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ABOUT STOCHASTIC DISCRETE NEURONETWORKS

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Abstract. A stochastic discrete neuronetwork is defined. In the investigation of discrete neuronetworks probability methods are applied – a weak convergence of probability measures. Limit theorems (the strong law of large number and normal law) are proved for the stream of signals, going out of neurons.

Key words: neuronetworks, signal size, limit theorems.

1. Introduction. A stochastic discrete neuronetwork is defined in this paper, and limit theorems, characterizing the size of signals going out of neurons, are proved. The model of a discrete neuron is used defined in the review [1]. In the investigation of discrete neuronetworks probability methods are applied, in the concrete, a weak convergence of probability measures. The case is considered, when streams of signals comprising the identically distributed random variables. The strong law of large numbers and normal law are proved for the stream of signals going out of neurons. All the random variables considered in this paper are defined in one basic probability space (Ω, β, P). Now we present some definitions of the weak convergence of measures theory (Billingsley, 1977). Let S be a metric space. We consider probability measures defined in the class of Borel sets in space S. If probability measures P_n and P satisfy the relation $\int_S f\,dP_n \to \int_S f\,dP$ for each really defined continuous function f in space S, then we consider P_n to be weakly convergent to P, and we write $Pn \Rightarrow P$. Let X map the probability space (Ω, β, P) into the matrix space S. If X is measurable (in the sense $X^{-1}\Phi \subset \beta$), then we call X as a random element.

The distribution of the random element X is a probability measure $P = PX^{-1}$ in pair (S, Φ) $P(A) = P(\omega;\ X(\omega) \in A)$. We suppose, that a sequence of random elements $\{X_n\}$ weakly converges in distribution to the random element X and we write that $X_n \Rightarrow X$, if distributions P_n of the elements X_n weakly converge to the distribution P of the element X. At first we consider a simple case, when we have two neurons with two signals going into the each neurons and one signal going out of the neurons.

2. A mathematical model of the discrete neurons. So we have two discrete neurons and relations between them (Vedenov, 1990). This model is presented in Fig. 1.

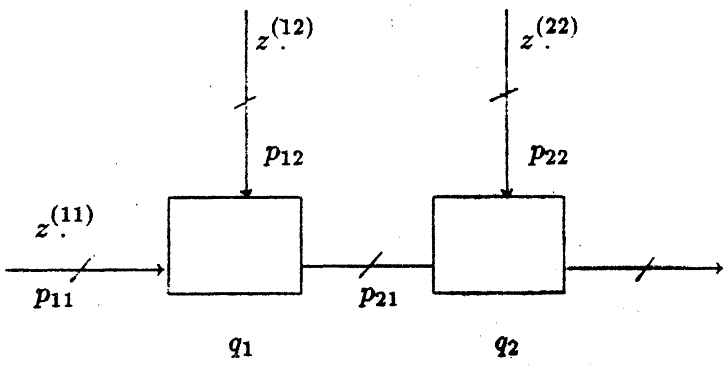

Fig. 1. The model of two discrete neurons.

The neurons play an identical ordinary function – they sum up the weight sizes of signals arriving at the neuron. The size of the signal going out of the neuron is determined by the calculated sum and the threshold of a neuron. The size of signals arriving from outside are independent identically distributed random variables with the given distribution function. The intervals between signals are the same. The sizes of signals are filtered by the function

$$\lambda(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0. \end{cases}$$

That is a threshold element of McCalloch W., Pitts W. (Vedenov, 1990).

By $p_{11}, p_{12}, p_{21}, p_{22}$ denoted the weights of the signals leading to neurons 1 and 2 (links between the neurons); by q_1, q_2 denote thresholds in neurons; by $z_n^{(11)}, z_n^{(12)}, z_n^{(22)}$ denote sizes of signals arriving at neurons 1 and 2 from outside, by β_{1n}, β_{2n} – the sizes of signals entering neurons 1 and 2; by u_{1n}, u_{2n} – the sizes of signals going out of neurons 1 and 2, $n \geqslant 1$.

Note that

$$\beta_{1n} = p_{11} \cdot z_n^{(11)} + p_{12} \cdot z_n^{(12)} - q_1,$$
$$\beta_{2n} = p_{21} \cdot \lambda(\beta_{1n}) + p_{22} \cdot z_n^{(22)} - q_2, \ n \geqslant 1.$$

Then $u_{1n} = \lambda(\beta_{1n})$ and $u_{2n} = \lambda(\beta_{2n})$. Note that u_{1n} are independent identically distributed random variables with the distribution function $f_1(x)$, where $F_1(x) = P(u_{1n} \leqslant x)$, $x \in R$, $n \geqslant 1$ (Billingsley, 1977). Analogously, u_{2n} are also independent identically distributed random variables with the distribution function $F_2(x)$, where $F_2(x) = P(u_{2n} \leqslant x)$, $x \in R$, $n \geqslant 1$. Determine the sum of size of the signals going out of neurons 1 and 2 up to order n $S_{1n} = \sum_{l=1}^{n} u_{1l}$, $S_{2n} = \sum_{l=1}^{n} u_{2l}$. We prove the theorem, which deals with the summary sizes of signals S_{1n} and S_{2n}:

Theorem 1. (The strong law of large numbers and the normal law). *If* $Mu_{11} \neq 0$ *or* $Mu_{11} \neq 1$, *then*

$$\frac{\sup\limits_{1 \leqslant l \leqslant n} S_{1l}}{n} \Rightarrow Mu_{11}; \qquad \frac{\sup\limits_{1 \leqslant l \leqslant n} S_{2l}}{n} \Rightarrow Mu_{21};$$

$$\frac{S_{1n} - nMu_{11}}{\sqrt{n} \cdot \sqrt{Mu_{11} \cdot (1 - Mu_{11})}} \Rightarrow N(0,1);$$

$$\frac{S_{2n} - nMu_{21}}{\sqrt{n} \cdot \sqrt{Mu_{21} \cdot (1 - Mu_{21})}} \Rightarrow N(0,1);$$

Proof. Note that $DS_{1n} = \sum_{l=1}^{n} Du_{1l}$, $DS_{2n} = \sum_{l=1}^{n} Du_{2l}$; therefore to prove the theorem, it sufficiently to calćulate Mu_{1n}, Mu_{2n}, Du_{1n}, Du_{2n}, $n \geqslant 1$ (Shiriaev, 1980). Put $\Phi_{11}(x) = P(p_{11} \cdot z_n^{(11)} + p_{12} \cdot z_n^{(12)} \leqslant x)$, $\Phi_{22}(x) = P(z_n^{(22)} \leqslant x)$, $x \in R, n \geqslant 1$.

Note that

$$Mu_{1n} = P(\beta_{1n} \geqslant 0) = 1 - \Phi_{11}(q_1). \tag{1}$$

We also obtain that

$$\begin{aligned} Mu_{2n} =& P(\beta_{2n} \geqslant 0) = P(\beta_{2n} \geqslant 0 \backslash \lambda(\beta_{1n}) = 1) \cdot P(\lambda(\beta_{1n}) = 1) \\ &+ P(\beta_{2n} \geqslant 0 \backslash \lambda(\beta_{1n}) = 0) \cdot P(\lambda(\beta_{1n}) = 0) \\ =& P(p_{21} + p_{22} \cdot z_n^{(22)} - q_2 \geqslant 0) \cdot P(\beta_{1n} \geqslant 0) \\ &+ P(p_{22} \cdot z_n^{(22)} - q_2 \geqslant 0) \cdot (1 - P(\beta_{1n} \geqslant 0)) \\ =& \Big(1 - \Phi_{22}\Big(\frac{q_2 - p_{21}}{p_{22}}\Big)\Big) \cdot Mu_{1n} \\ &+ \Big(1 - \Phi_{22}\Big(\frac{q_2}{p_{22}}\Big)\Big) \cdot (1 - Mu_{1n}). \end{aligned} \tag{2}$$

We calculate

$$\begin{aligned} Du_{1n} &= M\lambda^2(\beta_{1n}) - (M\lambda(\beta_{1n}))^2 \\ &= M\lambda(\beta_{1n}) - (M\lambda(\beta_{1n}))^2 = Mu_{1n} - (Mu_{1n})^2. \end{aligned} \tag{3}$$

Analogously we calculate

$$Du_{2n} = Mu_{2n} - (Mu_{2n})^2. \tag{4}$$

Eventually we obtain that

$$\begin{aligned} DS_{1n} &= n \cdot (Mu_{11} - (Mu_{11})^2), \\ DS_{2n} &= n \cdot (Mu_{21} - (Mu_{21})^2), \end{aligned}$$

Note that $Mu_{2n} \neq 0$, $DS_{1n} \neq 0$, $DS_{2n} \neq 0$, if $Mu_{1n} \neq 0$ or $Mu_{1n} \neq 1$.

Theorem 1 is proved.

3. A mathematical model of network of discrete neurons. Generalizing the first model we prove Theorem 2. This model is presented in Fig. 2. Theorem 1 will be a special case of this theorem. We consider a stochastic discrete neuronetwork. The network consists of k neurons connected in chain. Each neuron contains r signals. Neuron 1 includes r signals from outside, neuron $2,\ldots,k$ includes r signals from outside, neuron $2,\ldots,k$ includes a signal from the previous neuron and $r-1$ signal from outside. Only one signal goes out of each neuron, and the signal going out of the kth neuron, falls out of the network. The signals from outside arrive at discrete time moments and their sizes are independent identically distributed random variables. The signals in neurons are processed just like in the previous model.

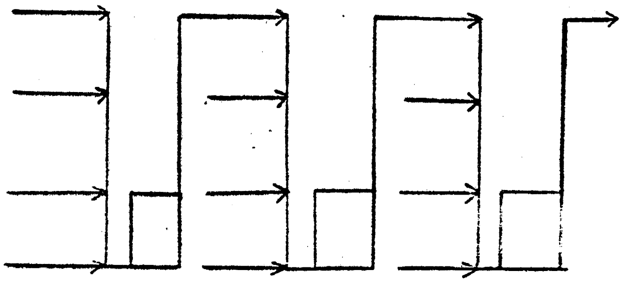

Fig. 2. The model of network of neurons.

