INFORMATICA, 1993, Vol.4, No.1-2, 148-162

OPTIMALITY OF THE REALIZATION OF DIFFERENCE SCHEMES

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Abstract. This paper is devoted to the investigation of the optimality of difference schemes. Some general methods are proposed to accelerate the computations.

Key words: difference schemes, optimality of the algorithm, adaptable difference grids.

Introduction. A new method of theoretical study-computational experiment – is more and more often applied in various fields of nature sciences. Its realization leads to a situation, where we must solve $r > 1$ similar equations and r can be very large. Therefore the minimization of computational costs of proposed algorithms is very important problem. In some cases the minimization of CPU – time is investigated instead of computational costs. We note there that the solution of this new problem depends not only on proposed algorithms, but also on the technical parameters of computers used to execute the numerical simulation. In this paper we will consider some general methods proposed to accelerate the computations. Our aim is to state appropriate optimization problems and to investigate numerical algoritms for their solving. Note that the position of grid points, the number of iterations, or the data storage schemes can be used as free parameters for solving optimization problem. Some closely related questions are investigated by Bajarūnas and Čiegis (1991).

1. Optimality of iterative methods. We begin our analysis with the following example. Iterative methods are implemented

for solving many nonlinear problems. Let assume that we approximated some nonlinear nonstationary differential problem by a difference scheme

$$\frac{\hat{y}-y}{\tau} = f(\hat{y}). \tag{1.1}$$

Suppose for simplicity that for the nonlinear function $f(y)$ the following estimates hold

$$\|f(u)-f(v)\| \leqslant C_1\|u-v\|, \qquad \|f\|_c \leqslant C_0. \tag{1.2}$$

An iterative method is used for solving (1.1)

$$\frac{{}^{s}y-y}{\tau} = f({}^{s-1}y). \tag{1.3}$$

Let ${}^{s}z = \hat{y} - {}^{s}y$ be the error function. Subtracting (1.3) from (1.1) and taking (1.2) into account we obtain the relation

$$\|{}^{s}z\| \leqslant q\|{}^{s-1}z\|, \qquad q = \tau C_1.$$

Now we discuss the stopping criterion for the iterative method (1.3). The process is continued until the iteration error is sufficiently small $\|{}^{s}z\| \leqslant \varepsilon_1\tau$. We have from (1.2) that $|y_t| \leqslant C_0$, therefore, if we take ${}^{0}y = y$, then $\|{}^{0}z\| \leqslant C_0\tau$. This leads to the stopping criterion

$$\|{}^{s}z\| \leqslant q^s\|{}^{0}z\| \leqslant q^sC_0\tau \leqslant \varepsilon_1\tau$$

and we have that the number of iterations can be estimated as

$$s \geqslant s_0 = \ln\varepsilon/\ln q, \qquad \varepsilon = \varepsilon_1/C_0.$$

In order to solve (1.1) for $t_0 \leqslant t \leqslant t_0+1$ we have to perform $1/\tau$ steps and the total amount of iterations is equal to $Q(\tau) = \ln\varepsilon/(\tau\ln q)$. The optimal value of the parameter τ is found from the minimization problem

$$\min_{\tau} Q(\tau) = Q(\tau^*). \tag{1.4}$$

Using a necessary condition $Q'(\tau) = 0$ for τ to be a local minimizer we get the optimal value of the parameter $\tau^* C_1 = 1/e \approx 0.36$.

COROLLARY 1.1. A pratical value of this result is that in the case of slow convergence of (1.3) ($0.5 < q < 1$) we must reduce the parameter τ instead of trying to achieve the stopping criterion with the given value of τ. The convergence of (1.3) is optimal if after each three iterations the error reduces by a factor $\varepsilon \approx 0.05$.

