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ON SET-VALUED MEASURES

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Abstract. As a rule, a measure is a mapping from a σ-field of sets into the set of reals, or more generally, into some Banach space. A concept of set-valued measure (SV-measure) is introduced in the paper being a specific mapping from a σ-field of sets into a power set of a set. Properties of SV-measures are analyzed and illustrated on examples. Close relationship between SV-measures and a new nonstandard approach in artificial intelligence (AI) is explained. Then, the construction of factorization of the measures is mentioned, a special class of σ-quasiatomic SV-measures is defined and corresponding characterization theorem is proved. This class involves SV-measures ranging in a countable set which were used in modelling uncertainty in AI. It enables to answer one question arising in connection with this application.

Key words: set-valued measure, factorization, quasiatom of set-valued measure.

Introduction. Presented paper deals with a notion of set-valued measure (SV-measure). Domains of SV-measures coincide with domains of "ordinary" measures, i.e., they are σ-fields of subsets of a set. In contrast with classical measure theory, values of an SV-measure are members of a power set of a set called target below.

This article has two basic sources of motivation. The first one is in artificial intelligence. Bundy (1985) suggested a new approach to description of uncertainty in expert systems: degrees of uncertainty of propositions are described by means of subsets of certain basic set (instead of numbers). This approach was followed by Kramosil (1991) who introduced the notion of nonstandard B-valued probability measure for similar purposes.

The second motivation source is measure theory. We present here principal features of certain analogy of this theory. Let us recall that a measure is a mapping ascribing numbers to sets (namely to the elements of a σ-fields of sets). There are some generalizations, for example Banach space-valued measures in functional analysis discussed by Dunford and Schwartz (1958) or (orthogonal) stochastic measures in the theory of stochastic processes; see Cramér and Leadbetter (1967). Our approach is similar, but we consider a mapping ranging in subsets of another set.

We feel that there is some void which should be filled up by corresponding theory. Our concept can easily relate the above mentioned apparently remote areas. Nevertheless, it can be also considered as further alternative model of probability (as reviewed in Fine (1973)) or another attempt to change quantities of measure values cf. Pfanzagl (1971).

We shall show, that an SV-measure is a σ-homomorphism of σ-fields. Maybe such a concept is treated somewhere in literature but we have no information about it.

A principal feature of SV-measures is that they are *extensional;* it means that the measure of the union of two sets can be obtained explicitly from values of measures of these components (similarly for the difference and other set-theoretical operations). This is not valid in case of "ordinary" measures and probably it is the main motivation point of above mentioned approaches in artificial intelligence.

The aim of the article is not only to define the concept of SV-measure but also to deepen it by further more advanced concepts and some results arising in connection with new nonstandard approaches to modelling uncertainty in expert systems.

A concept of SV-measure is introduced in the first section, various example of SV-measures are stated. It is shown that an SV-measure is a σ-homomorphism of σ-fields. Properties of coverings of a set applied in the paper are summarized in the second (auxiliary) section. As a rule, both description and characterization of an SV-measure can be simplified using a simple factorization procedure stated in the third section. Further, notions useful for analysis of SV-measures, like null-sets and quasiatoms, are introduced in the fourth section. Our null-sets are similar to null-sets of ordinary measures, while quasiatoms resemble atoms of ordinary measures.

The last two sections and appendixes are dealt with specific classes of SV-measures closely related to description of uncertainty in expert systems mentioned above. Namely, a concept of σ-quasiatomic SV-measure is introduced and discussed in the fifth section. Complete characterization of σ-quasiatomic SV-measures is given. It enables us to derive a complete characterization of SV-measures ranging in a countable target; see the last (sixth) section. In Appendices we related our theory to the concept of nonstandard $\mathcal{B}$-valued probability measure introduced by Kramosil (1991) and give an affirmative answer of a question from that work.

1. Set valued measures. In this section we introduce the concept of set valued measure (SV-measure). Some examples of SV-measures are given and basic properties are mentioned. We show that an SV-measure is a σ-homomorphism to σ-fields. At the end of the section we discuss a distinction between SV-measures and "ordinary" measures. Namely, in contrast with ordinary measures, the SV-measures could "save" the structure of underlying σ-field and they are extensional with respect to set-theoretical operations.

DEFINITION 1. A tetrad $\mathfrak{R} = (\Omega, \mathcal{A}, H, \mu)$ is called *a space with a set-valued measure* μ iff the following conditions are satisfied:

$(\Omega, \mathcal{A})$ is a measurable space, (1)

H is a set, (2)

$\mu : \mathcal{A} \to \exp H,$ (3)

for every $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$ it holds $\mu(A) \cap \mu(B) = \emptyset$, (4)

for every countable collection $\{A_i \mid i \in I\}$ (5)

of mutually disjoint measurable sets it holds:

$$\mu\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} \mu(A_i).$$

The set H will be called the *target* of $\mathfrak{M}$.

We give five examples of spaces with SV-measures here.

Example 1. (trivial SV-measure). Let $(\Omega, \mathcal{A})$ be a measurable space and H be a set. By a *trivial* SV-measure we understand the mapping $\mu : \mathcal{A} \to \exp H$ ascribing empty set to each measurable set.

Example 2. (identical SV-measure). Let $(\Omega, \mathcal{A})$ be a measurable space. Put $H = \Omega$ and define μ as the identity mapping on $\mathcal{A}$:

$$\mu(A) = A \qquad \text{for each } A \in \mathcal{A}.$$

Example 3. Let $\mathcal{A}$ be the system of all at most countable subsets of the interval $\langle 0, 1)$ and their complements. Take any nonempty set H and define $\mu : \mathcal{A} \to \exp H$ as follows:

$$\mu(A) = \begin{cases} \emptyset & \text{if A is at most countable,} \\ H & \text{otherwise.} \end{cases}$$

It makes no problem to see that $(\langle 0, 1), \mathcal{A}, H, \mu)$ is a space with an SV-measure.

