliary assertion follows immediately from well-known properties of probability measures.

Lemma 7. *Let $\delta > 0$ be arbitrary. Let, moreover,*

1. *the probability measure $P_F(\cdot)$ be absolutely continuous with respect to the Lebesgue measure in E_1,*
2. *the support Z of $P_F(\cdot)$ be an interval $\langle a, b\rangle$ for some $a, b \in E_1,\ a < b$,*
3. *$\vartheta_1 > 0$ be a real-valued constant such that $\vartheta_1 \leqslant \bar{f}(z)$ for every $z \in Z$, ($\bar{f}(z)$ denotes the probability density corresponding to the distribution function $F(z)$).*

If $G(z)$ is an arbitrary one-dimensional distribution function with support Z such that

$$\sup \left|F(z) - G(z)\right| \leqslant \delta\vartheta_1, \tag{18a}$$

then

$$G(z) \in \left\langle \underline{F}_\delta(z), \overline{F}_\delta(z)\right\rangle,\ z \in E_1.$$

Proof. Let $\delta > 0$ be arbitrary. Since it follows from the Assumption 3 of Lemma 7 that

$$\left|F(z+\delta) - F(z)\right| \geqslant \delta\vartheta_1$$

for an arbitrary $z \in Z$, we can see that the implication

$$\sup \left|F(z) - G(z)\right| \leqslant \vartheta_1\delta \Rightarrow G(z) \in \left\langle \underline{F}_\delta(z), \overline{F}_\delta(z)\right\rangle$$

holds for every $z \in Z$.

It follows from the relations (4) and (5) that the results of Sections 2 and 3 can be employed for the stability of the problem (2). At the end of this section we shall present a result dealing with the problem (6).

To this end let $F_i(\cdot),\ i = 1, 2, \ldots, l$ denote the one-dimensional marginal distribution functions corresponding to $F(\cdot)$.

If $\delta_i > 0,\ i = 1, 2, \dots, l$ are arbitrary, then we can define the functions $\underline{F}_{\delta_i}(z_i),\ \overline{F}_{\delta_i}(z_i),\ z_i \in E_1$ by the relations

$$\begin{aligned} \underline{F}_{\delta_i}(z_i) &= F_i(z_i - \delta_i), \\ \overline{F}_{\delta_i}(z_i) &= F_i(z_i + \delta_i). \end{aligned} \tag{19}$$

Now we can present the corresponding stability assertion.

Lemma 8. *Let $\delta_i > 0,\ i = 1, 2, \dots, l$ be arbitrary, X be a compact set. If*

1. a. *for every $x \in X$, $\varphi_i(x, z_i)$ are Lipschitz functions of $z_i \in E_1$ with the Lipschitz constants L^{φ_i}, $i = 1, \dots, l$ independent of $x \in E_n$,*
   b. *$\varphi_i(x, z_i)$, $i = 1, \dots, l$ are uniformly continuous functions on $E_n \times E_1$,*
2. *there exists a finite $E_F \xi(\omega)$,*
3. *the functions $\underline{F}_{\delta_i}(z_i),\ \overline{F}_{\delta_i}(z_i),\ z_i \in E_1,\ i = 1, 2, \dots, l$ are defined by (19),*

and if $G(z)$ is an arbitrary l-dimensional distribution function with one-dimensional marginal ones $G_i(z_i)$, $i = 1, \dots, l$ fulfilling the relations

$$G_i(z_i) \in \langle \underline{F}_{\delta_i}(z_i), \overline{F}_{\delta_i}(z_i) \rangle, \quad i = 1, 2, \dots, l,$$

then

$$\Big| \max_X E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) - \max_X E_G \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) \Big| \leqslant \sum_{i=1}^{l} \delta_i L^{\varphi_i}.$$

Proof. Since it follows from the triangular inequality and from Lemma 6 that

$$\Big| E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) - E_G \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) \Big| \leqslant \sum_{i=1}^{l} \delta_i L^{\varphi_i}$$

for every $x \in X$, we can see that the assertion of Lemma 8 is valid.

4. Stability results. In this part of this paper we shall try to utilize the results of the previous parts to obtain some new stability results for the problem given by the relations (2), (3).

