

ON THE ITERATIVE METHODS FOR LINEAR PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

Tatyana ALEINIKOVA

Institute of Mathematics Belarussian Academy of Sciences
220604 Minsk, Surganeva St. 11, Belarus'

Raimondas ČIEGIS

Institute of Mathematics and Informatics
2600 Vilnius, Akademijos St. 4, Lithuania

Abstract. This paper is devoted to the investigation of the investigation of the convergence of iterative methods for solving boundary value problems with discontinuous coefficients. The dependence of the rate of convergence on the size of the discontinuity of coefficients is analyzed for three popular general iterative methods. A new criterion on the applicability of such methods is proposed and investigated. The efficiency of this criterion is demonstrated for a model problem.

Key words: iterative methods, boundary value problems, discontinuous coefficients, numerical simulation.

1. Introduction. Consider the problem of selection of numerical algorithms for a computational experiment. There one must have in mind two conflicting tendencies, which make it difficult to determine a solution. Firstly, we require the algorithm to be simple (low costs of a realization) and, secondly, it must be efficient for the kind of problem on investigation. The most simple solution is to use the well-known general algorithms (or their modifications), efficient subroutine implementations of which are given in most software packages. Such a possibility enables us to accelerate a preparatory stage of a computational experiment considerably. Therefore, before constructing special methods for a new problem it is very important to investigate the efficiency of general algo-

rithms for the same class of problems. We consider these questions for the following problem of linear algebra

$$A_\alpha y \equiv - \sum_{\alpha=1}^2 (a_\alpha y_{x_\alpha})_{x_\alpha} = f(x), \quad x \in \bar{\omega}_h, \gamma = g, \quad (1.1)$$

which is the finite - difference approximation of the differential problem

$$- \sum_{\alpha=1}^2 \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x), \quad x \in D, \quad u|_\Gamma = g,$$

where we denote

$$\begin{aligned} \bar{D} &= D \cup \Gamma, \quad D = \{(x_1, x_2), 0 \leq x_\alpha \leq 1\}, \quad 0 \leq k_1 \leq k(x) \leq k_2, \\ \bar{\omega}_h &= \omega_h \cup \gamma = \{(x_{1i}, x_{2j}), x_{\alpha i} = (i-1)h_\alpha, N_\alpha h_\alpha = 1\}, \quad \omega_\alpha = \{x_{\alpha i}\}. \end{aligned}$$

The coefficient $k_\alpha(x)$ may be discontinuous. Boundary value problems with discontinuous coefficients arise in the solution of a number of important in practise problems, for example, in calculation of magnetostatic fields. One more interesting example of problem (1.1) with discontinuous coefficients is given by the method of fictitious domains (see, Vabishchevich, 1991). It is well-known that for large discontinuities of the coefficients the rate of convergence of many iterative methods becomes much worse (see, Samarskij and Nikolayev, 1978). Therefore this problem have been under fixed attention during the last few years and some special numerical algorithms are proposed to solve it. We shall mention only two of them: the domain decomposition method (e.g. Dryja and Widlund, 1989, Bramble et. al., 1988) and the method proposed by Bakhvalov and Kniazev (1990), the rate of convergence of which depends very weakly on the size of the discontinuity of the coefficients.

The basic ideas of our analysis we shall demonstrate on two typical model problems. Let $\bar{D} = D_1 \cup D_2$ be a union of two subsets D_α and assume that $k_\alpha(x)$ is constant on D_α

$$k(x) = c_j, \quad x \in D_j, \quad c_1 = 1, \quad c_2 = c.$$

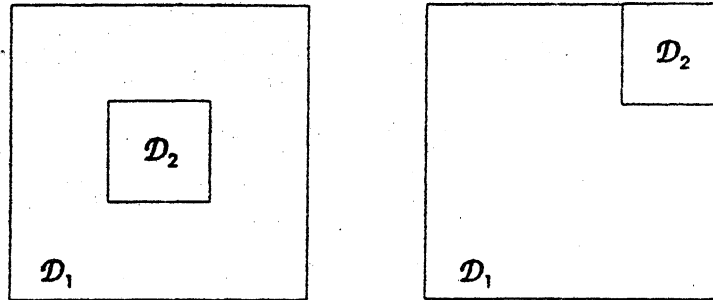


Fig. 1. The regions for model problems P1, P2.

PROBLEM P1. $D_2 = \{(x_1, x_2) : 1/3 \leq x_\alpha \leq 2/3, \alpha = 1, 2\}$.