By p_{1l} denote the weights of signals $l = 1,2,\ldots,r$ entering neuron i (links between the neurons); by $z_n^{(il)}$ denote the size of signals arriving from outside at neuron i (independent identically distributed random variables); by β_{in} – sizes of signals entering neuron i; by u_{in} – the sizes of signals going out of neuron i; q_i stand for the thresholds in neuron i, $i = 1,2,\ldots,k$, $n \geqslant 1$.

$S_{in} = \sum_{l=1}^{n} u_{il}$, $i = 1,2,\ldots,k$ are summary sizes of signals going out of neurons. We prove a theorem considering these sizes.

Theorem 2. (The strong law of large numbers and the normal law.) *If* $Mu_{11} \neq 0$ *or* $Mu_{11} \neq 1$, *then*

$$\frac{\sup\limits_{1\leqslant l\leqslant n} S_{i,l}}{n} \Rightarrow Mu_{i1};$$

$$\frac{S_{in} - n\cdot Mu_{i1}}{\sqrt{n}\cdot\sqrt{Mu_{i1}\cdot(1-Mu_{i1})}} \Rightarrow N(0,1), \quad i = 1,2,\ldots,k.$$

Proof. The proof of Theorem 2 is analogous to that of Theorem 1.

Note that $DS_{in} = \sum_{l=1}^{n} Du_{il}$, $i = 1, 2, \ldots, k$.

Put $\Phi_i(x) = P\Big(\sum_{l=1}^{r} p_{il} z_n^{(il)} \leqslant x\Big)$, $x \in R$, $n \geqslant 1$, $i = 1, 2, \ldots, k$.

Notice that

$$\beta_{1n} = \sum_{l=1}^{r} p_{1l} z_n^{(1l)} - q_1,$$

$$\beta_{2n} = p_{21} \cdot \lambda(\beta_{1n}) + \sum_{l=2}^{r} p_{2l} \cdot z_n^{(2l)} - q_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\beta_{kn} = p_{k1} \cdot \lambda(\beta_{k-1}n) + \sum_{l=2}^{r} p_{kl} \cdot z_n^{(kl)} - q_k, \quad n \geqslant 1.$$

Note also that u_{in} are independent identically distributed random variables $i = 1, 2, \ldots, k$, $n \geqslant 1$ (see Theorem 1).

Analogously as in Theorem 1 we prove that

$$Mu_{in} = \big(1 - \Phi_i(q_i - p_{i1})\big) \cdot Mu_{i-1n} + \big(1 - \Phi_i(q_i)\big) \cdot \big(1 - Mu_{i-1n}\big),$$
$$Mu_0 \equiv 0; \quad Du_{in} = Mu_{in} - (Mu_{in})^2, \quad i = 1, 2, \ldots, k, \; n \geqslant 1. \qquad (5)$$

Finally we obtain that $DS_{in} = n \cdot [Mu_{in} - (Mu_{in}^2]$. There Mu_{in} is determined by a (5) recurrent equation.

Note that $Mu_{in} \neq 0$, $DS_{in} \neq 0$, $i = 1, 2, \ldots, k$, if $Mu_{1n} \neq 0$ or $Mu_{1n} \neq 1$ (see (5)).

The proof of Theorem 2 is completed.

4. Discussion.

4.1. We mark, that if the conditions of Theorem 1 are satisfied, then the law of double logarithm is true, i.e.

$$P\Big(\overline{\lim_{n}} \frac{S_{1n} - nMu_{11}}{\sqrt{Mu_{11} \cdot (1 - Mu_{11}) \cdot 2n \ln \ln n}} = 1\Big) = 1$$

and

$$P\Big(\overline{\lim_{n}} \frac{S_{2n} - nMu_{11}}{\sqrt{Mu_{21} \cdot (1 - Mu_{21}) \cdot 2n \ln \ln n}} = 1\Big) = 1.$$

It means that with probability 1 S_{1n} does not exceed $\sqrt{Mu_{11}\cdot(1-Mu_{11})\cdot 2n\ln\ln n}+n\cdot Mu_{11}$ and does not exceed $\sqrt{Mu_{21}\cdot(1-Mu_{21})\cdot 2n\ln\ln n}+Mu_{21}$ (Shiriaev, 1980).

4.2. We give an example, how it is possible to apply the central limit theorem in practise, while considering this neuron model. Let us define the average size $\widehat{S}_n = \frac{S_{1n}}{n}$, $n \geqslant 1$ of a signal going out of the first neuron by the n-th time moment. Let the size of the signal, entering the first neuron, be random uniformly distributed in the interval from 70 mV to 80 mV. Then, almost surely 0.99947 (i.e., in fact with probability 1), $\widehat{S}_{100}$ will be no less than 74.9 mV and no more than 75.1 mV (Sevastjanov, 1982). One may draw an analogous conclusion for the signals going out of the second neuron.

4.3. Other, much move subtle practical conclusions are possible, when applying asymptotical expansions of the central limit theorem and statistical methods, but that is not the topic of this paper.

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