Theorem 1. *Let $\delta_i > 0$, $i = 1, 2, \ldots, l$ be arbitrary , X be a compact set. If*

1. *for every $i = 1, 2, \ldots, l$ one of the following assumptions is valid*

   a.
$$\sup_{y_i^+ \in E_1^+} [h_i^+(y_i^+)]' \leqslant \inf_{y_i^- \in E_1^+} \left| [h_i^-(y_i^-)]' \right|,$$
$$[h_i^-(y_i^-)]' \leqslant 0, \quad [h_i^+(y_i^+)]' \geqslant 0$$
$$\text{for } \ y_i^-, y_i^+ \in E_1^+,$$

   b.
$$\sup_{y_i^- \in E_1^+} [h_i^-(y_i^-)]' \leqslant \inf_{y_i^+ \in E_1^+} \left| [h_i^+(y_i^+)]' \right|,$$
$$[h_i^+(y_i^+)]' \leqslant 0, \quad [h_i^-(y_i^-)]' \geqslant 0$$
$$\text{for } \ y_i^-, y_i^+ \in E_1^+,$$

   c.
$$[h_i^+(y_i^+)] \leqslant 0, \quad [h_i^-(y_i^-)]' \leqslant 0,$$
$$\text{for } \ y_i^+, y_i^- \in E_1^+,$$

2. $h_i^+(0) = h_i^-(0) = 0$, $i = 1, 2, \ldots, l$,
3. *for every $i = 1, \ldots, l$,* $g_i(x, z_i)$ *are*
   a. *Lipschitz functions of $z_i \in E_1$ with Lipschitz constants L_i independent of $x \in X$,*
   b. *uniformly continuous functions on $E_n \times E_1$,*
4. $h_i^+(\cdot)$, $h_i^-(\cdot)$, $i = 1, 2, \ldots, l$ *are Lipschitz functions on E_1 with Lipschitz constants L_i^+, L_i^-,*
5. $f_i(x)$, $i = 1, \ldots, l$ *are continuous functions on X,*

6. *there exists a finite* $E_{F_i}\xi_i(\omega),\ i = 1,2,\dots,l,$
7. $\underline{F}_{\delta_i}(z_i),\ \overline{F}_{\delta_i}(z_i),\quad z_i \in E_1,\ i = 1,2,\dots,l$ *are defined by* (19),

and if $G(z)$ *is an arbitrary* l*-dimensional distribution function with one-dimensional marginal ones* $G_i(z_i),\ i = 1,\dots,l$ *fulfilling the relations*

$$G_i(z_i) \in \langle \underline{F}_{\delta_i}(z_i), \overline{F}_{\delta_i}(z_i)\rangle,\ i = 1,2,\dots,l,\ z_i \in E_1,$$

then

$$\left|\varphi(F) - \varphi(G)\right| \leqslant \sum_{i=1}^{l} \delta_i \max(L_i, L_i^+, L_i^-).$$

Proof. Since it follows from the assumptions of Theorem 1 and from Lemma 2 and Lemma 3 that the functions $\varphi_i(x, z_i) = g_i(x, z_i) + \psi_i(x, z_i),\ i = 1,\dots,l$ (defined by (6)) fulfil the assumptions of Lemma 8 ($L^{\varphi_i} := \max(L_i, L_i^+, L_i^-)$), we can see that the assertion of Theorem 1 is valid.

If we define the point $x(F)$ by the relation

$$x(F) = \arg\max_X E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big), \tag{20}$$

then the next assertion follows from Theorem 1.

COROLLARY 1. Let the assumptions of Theorem 1 be fulfilled and let X be a convex set. If for every $z_i \in E_1,\ i = 1,2,\dots,l$

$$\max_{y_i \in \mathcal{K}_i(x, z_i)} \left[h_i^+(y_i^+) + h_i^-(y_i^-)\right]$$

are concave functions on E_n and if for $z_i \in E_1,\ i = 1,2,\dots,l$, $\sum_{i=1}^{l} g_i(x, z_i)$ is a strongly concave with a parameter $\rho > 0$ function on E_n, then

$$\left\|x(F) - x(G)\right\|^2 \leqslant \frac{4}{\rho} \sum_{i=1}^{l} \delta_i \max(L_i, L_i^+, L_i^-).$$

Proof. . First, it follows from Lemma 5 that

$$E_F \sum_{i=1}^{l} \varphi_i(x, \xi_i(\omega))$$

is a strongly concave with a parameter ρ function on E_n. Consequently, employing Lemma 4, Theorem 1 and the triangular enequality we get successively

$$\begin{aligned}
&\|x(F) - x(G)\|^2 \\
&\leqslant \frac{2}{\rho}\Big|E_F \sum_{i=1}^{l} \varphi_i(x(F), \xi_i(\omega)) - E_F \sum_{i=1}^{l} \varphi_i(x(G), \xi_i(\omega))\Big| \\
&\leqslant \frac{2}{\rho}\Big\{\Big|E_F \sum_{i=1}^{l} \varphi_i(x(F), \xi_i(\omega)) - E_G \sum_{i=1}^{l} \varphi_i(x(G), \xi_i(\omega))\Big| \\
&\quad + \Big|E_G \sum_{i=1}^{l} \varphi_i(x(G), \xi_i(\omega)) - E_F \sum_{i=1}^{l} \varphi_i(x(G), \xi_i(\omega))\Big|\Big\}.
\end{aligned}$$

Now already the assertion of Corollary 1 follows from the relation (20), Theorem 1, Lemma 6 and the last system of inequalities.