Using this methodology we can take into account various details about the algorithm. For example, if Runge's method is used to estimate the accuracy of the time integration, then after simple calculations we get

$$\min_{\tau} Q_R(\tau) = Q_R(\tau^*), \qquad Q_R(\tau) = \frac{1}{\tau}\left(\frac{\ln\varepsilon}{\ln(\tau C_1)} + \frac{2\ln\varepsilon}{\ln(0.5\tau C_1)}\right).$$

Now the optimal value of the parameter τ is equal to $\tau^* C_1 = \exp(A)$ where A satisfies the cubic equation

$$3A^3 + (3 - 4\ln 2)A^2 + (\ln^2 2 - 2\ln 2)A + \ln^2 2 = 0.$$

Next we consider a typical situation when the choice of τ is restricted by the convergence analysis but not by the accuracy demands. Then the stopping criterion $\| \dot{z} \| \leqslant \varepsilon_1 \tau_2$ depends on τ_2, defined according to the accuracy analysis and the computations are implemented with the parameter $\tau = p\tau_2, p \leqslant 1$. The optimal value of τ is found from the minimization problem

$$\min_{\tau} Q_1(\tau) = Q_1(\tau_1), \qquad Q_1(\tau) = \frac{\ln\varepsilon - \ln p}{p\tau_2 \ln(p\tau_2 C_1)}, \qquad \varepsilon < p.$$

Using a necessary condition $Q_1'(\tau) = 0$ we get the equation

$$A^2 - (\ln\varepsilon - D)A - (D + \ln\varepsilon(D+1)) = 0,$$

where we noted $A = \ln p_1$, $D = \ln(\tau_2 C_1)$. A simple analysis shows that $\tau_1 C_1 < \tau^* C_1 = 1/e$, therefore the statement of Corollary 1.1 becomes even stronger in this case. In practice we can use an iterative process of predictor-corrector type as an approximate solution of the minimization problems stated above.

2. Optimality of the data storage scheme. In this section we consider one interesting example, which is connected with the optimality of numerical methods. The method of a computational experiment was used by Čiegis and Dement'ev (1992) for the theoretical investigation of backward transient stimulated scattering (SBS).The dimensionless equations for the mathematical description of SBS, considering diffractive broadening of beams, waves and material instationarities, self-focussing nonlinearity, are given as

$$\frac{\partial e_L}{\partial t}+\frac{\partial e_L}{\partial Z}-i\mu\Delta_\perp e_L+\frac{\alpha_L Z_0}{2}e_L = -\Gamma_L\sigma e_S+i\eta_L(|e_L|^2+2|e_S|^2)e_L, \tag{2.1a}$$

$$\frac{\partial e_S}{\partial t}-\frac{\partial e_S}{\partial Z}-i\mu\Delta_\perp e_S+\frac{\alpha_S Z_0}{2}e_S = -\Gamma_S\sigma^* e_L+i\eta_S(2|e_L|^2+|e_S|^2)e_S, \tag{2.1b}$$

$$\frac{iT_0}{2\pi}\left(\frac{\partial^2\sigma}{\partial t^2}+\frac{1}{\tau^*}\frac{\partial\sigma}{\partial t}\right)+2\frac{\partial\sigma}{\partial t}+\left(\frac{1}{\tau^*}\frac{i(\omega_B^2-\omega^2)}{2\pi}\right)\sigma = \Gamma_\sigma e_L e_S^*+\frac{\sigma_N}{\tau^*}, \tag{2.1c}$$

where $e_{L,S}$, σ are complex amplitudes of laser, Stokes and hypersonic waves, respectively. In the case of cylindrically symmetric beams

$$\Delta_\perp u=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}u\,,\quad 0\leqslant r\leqslant R\,,\ 0\leqslant Z\leqslant L\,.$$

The appropriate initial and boundary conditions are given at the boundary of the region $\overline{Q}_T$

$$e_L(0,r,t)=e_L^1(r,t),\quad e_S(L,r,t)=e_S^1(r,t), \tag{2.1d}$$

$$e_{L,S}(Z,R,t)=0,\quad \frac{\partial}{\partial r}e_{L,S}(Z,0,t)=0, \tag{2.1e}$$

$$e_{L,S}(Z,r,0)=e_L^0(Z,r),\quad \sigma(Z,r,0)=\sigma^0(Z,r),\quad \sigma_t'(Z,r,0)=\sigma_t^0(Z,r).$$

It is well known that the conservation of some energy quantities of $e_{L,S}(Z,r,t)$ is a very important feature of nonlinear (linear) optics problems. Simulating of the main qualitative features of the solution is a desirable attribute of difference schemes (see, e.g.

