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MAXIMIZING THROUGHPUT OF FINITE POPULATION SLOTTED ALOHA

Rudolf MATHAR

Institute of Statistics

Aachen University of Technology

Antanas ŽILINSKAS*

Institute of Mathematics and Informatics &

Pedagogical University, Vilnius

Abstract. We consider finite population slotted ALOHA where each of n terminals may have its own transmission probability p_i. Given the traffic load λ, throughput is maximized via a constrained optimization problem. The results of Abramson (1985) are obtained as special case.

Key words: performance analysis, data transmision systems, throughput maximization.

The throughput d_A of the ALOHA–protocol is well known to be $d_A = \lambda e^{-2\lambda}$ (see e.g.,Abramson, 1970; Tanenbaum 1988). To obtain this result it is assumed that data packets arrive according to a homogeneous Poisson process with constant intensity $\lambda > 0$. Without loss of generality it may be assumed that data packets have a length of one time unit, i.e. the time scale is determined by the packet length. Throughput is defined as the average number of successfully received packets per time unit.

The maximum throughput of ALOHA is $1/2e \approx 0.184$, which is achieved at $\lambda = 1/2$ (one packet arriving each two time units on the average). This is a relatively small value, and Roberts (1972) published a protocol for doubling the capacity. His method is known as *slotted* ALOHA and works as follows. Time is devided into slots

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of just one packet length. A station is not allowed to send whenever it wishes, but instead has to wait for the beginning of the next slot. Slotted ALOHA has been analyzed in an approximate model to have throughput $d_{sA} = \lambda e^{-\lambda}$ which is maximum for $\lambda = 1$ with value $1/e$. Analogously to the continuous model λ means the expected number of packets transmitted per slot. If two or more packets are transmitted in the same slot collisions occur and all packets involved are destroyed by superposition. The corresponding model assumes a large number of users K, each independently transmitting with equal probability λ/K. Throughput is obtained by considering the limit with $K \to \infty$ (see e.g., Roberts, 1972; Tanenbaum, 1988).

In this note we consider finite population slotted ALOHA with different access probabilities. Abramson (1985) has investigated a particular model of this type with fixed traffic load $\lambda = 1$. He considered two groups of n_1 and n_2 users, respectively, each with different access probabilities g_1 and g_2. He concludes that the asymmetric case (g_1 large and g_2 small such that $n_1 g_1 + n_2 g_2 = \lambda = 1$) achieves large overall throughput given in (1).

We extend the results of Abramson by maximizing throughput over all access probabilities of n individual users such that the traffic load is fixed. As a special case we observe the claim of Abramson concerning asymmetricity, but only for small values of λ. For λ approximately larger than $e = 2.718...$ again a symmetric distribution of traffic load turns out to be most favorable.

Let us assume a finite community of n users. Each of them transmits in a slot independently of each other with probability p_i, $0 \leqslant p_i \leqslant 1$, $i = 1, \ldots, n$. The expected traffic load, i.e. the average number of packets transmitted in a slot is $\lambda = \sum_{i=1}^{n} p_i$. The probability that a packet will be successfully transmitted in a particular slot obviously coincides with the throughput, and is given by

$$d(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i \prod_{j \neq i} (1 - p_j). \tag{1}$$

Maximizing throughput w.r.t. a fixed traffic load $0 < \lambda \leqslant n$ may be formulated as to

$$\text{maximize } d(p_1, \ldots, p_n) \quad \text{such that} \quad 0 \leqslant p_i \leqslant 1, \ \sum_{i=1}^{n} p_i = \lambda > 0.$$

By substituting $x_i = 1 - p_i$, $i = 1, \ldots, n$, it is easily seen that

$$d(p_1, \ldots, p_n) = \bar{d}(x_1, \ldots, x_n) = S_{n-1}(x) - nS_n(x),$$

where $x = (x_1, \ldots, x_n)$. $S_{n-1}(x) = \sum_{i=1}^{n} \prod_{j \neq i} x_j$ and $S_n(x) = \prod_{j=1}^{n} x_j$ are the elementary symmetric functions of order $n-1$ and n, respectively. Both are Schur-concave and increasing (cf. Marshall and Olkin, 1979), such that maximization of $d(p)$ means to maximize the difference of two Schur-concave functions. Indeed, the theory of majorization does not help much in this case.

Let $p = (p_1, \ldots, p_n)$. In case $\lambda \leqslant 1$ the solution $p^* = (p_1^*, \ldots, p_n^*)$ is obvious since

$$d(p) \leqslant \sum_{i=1}^{n} p_i = \lambda,$$

and equality $d(p^*) = \lambda$ holds whenever $p_k^* = \lambda$ for some k, and $p_i^* = 0$ for $i \neq k$.