REMARK. It follows from Lemma 5 that for $z = (z_1, \ldots, z_l)$, $\sum_{i=1}^{l} g_i(x, z_i)$ is a strongly concave function with a parameter ρ on E_n if, for example, $g_i(x, z_i)$, $i = 1, \ldots, l$ are concave functions and for at least one $i \in \{1, \ldots, l\}$, $g_i(x, z_i)$ is a strongly concave with a parameter ρ function.

Theorem 1 and Corollary 1 present the stability results for the deterministic equivalent given by the relations (2) and (3). In detail, employing these assertions, for an arbitrary $\varepsilon > 0$ we can determine the stability region such that the error arose by the substitution theoretical distribution function by some another one (from stability region) is less than this ε. Further, we shall deal with the case of a continuous $F(z)$. Consequently, Kolmogorov metric can be employed in this case.

Theorem 2. *Let $X \subset E_n$ be a compact set. Let, moreover, the Assumptions* 1, 2, 3, 4, 5 *of Theorem 1 be fulfilled. If*

1') *the probability measures corresponding to all one-dimensional marginal distribution functions $F_i(z_i)$, $i = 1, 2, \ldots, l$ are absolutely continuous with respect to the one-dimensional Lebesgue measure.*
We denote by $\bar{f}_i(z_i)$, $i = 1, 2, \ldots, l$, $z_i \in E_1$ probability densities corresponding to distribution functions $F_i(z_i)$, $i = 1, 2, \ldots, l$,
2') *the supports Z_i of the probability measures $P_{F_i}(\cdot)$ are compact intervals, $i = 1, 2, \ldots, l$,*
3') *constants $\vartheta_i > 0$, $i = 1, 2, \ldots, l$ fulfil the inequalities*

$$\vartheta_i \leqslant \bar{f}_i(z_i), \quad i = 1, 2, \ldots, l, \ z_i \in Z_i,$$

and if $G(z)$ is an arbitrary l-dimensional distribution function with one-dimensional marginal ones $G_i(\cdot)$ for which

$$P_{G_i}\{\omega\colon\ \xi_i(\omega) \in Z_i\} = 1, \quad i = 1, 2, \ldots, l,$$

then

$$\left|\varphi(F) - \varphi(G)\right| \leqslant \sum_{i=1}^{l} \frac{\max(L_i, L_i^+, L_i^-)}{\vartheta_i} \sup \left|F_i(z_i) - G_i(z_i)\right|.$$

Proof. The proof of Theorem 2 follows from Theorem 1 and Lemma 7.

We have finished the stability results corresponding to the deterministic equivalent given by the relations (2) and (3). The presented results are fully determined by the behaviour of the corresponding one-dimensional marginal distribution functions. Surely this fact is very pleasant for practice. Moreover, the stability properties of these problems don't depend on the components dependence of the random vector $\xi(\omega)$.

5. Applications to empirical estimates. It follows from the previous parts of the paper that the statistical behaviour of the generalized simple recourse problem given by the relations (2), (3) depends on the probability measure only through the corresponding one-dimensional marginal distribution functions. Consequently, the empirical estimates of the optimal value and the optimal solution can be based also on one-dimensional marginal empirical distribution functions.

Let for $i = 1, 2, \ldots, l$, $\xi_i^k(\omega)$, $k = 1, 2, \ldots$ be a sequence of random values defined on (Ω, S, P) such, that for every $k = 1, 2, \ldots$ the random value $\xi_i^k(\omega)$ has the same distribution function as the i-component $\xi_i(\omega)$ of the random vector $\xi(\omega)$. We denote by the symbol $F_i^{N_i}(z_i) = F_i^{N_i}(z_i, \omega)$, $N_i = 1, 2, \ldots$ the empirical one-dimensional distribution function determined by $\xi_i^1(\omega), \xi_i^2(\omega), \ldots, \xi_i^{N_i}(\omega)$, $i = 1, 2, \ldots, l$.

Evidently, under very general conditions

$$\max_X \sum_{i=1}^{l} E_{F_i^{N_i}} \varphi_i\big(x, \xi_i(\omega)\big)$$

estimates the theoretical value

$$\max_X E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big).$$

The next theorem follows immediately from Theorem 2.