PROBLEM P2. $D_2 = \{(x_1, x_2) : 2/3 \leq x_\alpha \leq 1, \alpha = 1, 2\}$.

Fig. 1 (a,b) shows the regions D_α for these model problems.

2. Iterative methods. First we shall briefly present some results on general two stage iterative process (see Samarskij and Nikolayev, 1978)

$$B \frac{y^{k+1} - y^k}{\tau_k} + Ay^k = f, \quad B = B^* > 0. \quad (2.1)$$

Let assume that A and B are spectrally equivalent $\gamma_1 B \leq A \leq \gamma_2 B$ and parameters τ_k are defined according Richardson's method. An upper bound for the number of iterations $n \geq n_0(\varepsilon)$, which suffices for fulfilling of the inequality $\|z^k\| \leq \varepsilon \|z^0\|$, $z^k = y^k - y$ is given by

$$n_0(\varepsilon) = \ln(2/\varepsilon) / \ln(1/\rho_1), \quad \rho_1 = \frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}}, \quad \eta = \gamma_1 / \gamma_2.$$

For $\eta \ll 1$ we can use a simpler formula $n_0(\varepsilon) = 0.5 \ln(2/\varepsilon) \sqrt{\xi}$, where $\xi = \gamma_2 / \gamma_1$ is a spectral condition number. Our main goal is to investigate the dependence of the rate of convergence on the size of the discontinuity of the coefficients for three popular general iterative methods:

a) Richardson's method with the diagonal preconditioning matrix (RDP)

$$B_R = \text{diag} A, \quad B_R y = b(x)y,$$

b) an implicit alternately triangular method (IAT)

$$B_T = (D + \omega R_1)D^{-1}(D + \omega R_2), \quad Dy = d(x)y,$$

$$R_1 y = \sum_{\alpha=1}^2 (a_\alpha y_{x_\alpha} + 0.5 a_{\alpha x_\alpha} y) / h_\alpha,$$

$$R_2 y = - \sum_{\alpha=1}^2 (a_\alpha^{+1} y_{x_\alpha} + 0.5 a_{\alpha x_\alpha} y) / h_\alpha,$$

$$d(x) = \sum_{\alpha=1}^2 \left(a_\alpha^{+1} / (h_\alpha^2 \sqrt{b_\alpha}) + 0.5 |a_{\alpha x_\alpha}| / h_\alpha \right) / (c_\alpha \sqrt{b_\alpha}),$$

where c_α, b_α will be defined later,

c) the method of alternating directions (AD)

$$B_{AD} = (E + \omega_1 A_1)(E + \omega_2 A_2), \quad A_\alpha y = -(a_\alpha y_{x_\alpha})_{x_\alpha}.$$

For the diagonal preconditioning matrix B_R a spectral equivalence number $\xi = \gamma_2 / \gamma_1$ can be calculated in the following way (see Samarskij and Nikolayev, 1978):

$$\gamma_1 + \gamma_2 = 2, \quad \gamma_1 = \sum_{\alpha=1}^2 \min_{x_\beta \in \omega_\beta} 1 / \kappa_\alpha(x_\beta), \quad \beta = 3 - \alpha, \quad (2.2a)$$

$$\kappa_\alpha(x_\beta) = \max_{x_\alpha \in \omega_\alpha} v^\alpha(x_\alpha), \quad \alpha = 1, 2$$

and $v^\alpha(x)$ is a solution of the difference problem

$$-(a_\alpha v_{x_\alpha}^\alpha)_{x_\alpha} = b(x), \quad v^\alpha(0) = 0, \quad v^\alpha(1) = 0. \quad (2.2b)$$

REMARK 2.1. In order to simplify the calculations it is proposed by Čiegis and Šeibak (1985) to replace (2.2) with the differential problem

$$-\frac{d}{dx_\alpha} \left(a_\alpha(x) \frac{dv^\alpha}{dx_\alpha} \right) = 2a_\alpha(x) \frac{1}{h_1^2 + h_2^2}, \quad v^\alpha(0) = 0, \quad v^\alpha(1) = 0. \quad (2.3)$$