Example 4. (direct product of spaces with SV-measures). Let us consider a nonempty system

$$(\Omega_j, \mathcal{A}_j, H_j, \mu_j), \qquad j \in J$$

of spaces with SV-measures and suppose that Ω_j, $j \in J$ are mutually disjoint and H_j, $j \in J$ are mutually disjoint.

Put

$$\Omega = \bigcup_{j\in J} \Omega_j$$

and define the σ-field $\mathcal{A}$ on Ω by

$$\mathcal{A} = \left\{ \bigcup_{j\in J} A_j \mid \forall j \in J : A_j \in \mathcal{A}_j \right\} = \{A \subseteq \Omega \mid \forall j \in J : A \cap \Omega_j \in \mathcal{A}_j\}.$$

The target H is defined by

$$H = \bigcup_{j\in J} H_j.$$

Finally, for any $A \in \mathcal{A}$ put

$$\mu(A) = \bigcup_{j\in J} \mu_j(A \cap \Omega_j).$$

As can be easily verified $(\Omega, \mathcal{A}, H, \mu)$ is a space with SV-measure.

In fact, Example 5 is a combination of the preceding ones.

Example 5. Let $\varkappa_1 < \varkappa_2$ be two infinite cardinals.

Let $T_1, T_2, T_3, \dots$ be mutually disjoint sets having the cardinality $\varkappa_2$. We put

$$\Omega = \bigcup_{j=1}^{\infty} T_j,$$

$$\mathcal{A}_j = \sigma\left(\{D \subseteq T_j \mid \text{ card } D \leqslant \varkappa_1\}\right)$$

and

$$\mathcal{A} = \sigma\left(\{T_1, T_2, T_3, \dots\} \cup \{D \subseteq \Omega \mid \text{ card } D \leqslant \varkappa_1\}\right).$$

Let $H_1, H_2, H_3, \dots$ be mutually disjoint sets and

$$H = \bigcup_{j=1}^{\infty} H_j.$$

At first, for each $j = 1, 2, \dots$ define an SV-measure $\mu_j : \mathcal{A}_j \to H_j$ similarly as in Example 3:

$$\mu_j(A_j) = \begin{cases} \emptyset & \text{if card } A_j \leqslant \varkappa_1, \\ H_j & \text{otherwise.} \end{cases}$$

To define μ use the procedure from Example 4:

$$\mu(A) = \bigcup_{j=1}^{\infty} \mu_j(A \cap T_j), \qquad \text{whenever } A \in \mathcal{A}.$$

Clearly $(\Omega, \mathcal{A}, H, \mu)$ is a space with SV-measure.

In the rest of the presented paper we suppose that a space with SV-measure

$$\mathfrak{R} = (\Omega, \mathcal{A}, H, \mu)$$

is given. The only exception concerns Proposition 2 in the section 5.

We shall show that an SV-measure preserves basic set-theoretical operations.

Lemma 1. *Let A, $B \in \mathcal{A}$. Then*

$$\mu(B \setminus A) = \mu(B) \setminus \mu(A) \quad \textit{whenever } A \subseteq B, \tag{6}$$

$$\mu(A \cup B) = \mu(A) \cup \mu(B), \tag{7}$$

$$\mu(A \cap B) = \mu(A) \cap \mu(B). \tag{8}$$

Proof.

a) It holds $\mu(\emptyset) \cap \mu(\emptyset) = 0$ by (4), i.e., $\mu(\emptyset) = \emptyset$.

b) We prove that $\mu(A \cup B) = \mu(A) \cup \mu(B)$ whenever $A \cap B = \emptyset$. We have

$$\mu(A \cup B) = \mu(A \cup B \cup \emptyset \cup \emptyset \cup \ldots) = \mu(A) \cup \mu(B) \cup \mu(\emptyset) \cup \mu(\emptyset) \cup \ldots$$

by (5), thus $\mu(A \cup B) = \mu(A) \cup \mu(B)$ by a).

c) We prove (6). The sets $\mu(A)$ and $\mu(B \setminus A)$ are mutually disjoint according to (4), $\mu(B) = \mu(A) \cup \mu(B \setminus A)$ according to b). Therefore $\mu(B \setminus A) = \mu(B) \setminus \mu(A)$.

d) We prove (7). Using b) twice we obtain

$$\begin{aligned} \mu(A \cup B) &= \mu(A \setminus B) \cup \mu(A \cap B) \cup \mu(B \setminus A) = \\ &= [\mu(A \setminus B) \cup \mu(A \cap B)] \cup [\mu(A \cap B) \cup \mu(B \setminus A)] = \\ &= \mu(A) \cup \mu(B). \end{aligned}$$

e) Let us consider the remaining case (8). It holds $A \cap B = \Omega \setminus \{[\Omega \setminus A] \cup [\Omega \setminus B]\}$. Using it together with (6) and (7) we get

$$\mu(A \cap B) = \mu(\Omega) \setminus \{[\mu(\Omega) \setminus \mu(A)] \cup [\mu(\Omega) \setminus \mu(B)]\} = \mu(A) \cap \mu(B).$$

Precisely, μ is a homomorphism of $\mathcal{A}$ into $\exp \mu(\Omega)$. Such mappings have been studied in lattice theory. Therefore, μ holds the following properties, cf. Birkhoff (1940), Sikorski (1960).

Lemma 2. *Let $A, B \in \mathcal{A}$. Then*

$$\mu(\emptyset) = \emptyset, \tag{9}$$

$$A \subseteq B \textit{ implies } \mu(A) \subseteq \mu(B), \tag{10}$$

$$\mu(B \setminus A) = \mu(B) \setminus \mu(A), \tag{11}$$

$$\mu(A) = \mu(B) \quad iff \quad \mu(A \Delta B) = \emptyset. \tag{12}$$

Theorem 1. *An SV-measure μ is a σ-homomorphism of $\mathcal{A}$ into $\exp \mu(\Omega)$, i.e., μ fulfils (6) and whenever $A_n \in \mathcal{A}$ for all $n \in N$ then both*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \mu(A_n) \tag{13}$$

and

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} \mu(A_n) \tag{14}$$

hold.

