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HOW TO CHOOSE A PIVOT ELEMENT IN SIMPLEX METHOD?

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Abstract. New computational rules of the simplex method are represented. They differ from classical rules in the sense that the column corresponding to the objective function is also transformed and first the pivot row and then the pivot column is determined. In this case the most negative element in pivot row can be chosen for pivot element. In evaluating procedure systems equivalent to systems appearing in classical simplex method are used and theoretically they determine the same sequence of basic solutions. The calculations are more precise due to such choice of pivot element and it is assured also by the results of test-problems.

Key words: pivot element, simplex method, ill-conditioned problem of linear programming.

1. Description of the algorithm of the simplex method. Let us consider the linear programming problem in the following form

$$\begin{gathered} a_{11}x_1 + \ldots + a_{1n}x_n + a_{1n+1}z = b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m+11}x_1 + \ldots + a_{m+1n}x_n + a_{m+1n+1}z = b_{m+1} \\ z \to \max \\ x \geqslant 0. \end{gathered} \tag{1}$$

The problem obtains such form after the 0th step when from the last row corresponding to the objective function the $(m+1)$th variable has been entered to the basis (see Example 1). Assume that $a_{m+1n+1} > 0$ and $b_{m+1} = 0$ (see Remark v below). Describe the algorithm SIMP for solution of the problem (1) with the simplex method. Assume that A is an $(m+1) \times (n+1)$ matrix and b is an

$(m+1)$ vector. Use for the transformed elements of A and b the same notations.

Algorithm SIMP $(A, b, x, z, \varepsilon, m, n)$.

1. Determine the initial basis.
2. Initialize $z = 0$.
3. Assign to the index of the pivot row the value $l = m + 1$.
4. Find $RE = \min_{j=1,\dots,n} a_{lj} = a_{lk}$.
5. If $-\varepsilon \leqslant RE$ then go to step 13.
6. Enter variable x_k into the basis.
7. Make Gauss eliminations with the pivot element a_{lk}.
8. Find $\min_{a_{in+1} > \varepsilon} b_i / a_{in+1} = b_l / a_{ln+1}$.
9. Delete the variable x_{j_l} corresponding to the lth row from the basis.
10. If $a_{in+1} \leqslant \varepsilon$, $i = 1, \dots, m+1$, then the linear programming problem has no finite optimal solution, stop.
11. Let $z = b_l / a_{ln+1}$.
12. Go to step 4.
13. Find the optimal solution

$$x_{j_i} = b_i - a_{in+1} z, \quad i = 1, \dots, m+1.$$

14. Stop.

REMARKS.

i) at each step at least one of the basic variables is equal to 0. Contrary to the ordinary simplex method z is the only nonbasic variable which value differs from 0.

ii) at every step the pivot element is negative (according to the step 8 in pivot row $a_{in+1} > 0$).

iii) in non-degenerate case the classical simplex method and the method proposed here determine the same sequence of basic solutions, because their estimates differ according to a constant coefficient. For instance, in the example proposed in this article a new sequence of estimates is evaluated according to the formula $A_3 = A_3 - A_4/3$ and in the classical simplex method according to the formula $A_4 - 3A_3$, the coefficient is $-1/3$. Therefore, the method proposed in the present article and the classical simplex method are two different descriptions of one method and there is no need to prove statements and criterion established in this article.

iv) as in our method pivot element is most negative element in the pivot row then it enables us to solve the problem more precisely compared with the classical method, see, e.g., the example with Hilbert matrix.

v) in the description of SIMP it is assumed that the initial value of the objective function $z^0 = 0$. If this assumption is not fulfilled the SIMP must be slightly changed, see, e.g., Example 2. However, it is more convenient to shift z then assign $z^0 = -b_{m+1} = 0$ and after finding the optimal solution change the maximum value z^* by the same quality.

vi) if all the elements in the column corresponding to the variable z $a_{in+1} \leqslant 0$, $i = 1, \dots, m+1$ then the objective function is unbounded.

