INFORMATICA, 1993, Vol.4, No.3–4, 255–266

STABILIZABILITY OF A CLASS OF LINEAR DELAY-DIFFERENTIAL SYSTEMS

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Abstract. Control laws' design strategies in order to stabilize a class of linear delay-differential systems are developed by using the matrix measure. A new measure, the delay measure, is introduced in order to clarify and formalize the results.

Key words: point delayed systems, matrix measure, delay measure, stabilizability.

1. Introduction and problem statement. During the last decades increasing interest has been focused on delay systems, also called hereditary systems. Mathematical models with time-delay constitute a natural way to represent a wide class of physical systems such as transportation problems, population growth laws and economic systems with multiple non-instantaneous dynamical effects (Churakova, 1969). The presence of time-delay in dynamical discrete linear equations can be overcome, when necessary, by using extended systems (Franklin and Powell, 1981). In the continuous system it is not possible, in general, the use of a linear space of sate vectors because of the post-effect due to the time-delay, and a function space must be alternatively used. The stabilizability conditions for systems with general delays in state were extended by Pandolfi (1975) and also by Bhat and Koivo (1976). In a work due Olbrot (1978) open-loop stabilizability problems for systems with control and state delays were defined. In that work, by introducing a dual observed system characterised by matrices which are transposes of given system matrices, a duality theorem between stabilizability and detectability was demonstrated. Also, for systems with input delays only, the fact that spectrum assignability is equivalent to a defined controllability of the system was shown. A characterisation of trajectory-stabilizable systems, and of the relations between state- and trajectory-stabilizability were given by Tadmor (1988); trajectory stabilizability is an appropriate notion in the presence of delays. Conditions for the delay-independent stabilization of linear systems were given by Amemiya *et al.* (1986), being the upper bound or the lower bound of the decay rates assignable, and (Akazawa *et al.*, 1987), by using in the proof matrices with some of their elements being arbitrary. In addition, Fiagbedzi and Pearson (1986, 1990) introduced techniques for the feedback and output feedback stabilization of delay systems by using a generalization of the transformation method. Chen *et al.* (1988) introduced state feedback and sampled-state feedback stabilizing laws for time-delay systems containing saturating actuators. Kamen *et al.* (1986) introduced an explicit procedure for computing proper stable Bezout factorizations in terms of a special ring of pure and distributed delays. This procedure can be used to construct finite-dimensional stabilizing compensators and to construct feedback systems which assign the characteristic polynomial of the closed-loop system. Furthermore, Mori *et al.* (1983) developed a way to stabilize linear systems with delayed state.

The stability of a linear delay-differential system with a point delay in its state has been studied in different works (Mori *et al.*, 1982; Hmamed, 1985; 1986 a-b; Mori, 1986; Bourlès, 1987; Mori and Kokame, 1989). Mainly, stability criteria for delay-differential systems can be classified into two categories depending on if they include information on delays or not (Amemiya, 1989). In this note, several criteria in order to design stabilizing control laws for linear delay-differential systems with a point delay in their state are introduced by using the matrix measure and a new measure called delay measure.

The paper is organized as follows. Section 2 introduces the concepts of measure and delay measure and points out some properties of the delay measure. Section 3 introduces some stability results by using the delay measure notation. Section 4 presents the main results concerning the stabilizability of a class of hereditary systems. Finally, conclusions end the paper.