The results of a numerical simulation show that the rate of convergence of the preconditioned $B = B_R$ iterative method (2.1) depends weakly on the extremal values of continuous coefficient $k(x)$. We have that $\xi = k_1 \xi^0 / k_0$, $\xi^0 = O(1/h^2)$ in the case of explicit iterative method $B = E$. For $B = B_R$ the equivalence spectral number ξ depends on some integral expression of $k(x)$. This follows from a general form of the exact solution of problem (2.3)

$$v^\alpha(x) = \frac{2}{h^2} \left(\int_0^1 \frac{1}{k(x)} \int_0^x k(t) dt dx / \int_0^1 \frac{dx}{k(x)} \right) \int_0^x \frac{dt}{k(t)} dt - \frac{2}{h^2} \int_0^x \frac{1}{k(z)} \int_0^z k(t) dt dz.$$

At the end of this section we consider a model problem with greatly varying but continuous functions $k_\alpha(x)$

$$k_1(x) = 1 + c[(x_1 - 0.5)^2 + (x_2 - 0.5)^2], \quad 0 \leq x_\alpha \leq 1, \quad \alpha = 1, 2, \quad (2.4)$$

$$k_2(x) = 1 + c[0.5 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2], \quad 1 \leq k_1(x) + k_2(x) \leq \gamma(c).$$

This problem (1.1) is widely used by Samarskij and Nikolayev (1978) to compare the convergence rates of iterative methods. For example, they investigated (2.1) with the preconditioning matrix $B = B_R$ (as in the RDP method) and with $B = E$ (an explicit method). We represent in Table 1 the number of iterations $n_0(\epsilon)$ obtained for $\epsilon = 10^{-4}$, $N = 32$ and various values of c (see Samarskij and Nikolayev, 1978).

Table 1. The number of iterations

$\gamma(c)$	2	8	32	128	512
$B = E$	143	286	571	1141	2281
$B = B_R$	123	149	175	192	202

In accordance with Remark 2.1 the number of iterations for (2.1) with the preconditioning matrix $B = B_R$ depends very weakly

on the value of $\gamma(c)$ and, hence, the RDP method is superior to the explicit one with $B = E$. Now we will show that for this model problem the difference between two preconditioning matrix $B = B_R$ and $B = E$ is not so serious. The number of iterations for the explicit iterative process was obtained by using following simple estimates (Samarskij and Nikolayev, 1978)

$$\overset{\circ}{\gamma}_1 E \leq \overset{\circ}{A} \leq A \leq \gamma \overset{\circ}{A} \leq \gamma \overset{\circ}{\gamma}_2 E, \tag{2.5}$$

where $\overset{\circ}{A}y = - \sum_{\alpha=1}^2 y_{\bar{x}_\alpha x_\alpha}$ and $\overset{\circ}{\gamma}_1, \overset{\circ}{\gamma}_2$ are minimal and maximal eigenvalues of $\overset{\circ}{A}$

$$\overset{\circ}{\gamma}_1 = \sum_{\alpha=1}^2 \frac{4}{h_\alpha^2} \sin^2 \frac{\pi h_\alpha}{2}, \quad \overset{\circ}{\gamma}_2 = \sum_{\alpha=1}^2 \frac{4}{h_\alpha^2} \cos^2 \frac{\pi h_\alpha}{2}.$$

We can replace (2.5) with more accurate estimates of parameters γ_1, γ_2 . In fact it is sufficient to get a better estimate for γ_1 . To this goal we use the technique defined by (2.2a), but we replace the auxiliary problem (2.2b) with the following one

$$-(a_\alpha v_{\bar{x}_\alpha}^\alpha)_{x_\alpha} = 1, \quad v^\alpha(0) = 0, \quad v^\alpha(1) = 0.$$

The number of iterations for the explicit iterative method (2.1) obtained by this technique is presented in Table 2.

Table 2. Improved number of iterations

$\gamma(c)$	2	8	32	128	512
$B = E$	141	188	240	266	288

3. The case of discontinuous coefficients. In this section we investigate the rate of convergence of the RDP, IAT, AD methods for model problems P1, P2 with discontinuous coefficients. Using the symmetry of the solution we obtain that

$$\xi = 2/\gamma_1 - 1 = 1/\max_{x_\beta \in \omega_\beta} (1/\kappa_\alpha(x_\beta)) - 1 = \max_{x_\beta \in \omega_\beta} \kappa_\alpha(x_\beta) - 1.$$

It is easy to show that for $x_\beta \in D_1$

$$v^\alpha(x_\alpha) = \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right) x_\alpha(1 - x_\alpha),$$

$$\kappa_\alpha(x_\beta) = \max_{x_\alpha \in \omega_\alpha} v^\alpha(x_\alpha) = \frac{1}{4} \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right).$$

If $x_\beta \in D_2$ then the exact solution of (2.2) for problem P1 is the function

$$v^\alpha(x_\alpha) = a_j x_\alpha^2 + b_j x_\alpha + c_j, \quad x_{Lj} \leq x_\alpha \leq x_{Rj}, \quad j = 1, 2, 3,$$

where explicit expressions for the parameters a_j, b_j, c_j are given by Šeibak and Čiegis (1986).