EXAMPLE 1.

$$\begin{aligned} -x_1 + x_2 + x_3 + x_4 \qquad\qquad &= 2 \\ x_1 + x_2 + x_3 \qquad + x_5 \qquad &= 4 \\ x_3 \qquad\qquad + x_6 &= 1 \\ x_1 + 2x_2 + 3x_3 = z \to \max & \\ x \geqslant 0. & \end{aligned}$$

The solution of this problem with the new rule for choosing the pivot element is presented in the Table below.

At the initial step in addition to the basis x_4, x_5, x_6 one more variable is entered. The last row which corresponds to the objective function is chosen for the pivot row, pivot element is $-c_3 = -3$. First the variable leaving the basis and then variable entering the basis are determined. At the first step let us express the basis variables through z:

$$x_4 = 2 - z/3, \; x_5 = 4 - z/3, \; x_6 = 1 - z/3, \; x_3 = z/3.$$

Table 1.

Step	x_1	x_2	x_3	x_4	x_5	x_6	z	b
	−1	1	1	1	0	0	0	2
0	1	1	1	0	1	0	0	4
	0	0	1	0	0	1	0	1
	−1	−2	[-3]	0	0	0	1	0
	−4/3	1/3	0	1	0	0	1/3	2
1	2/3	1/3	0	0	1	0	1/3	4
	−1/3	[-2/3]	0	0	0	1	1/3	1
	1/3	2/3	1	0	0	0	−1/3	0
	[-3/2]	0	0	1	0	1/2	1/2	5/2
2	1/2	0	0	0	1	1/2	1/2	9/2
	1/2	1	0	0	0	−3/2	−1/2	−3/2
	0	0	1	0	0	1	0	1
	1	0	0	−2/3	0	−1/3	−1/3	−5/3
	0	0	0	1/3	1	2/3	2/3	16/3
3	0	1	0	1/3	0	−4/3	−1/3	−2/3
	0	0	1	0	0	1	0	1

For the initial basis $z = z_0 = 0$. If z increases and $z = 3$ then $x_6 = 0$. In general, the variable leaving the basis is determined by the minimal ratio of the elements of two last columns, taking into account only positive elements of the z-columns. At the first step x_6 leaves the basis, the estimates of variables are on the row corresponding to x_6. As z has there a positive coefficient then for finding a new pivot element it is necessary to find the minimal element in this row, i.e., $a_{32} = -2/3$. At the second step the minimal ratio of the elements of the two last columns is on the first row, z is increasing up to 5, x_4 leaves the basis and x_1 enters the basis, the first row is pivoting. At the last step z increases up to 8, $x_5 = (16-2z)/3$, the second row is pivoting and as all the coefficients in this row are nonnegative then the solution found is optimal, $x^* = (1,2,1,0,0,0)^T$, $z^* = 8$.

Optimality criterion: all elements in the pivot row are non-negative.

vii) let us consider now an example on two-phase simplex method.

EXAMPLE 2.

$$
\begin{aligned}
x_1 + 2x_2 - x_3 \qquad &= 4 \\
x_1 + x_2 \qquad + x_4 &= 6 \\
-x_1 - 3x_2 = z &\to \max \\
x &\geqslant 0.
\end{aligned}
$$

To the first restriction we add an artificial basic variable x_5 and use the objective function $-x_5 = z \to \max$

After elimination x_5 from the objective function the initial value of z at the 0th step $z^0 = -4$ and x_2 will enter the basis. At the first step z increases up to 0, x_5 is leaving the basis and the first phase of the simplex method is completed. In the table the column corresponding to x_5 and the first row are deleted. The first row is replaced by the coefficients of the objective function of the initial problem and the coefficients of the z column are changed. At the 2nd step in the first row of the table x_2 is deleted. At the 3rd step an initial solution $x_2 = 2,\ x_4 = 4,\ z = -6$ is found. At the 4th step $x_2 = -4 - z,\ x_4 = -2 - z,\ x_1 = 12 + 2z,\ z$ increases up to -4, the third row is the pivot row. The criterion of optimality is fulfilled $z^* = -4,\ x_2^* = 0,\ x_4^* = 2,\ x_1^* = 4$.