2. Matrix measure and delay measure. Matrix measure has been widely used in the literature when dealing with stability of delay-differential systems (see, for instance, Mori). The matrix measure μ for matrix X is defined as follows:

$$\mu(X) \equiv \lim_{\varepsilon \to 0} \frac{\|I + \varepsilon X\| - 1}{\varepsilon}. \tag{1}$$

The following lemma provides some properties for the matrix measure $\mu(\cdot)$:

Lemma 2.1 (Desoer and Vidyasagar, 1975). *For any matrices $X, Y \in C^{n\times n}$ the following inequalities hold:*

$$\text{(i)} \quad \operatorname{Re}\lambda_i(X) \leqslant \mu(X), \tag{2}$$

$$\text{(ii)} \quad -\mu(jX) \leqslant \operatorname{Im}\lambda_i(X) \leqslant \mu(-jX), \tag{3}$$

$$\text{(iii)} \quad \mu(X+Y) \leqslant \mu(X) = \mu(Y), \tag{4}$$

$$\text{(iv)} \quad \mu(X) \leqslant \|X\|. \tag{5}$$

In addition, the following equality holds:

$$\text{(v) } \mu(\varepsilon X) = \varepsilon\mu(X), \quad \textit{for any} \quad \varepsilon \geqslant 0. \tag{6}$$

The matrix measure defined in equation (1) can be subdefined in three different ways according to the norm utilized in its definition. Therefore, there are three different ways to compute the matrix measure.

(i) If one considers 1-norms, then the matrix measure can be computed as follows:

$$\mu_1(X) = \max_k \left(\operatorname{Re}(x_{kk}) + \sum_{\substack{i=1 \\ \neq k}}^{n} |x_{ik}| \right). \tag{7}$$

(ii) Alternatively, considering 2-norms, one gets:

$$\mu_2(X) = \frac{1}{2}\max_i \lambda_i(X^* + X), \tag{8}$$

where $*$ denotes transpose conjuate.

(iii) Finally, with ∞-norms, one gets:

$$\mu_\infty(X) = \max_i \left(\mathrm{Re}, (x_{ii}) + \sum_{\substack{k=1\\ \neq i}}^{n} |x_{ki}|\right). \tag{9}$$

Consider the following class of linear delay-differential systems with two point delays in the state and in the control variables:

$$\dot{x}(t) = Ax(t) + A_0x(t-h) + Bu(t) + B_0u(t-q), \quad h, q \in R^+ \tag{10}$$

where $A, A_0, B, B_0 \in W \subset R^{n\times n}$, being W the set of n-matrices Q such that $\|Q\| < \infty$.

DEFINITION 2.1. The Delay Measure for system (10) is defined as follows:

$$\xi(h,q) \equiv \frac{\|A_0\|h + \|B_0\|q}{\mu(A) + \mu(B)}. \tag{11}$$

REMARK 2.1. If there is no delay (i.e., $h, q = 0$, or A_0 and b_0 are null matrices), then the delay measure is zero. On the other hand, if the point delays h and q verify $0 < h < \infty$, $0 < q < \infty$, and there is not a delay-free term (i.e., A, B are matrices of zeros) then the delay measure is infinite. Therefore, the delay measure can be considered, intuitively, as a way to evaluate the effect of delay terms in a system compared with its delay free terms.

Some properties of the delay measure are outlined.

PROPERTY 2.1. Lower bounds for the first derivatives of the delay measure:

$$\rho_h \equiv \frac{\partial \xi(h,q)}{\partial h} = \frac{\|A_0\|}{\mu(A) + \mu(B)} \geqslant \frac{\|A_0\|}{\|A\| + \|B\|}, \tag{12}$$

$$\rho_q \equiv \frac{\partial \xi(h,q)}{\partial q} = \frac{\|B_0\|}{\mu(A) + \mu(B)} \geqslant \frac{\|B_0\|}{\|A\| + \|B\|}. \tag{13}$$

PROPERTY 2.2. Ratio of the first derivatives of the delay measure:

$$\frac{\rho_h}{\rho_q} = \frac{\|A_0\|}{\|B_0\|}. \tag{14}$$

PROPERTY 2.3. Absolute lower bound for the delay measure (supposing h, q variables):

$$\xi(h,q) = \frac{\|A_0\|h + \|B_0\|q}{\mu(A) + \mu(B)} \geqslant \frac{\|A_0\|\hat{h} + \|B_0\|\hat{q}}{\|A\| + \|B\|}$$
$$\text{with } \hat{h} = \min h \text{ and } \hat{q} = \min q. \tag{15}$$

REMARK 2.2. Observed that Property 2.1 helps to estimate boundedness conditions for the variations in value of the delay measure; at the same time, Property 2.2 permits to deduce the value of a partial derivative of the delay measure when the other one is known. In addition, Property 2.3 gives absolute boundedness conditions for the delay measure, provided that n-matrices appearing in (10) belong to the class W.