We now consider the more complicated case $\lambda > 1$, and first state some preliminary results concerning boundary points of the constraining set

$$C = \left\{ p = (p_1, \ldots, p_n) \mid 0 \leqslant p_i \leqslant 1, \ \sum_{i=1}^{n} = \lambda \right\}.$$

If $p_i = 1$ for at least two different components of p we have $d(p) = 0$, which excludes p as a maximum point. If just one component of p equals 1, without loss of generality we may assume $p_n = 1$, then

$$d(p_1, \ldots, p_n) = \prod_{j=1}^{n-1} (1 - p_j).$$

From (Marshall and Olkin, 1979, p.79) it is easily concluded that $\prod_{j=1}^{n-1}(1-p_j)$ is a Schur–concave function such that the maximum of $d(p_1,\ldots,p_{n-1},1)$ over C is attained at $p_i^* = \frac{\lambda-1}{n-1}$, $i=1,\ldots,n-1$, and by symmetry of d, for any vector with permuted components.

This discussion shows that $p^* = (p_1^*,\ldots,p_n^*)$ with $p_k^* = 1$ for just one component and $p_i^* = \frac{\lambda-1}{n-1}$ for $i \neq k$ is a candidate for a maximum point over the boundary of C with value $d(p^*) = (\frac{n-\lambda}{n-1})^{n-1}$.

We now investigate interior points of C by a Langrangian setup. We search for stationary points by solving the system of equations

$$\nabla d(p) + \gamma \nabla g(p) = 0,$$

where $g(p) = \sum_{i=1}^n p_i - \lambda = 0$ describes the restrictions. Carrying out differentiation yields the following system

$$\prod_{i\neq j}(1-p_i) - \sum_{k\neq j} p_k \prod_{i\neq j, i\neq k}(1-p_i) + \gamma = 0, \quad j=1,\ldots,n, \tag{2}$$
$$g(p) = 0.$$

After multiplying the j-th equation by $(1-p_j)$ we get

$$\prod_{i=1}^n(1-p_i) - \sum_{k\neq j} p_k \prod_{i\neq k}(1-p_i) + \gamma(1-p_j) = 0.$$

This leads to

$$\prod_{i\neq j}(1-p_i) + \gamma(1-p_j) = d(p) \quad \text{for all } j=1,\ldots,n, \tag{3}$$

and the differences of the j–th and k–th equation give

$$\Big(\prod_{i\neq j, i\neq k}(1-p_i) - \gamma\Big)(p_j - p_k) = 0 \quad \text{for all } j,k=1,\ldots,n.$$

If $p_j \neq p_k$ for some $j \neq k$ it follows that $\prod_{i\neq j, i\neq k}(1-p_i) = \gamma$, and from $p_k \neq p_\ell$ for some $\ell \neq k$ it follows that $\prod_{i\neq k, i\neq \ell}(1-p_i) = \gamma$, which yields $p_j = p_\ell$. This shows that stationary points p have at most two different components. Thus, each stationary point p may be represented as $p = (a, \dots, a, b, \dots, b)$ with k entries a and $n-k$ of them equal to b, where $ka + (n-k)b = \lambda$, $k \in \{0, \dots, n\}$.

$k = 0$ and $k = n$ means $a = b = \frac{\lambda}{n}$. By (2), with $\gamma = (\lambda - 1)(1 - \frac{\lambda}{n})^{n-2}$, it is easily verified that the corresponding point $p = (\frac{\lambda}{n}, \dots, \frac{\lambda}{n})$ is stationary.

Now let $1 \leqslant k \leqslant n-1$, $0 \leqslant a, b < 1$, $a \neq b$, and consider the system (3) for corresponding stationary points p:

$$(1-a)^{k-1}(1-b)^{n-k} + \gamma(1-a) = d(p),$$
$$(1-a)^{k}(1-b)^{n-k-1} + \gamma(1-b) = d(p).$$

By elementary transformations it follows that

$$\gamma = (1-a)^{k-1}(1-b)^{n-k-1}.$$

ubstituting γ in (3) and observing that

$$d(p) = (1-a)^{k-1}(1-b)^{n-k-1}(\lambda - nab) = \gamma(\lambda - nab),$$

we obtain the following system for stationary points with two different components.

$$\begin{aligned} nab - a - b &= \lambda - 2, \\ ka + (n-k)b &= \lambda, \\ 0 \leqslant a, b &< 1, a \neq b. \end{aligned} \tag{4}$$