Čiegis (1990)). We use a splitting method to construct the basic difference scheme (Čiegis, 1989)

$$\frac{\tilde{u}-u(-1)}{\tau}-i\mu_1\Lambda\frac{\tilde{u}+u(-1)}{2}+\frac{\alpha_L Z_0}{2}\cdot\frac{\tilde{u}+u(-1)}{2}=0, \tag{2.2a}$$

$$\frac{\sigma^p-\sigma}{\tau}+a\frac{\sigma^p+\sigma}{2}=\Gamma_\sigma uv^*+\frac{\sigma_N}{\tau^*}, \tag{2.2b}$$

$$\frac{\mathring{u}-u}{\tau}=-\Gamma_L\overline{\sigma^p}\overline{v},\quad \frac{\mathring{v}-v}{\tau}=-\Gamma_S\overline{\sigma^{*p}}\overline{u}, \tag{2.2c}$$

$$\overline{\sigma^p}=\frac{1}{2}(\sigma^p+\sigma),\quad \overline{u}=\frac{1}{2}(\mathring{u}+u),\quad \overline{v}=\frac{1}{2}(\mathring{v}+v),$$

$$\frac{\hat{\sigma}-\sigma}{\tau}+a\frac{\hat{\sigma}+\sigma}{2}=\Gamma_\sigma\overline{uv}^*+\frac{\sigma_N}{\tau^*}, \tag{2.2d}$$

$$\hat{u}=\exp\left(i\eta_L(|\mathring{u}|^2+2|\mathring{v}|^2)\tau\right)\mathring{u}, \tag{2.2e}$$

$$\tilde{v}=\exp\left(i\eta_s(|\mathring{v}|^2+2|\mathring{u}|^2)\tau\right)\mathring{v}, \tag{2.2f}$$

$$\frac{\hat{v}(-1)-\tilde{v}}{\tau}-i\mu_s\Lambda\frac{\hat{v}(-1)+\tilde{v}}{2}+\frac{\alpha_S Z_0}{2}\cdot\frac{\hat{v}(-1)+\tilde{v}}{2}=0. \tag{2.2g}$$

The implementation of this scheme is noniterative and only the tridiagonal systems of equations are required to be solved at each time step. It is easy to prove that the difference scheme (2.2) preserves the conservation of discrete energy

$$\|\tilde{u}\|^2+\|\hat{v}(-1)\|^2=\|u(-1)\|^2+\|v\|^2. \tag{2.3}$$

In practice the dimension of the difference system $N\times M$ (N and M are the numbers of points of ω_r,ω_s, respectively) is too large for saving all information in the fast memory of the computer due to the limited amount of this memory. Therefore the implementation of the algorithm involves swapping and this operation takes a bigger part of all computational time. Suppose that the processor time, needed for the realization of (2.2) per time step is t_1, and the data exchange time is t_2 ($t_1 << t_2$). Then the following modification is proposed. Each partial problem of the dimension $N\times M_1, M_1 < M$ is solved for the time interval $t_j \leqslant t \leqslant t_{j+p}$ before the next exchange of data with exterior memory ($p \geqslant 2$). The recalculation is performed

in some overlapping boundary domains, but the share of this extra work is not big. The total realization time can be estimated as

$$T(p) = C\left(t_1 + \frac{t_2}{p} + \frac{(p-1)}{M_1}(t_1 + t_2)\right),$$

where $N \times M_1$ is the dimension of data, that can be stored in the fast memory of the computer. The optimal value of p is obtained from the minimization problem

$$\min_p T(p) = T(p^*).$$

This modification proved to be very efficient for solving applied equations (see Fig. 1).