3. Stability conditions expressed by using delay-measure notation. In this section, some stability results for a class of free delay-differential systems are introduced under delay-measure notation. This representation will be useful in order to deduce the main stabilizability results that are to be presented in Section 4.

Consider the free linear delay-differential system:

$$\dot{x}(t) = Ax(t) + A_0x(t-h), \quad \text{with} \quad A, A_0 \in W. \tag{16}$$

Lemma 3.1. *Provided $h \geqslant 1$, a sufficient condition for system (16) to be stable is*

$$\xi(h) < -1. \tag{17}$$

Proof. Observe that for system (16) the delay measure is reduced to

$$\xi(h,q) = \xi(h) = \frac{\|A_0\|h}{\mu(A)}. \tag{18}$$

Suppose $\xi(h) < -1$, then

$$\frac{\|A_0\|h}{\mu(A)} < -1 \Rightarrow -\mu(A) > \|A_0\|h \Rightarrow \mu(A) < -\|A_0\|h. \tag{19}$$

As $h \geqslant 1$ then

$$\mu(A) < -\|A_0\| \Rightarrow \mu(A) + \|A_0\| < 0, \tag{20}$$

that is one of the simplest conditions for stability in system (18) (Mori *et al.* 1982).

Lemma 3.2 (Mori and Kokame 1989). *Consider system* (16). *assume that* $L1 := \mu(A) + \|A_0\| \geqslant 0$ (otherwise system (16) is stable because of (20)) *and* $L2 := \mu(-jA) + \|A_0\|$ ($j^2 = -1$). *If no solutions of the characteristic equation of* (16)

$$\det\left(sI - A - A_0 \exp(-hs)\right) = 0 \tag{21}$$

lie in the rectangular region Σ *shown in Fig. 3.1, then system* (16) *is asymptotically stable.*

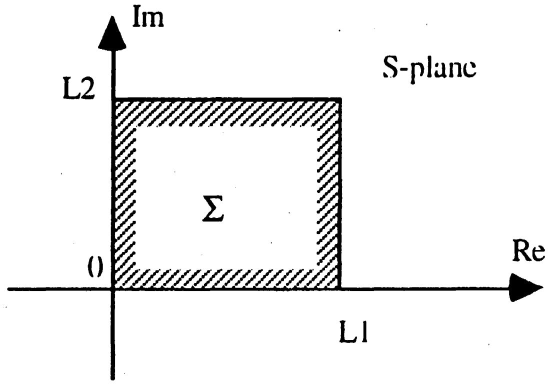

Fig. 3.1. Existence region of unstable characteristic roots in the S-plane for Lemma 3.2.

An equivalence of Lemma 3.2 under Delay-Measure notation can be established as follows:

Lemma 3.3. *Consider system* (16) *and suppose* $-h \leqslant \xi(h)$. *Consider the auxiliary complex system:*

$$\dot{x}(t) = -jAx(t) + A_0x(t-h). \tag{22}$$

Assume

$$M1 := \|A_0\|\left[\frac{h}{\xi(h)} + 1\right] \quad \text{and} \quad M2 := \|A_0\|\left[\frac{h}{\xi_{\text{complex}}(h)} + 1\right]. \tag{23}$$

Then, if no solutions of the characteristic equation of (16) *lie in the rectangular region* Λ *shown in Fig. 3.2, system* (16) *is asymptotically stable.*