Proof.

a) Let us prove (13). Since the sets $A_n \setminus \bigcup_{k=1}^{n-1} A_k$ are mutually disjoint the last equality in

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \left[A_n \setminus \bigcup_{k=1}^{n-1} A_k\right]\right) = \bigcup_{n=1}^{\infty} \mu\left(A_n \setminus \bigcup_{k=1}^{n-1} A_k\right)$$

follows from (5). Hence by (11) and (7) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty}\left[\mu(A_n) \setminus \mu\left(\bigcup_{k=1}^{n-1} A_k\right)\right]$$
$$= \bigcup_{n=1}^{\infty}\left[\mu(A_n) \setminus \bigcup_{k=1}^{n-1} \mu(A_k)\right] = \bigcup_{n=1}^{\infty} \mu(A_n).$$

b) Using (13) and (11) we get

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \mu\left[\Omega \setminus \bigcup_{n=1}^{\infty}(\Omega \setminus A_n)\right]$$
$$= \mu(\Omega) \setminus \bigcup_{n=1}^{\infty}\left[\mu(\Omega) \setminus \mu(A_n)\right] = \bigcap_{n=1}^{\infty} \mu(A_n)$$

which gives (12).

Corollary 1. The range $\mu(\mathcal{A})$ is a σ-ring of subsets of H, more precisely it is a σ-field of subsets of $\mu(\Omega)$.

Properties of SV-measures stated in Theorem 1 and Corollary 1 express our phrase that SV-measures could "save" the structure of an underlying σ-field.

Owing to SV-measures the properties, they are extensional. E.g., there exists a concrete formula (namely (7)) which express the value (of SV-measure) of the union of two sets by means of the values (of SV-measure) of these sets. Nevertheless, in case of "ordinary" measure μ the value $\mu(A \cup B)$ is not determined by values of $\mu(A)$ and $\mu(B)$ uniquely. Analogical situation concerns other set-theoretical operations, as follows from (11), (13) and (14). These formulas show extensionality of SV-measure with respect to set-theoretical operations. They have no counterparts in classical measure theory.

2. Equivalences determined by coverings. This section contains some auxiliary results on coverings of a set used later. We shall need special equivalence determined by a covering of the basic set Ω (resp. $\mu(\Omega)$) and the corresponding partitions of Ω (resp. $\mu(\Omega)$).

DEFINITION 2. Let X be a nonempty set. A set $\mathcal{M}$ is called a *covering* of X, iff $X = \bigcup_{M \in \mathcal{M}} M$.

Moreover, any covering $\mathcal{M}$ of X determines the following equivalence $\sim_{\mathcal{M}}$ on X : if $x, y \in X$, then

$$x \sim_{\mathcal{M}} y \iff [\forall M \in \mathcal{M} \text{ either } x, y \in M, \text{ or } x, y \notin M] \ .$$

The set of all $\sim_{\mathcal{M}}$ classes will be denoted by $\widehat{X}_{\mathcal{M}}$.

(Cf. Roblin (1947), Samorodnickij (1990) p.18–21 for details.)

The classes of $\sim_{\mathcal{M}}$ equivalence have great importance for our study. Clearly, $\widehat{X}_{\mathcal{M}}$ is also a covering of X and $\sim_{\mathcal{M}}$ equals to $\sim_{\widehat{X}_{\mathcal{M}}}$.

The following lemma summarizes basic properties of equivalence classes.

Lemma 3. *Let $\mathcal{M}$ be a covering of X.*

a) *The relation $\sim_{\mathcal{M}}$ is an equivalence on X.*

b) *Whenever $M \in \mathcal{M}$ and $u \in \widehat{X}_{\mathcal{M}}$, then either $u \subseteq M$, or $u \cap M = \emptyset$.*

c) *Let $\mathcal{M}$ be closed under the complement operation, i.e., $X \setminus M \in \mathcal{M}$ for any $M \in \mathcal{M}$. Then*

$$u = \bigcap_{\substack{M \in \mathcal{M} \\ u \subseteq M}} M = \bigcap_{\substack{M \in \mathcal{M} \\ x \in M}} M \qquad \text{whenever} \quad u \in \widehat{X}_{\mathcal{M}}, \ x \in u.$$

d) *$\sigma(\mathcal{M})$ is a covering of X and $\sim_{\mathcal{M}}$ equals to $\sim_{\sigma(\mathcal{M})}$.*

Proof. Proof of a), b), c) is left to the reader. For sketch of the proof one can consult Rohlin (1947) or Samorodnickij (1990), Chapter 1, §4.

For a proof of d) we consider $x, y \in X$.

If $x \sim_{\sigma(\mathcal{M})} y$ then $x \sim_{\mathcal{M}} y$ since $\sigma(\mathcal{M}) \supseteq \mathcal{M}$.

If $x \sim_{\mathcal{M}} y$ then there exists $u \in \widehat{X}_{\mathcal{M}}$ such that $x, y \in u$.

As $\{M \subseteq X \mid u \subset M \text{ or } u \subseteq X \setminus M\}$ is a σ-field, for each $M \in \sigma(\mathcal{M})$ we have either $u \subseteq M$ or $u \subseteq X \setminus M$. Therefore $x \sim_{\sigma(\mathcal{M})} y$.

3. Factorization of SV-measures. In this section we show that both SV-measure and target set could be modified in order to ensure that points of the new target are distinguishable by means of the modified SV-measure. The construction is called factorization. It is possible to reconstruct the original SV-measure from its factor-measure.

Factorization can be applied to simplify both description and characterization of a particular SV-measure.

It seems natural to call points $x, y \in \mu(\Omega)$ *μ-separable* if there exists $A \in \mathcal{A}$ such that $x \in \mu(A)$ and $y \notin \mu(A)$, and in the opposite case to call them *μ-inseparable.* Certainly, it defines an equivalence on $\mu(\Omega)$. Clearly, points outside $\mu(\Omega)$ form a special class which can be omitted (cf. Corollary 1).