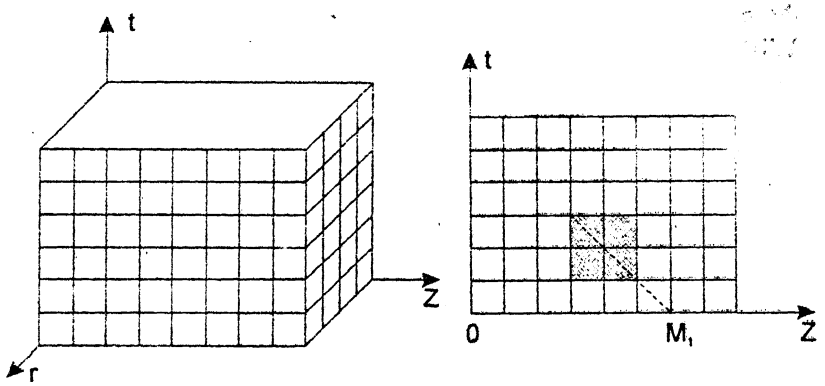

Fig. 1. The classical region of computation and the domain of realization of the proposed algorithm.

3. Adaptable grids and the optimality of difference schemes. The method of grid adaptation is usually used for solving problems with fastly varying solutions. In the case of nonlinear optics problems we deal with the specific situation of focussed beams. An adaptable modification of (2.2) is proposed by Čiegis (1990), Čiegis and Dement'ev (1992) to convey the rapid oscilations of the solution accurately.

Methods of variational calculus enable us to investigate the optimality of numerical algorithms by stating a general variational

problem. It is very important that the classification of various modifications of the basic algorithm can be based on this variational model. In this section we investigate the main steps of such a methodology. The class of problems considered by us has the reaction-diffusion form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t,u), \tag{3.1a}$$

$$u(x,0) = u_0(x), \qquad l_i u = 0, \quad i = 1,2. \tag{3.1b}$$

Let ω_τ, ω_h be a family of time and spatial grids. They are assumed to be nonuniform

$$\omega_\tau = \{t_j : \ t_j = t_{j-1} + \tau_j, \quad j = 0,1,\cdots,K, \quad t_K = T\},$$

$$\omega_h = \{x_i : \ x_i = x_{i-1} + h_i, \quad i = 0,1,\cdots,N, \quad x_N = 1\}.$$

The position of x_i, τ_j can be used as free parameters for solving the optimization problem. But first we define a difference scheme. Aditional requirements, such as the conservativity of the difference scheme, the order of the approximation accuracy or the possibility of economical realization, are usually considered for solving this problem. We note that some of these properties can be included into the variational model and a selection of the best difference scheme then depends on the obtained solution. We restrict there to the implicit difference scheme, which is extensively used for solving problems of reaction-diffusion type:

$$y_t = \hat{y}_{\bar{x}x} + f(x_i, t_{j+1}, \hat{y}), \tag{3.2a}$$

$$y(x_i,0) = u_0(x_i), \quad l_j^h y = 0, \quad j = 1,2. \tag{3.2b}$$

Let $z = u - y$ be the error function. It satisfies the following equation

$$z_t = \hat{z}_{\bar{x}x} + f(x_i,\hat{u}) - f(x_i,\hat{y}) + \psi, \tag{3.3}$$

where ψ_i is the truncation error, which can be represented as

$$\psi_i^{j+1} = 0.5\tau_{j+1}\ddot{u}^{j+1} + \frac{h_{i+1} - h_i}{3}\hat{u}_i''' + \frac{h_{i+1}^3 + h_i^3}{24h_i}\hat{u}^{IV} + O(\tau^2 + h^3). \tag{3.4}$$

We note that the second order accuracy is still obtained for the solution of (3.2), if we use the negative norm to estimate ψ .