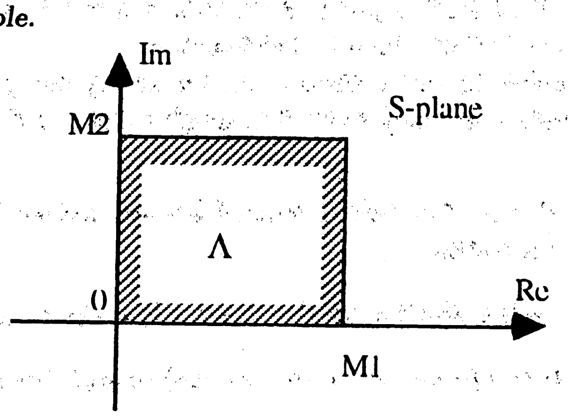

Fig. 3.2. Existence region of unstable characteristic roots in the S-plane for Lemma 3.3.

Proof. Firstly, observe that condition $-h \leqslant \xi(h)$ implies that:

$$-h \leqslant \frac{h\|A_0\|}{\mu(A)} \Rightarrow -1 \leqslant \frac{\|A_0\|}{\mu(A)} \Rightarrow 1 \geqslant \frac{-\|A_0\|}{\mu(A)}. \tag{24}$$

Then one gets: $\mu(A) \geqslant -\|A_0\| \Rightarrow \mu(A) + \|A_0\| \geqslant 0$, that is the same pre-condition than in Lemma 3.2.

Furthermore, quantities $M1$ and $M2$ verify:

$$\begin{aligned} M1 &= \|A_0\|\left[\frac{h}{\xi(h)} + 1\right] = \|A_0\|\left[\frac{\mu(A)\cdot h}{\|A_0\|h} + 1\right] \\ &= \mu(A) + \|A_0\| \geqslant 0, \end{aligned} \tag{25}$$

$$\begin{aligned} M2 &= \|A_0\|\left[\frac{h}{\xi_{\text{complex}}(h)} + 1\right] = \|A_0\|\left[\frac{\mu(-jA)\cdot h}{\|A_0\|h} + 1\right] \\ &= \mu(-jA) + \|A_0\|. \end{aligned} \tag{26}$$

But $M1 = L1 \geqslant 0$ and $M2 = L2$.

4. Stabilizability of linear delay-different systems. In this section conditions for a control law to stabilize a linear system with delayed state will be discussed and several results are to be introduced. Consider the linear delayed system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0 x(t-h) + Bu(t) + B_0 u(t-h), \\ x(t) &= \varphi(t), \ t \in [-h, 0], \end{aligned} \tag{27}$$

where $x \in R^n, u \in R^m, A_0, A \in R^{n\times n}, B, B_0 \in R^{n\times m}, h \geqslant 0$ and $\varphi(t)$ is a continuous vector-valued initial function.

The following result refers to stabilizability for system (27) by using a control law $u(t)$ defined through a delay-differential equation.

Result 4.1. *Consider a control law $u(t)$ defined by the delay-differential equation*

$$\dot{u}(t) = Dx(t) + Eu(t) + D_0 x(t-h) + E_0 u(t-h). \tag{28}$$

A sufficient condition for control law (28) *to stabilize system* (27) *is given by*

$$\xi(h) < -h, \tag{29}$$

where the delay measure is refered to the extended system

$$\dot{z} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} z(t) + \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix} z(t-h). \tag{30}$$

Proof. Firstly observe that the two delay-differential equations (27) and (28) can be rewritten as one single delay-differential equation as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix} \begin{bmatrix} x(t-h) \\ u(t-h) \end{bmatrix}. \tag{31}$$

Define

$$z(t) = [x^T(t)\ u^T(t)]^T; \ \widetilde{A} = \begin{bmatrix} A & B \\ D & E \end{bmatrix}; \qquad \widetilde{A}_0 = \begin{bmatrix} A_0 & B_0 \\ D_0 & E_0 \end{bmatrix}. \tag{32}$$