Definition 3. The factor space $\widehat{\mu(\Omega)}_{\mu(\mathcal{A})}$ is denoted by $\widehat{H}(\mu)$. The class in $\widehat{H}(\mu)$ containing $x \in \mu(\Omega)$ is denoted by $\hat{x}$. For any $A \in \mathcal{A}$ we set

$$\widehat{\mu}(A) = \{\hat{x} \mid x \in \mu(A)\}.$$

It makes no problems to see from preceding results that the following holds.

Proposition 1. The tetrad $(\Omega, \mathcal{A}, \widehat{H}(\mu), \widehat{\mu})$ is a space with SV-measure. Moreover, each different points $u, v \in \widehat{H}(\mu)$ are $\widehat{\mu}$-separable.

Further, it easily follows from Lemma 2.

Lemma 4. a) *Let $A \in \mathcal{A}$ and $u \in \widehat{H}(\mu)$. Then either $u \subseteq \mu(A)$, or $u \cap \mu(A) = \emptyset$.*

b) *If $u \in \widehat{H}(\mu)$, then $u = \bigcap_{\substack{A \in \mathcal{A} \\ u \subseteq \mu(A)}} \mu(A)$.*

In general, the members of new target $\widehat{H}(\mu)$ may not be $\mu(\mathcal{A})$-measurable, i.e., the inclusion

$$\widehat{H}(\mu) \subseteq \mu(\mathcal{A})$$

may not hold. Namely, consider a measurable space $(\Omega, \mathcal{A})$ such that some classes of $\sim_{\mathcal{A}}$ equivalence are not $\mathcal{A}$-measurable. If we define the space $(\Omega, \mathcal{A}, H, \mu)$ with identical SV-measure μ according to Example 2, then $\widehat{H}(\mu)$ equals the set of all $\sim_{\mathcal{A}}$-classes and $\mu(\mathcal{A}) = \mathcal{A}$, i.e., $\widehat{H}(\mu) \subseteq \mu(\mathcal{A})$ does not hold.

The problem how to ensure that elements of $\widehat{H}(\mu)$ belong to $\mu(\mathcal{A})$ is postponed to the end of the next section.

4. Null-sets, atoms and quasiatoms. Here, the class of null-sets on an SV-measure is introduced and shown to be a σ-ring. Our null-sets are parallels of null-sets in classical measure theory. Further, the concepts of an atom and a quasiatom are defined. Our quasiatoms resemble atoms of ordinary measures. Several equivalent characterizations of quasiatoms are given. In the rest of the section it is shown that under the assumption that the underlying σ-field $\mathcal{A}$ is countably generated, all classes in the factor space $\widehat{H}(\mu)$ introduced above belong to $\mu(\mathcal{A})$.

DEFINITION 4. A set $A \in \mathcal{A}$ is called a *null-set* iff $\mu(A) = \emptyset$. The class of all null-sets will be denoted by $\mathcal{N}(\mu)$. i.e.,

$$\mathcal{N}(\mu) = \{A \in \mathcal{A} \mid \mu(A) = \emptyset\}.$$

The structure of $\mathcal{N}(\mu)$ is characterized by the following lemma.

Lemma 5. *$\mathcal{N}(\mu)$ is a σ-ring.*

Proof. a) Let $A, B \in \mathcal{N}(\mu)$. Then $\mu(A \setminus B) \subseteq \mu(A) = \emptyset$ by (10) and consequently $A \setminus B \in \mathcal{N}(\mu)$.

b) Let $A_n \in \mathcal{N}(\mu)$, $n \in N$. Then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} \mu(A_n) = \emptyset$ according to (12). Thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}(\mu)$.

DEFINITION 5. Let $(X, \mathcal{X})$ be a measurable space. A measurable set $u \in \mathcal{X}$ is called an *atom* of $\mathcal{X}$ iff $u \neq \emptyset$ and the only proper measurable (i.e., belonging to $\mathcal{X}$) subset of u is the empty set.

DEFINITION 6. A measurable set $A \in \mathcal{A}$ is called a *quasiatom* of SV-measure μ iff $\mu(A)$ is an atom of $\mu(\mathcal{A})$.

If A is an atom of $\mathcal{A}$ and $\mu(A) \neq \emptyset$ then A is a quasiatom of μ. (Namely, if $u \in \mu(\mathcal{A})$, then there is $B \in \mathcal{A}$ with $\mu(B) = u$. Thus either $A \subseteq B$, or $A \cap B = \emptyset$, i.e., either $\mu(A) \subseteq \mu(B) = u$ or $\emptyset = \mu(A) \cap \mu(B) = \mu(A) \cap u$).

On the other hand a quasiatom A of μ may exist such that $\mu(A) \neq \mu(B)$ holds for all atoms B of $\mathcal{A}$. E.g., in Example 3 all atoms of $\mathcal{A}$ are singletons of Ω and $\mu(\{x\}) = \emptyset \neq H = \mu(A)$ holds for all $x \in \Omega$ and for all quasiatoms A of μ.

There are several equivalent characterizations of quasiatoms:

Lemma 6. *Let $A \in \mathcal{A}$. The following statements are equivalent*

$$A \quad \text{is a quasiatom of} \quad \mu \tag{15}$$

$$\mu(A) \neq \emptyset \quad \textit{and for any} \quad A \supseteq B \in \mathcal{A} \quad \textit{we have either} \tag{16}$$
$$\mu(B) = \mu(A), \quad \textit{or} \quad \mu(B) = \emptyset$$

$$\mu(A) \neq \emptyset \quad \textit{and for any} \quad C \in \mathcal{A} \quad \textit{we have either} \tag{17}$$
$$\mu(A) \subseteq \mu(C) \quad \textit{or} \quad \mu(A) \cap \mu(C) = \emptyset$$

$$\mu(A) \in \widehat{H}(\mu). \tag{18}$$

Proof. a) Using (10) we easily derive that (15) implies (16).

b) Suppose that A satisfies (16) and consider a set $C \in \mathcal{A}$. Hence the statement (16) yields (17) taking $B = C \cap A$ and using (8).