$$||\psi||_{-1} = \left(\sum_{i=1}^{N-1} h_i (\sum_{k=i}^{N-1} \hbar_k \psi_k)^2 \right)^{0.5} \leqslant C(\tau + h^2).$$

Now the definition of the optimization problem depends on our specific goals. Assume that the number of grid ω_h points is fixed to N and our aim is to achieve the highest possible accuracy of the difference scheme solution . Then we have the minimization problem

$$\min_{\omega_\tau, \omega_h(t_j,N)} \max_{t_j} ||z||_1 = ||z^*||_1. \tag{3.5}$$

The problem of determining the optimal time grid ω_τ is much simpler then the one connected with the space grid ω_h. We can replace it by the following optimization problem, which is very usefull in computational experiment

$$\begin{gathered} \max_{\|z^t\| \in Z(t_{j+1})} \tau = \tau_{j+1}, \\ Z(t_{j+1}) = \{ z^t : \|z^t(t_{j+1})\| \leqslant \varepsilon \|z^x(t_{j+1})\| \}, \end{gathered} \tag{3.6}$$

where $z = z^t + z^x$ and z^t represents a time-dependent part of the error. Numerical methods for solving (3.6) are investigated in many papers (see Lauson *et al.*, 1991). The global problem (3.5) is very complex for solving. Therefore we define the optimal grid $\omega_h(t_{j+1})$ from the local error per time step

$$\min_{\omega_h(t_{j+1},N)} ||z^{j+1}||_1 = ||z^{**}||_1. \tag{3.7}$$

The exact value of z_i^{j+1} can be find only for model problems. In practice we determine it approximately by solving (3.3). The truncation error is estimated from $(\Lambda^* \hat{y} - \Lambda \hat{y})$, where $\Lambda^* \hat{y}$ is a second finite-difference operator of high order accuracy . Expression (3.4) can be also used for this purpose. Note that we tacitly assume that substitution of the numerical solution into the theoretical error expressions is allowed in the sence that the numerical estimate is also asymptotically correct (see Babuška and Yu (1986), Adjerid

and Flaherty (1988)). Thus we must solve an additional boundary value problem to compute the global error z_i^{j+1}. In order to avoid this extra work we can use the estimate, which follows from the stability of the difference scheme

$$\|z^j\|_1 \leqslant C\|\psi\|_2 \leqslant C_1\|\psi^t\|_2 + C_2\|\psi^x\|_2.$$

We now define the approximate minimization problem

$$\min_{\omega_h(t_{j+1},N)} \|\psi^x\|_2 = \|\psi^*\|_2. \tag{3.8}$$

The problem (3.8) is still a difficult mathematical problem.

REMARK 3.1. A more simple is the adaptive local grid refinement algorithm, for which the number of grid points N is not fixed and the local fine grids are introduced in regions where the error indicator exceeds a prescribed tolerance (see Moore and Flaherty (1990), Verwer *et al.* (1992)).

Next we shortly consider numerical methods for solving (3.8). In order to illustrate our analysis we examine the truncation error (3.4). Assume that the grid satisfies the condition $h_{i+1} - h_i = O(h_i^2)$. This further implies that (3.8) can be approximated as

$$\min_{\omega_h(N)} \sum_{i=1}^{N} h_i|\psi^x|^2 = \min_{\omega_h(N)} \sum_{i=1}^{N} h_i h_i^4 |u_i^{IV}|^2 = \|\psi^*\|^2.$$

In general we obtain the minimization problem

$$\min_{\omega_h(N)} \sum_{i=1}^{N} h_i d_i^2(u,h) = \|\psi^*\|_3^2, \quad d_i(u,h) = h^n g_i(u), \tag{3.9}$$

where the selection of $d_i(u,h)$ depends on the difference scheme, the norm, for which the stability inequality is proved, and on the other additional information. Since we don't know the exact solution $u(x,t)$ the difference scheme solution $\hat{y}$ is used to define $d_i(\hat{y},h)$. The constrained minimization problem (3.9) is nonlinear due to the dependence of $\hat{y}$ on the unknow solution vector $h = (h_1, h_2, \cdots, h_N)$.