There is *no feasible solutions* if after the first phase the maximum of the objective function is negative.

viii) To find an initial basis one can use also the algorithm VRMA (Ubi, 1991a), where orthogonal transformations of columns are used instead of Gauss eliminations. This algorithm takes into account also the coefficients of the objective function and so it may occur that an initial solution is optimal.

ix) as in the method proposed negative pivot elements and pivot row is determined before determining the pivot column so in the case of a degeneracy basis there appear differences compared

Table 2.

Step	x_1	x_2	x_3	x_4	x_5	z	b
	1	2	−1	0	1	0	4
	1	1	0	1	0	0	6
	0	0	0	0	1	1	0
	1	2	−1	0	1	0	4
0	1	1	0	1	0	0	6
	−1	−2	1	0	0	1	−4
	0	0	0	0	1	1	0
1	1/2	0	1/2	1	0	1/2	4
	1/2	1	−1/2	0	0	−1/2	2
	1	3	0	0		1	0
2	1/2	0	1/2	1		0	4
	1/2	1	−1/2	0		0	2
	−1/2	0	3/2	0		1	−6
3	1/2	0	1/2	1		0	4
	1/2	1	−1/2	0		0	2
	1	0	−3	0		−2	12
4	0	0	2	1		1	−2
	0	1	1	0		1	−4

with the classical method. Solving problems with the degenerate basis cycle was never arised if the following rule was followed: if at the 8th step the pivot row is not uniquely determined then divided elements of these rows with respective a_{in+1} and take for a pivot row the lexicograhically minimal one.

2. Description of the algorithm of revised simplex method. In the following table the solution of the first example with the revised simplex method is presented. It has begun from the system at the first step (when x_3 is entered the basis).

Table 3.

Step		b	P^{-1}				a_{n+1}	λ
	a_4	2	1	0	0	0	1/3	1/3
	a_5	4	0	1	0	0	1/3	1/3
1	a_6	1	0	0	1	0	1/3	**[-2/3]**
	a_3	0	0	0	0	1	−1/3	2/3
	a_4	5/2	1	0	1/2	0	1/2	**[-3/2]**
	a_5	9/2	0	1	1/2	0	1/2	1/2
2	a_2	−3/2	0	0	−3/2	0	−1/2	1/2
	a_3	1	0	0	1	1	0	0
		−5/3	−2/3	0	−1/3	0	−1/3	
3		16/3	1/3	1	2/3	0	2/3	
		−2/3	1/3	0	−4/3	0	−1/3	
		1	0	0	1	1	0	

Here P^{-1} denotes the inverse of the basic matrix, a_{n+1} the $(n+1)$th column of the transformed matrix A (usually a_{n+1} is not used in the revised simplex method), a_k the column corresponding to the variables entering the basis, $\lambda = P^{-1}a_k$, see the description of the algorithm below. At the first step according to the 11th step of the algorithm MSIMP for nonbasic variables find $P_3^{-1}a_1 = -1/3$, $P_3^{-1}a_2 = -2/3$. Therefore, into the basis instead of x_6 the variable x_2 is entered. At the second step $P_1^{-1}a_1 = -3/2$, $P_1^{-1}a_6 = 1/2$ instead of x_4 the variable x_1 will enter the basis. At the third step the criterion of optimality is fulfilled, optimal values of the variables are found according to the formulas introduced at the 17th step.

Describe the algorithm MSIMP for solving the problem (1) with revised simplex method. Besides $(m+1) \times (n+1)$ matrix A and $(m+1)$ vector b an $(m+1) \times (m+1)$ matrix P^{-1} which is the inverse of the basic matrix P and $(m+1)$ vector λ are needed. Assume that $a_{m+1n+1} > 0$, $b_{m+1} = 0$.

Algorithm MSIMP $(A, b, P^{-1}, \lambda, x, m, n, z, \varepsilon)$.