c) We prove that (17) imply (18). Let $x \in \mu(A)$. We shall show that $\hat{x} = \mu(A)$. Immediately we have $\hat{x} \subseteq \mu(A)$ according to Lemma 4. Conversely take another point $y \in \mu(A)$. Using (17) one has either $\{x, y\} \subseteq \mu(A) \subseteq \mu(C)$ or $\{x, y\} \cap \mu(C) \subseteq \mu(A) \cap \mu(C) = \emptyset$ for each $C \in \mathcal{A}$. Hence $y \sim_{\mu(A)} x$, i.e., $y \in \hat{x}$ and consequently $\mu(A) \subseteq \hat{x}$.

d) Let $\mu(A) \in \widehat{H}(\mu)$. We shall show that A is a quasiatom of μ. Necessarily $\mu(A) \neq \emptyset$. Suppose that $B \in \mathcal{A}$, $\mu(B) \neq \emptyset$ and $\mu(B) \subseteq \mu(A)$. Taking $x \in \mu(B)$ we have $\hat{x} \subseteq \mu(B) \subseteq \mu(A) = \hat{x}$ according to Lemma 4. Hence $\mu(B) = \mu(A)$ and A is a quasiatom of μ.

The preceding lemma easily implies the following corollary.

Corollary 2. If $A, B \in \mathcal{A}$ are quasiatoms of μ, then we have

$$\text{either} \qquad \mu(A) = \mu(B) \qquad \text{or} \qquad \mu(A) \cap \mu(B) = \emptyset \tag{19}$$

and

$$\text{either} \qquad \mu(A\Delta B) = \emptyset \qquad \text{or} \qquad \mu(A \cap B) = \emptyset\,. \tag{20}$$

The following lemma enables us to derive an easy sufficient conditions for measurability of all classes of the factor space $\widehat{H}(\mu)$.

Lemma 7. *Let $X, \mathcal{X})$ be a measurable space. If a σ-field $\mathcal{X}$ is countably generated, then X is the union of all atoms of $\mathcal{X}$.*

Proof. Let $\mathcal{D}$ be a countable collection of generators of $\mathcal{X}$. Assume without any loss of generality that $\mathcal{D}$ is a covering of X which is closed under the complement operation (i.e., $X \setminus D \in \mathcal{D}$ for any $D \in \mathcal{D}$). Hence, it suffices to show that $\widehat{X}_{\mathcal{D}} \subseteq \mathcal{X}$ since $X = \bigcup_{A \in \hat{X}_{\mathcal{D}}} A$.
Take $A \in \widehat{X}_{\mathcal{D}}$. By Lemma 3, parts c and d, we get $A = \bigcap_{\substack{D \in \mathcal{D} \\ A \subseteq D}} D$.
But $\mathcal{D}$ is a countable set, thus consequently $A \in \mathcal{X}$.

COROLLARY 3. Let $\mathcal{A}$ be countably generated. Then $\widehat{H}(\mu) \subseteq \mu(\mathcal{A})$. Moreover. there exists a collection $\{T_i \mid i \in I\}$ of atoms of $\mathcal{A}$ such that $\mu(T_i) \neq \emptyset$ holds for any $i \in I$ and

$$\mu(A) = \bigcap_{\substack{i \in I \\ T_i \subseteq A}} \mu(T_i)$$

takes place for each $A \in \mathcal{A}$.
Proof. By Lemma 7, $\Omega = \bigcup_{i \in \hat{I}} u_i$ for the collection $\{u_i \mid i \in \hat{I}\}$ of all atoms $\mathcal{A}$. Let us select a collection of atoms $\mathcal{A}$ such that

$$\{T_i \mid i \in I\} = \{u_i \mid \mu(u_i) \neq \emptyset,\ i \in \hat{I}\}.$$

Consider $x \in \mu(\Omega)$. Let us denote $\mathcal{W} = \{D \in \mathcal{D} \mid x \in \mu(D)\}$ and

$$B = \bigcap_{D \in \mathcal{W}} D. \tag{21}$$

We prove that $B \in \mathcal{A}$ and $x \in \mu(B)$ hold. If $D \in \mathcal{D}$, then $\Omega \setminus D \in \mathcal{D}$, thus $x \in \mu(\Omega) = \mu(D) \cup \mu(\Omega \setminus D)$, i.e., $\mathcal{W} \neq \emptyset$. The set $\mathcal{D}$ is countable and $\mathcal{W} \subseteq \mathcal{D}$, i.e., $B \in \mathcal{A}$ by (21), thus $\mu(B) = \bigcap_{\substack{D \in \mathcal{D} \\ x \in \mu(D)}} \mu(D) \supseteq \{x\}$.

We prove that B is an atom of $\mathcal{A}$. It holds $\sim_{\mathcal{A}}=\sim_{\mathcal{D}}$ by Lemma 3d, thus it suffices to prove that if $D \in \mathcal{D}$, then either $B \subset D$, or $B \cap D = \emptyset$. If $x \in \mu(D)$, then $D \in \mathcal{W}$, i.e., $B \subseteq D$ by (21). If $x \notin \mu(D)$, then $x \in \mu(\Omega \setminus D)$, i.e., $\Omega \setminus D \in \mathcal{W}$, so that $B \subseteq \Omega \setminus D$, i.e., $B \cap D = \emptyset$. Thus $B \in \{T_i \mid i \in I\}$ and $x \in \bigcup_{i\in I} \mu(T_i) = \mu(\Omega)$.