Theorem 3. *Let X be a compact set. Let, moreover, the Assumptions* 1, 2, 3, 4, 5, *of Theorem 1 and the Assumptions* 1', 2', 3' *of Theorem* 2 *be fulfilled. If $F_i^{N_i}(\cdot)$, $N_i = 1, 2, \ldots$ are* *one-dimensional empirical distribution functions determined by* $\xi_i^1(\omega), \ldots, \xi_i^{N_i}(\omega)$, $i = 1, 2, \ldots, l$, *then*

$$P\Big\{\omega\colon \Big|\max_X E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) - \max_X \sum_{i=1}^{l} E_{F_i^{N_i}} \varphi_i\big(x, \xi_i(\omega)\big)\Big| > t\Big\} \leqslant \sum_{i=1}^{l} P\Big\{\omega\colon \sup \big|F_i^{N_i}(z_i) - F_i(z_i)\big| > \frac{\vartheta_i}{\max(L_i, L_i^+, L_i^-)} \frac{t}{l}\Big\},$$

for an arbitrary $t > 0$, $N_i = 1, 2, \ldots,$.

Proof. The assertion of Theorem 3 follows immediately from the assertion of Theorem 2 and elementary properties of probability measures.

If, moreover, for every $i = 1, 2, \ldots, l$ the random sequence $\xi_i^k(\omega)$, $k = 1, 2, \ldots$ is a sequence of independent random values, then we can employ Kolmogorov's limit theorem.

Theorem 4. *Let the assumptions of Theorem 3 be fulfilled. If for every* $i = 1, 2, \ldots, l$, *random sequence* $\{\xi_i^k(\omega)\}_{k=1}^{\infty}$ *is a sequence of independent random values, then*

$$\overline{\lim_{\substack{N_i \to \infty \\ i=1,\ldots,l}}} P\Big\{\omega\colon \sqrt{\min N_i}\Big|\max_X E_F \sum_{i=1}^{l} \varphi_i\big(x, \xi_i(\omega)\big) - \max_X \sum_{i=1}^{l} E_{F_i^{N_i}} \varphi_i\big(x, \xi_i(\omega)\big)\Big| > t\Big\} \leqslant \sum_{i=1}^{l} \Big[1 - \sum_{k=-\infty}^{\infty} (-1)^k \exp\Big\{-2k^2\Big(\frac{\vartheta_i t}{\max(L_i, L_i^+, L_i^-) l}\Big)^2\Big\}$$

for an arbitrary $t > 0$.

Proof. First, it follows from Theorem 3 that

$$P\Big\{\omega\colon \sqrt{\min N_i}\,\Big|\max_X E_F \sum_{i=1}^{l} \varphi_i(x, \xi_i(\omega)) - \max_X \sum_{i=1}^{l} E_{F_i^{N_i}} \varphi_i(x, \xi_i(\omega))\Big| > t\Big\}$$
$$\leqslant \sum_{i=1}^{l} P\Big\{\omega\colon \sqrt{N_i} \sup \big|F_i^{N_i}(z_i) - F_i(z_i)\big| > \frac{\vartheta_i}{\max(L_i, L_i^+, L_i^-)} \frac{t}{l}\Big\}$$

for an arbitrary $t > 0$.

However, since it follows from Kolmogorov's limit theorem that

$$\lim_{N_i \to \infty} P\{\omega\colon \sqrt{N_i} \sup \big|F_i^{N_i}(z_i) - F_i(z_i)\big| > t'\Big\}$$
$$\leqslant 1 - \sum_{k=-\infty}^{+\infty} (-1)^k \exp\{-2k^2 (t')^2\}$$

for an arbitrary $t' > 0$ and every $i = 1, 2, \ldots, l$, we can see that the assertion of Theorem 4 is valid.

REMARK. Evidently, the similar assertions to Corollary 1 for the optimal solution can be presented also in the case of Theorem 2, 3, 4.

6. Conclusion. The stability of a specific type of stochastic programming problem has been discussed in the paper. In detail, it was the generalized simple recourse problem. The stability has been considered there with respect to the distribution function space. It was shown that this problem can be transformed to several one-dimensional cases. Namely, the stability of the original problem is practically determined by the behaviour of one-dimensional marginal distribution functions, from mathematical point of view. Evidently, this fact is very "pleasant" for practice. Surely to work with one-dimensional probability measures is much more simple, then to work with general case. Obviously, it appears also in the applications to the empirical estimates. There Kolmogorov's limit distribution can be utilized.

The similar access has been already taken by Gröwe and Römisch in [9] for the linear and quadratic recourse case. However there another type of metric in the space of distribution function has been employed. Surely, it would be valuable to compare both these results with the ones achieved by a simulation method. However this problem will not be more discussed in this paper.

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