$a = \frac{1}{n}$ easily yields $\lambda = 2 - \frac{1}{n}$. So, if $\lambda \neq 2 - \frac{1}{n}$ some elementary algebra shows that the solutions of system (4) are given by

$$\begin{aligned} a_k &= \frac{1}{2kn}\Big(\sqrt{\Delta} + 2k + n(\lambda - 1)\Big), \\ \bar{a}_k &= \frac{1}{2kn}\Big(-\sqrt{\Delta} + 2k + n(\lambda - 1)\Big), \\ b_k &= \frac{\lambda - ka_k}{n-k}, \\ \bar{b}_k &= \frac{\lambda - k\bar{a}_k}{n-k}, \\ 0 \leqslant a_k, \bar{a}_k, b_k, \bar{b}_k &< 1, a_k \neq b_k, \bar{a}_k \neq \bar{b}_k, \end{aligned} \tag{5}$$

where

$$\Delta = \Delta(n,k) = 4k(k-n)(\lambda n - 2n + 1) + n^2(\lambda - 1)^2.$$

Obviously $\Delta(n,k) = \Delta(n,n-k)$ holds for all $k = 1,\ldots,n-1$. Furthermore it is easily seen that

$$a_k = \bar{b}_{n-k} \quad \text{and} \quad b_k = \bar{a}_{n-k}, \quad k = 1,\ldots,n-1.$$

Thus, changing the sign of $\sqrt{\Delta}$ in (5) eventually gives a symmetric solution $\bar{p}$ with coordinates in reversed order, $\bar{p} = (b,\ldots,b,a,\ldots,a)$.

The case $\lambda = 2 - \frac{1}{n}$ needs some extra consideration. If $a \neq \frac{1}{n}$, from (4) we obtain $b = (a - \frac{1}{n})/(na-1) = \frac{1}{n}$. Thus by symmetry of d,

$$a_k = \frac{1}{n}, \quad b_k = \frac{1}{n-k}\Big(2 - \frac{k+1}{n}\Big), \quad k = 1,\ldots,n-1,$$

are the only stationary points, up to rearranging the components of the corresponding p. If $\lambda = 2 - \frac{1}{n}$ we have $\sqrt{\Delta} = n(\lambda - 1)$, and this in turn gives

$$b_{n-k} = \frac{1}{k}\Big(2 - \frac{n-k+1}{n}\Big) = \frac{n+k-1}{kn} = \frac{1}{2kn}\big(\sqrt{\Delta} + 2 + n(\lambda - 1)\big),$$

which coincides with a_k in (5). This shows that for $\lambda = 2 - \frac{1}{n}$ the corresponding stationary points are already characterized by (5).

The only case missing is $0 < p_1,\ldots,p_r < 1$ and $p_{r+1} = \cdots = p_n = 0$ for some $r \in \{1,\ldots,n-1\}$. Obviously, in this case we encounter the same problem of maximizing (1) over the interior points of C in an r–dimensional space. Following the above considerations stationary points are given by

$$(\lambda/r,\ldots,\lambda/r) \quad \text{and} \quad (a,\ldots,a,b,\ldots,b), \quad k = 1,\ldots,r-1,$$

with $ka + (r-k)b = \lambda$. In general d is not monotone in r at such stationary points. So we have to search for maxima over all corresponding stationary points with dimension ranging from $\lceil\lambda\rceil$ to n. $\lceil\lambda\rceil$ denotes the smallest integer larger than λ.

Summarizing the results obtained so far, the following algorithm gives a solution of (1) for any $1 < \lambda < n$. Let

$$d(n,k,a,b) = (1-a)^{k-1}(1-b)^{n-k-1}(\lambda - nab)$$

denote the value of d at stationary points $p^* \in \mathbb{R}$ with just two different components a and b and multiplicities k and $n-k$, respectively.

We use a PASCAL–like notation for our algorithm. Two essential blocks are easily recognized. In a nested loop we first search for stationary points over the interior of C with dimension ranging from $\lceil\lambda\rceil$ to n. Then the actual maximum value is compared with the value of d at boundary points with just one component 1 and all other $\frac{\lambda-1}{n-1}$. Let $n \geqslant 2$ and $1 < \lambda < n$ be given.

$d_{\max} = -1$;
for $n_1 = \lceil\lambda\rceil$ **to** n **do**
    begin
    for $k = 1$ **to** $n_1 - 1$ **do**
        begin
        $d = -1$; **calculate** $\Delta = \Delta(n_1, k)$;
        if $\Delta \geqslant 0$ **then**
            begin
            $a = \frac{1}{2kn_1}(\sqrt{\Delta} + 2k + n_1(\lambda - 1))$;
            $b = \frac{\lambda - ka}{n_1 - k}$;
            if $0 \leqslant a < 1$ **and** $0 \leqslant b < 1$ **then**
                $d = (1-a)^{k-1}(1-b)^{n_1-k-1}(\lambda - n_1ab)$;
            end;
        if $d > d_{\max}$ **then** $d_{\max} = d$;
        end
    if $d_{\max} < \lambda\left(\frac{n_1-\lambda}{n_1}\right)^{n_1-1}$ **then** $d_{\max} = \lambda\left(\frac{n_1-\lambda}{n_1}\right)^{n_1-1}$;
    end;
if $d_{\max} < \left(\frac{n-\lambda}{n-1}\right)^{n-1}$ **then** $d_{\max} = \left(\frac{n-\lambda}{n-1}\right)^{n-1}$.