Using these formulas and the symmetry of the solution $v^\alpha(x)$ we have

$$\kappa_\alpha(x_\beta) = \max_{x_\alpha} v^\alpha(x_\alpha) = v^\alpha(0.5)$$

$$= \frac{1}{4} \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right) \left(1 + \frac{4}{9}(c-1) \left(1 + \frac{3h_1}{1 + h_2^2/h_1^2} \right) \right).$$

Combining the estimates above we obtain that

$$\xi = \max_{x_\beta \in \omega_\beta} \kappa_\alpha(x_\beta) - 1 = \frac{1}{4} \left(\frac{1}{h_1^2} + \frac{1}{h_2^2} \right) \max \left(1, 1 + \frac{4}{9}(c-1) \left(1 + \frac{3h_1}{1 + h_2^2/h_1^2} \right) \right).$$

We have proved that the number of iterations required by the RDP method is proportional to $n_0(\epsilon) = O(\sqrt{c}/h)$ for $c \gg 1$ and depends very weakly on the size of the discontinuity of the coefficients for $c \ll 1$.

Next we investigate analogically problem P2. It is sufficient to find a solution of (2.2b) for $x_\beta \in D_2$, only. After simple calculations we have

$$v^\alpha(x_\alpha) = a_j x_\alpha^2 + b_j x_\alpha + c_j, \quad x_{Lj} \leq x_\alpha \leq x_{Rj}, \quad j = 1, 2,$$

$$a = a_1 = a_2 = -(1/h_1^2 + 1/h_2^2), \quad c_1 = 0, \quad b_2 = 3a - 2b_1, \quad c_2 = 2(b_1 + a),$$

$$b_1 = a \frac{-5c - 4 + 3h(c+1) - ((c+1)/h_1^2 + 2c/h_2^2)/a}{3(2c+1)} = O\left(\frac{1}{h_1^2} + \frac{1}{h_2^2}\right).$$

Therefore the number of iterations required by the RDP method is not dependent on the size of the discontinuity of the coefficients for any value of c ($c \geq 1, c < 1$).

Now we consider the behaviour of the IAT method. In this case the number of iterations can be estimated as

$$\Delta = \max_{\alpha=1,2} \left(\max_{x_\beta} \left(c_\alpha(x_\beta) + \sqrt{b_\alpha(x_\beta)} \right)^2 \right)$$

and functions $b_\alpha(x_\beta), c_\alpha(x_\beta)$ are defined by solutions of auxiliary tridiagonal problems

$$\begin{aligned} b_\alpha(x_\beta) &= \max_{x_\alpha} v^\alpha(x_\alpha), & c_\alpha(x_\beta) &= \max_{x_\alpha} w^\alpha(x_\alpha), \\ -(a_\alpha v_{x_\alpha}^\alpha)_{x_\alpha} &= \frac{a^{+1}}{h_\alpha^2}, & v^\alpha(0) &= 0, \quad v^\alpha(1) = 0, \end{aligned} \quad (3.1)$$

$$-(a_\alpha w_{x_\alpha}^\alpha)_{x_\alpha} = \frac{|a_{x_\alpha}|}{2h_\alpha}, \quad w^\alpha(0) = 0, \quad w^\alpha(1) = 0. \quad (3.2)$$

If $x_\beta \in D_1$ then it is easy to show that

$$\begin{aligned} w^\alpha(x_\alpha) &= 0, & c_\alpha(x_\beta) &= 0, \\ v^\alpha(x_\alpha) &= \frac{1}{2h_\alpha^2} x_\alpha(1-x_\alpha), & b_\alpha(x_\beta) &= \frac{1}{8h_\alpha^2}. \end{aligned}$$