Equation (3.9) must be solved by some iterative method. As one of such an iterative method we can use the inverse interpolation procedure of de Boor (Sanz-Serna and Christie, 1986). The other simple iterative method is defined by

$$\min_{\omega_h(N)} \sum_{i=1}^{N} \overset{s}{h}_i^{2n+1} g_i^2(\overset{s-1}{y}) = \|\overset{s}{\psi}{}^*\|_3^2. \tag{3.10}$$

The solution of (3.10) is well-known, it is based on equidistributing a local truncation error indicator

$$(\overset{s}{h}_i)^n g_i(\overset{s-1}{y}) = c, \quad c \doteq \text{const}.$$

This solution can be obtained from the boundary-value problem

$$-(p(\overset{s-1}{y})\,\overset{s}{x}_{\bar{r}})_r = 0, \quad x_0 = 0, \quad x_N = 1, \tag{3.11a}$$
$$p(y) = (g(y))^{1/r}, \quad x_r = x_{i+1} - x_i = h_{i+1}$$

Equation (3.11a) can be replaced with a more general iterative method

$$\frac{\overset{s}{x} - \overset{s-1}{x}}{\gamma_s} = (p(\overset{s-1}{y})\,\overset{s}{x}_{\bar{r}})_r. \tag{3.11b}$$

Though there is no theoretical convergence results, iterative method (3.11) is investigated numerically (see Daripa (1991) and the references therein). For some test problems these authors encountered convergence problems, therefore developing better methods for solving (3.9) is still a challenging task.

Now we cite some typical forms of the adaptive (or monitor) functions $g(y)$.

AF1. *Geometrical adaptation* (see Sanz-Serna and Christie (1986), Dew (1992)).

$$n = 1, \qquad g(y) = (\alpha + y_{\bar{x}}^2)^{0.5}, \quad \alpha \geqslant 1,$$

where α is the regularization parameter. This adaptive function $g(y)$ leads to equidistribution of arclength. We note that such a selection of $g(y)$ is not based on any truncation error estimate.

AF2. *Minimization of finite-element truncation error* (see Sanz-Serna *et al.*,1988).

$$n = 2, \qquad g(y) = \alpha + |y_{\xi\xi}|, \quad \alpha > 0.$$

AF3. *High order truncation error* (Degteriov *et al.*, 1987).

$$n = 4, \qquad g(y) = \alpha + |y_{\xi\xi\xi}|, \quad \alpha > 0.$$

Assume that $x = x(\alpha) \in C[0,1]$ and $x_i = x(\alpha_i), \alpha_i = ih_\alpha$. Then it follows from (3.4) that we can write the truncation error ψ^* as

$$\begin{aligned}\psi_i^* &= h_\alpha^2(\frac{1}{3}x_{\alpha\alpha}u_i''' + \frac{1}{12}x_\alpha^2 u_i''') + O(h_\alpha^4) \\ &= \frac{1}{3}(u''')^{3/4}((u_i''')^{1/4}x_\alpha)_\alpha + O(h_\alpha^4).\end{aligned}$$

The truncation error ψ^x is of order $O(h_\alpha^4)$ if $x(\alpha)$ satisfies the boundary value problem

$$-((u_i''')^{1/4}x_\alpha)_\alpha = 0, \quad x(0) = 0, \quad x(1) = 1.$$

AF4. *Minimization of the difference scheme trucation error.*

$$n = 2, \qquad g(y) = \alpha + |y_{\xi\xi\xi\xi}|, \quad \alpha > 0.$$

This monitor function follows from (3.4).

4. Adaptable grids for time-dependent PDE. We can include into our analysis various other modifications of the basic algorithm. For time-dependent PDE one may distinquish two main categorics of adaptive-grid methods, viz. dynamic and static ones. While dynamic methods adapt the grid in a continuous manner, like classical Lagrangian methods (Miller and Miller (1981), Mazhukin and Takoyeva (1990)), static methods adapt the grid only at discrete times (Verwer and Trompert (1992), Moore and Flaherty (1990)). Some methods may be called intermediate between these two groups (Verwer *et al.*, 1988). In all cases we obtain additional terms of the truncation error, which arise after the discretization step. As an example we consider the following well-know static regridding method (see Bajarūnas and Čiegis, 1991).