We have proved that $\{T_i \mid i \in I\}$ is a collection of atoms $\mathcal{A}$ with property $\mu(\Omega) = \bigcup_{i\in I} \mu(T_i)$. Let $A \in \mathcal{A}$. Obviously, $\mu(A) \supseteq \bigcup_{\substack{i\in I\\ T_i\subseteq A}} \mu(T_i)$ and $\mu(\Omega \setminus A) \supseteq \bigcup_{\substack{i\in I\\ T_i\subseteq \Omega\setminus A}} \mu(T_i)$ take place. Moreover $\mu(A) \cup \mu(\Omega \setminus A) = \mu(\Omega) = \bigcup_{i\in I} \mu(T_i)$ and $\mu(A) \cap \mu(\Omega \setminus A) = \emptyset$. Consequently, $\mu(A) = \bigcap_{\substack{i\in I\\ T_i\subseteq A}} \mu(T_i)$. Hence we have immediately $\widehat{H}(\mu) = \{\mu(T_i) \mid i \in I\} \subseteq \mu(A)$ which concludes the proof.

The assumption that $\mathcal{A}$ is countable generated cannot be omitted in Corollary 3. E.g., in Example 3 atoms of $\mathcal{A}$ coincide with singletons of Ω and $\mu(\{x\}) = \emptyset$ for any singleton $x \in \Omega$. Thus if $\mu(A) \neq \emptyset$, then $\mu(A) = \bigcup_{\substack{i\in I\\ T_i\subseteq A}} \mu(T_i)$ cannot hold for any collection $\{T_i \mid i \in I\}$ of atoms of $\mathcal{A}$.

5. The characterization of σ-quasiatomic SV-measures. A special class of SV-measures called σ-quasiatomic SV-measures is introduced. A complete characterization of σ-quasiatomic measures is given. This type of SV- measures is closely related to description of uncertainty in expert systems (see Introduction). They are, in fact, generalizations of measures introduced by Kramosil (1991), cf. Appendix A.

DEFINITION 7. An SV-measure μ is called *σ-quasiatomic* iff there is an at most countable collection $\{T_i \mid i \in I\}$ of quasiatoms of μ satisfying

$$\mu(\Omega) = \bigcup_{i\in I} \mu(T_i). \tag{22}$$

DEFINITION 8. A collection $\{T_i \mid i \in I\}$ is called *μ- admissible* iff $\{T_i \mid i \in I\}$ is an at most countable collection of mutually disjoint quasiatoms of μ and (22) takes place.

Lemma 8. *The SV-measure μ is σ-quasiatomic iff a μ-admi-*

ssible collection exists.

Proof. Let $\{T_i' \mid i \in I\}$ be at most countable collection of quasiatoms of μ satisfying $\bigcup_{i\in I} \mu(T_i') = \mu(\Omega)$. Without a loss of generality we can assume that $\mu(T_i')$, $i \in I$ are different sets. Thus if $i \neq j$, then $\mu(T_i') \cap \mu(T_j') = \emptyset$ by Corollary 2. We set $T_i = T_i' \setminus \bigcup_{\substack{j\in I\\ j\neq i}} T_j'$ for all $i \in I$. Then T_i, $i \in I$ are mutually disjoint. Moreover $T_i \subseteq T_i'$ and

$$\mu(T_i) = \mu(T_i') \setminus \bigcup_{\substack{j\in I\\ j\neq i}} \mu(T_j') = \mu(T_i'),$$

i.e., T_i are quasiatoms of μ and (22) takes place.

A full characterization of σ-quasiatomic SV-measures is given in the rest of the section.

Theorem 2. *Let $(\Omega, \mathcal{A}, H, \mu)$ be a space with σ-quasiatomic SV-measure μ and $\{T_i \mid i \in I\}$ be a μ-admissible collection.*

Let us denote $\mathcal{B} = \mathcal{N}(\mu)$ and $H_i = \mu(T_i)$ for all $i \in I$. Then

$$T_i \notin \mathcal{B} \quad \textit{for any} \quad i \in I, \tag{23}$$

$$\Omega \setminus \bigcup_{i\in I} T_i \in \mathcal{B}, \tag{24}$$

$$\textit{for any} \quad B \in \mathcal{B},\ i \in I \quad \textit{we have} \quad B \cap T_i \in \mathcal{B}, \tag{25}$$

$$\mathcal{A} = \sigma\left(\mathcal{B} \cup \{T_i \mid i \in I\}\right), \tag{26}$$

$$\forall A \in \mathcal{A} : \mu(A) = \bigcup_{i\in I_A} H_i, \quad \textit{where} \quad I_A = \{i \in I \mid A \cap T_i \notin \mathcal{B}\}. \tag{27}$$

Proof. (23) and (24) immediately follow from the definitions of σ-quasiatomic SV-measure and of null-sets. (25) is obvious by (8). Thus (26) and (27) remain for a proof. Let us put

$$\sum = \sigma\left(\mathcal{N}(\mu) \cup \{T_i \mid i \in I\}\right)$$

for that goal. Of course, $\sum \subseteq \mathcal{A}$ takes place.

Take $A \in \mathcal{A}$. For each $i \in I$ we have either $\mu(A \cap T_i) = \mu(T_i)$ or $\mu(A \cap T_i) = \emptyset$ because of T_i is a quasiatom. Since I is at most countable set it follows

$$\mu(A)=\bigcup_{i\in I}\mu(A\cap T_i)\cup\mu\left(A\setminus\bigcup_{i\in I}T_i\right)=\bigcup_{i\in I_A}\mu(A\cap T_i),$$

thus

$$\mu(A)=\bigcup_{i\in I_A}\mu(T_i)=\mu\left(\bigcup_{i\in I_A}T_i\right)$$

and (27) is true.

Using Lemma 2 we derive $A\Delta(\bigcup_{i\in I_A}T_i)\in\mathcal{N}(\mu)$ and hence $A\setminus\bigcup_{i\in I_A}T_i$, $\bigcup_{i\in I_A}T_i\setminus A\in\mathcal{N}(\mu)$ according to (10). Consequently $A\in\sum$ and therefore $\mathcal{A}\subseteq\sum$.

Theorem 2 and Lemma 5 describe σ-quasiatomic SV-measure completely as the following Proposition 2 shows. Proposition 2 builds σ-quasiatomic SV-measure with prescribed collection of quasiatoms, with given values on them and with given σ-ring of null-sets. Only in this Proposition we leave the assumption that $(\Omega,\mathcal{A},H,\mu)$ denotes a space with SV-measure.