If $x_\beta \in D_2$ then problem (3.1) is the same as problem (2.2), investigated above. Therefore we have that $b_\alpha(x_\beta) = O(1/h_\alpha^2)$ for problem P2 and $b_\alpha(x_\beta) = O((c+1)/h_\alpha^2)$ for problem P1. It remains to investigate (3.2). For problem P2 the exact solution of (3.2) is a function

$$w^\alpha(x) = \begin{cases} a_1 x, & 0 \leq x \leq 2/3 \\ 2a_1(1-x), & 2/3 \leq x \leq 1, \end{cases} \quad a_1 = \frac{|c-1|}{2h_\alpha(2c+1)}$$

and hence

$$c_\alpha(x_\beta) = w^\alpha(2/3) = |c-1|/(3h_\alpha(2c+1)) = O(1/h_\alpha).$$

A situation is different again for problem P1, where

$$\begin{aligned} w^\alpha(x) &= a_j x + b_j, & x_{Lj} &\leq x \leq x_{Rj}, \\ j &= 1, 2, 3, & b_1 &= 0, \quad a_2 = 0, \quad b_2 = a_1/3, \\ c_\alpha(x_\beta) &= \max_{x_\alpha} w^\alpha(x_\alpha) = |c-1|/6h_\alpha. \end{aligned}$$

We have proved that the number of iterations required by the IAT method is proportional to $n_0(\varepsilon) = O(((c+1)/h)^{0.5})$ for problem P1 and $n_0(\varepsilon) = O(1/h^{0.5})$ for problem P2. Therefore the number of iterations $n_0(\varepsilon)$ depends on the size of the discontinuity of the coefficients in the same manner as it depends for the RDP method.

REMARK 3.1. In the case of $D = \text{diag} A$, the preconditioning matrix B_T coincides with the preconditioning matrix defined by the symmetric successive overrelaxation method (SSOR), proposed by Sheldon (1955).

Next we consider very briefly the method of AD. An upper bound for the number of iterations is given by $n_0(\varepsilon) = 0.25 \ln(1/\varepsilon) (\Delta/\delta)^{0.5}$, where

$$\begin{aligned} \delta_\alpha &= \min_{x_\beta} 1/\kappa_\alpha(x_\beta), \quad \kappa_\alpha(x_\beta) = \max_{x_\alpha} w^\alpha(x_\alpha), \\ &- (a_\alpha w_{x_\alpha}^\alpha)_{x_\alpha} = 1, \quad w^\alpha(0) = 0, \quad w^\alpha(1) = 0, \\ \Delta_\alpha &= \left(2 - 1/\max_{x_\beta \in \omega_\beta} \rho_\alpha(x_\beta)\right) \max_{x \in \omega} d_\alpha(x), \quad \rho_\alpha(x_\beta) = \max_{x_\alpha \in \omega_\alpha} v_\alpha(x), \\ &- (a_\alpha v_{x_\alpha}^\alpha)_{x_\alpha} = d_\alpha(x), \quad v^\alpha(0) = 0, \quad v^\alpha(1) = 0, \\ d_\alpha(x) &= \left(a_\alpha(x_\alpha, x_\beta) + a_\alpha^{+1}(x_\alpha, x_\beta)\right)/h_\alpha^2. \end{aligned} \quad (3.3)$$

Now it suffices to solve a new problem (3.3). Using the method described above we get that in all cases $w^\alpha(x_\alpha) = O(1)$. Therefore the number of iterations for the method of AD is proportional to $n_0(\varepsilon)$ defined by the RDP method.

In summary, a wide class of problems with discontinuous coefficients have been defined, for which the rate of convergence of three popular general iterative methods depends very weakly on the size of the discontinuity of the coefficients. The analysis given above was based on approximate bounds for the exact values of spectral equivalence parameters γ_1, γ_2 from below and above, respectively (recall, that $\gamma_1 B \leq A \leq \gamma_2 B$.) Therefore, in the case of pessimistic estimates of the number of iterations $n_0(\varepsilon) = \sqrt{cn_1(N)}$, an additional analysis must be made. In the next section we will

use numerical methods to find exact values of the parameters γ_1, γ_2 for this purpose.

4. Formulation of new criterion. In this section we shall propose a more general criterion on the applicability of classical iterative methods. Some theoretical foundation of this criterion can be given. All previous estimates were based on the following lemma (see Samarskij and Nikolayev, 1978).