Then equation (31) can be rewritten as

$$\dot{z} = \widetilde{A}z(t) + \widetilde{A}_0 z(t-h). \tag{33}$$

If, by hypothesis, condition (29) holds for system (33) then

$$\frac{\|\widetilde{A}_0\| \cdot h}{\mu(\widetilde{A})} < -h \Rightarrow \|\widetilde{A}_0\| < -\mu(\widetilde{A}) \Rightarrow \|\widetilde{A}_0\| + \mu(\widetilde{A}) < 0. \tag{34}$$

Therefore, system (33) is stable, which implies that control law (28) stabilizes system (27).

The two following corollaries are immediatly deduced from Result 4.1.

Corollary 4.1. System (27) is stabilizable by control law (28) if

$$\xi(h) < -1. \tag{35}$$

Corollary 4.2. System (27) is stabilizable by control law (28) if

$$\|\widetilde{A}_0\| < \frac{1}{h} \lim_{\varepsilon \to 0} \frac{1 - \|I + \varepsilon \widetilde{A}\|}{\varepsilon}, \tag{36}$$

provided $h \geqslant 1$.

Proof of Corollary 4.2. By hypothesis, condition (36) holds. Then

$$\|\widetilde{A}_0\| < -\frac{1}{h} \cdot \mu(\widetilde{A}) \Rightarrow \xi(h) = \frac{\|\widetilde{A}_0\| h}{\mu(\widetilde{A})} < -1. \tag{37}$$

But, by Lemma 3.1, provided $h \geqslant 1$, (37) is a sufficient condition for stability in a system like (33).

Consider now a system defined by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0 x(t-h) + Bu(t), \\ x(t) &= \varphi(t), \ t \in [-h, 0]. \end{aligned} \tag{38}$$

The following result refers to stabilizability for system (38) by using a control law $u(t)$ which is proportional to the state vector $x(t)$.

Result 4.2. Consider a control law $u(t)$ defined as

$$u(t) = kx(t), \quad k \ \text{ real.} \tag{39}$$

Then system (40) is stabilizable by control law (41) if

$$\xi(h) > \frac{\|A_0\| h}{(k-1)\cdot\mu(B) - \|A_0\|}. \tag{40}$$

Proof. The following implications hold:

$$\begin{aligned} &\frac{\|A_0\| h}{\mu(A)+\mu(B)} > \frac{\|A_0\| h}{(k-1)\cdot\mu(B) - \|A_0\|} \\ &\Rightarrow \mu(A)+\mu(B) < (k-1)\cdot\mu(B) - \|A_0\| \\ &\Rightarrow \mu(A) + k\cdot\mu(b) + \|A_0\| < 0 \,. \end{aligned} \tag{41}$$

By using property (v) of Lemma 2.1 one gets

$$\mu(A) + \mu(kB) + \|A_0\| < 0, \tag{42}$$

but property (iii), Lemma 2.1, leads to

$$\mu(A+kB) + \|A_0\| \leqslant \mu(A) + \mu(kB) + \|A_0\| < 0. \tag{43}$$

Finally, provided $u(t)$ defined in Eq. (39), let's see that system (38) can be rewritten as

$$\begin{aligned} &\dot{x}(t) = (A+kB)x(t) + A_0x(t-h) = \overline{A}x(t) + A_0x(t-h), \\ &x(t) = \varphi(t), \quad t \in [-h, 0], \end{aligned} \tag{44}$$

where $\overline{A} = A + kB$. But Eq. (43) becomes $\mu(\overline{A}) + \|A_0\| < 0$, which is a sufficient condition for stability in system (44) (Mori *et al.* 1982).

5. Conclusions. This note has introduced two important results for stabilizability of a class of linear delay-differential systems, by using the delay-measure notation. The concept of delay-measure allows to express stabilizability results in a very simple way. The delay-measure function can be implemented for computational purposes and permits to establish a study about in what measure the stability depends on the delay terms.

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Received April 1993

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