After a run of this subroutine $d_{\max}$ contains the maximum value of (1). Of course, the argument where $d_{\max}$ is attained should be stored somewhere.

In certain cases we may exclude points with two different components as stationary points. If $\lambda \neq 2 - \frac{1}{n}$ there are no points with real coordinates satifying (5) whenever $\Delta(k) < 0$. $\Delta(k)$, $k \in \mathbb{R}$, is a parabola which vanishes at

$$k_{1,2} = n\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{(\lambda-2)(n-\lambda)}{(\lambda-2)n+1}}\right).$$

$\Delta(k)$ is concave if $\lambda < 2 - \frac{1}{n}$, with $k_1 \leqslant 0$ and $k_2 \geqslant n$, and convex if $\lambda > 2 - \frac{1}{n}$. $k_{1,2}$ is complex for $2 - \frac{1}{n} \leqslant \lambda < 2$. For any $\lambda \geqslant 2$ it holds that $0 \leqslant k_{1,2} \leqslant n$. In summary, $\Delta(k) \geqslant 0$ for all $k = 1, \ldots, n-1$, whenever $1 < \lambda < 2$, and $\Delta(k) \leqslant 0$ for $k_1 \leqslant k \leqslant k_2$, whenever $\lambda \geqslant 2$.

Now assume that $\lambda \geqslant 3 - \frac{2}{n}$, and $n \geqslant 2$. The first equation of (4) can be easily transformed to

$$nab - (k+1)a - (n-k+1)b + 2 = 0.$$

The assumptions $k = 1$ and w.l.o.g. $a < b$ yield $(nb-2)(a-1) = 0$. If $b = \frac{2}{n}$ then $\lambda = a + \frac{2(n-1)}{n} < 2$ which is a contradiction. Thus, $b \neq \frac{2}{n}$ and $a = 1$, which contradicts $a < 1$. Along the same lines we obtain a contradiction to $k = n-1$ and $a < b$. Consequently, for $\lambda \geqslant 3 - \frac{2}{n}$ we don't have stationary points at $k = 1$ and $k = n-1$.

Moreover, $k_1 < 2$ and $k_2 > n-2$ holds iff $\frac{(\lambda-2)(n-\lambda)}{(\lambda-2)n+1} > (\frac{n-4}{n})^2$. Solving this inequality for λ we get $\Delta(k) < 0$ if $5 - 2\sqrt{2} + \frac{6\sqrt{2}-8}{n} < \lambda < 5 + 2\sqrt{2} - \frac{6\sqrt{2}+8}{n}$..Thus, for large n and approximately $\lambda \in (3, 7.8)$ the only stationary point is given by $p = (\frac{\lambda}{n}, \ldots, \frac{\lambda}{n})$.

REFERENCES

Abramson, N. (1970). The ALOHA system — another alternative for computer communication. *AFIPS, Spring Joint Computer Conference*, **37**, 281–285.

Abramson, N. (1985). Development of the ALOHANET. *IEEE Transactions on Information Theory*, **IT–31**, 119–123.

Marshall, A.W., Olkin, I. (1979). *Inequalities, Theory of Majorization and its Applications*. Academic Press, New York.

Roberts, L.G. (1972). Extensions of packet communication technology to a hand held personal terminal. *AFIPS, Spring Joint Computer Conference*, **40**, 295–298.

Tanenbaum A.S. (1999). *Computer Netw ks*. Prentice Hall, Englewood Cliffs, NJ.

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R. Mathar was born in Kalterherberg, Germany, in 1952. He received his Dipl. Math. and Dr. rer. nat. degree in mathematics from Aachen University of Technology in 1978 and 1981, respectively. In 1985 he received his habilitation degree in mathematics.

In 1986–87 he worked at the European Business School as a lecturer in computer science, and in 1987–89 he joined a research group in applied optimization at the University of Augsburg. Since October 1989 he is Professor of stochastics, especially applications in computer science at Aachen University of Technology.

His research interests are probabilistic performance analysis of networks and mobile data transmission systems.

A. Žilinskas received the Degree of Candidate of Technical Sciences from Kaunas Polytechnic Institute, Lithuania, 1973, and the Degree of Doctor of Physical and Mathematical Sciences from Leningrad University, 1985. He is a senior researcher at the Department of Optimal Decision Theory, Institute of Mathematics and Informatics and a head of the Department of Informatics at Vilnius Pedagogical University.

His research interests are global and stochastic optimization. He is an author or co-author of seven books on optimization methods and applications.