1. Find an initial basis.
2. Let $z = 0$.
3. Evaluate $RE = \min_{j=1,\dots,n} a_{m+1j} = a_{m+1k}$.
4. If $\varepsilon \leqslant RE$ then go to step 17.
5. Enter x_k into the basis.
6. Fulfill Gauss eliminations with the pivot element a_{m+1k}.
7. Find $\min_{a_{in+1}>\varepsilon} b_i/a_{in+1} = b_l/a_{ln+1}$.
8. Delete the variable x_{j_l} corresponding to the lth row from the basis.
9. If $a_{in+1} \leqslant \varepsilon$, $i = 1, \dots, m+1$ then the objective function is unbounded, stop.
10. Let $z = b_l/a_{ln+1}$.
11. Find $RE = \min_j P_l^{-1} a_j = a_{lk}$ which is evaluated for all non-basic columns $a_j, j = 1, \dots, n$ and where P_l^{-1} denotes the lth row of the matrix P^{-1}.
12. If $-\varepsilon \leqslant RE$ then go to step 17.
13. Enter x_k into the basis.
14. Evaluate the vector $\lambda = P^{-1} a_k$.
15. Fulfill Gauss eliminations with the pivot element $a_{lk} = \lambda_l$ to b, P^{-1}, a_{n+1}.
16. Go to step 7.
17. Find the optimal solution

$$x_{j_i} = b_i - a_{in+1} z, \quad i = 1, \dots, m+1.$$

18. Stop.

REMARK. Contrary to the commonly used version of the simplex method here estimates basing on dual variables are not calculated. They are found at the 11th step with the aid of the inverse matrix taking into account that the place of estimates changes at each step.

3. Numerical experiments. The program of MSIMP has been written in FORTRAN-77 for ES-1055M with VM. For all variables double-exactness was used. In both of examples $\varepsilon = 10^{-15}$.

EXAMPLE 3. Let us consider a linear programming problem with Hilbert matrix, $a_{ij} = 1/(i+j)$, $b_i = \sum_{k=1}^{m} 1/(k+i)$, $c_j = b_j + 1/(j+1)$, $i, j = 1, \ldots, m$. Inequality constraints are transformed to equality constraints with the aid of slack variables which form an initial basis. The optimal solution $x_j^* = 1$ was found for $m = 8$ with the exactness $\Delta = 10^{-4}$. At the 11th step of MSIMP the pivot element $RE = -0.5 \cdot 10^{-9}$. For $m = 9$ the same quantities were $\Delta = 10^{-3}$, $RE = -0.4 \cdot 10^{-11}$, for $m = 10$ $\Delta = 10^{-2}$, $RE = -0.2 \cdot 10^{-12}$. The maximum value of the objective function for $m = 11$ is found with exactness 10^{-14} but at the 10th step of MSIMP the running values of z descreases due to miscalculations and it is impossible to solve the problem at all. Analogously, for $m > 11$ it is impossible to find x^* although values of the objective functions are determined with great exactness.

EXAMPLE 4.

$$
\begin{aligned}
(1+t)x_1 + x_2 + x_3 + x_4 &\leqslant 4 + t \\
x_1 \qquad + x_3 + x_4 &\leqslant 3 \\
x_1 \qquad\qquad + x_4 &\leqslant 2 \\
x_1 + x_2 + x_3 + x_4 &= z \to \max \\
x &\geqslant 0.
\end{aligned}
$$

The slack variables x_5, x_6, x_7 belong to the initial basis. For $t = 10^{-10}$ with this algorithm the optimal solution $x = (0{,}000\,000\,9572;$ $2{,}000\,000\,0001;$ $0{,}000\,000\,0000;$ $1{,}999\,999\,0418;$ $0{,}000\,000\,0000;$ $1{,}000\,000\,0000;$ $0{,}000\,000\,0000)^T$ was found. It is close to one of the optimal solution $x^* = (0, 2+t, 0, 2, 0, 1, 0)^T$.

Therefore, with MSIMP one can solve problems more precisely than many other widely know program packages but still not so exactly as VRMSIM (Ubi, 1991b). For example, well-know programs solve the problem with Hilbert matrix only for m in the interval from 4 to 8. Solving problems with VRMSIM the greater exactness is obtained due to greater labour consuming. Besides, basic matrix is used in the triangular form and evaluating simplex-tables only orthogonal transformations are used.

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