S1. *The grid prediction stage.* First we solve (3.2) on a fixed spatial grid. The obtained solution y^p then acts as input for a reqridding algorithm (e.g. (3.9)) which generates $\omega_h(t_{j+1}, N)$ at the advanced time-level by equidistributing a chosen monitor function.

S2. *The interpolation stage.*

$$\widetilde{y}^j = Py^j = P(\omega_h(t_j, N), \omega_h(t_{j+1}, N), y^j), \tag{4.1a}$$

where $\widetilde{y}^j$ is obtained by interpolating y^j and some interpolation formula is used for this porpose.

S3. *The integration stage.*

$$\frac{y^{j+1} - \widetilde{y}^j}{\tau_j} = \Lambda y^{j+1} + f(y^{j+1}). \tag{4.1b}$$

Assume that the difference scheme (3.2) is stable and the interpolation error can be bounded by $\|\widetilde{\psi}\|_3 \leqslant Ch^p$.

Theorem 4.1. *A solution of the adaptive method* (4.1) *converges to the solution of* (3.1) *and we have the asymptotic error estimate*

$$\|y^j - u(t_j)\|_1 \leqslant t_j(C_1\|\psi^j\|_2 + C_2\frac{1}{\tau}\|\widetilde{\psi}\|_3), \quad \tau_j \geqslant \tau. \tag{4.2}$$

The proof of this theorem is only technical and is given by Bajarūnas and Čiegis (1991). Only conditional convergence for $\tau \geqslant Ch^{p-1}$ follows from (4.2). Since (4.2) is only upper bound of the error to show that this result is sharp we will consider the model problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad 0 < x < 1,$$

where $f(x,t), u(x,0)$ and the Dirichlet boundary conditions are chosen so that the exact solution is $u(x,t) = t\sin \pi x$. Assume that the regridding step $S1$ generates consecutivly the grids

$$\omega_h(t_{2k}) = \{x_i : \ x_i = ih, \quad i = 0,1,\cdots,N, \quad Nh = 1\},$$
$$\omega_h(t_{2k+1}) = \{x_i : \ x_i = (i-0.5)h, \quad i = 1,2,\cdots,N,$$
$$x_0 = 0, \ x_{N+1} = 1\}.$$

Table 4.1. Errors of a computational experiment

τ	0.1	0.05	0.025	0.0125
h=0.100	3.21E-3	1.43E-2	3.57E-2	1.46E-1
h=0.050	7.27E-4	3.53E-3	9.09E-3	4.11E-2
h=0.025	1.71E-4	8.75E-4	2.27E-3	1.05E-2

Piecewise linear interpolation of the solution y^j is used in $S2$, therefore we have the interpolation error estimate $||\tilde{\psi}||_3 \leqslant Ch^2$. Errors $||z||_c$ at $t = 1$ as a function of parameters τ and h are shown in Table 4.1.

As it follows from the numerical results only conditional convergence is obtained in this case (see (4.2)). We can estimate the interpolation error $\tilde{\psi}$ more accurately

$$\frac{1}{\tau}||\tilde{\psi}||_c \leqslant |\tilde{\psi}|, \qquad |\tilde{\psi}| = \frac{0.5}{\tau} \max_l \min_i h_l^j |h_i^{j-1} - h_l^j| \, |u_l''|.$$

Balancing space discretization and interpolation error is obtained if we solve the minimization problem

$$\min_{\omega_h(N)} ||\psi^z||_2 + |\tilde{\psi}| = ||\bar{\psi}||.$$

Now the difference scheme solution converges unconditionally. The same method can be used for other adaptive procedures, too.

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Received January 1993

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