Proposition 2. Let $\Omega\neq\emptyset$, H be a set. Suppose that the following entities are given:

– I ... at most countable index set,

– $\{T_i, i\in I\}$ ... a collection of mutually disjoint nonempty subsets of Ω,

– $\{H_i, i\in I\}$ ... a collection of mutually disjoint nonempty subset of H,

– $\mathcal{B}\subseteq\exp\Omega$ ... a σ-ring of subsets of Ω.

Assume that conditions (23), (24) and (25) are fulfilled. Finally let $\mathcal{A}$ be defined by (26) and $\mu:\mathcal{A}\to\exp H$ by (27).

Then $(\Omega,\mathcal{A},H,\mu)$ is a space with σ-quasiatomic SV-measure μ. Moreover $\{T_i, i\in I\}$ is a μ-admissible collection, the σ-field $\mu(\mathcal{A})$ is generated by $\{H_i\mid i\in I\}$, and $N(\mu)=\mathcal{B}$.

Proof. Let us denote

$$\sum = \left\{ A \subseteq \Omega \mid A \setminus \bigcup_{i \in I} T_i \in \mathcal{B} \& \forall i \in I \right.$$
$$\left. [T_i \setminus A \in \mathcal{B} \quad \text{or} \quad T_i \cap A \in \mathcal{B}] \right\}. \tag{28}$$

We prove that $\sum$ is a σ-field (part a), $\sum = \mathcal{A}$ (part b), μ satisfies (4) (part d), μ satisfies (5) (part e) and $\mathcal{N}(\mu) = \mathcal{B}$ (part f).

a) Let $A \in \sum$. It holds $(\Omega \setminus A) \setminus \bigcup_{i \in I} T_i = (\Omega \setminus \bigcup_{i \in I} T_i) \setminus (A \setminus \bigcup_{i \in I} T_i) \in \mathcal{B}$, as follows from (24), (28) and the fact that $\mathcal{B}$ is a σ-ring. Moreover, $T_i \setminus (\Omega \setminus A) = T_i \cap A$ and $T_i \cap (\Omega \setminus A) = T_i \setminus A$ is true for all $i \in I$, so that $\Omega \setminus A \in \sum$ by (28).

Let $A_n \in \sum$ for all $n \in N$. Hence $\bigcup_{n \in N} A_n \setminus \bigcup_{i \in I} T_i = \bigcup_{n \in N} (A_n \setminus \bigcup_{i \in I} T_i) \in \mathcal{B}$ since $\mathcal{B}$ is a σ-ring. It remains to prove that $T_i \setminus \bigcup_{n \in N} A_n \in \mathcal{B}$ or that $T_i \cap \bigcup_{n \in N} A_n \in \mathcal{B}$. We distinguish two cases.

i) Let $T_i \cap A_n \in \mathcal{B}$ for all $n \in N$. Hence $T_i \cap \bigcup_{n \in N} A_n = \bigcup_{n \in N} (A_n \cap T_i) \in \mathcal{B}$ since $\mathcal{B}$ is a σ-ring.

ii) Let there be an index n_0 such that $T_i \setminus A_{n_0} \in \mathcal{B}$. It holds

$$T_i \setminus \bigcup_{n \in N} A_n = \bigcap_{n \in N_1} (T_i \setminus A_n) \setminus \bigcap_{n \in N_2} (T_i \cap A_n),$$

whenever N_1 is nonempty and $N_1 \cup N_2 = N$. Taking $N_1 = \{n \in N \mid T_i \setminus A_n \in \mathcal{B}\}$ and $N_2 = \{n \in N \mid T_i \cap A_n \in \mathcal{B}\}$ we find that $T_i \setminus \bigcup_{n \in N} A_n \in \mathcal{B}$.

b) Let us prove that $\sum = \mathcal{A}$. The σ-field $\sum$ contains any $A \in \mathcal{B}$ (cf. (24), (25) and (28)) as well as any T_i, $i \in I$. Thus $\sum \supseteq \sigma(\mathcal{B} \cup \{T_i \mid i \in I\}) = \mathcal{A}$.

Conversely, let us take $A \in \sum$. We have $A \setminus \bigcup_{i \in I} T_i \in \mathcal{B} \subseteq \mathcal{A}$. Moreover $T_i \cap A \in \mathcal{B} \subseteq \mathcal{A}$ or $T_i \setminus A \in \mathcal{B} \subseteq \mathcal{A}$, and always $T_i \in \mathcal{A}$, i.e., $T_i \cap A \in \mathcal{A}$ for all $i \in I$. Therefore $A = (A \setminus \bigcup_{i \in I} T_i) \cup \bigcup_{i \in I} (T_i \cap A) \in \mathcal{A}$.

c) We observe that if $A \in \sum$ and $i \in I$, then either $T_i \setminus A \in \mathcal{B}$, or $T_i \cap A \in \mathcal{B}$ (if $T_i \setminus A \in \mathcal{B}$ and $T_i \cap A \in \mathcal{B}$, then $T_i \in \mathcal{B}$, which contradicts to(23)).

c1) Moreover $T_i \cap A \in \mathcal{B}$ holds iff $\mu(T_i \cap A) = \emptyset$ takes place; $T_i \setminus A \in \mathcal{B}$ holds iff $\mu(T_i \cap A) = H_i$ takes place.

c2) Using these two facts we find that

$$\mu(A) = \bigcup_{i \in I} \mu(T_i \cap A) = \bigcup_{i \in I_A} H_i$$

holds for any $A \in \mathcal{A}$.

d) Let $A,\ B \in \mathcal{A}$ be disjoint. It suffices to prove that $I_A \cap I_B = \emptyset$. Let $i \in I$. It holds $T_i = T_i \setminus (A \cap B) = (T_i \setminus A) \cup (T_i \setminus B)$ and $T_i \notin \mathcal{B}$, i.e., at least one of $T_i \setminus A$ and $T_i \setminus B$ does not belong to $\mathcal{B}$, so that at least one of $T_i \cap A,\ T_i \cap B$ lies in $\mathcal{B}$, i.e., $i \notin I_A \cap I_B$. Thus $I_A \cap I_B = \emptyset$.