Lemma 4.1. Let $\rho_i \geq 0, a_i \geq c > 0$ be defined on ω_h . Then for any function y_i , such that $y_0 = y_N = 0$ the estimate (4.1) holds

$$\mu(\rho y, y) \leq (a, y_x^2), \quad 1/\mu = \max_{1 \leq i \leq N-1} v_i, \quad (4.1)$$

where v_i is a solution of the auxiliary problem

$$-(av_x)_x = \rho_i, \quad i = 1, 2, \dots, N-1, \quad v_0 = v_N = 0.$$

For simplicity we restrict our analysis to the RDP method. Using (4.1) we can obtain the inequality, from which a bound for the parameter γ_1 is found

$$\begin{aligned} (Ay, y) &\equiv \sum_{\alpha=1}^2 (a_\alpha, y_{x_\alpha}^2) \geq \left(\sum_{\alpha=1}^2 \frac{\rho}{\kappa_\alpha}, y^2 \right) \\ &\geq \min_{x \in \omega_h} \sum_{\alpha=1}^2 \frac{1}{\kappa_\alpha(x_\beta)} (\rho y, y) = \tilde{\gamma}_1 (\rho y, y), \quad (4.2) \\ \tilde{\gamma}_1 &= \min_{x \in \omega_h} \sum_{\alpha=1}^2 \frac{1}{\kappa_\alpha(x_\beta)} = \sum_{\alpha=1}^2 \min_{x_\beta \in \omega_\beta} \frac{1}{\kappa_\alpha(x_\beta)}. \end{aligned}$$

The inequality (4.2) holds unconditionally, but it may be too coarse in many cases. In the proof of (4.2) the worst case is achieved on fastly varying function y_i . But the exact value of γ_1 is defined from the Raleigh ratio and is equal to the first eigenvalue of the problem $Ay = \lambda \rho y$. The corresponding eigenvector μ_1 is a slowly varying continuous function. Therefore we propose to use the vector $y_i \equiv 1$ as a test vector. Modifying the derivation of (4.2) we get an approximate value of the parameter γ_1

$$\tilde{\gamma}_1 = \left(\rho, 1/\kappa_1(x_{2j}) + 1/\kappa_2(x_{1i}) \right) / (\rho, 1). \quad (4.3)$$

On the basis of all these results we propose the following criterion on the applicability of general iterative methods. In particular all cases investigated by us above, satisfy the demands of this criterion.

CRITERION 4.1. The rate of convergence of general iterative methods (such as RDP, IAT, AD, SSOR) depends weakly on the size of the discontinuity of the coefficients if for any region D_p with relatively large values of $k(x)$ (i.e., $k(x_p) \gg k(x_{p-1})$, where $x_{p-1} \in D_{p-1}$ and D_{p-1} is a neighbouring region for D_p):

- C1) either efficient boundary conditions (i.e., the Dirichlet or third type boundary conditions) are given on some part of the boundary Γ_p ,
- C2) or D_p has a common boundary with the other region D_{p+1} such that $k(x_{p+1}) \gg k(x_p)$.

We shall demonstrate the efficiency of our criterion (as well as formula (4.3)) for the following problem (1.1).

PROBLEM P3. Let $\bar{D} = D_1 \cup D_2 \cup D_3$ be a union of three subsets D_j and $k(x) = c_j$, $x \in D_j$, $j = 1, 2, 3$.

$$D_2 = \{(x_1, x_2) : 0.25 \leq x_\alpha \leq 0.75, \alpha = 1, 2\},$$

$$D_4 = \{(x_1, x_2) : 0.5 \leq x_\alpha \leq 1, \alpha = 1, 2\}, \quad D_3 = D_4 \setminus (D_2 \cap D_4).$$

Fig. 2 shows the regions D_α for this model problem.

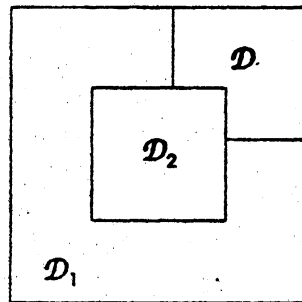


Fig. 2. The regions for model problem P3.

In Table 3 we present numerical results obtained for the following combinations of the constants c_j .

- V1) $c_1 = 1, c_2 = 1, c_3 = 1$; V2) $c_1 = 1, c_2 = 100, c_3 = 1$;
 V3) $c_1 = 1, c_2 = 10000, c_3 = 1$; V4) $c_1 = 1, c_2 = 1, c_3 = 10000$;
 V5) $c_1 = 1, c_2 = 100, c_3 = 10000$; V6) $c_1 = 1, c_2 = 10000, c_3 = 10000$.

Table 3. The number of iterations (numerical simulation)

V	B_R	\tilde{B}_R	B_R^*	B_T	B_{SSOR}^*	B_{IF}^*
1	70	70	63	14	13	12
2	498	431	463	78	96	20
3	4952	4276	4600	709	956	21
4	72	59	55	18	15	10
5	498	69	101	78	28	14
6	4952	129	143	709	31	16

Computations have been done for $N = 20$ (recall that $h = 1/N$ denote the mesh spacing in the $x_{1,2}$ directions). The column marked B_R provides the number of iterations defined by the algorithm (2.2) for the RDP method (a priori estimates). The column marked \tilde{B}_R gives the number of iterations, when formula (4.3) is used to estimate γ_1 . The column marked B_R^* gives the number of iterations defined from the exact values of spectral equivalence parameters γ_1, γ_2 . Analogically the columns marked B_T and B_{SSOR}^* provide the number of iterations defined by the IAT method (a priori estimations (3.1), (3.2)), and the SSOR method (exact values of γ_1, γ_2) respectively.

We get from the Criterion 4.1 that only in the cases $V2, V3$ the number of iterations must be sensitive to the size of the discontinuity of the coefficients. The numerical results confirm this conclusion. The number of iterations given by a priori estimates (3.2), (3.1) (the columns B_R, B_T) is too pessimistic in many cases. At the same time approximate formula (4.3) provides an accurate

estimation of the exact γ_1 value (see the columns \tilde{B}_R and B_R^*). The column marked B_{IF}^* provides the number of iterations defined by the method of Incomplete Factorization. We see that the rate of convergence depends very weakly on the size of the discontinuity for all investigated cases, even for the variants V2, V3. The theoretical investigation of the IF method is given by Axelsson and Barker (1984), it can be also used for our model problem.

REFERENCES

- Axelsson, O., and V.A. Barker (1984). *Finite element solution of boundary value problems. Theory and computation*. Academic press, New York London. 432pp.
- Bakhvalov, N.S., and A.V. Kniazev (1990). A new iterative algorithm for solution of problems emerging in the method of fictitious fluxes for elliptic equations. *Dokl. Akad. Nauk SSSR*. 312 (4), 783 - 786 (in Russian).
- Bramble, J.H., J.E. Pasciak and A.H. Schatz (1988). The construction of preconditioners for elliptic problems by substructuring. IV. *Math. Comput.* 53 (187), 1 - 24.
- Čiegis, R., and T. Šeibak (1985). On the efficiency of iterative methods for problems with discontinuous coefficients. In M. Sapagovas (Ed.), *Diferentsial'nye Uravneniya i ich Primenenya*, Vol. 37. Inst. Math. and Inform., Vilnius. pp. 68 - 81 (in Russian).
- Dryja, M., and O. Widlund (1989). Towards a unified theory of domain decomposition algorithms for elliptic problems. Ultracomputer Research Lab., Courant Institute, Technical Report 486.
- Samarskij, A.A., and E.S. Nikolayev (1978). *The methods of solution of grid equations*. Nauka, Moscow. pp. 589 (in Russian).
- Sheldon, J (1955). On the numerical solution of elliptic difference equations. *Math. Tables Aids Comput.* 9(2), 101 - 112.
- Šeibak, T., and R. Čiegis (1986). On iterative methods for a class of elliptic difference equations in a rectangular domain. In M. Sapagovas (Ed.), *Diferentsial'nye Uravneniya i ich Primenenya*, Vol. 38. Inst. Math. and Inform., Vilnius, pp. 100 - 111 (in Russian).
- Vabishchevich, P.N. (1991) *The method of fictitious domains in mathematical physics*. Moscow University, Moscow. 156 pp. (in Russian).

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T. Aleinikova graduated from the Vitebsk State Pedagogical Institute, receiving high-school diploma in mathematics, and made a postgraduate course at Belorussian State University. She is lecturer at Vitebsk State Pedagogical Institute. Her research interests include numerical methods for elliptic problems.

R. Čiegis received the Degree of Candidate of Physical and Mathematical Sciences from the Institute of Mathematics, Minsk, Acad. Sci. BSSR. He is a senior researcher at the Department of Numerical Methods, Institute of Mathematics and Informatics, Lithuanian Acad. Sci. His research interests include numerical methods for nonlinear problems.