e) Let $A_n \in \mathcal{A}$ for all $n \in N$. Then

$$\mu\left(\bigcup_{n \in N} A_n\right) = \bigcup_{i \in I} \mu\left(T_i \cap \bigcup_{n \in N} A_n\right),$$

and

$$\bigcup_{n \in N} \mu(A_n) = \bigcup_{i \in I} \bigcup_{n \in N} \mu(T_i \cap A_n),$$

as follows from c2. Thus it is enough to prove

$$\mu\left(T_i \cap \bigcup_{n \in N} A_n\right) = \bigcup_{n \in N} \mu(T_i \cap A_n) \tag{29}$$

for each $i \in I$. We distinguish the same two cases as in part a. In the case i) both sides of (29) equal $\emptyset$, as follows from c1. In the case ii) both sides of (29) equal H_i, as follows from c1.

f) If $A \in \mathcal{B}$ then $I_A = \emptyset$ by (25). Thus $A \in \mathcal{N}(\mu)$ by (27). If $A \in \mathcal{N}(\mu)$ then $I_A = \emptyset$ by (27), so that $A \cap T_i \in \mathcal{B}$ holds for all $i \in I$. Therefore $A \cap (\bigcup_{i \in I} T_i) \in \mathcal{B}$ and $A \setminus \bigcup_{i \in I} T_i \in \mathcal{B}$ since $A \in \mathcal{A} = \sum$. So $A \in \mathcal{B}$.

6. SV-measure ranging in at most countable target. SV-measures ranging in a countable target H and satisfying $\mu(\Omega) = H$ are equivalent to the so-called nonstandard $\mathcal{B}$-valued probability measures introduced in Kramosil (1991), cf. Appendix A for details.

An aim of the chapter is to completely characterize SV-measures ranging in at most countable target.

REMARK 1. Let $(X, \mathcal{X})$ be a measurable space with X at most countable. Then X is a union of at most countable collection of atoms of $\mathcal{X}$, thus $\mathcal{X}$ is countably generated.

Recall that $\mathcal{R} = (\Omega, \mathcal{A}, H, \mu)$ denotes a space with SV-measure.

Lemma 9. *Let H be at most countable. Then μ is σ-quasiatomic.*

Proof. We have $\mu(\Omega) \subseteq H$, thus $\mu(\Omega)$ is at most countable, i.e., $\mu(\Omega)$ is a union of at most countable collection $\{H_i \mid i \in I\}$ of atoms of $\mu(\mathcal{A})$ according to Remark 1 (in fact $\{H_i \mid i \in I\} = \widehat{H}(\mu)$). For any $i \in I$ there is $T_i \in \mathcal{A}$ such that $\mu(T_i) = H_i$. Of course, T_i, $i \in I$ are quasiatoms of μ and (22) takes place.

COROLLARY 4. An SV-measure μ is σ-quasiatomic iff $(\Omega, \mathcal{A}, \widehat{\mu}, \widehat{H}(\mu))$ is a space with SV-measure ranging in at most countable target.

Proof. a) Let μ be a σ-quasiatomic SV-measure. There is a μ-admissible collection $\{T_i \mid i \in I\}$. Thus $\widehat{H}(\mu) = \{\mu(T_i) \mid i \in I\}$, as follows from Lemma 6, and hence $\widehat{H}(\mu)$ is at most countable set.

b) Let the set $\widehat{H}(\mu)$ be at most countable. Then $\widehat{\mu}$ is a σ-quasiatomic SV-measure according to Lemma 9 and thus μ is σ-quasiatomic .

We derived a complete description of σ-quasiatomic SV-measures in the preceding Chapter 5 (namely in Theorem 2 and Proposition 2). It can be easily concretized to get a complete characterization of SV-measures ranging in a countable target. Details are left to the reader.

If we assume that the σ-field $\mathcal{A}$ is countably generated then we obtain a special type of SV-measures.

Lemma 10. *Let H be at most countable set and $\mathcal{A}$ be countably generated σ-field. Then there is at most countable collection $\{T_i \mid i \in I\}$ atoms of $\mathcal{A}$ such that $\mu(T_i) \neq \emptyset$ holds for any $i \in I$ and $\mu(\Omega) = \bigcup_{i \in I} \mu(T_i)$. Moreover $\mathcal{N}(\mu)$ is a σ-field of subsets of $\Omega \setminus \bigcup_{i \in I} T_i$.*

Proof. We have derived at Corollary 4 that there is a collection $\{T_i \mid i \in I\}$ of mutually disjoint atoms of $\mathcal{A}$ such that

$$\mu(\Omega) = \bigcup_{i \in I} \mu(T_i), \qquad \mu(T_i) \neq \emptyset.$$

Hence I is at most countable since $\mu(\Omega)$ is at most countable set and $\mu(T_i)$, $i \in I$ are mutually disjoint. Thus $\mu(\Omega \setminus \bigcup_{i \in I} T_i) = \emptyset$ and $\Omega \setminus \bigcup_{i \in I} T_i$ is the maximal null-set in $\mathcal{A}$.

Thus if H is at most countable and $\mathcal{A}$ is countable generated, then Ω has the following structure. There is the null-set B and at most countable collection $\{T_i \mid i \in I\}$ of atoms of $\mathcal{A}$ such that

$$\Omega = B \cup \bigcup_{i \in I} T_i,$$

B and $\bigcup_{i \in I} T_i$ are disjoint and $\mu(T_i) \neq \emptyset$ for all $i \in I$.

Appendix A. The concept of *nonstandard B-valued probability measure* (n. B-v.p.m.) is introduced in Kramosil (1991), Definition 2, and recalled below.

We shall show that such a measure can be interpreted as SV-measure ranging in a countable target set. Namely, we represent any n. B-v.p.m. by some SV-measure.

Let us consider a measurable space $(\Omega, \mathcal{A})$ and a nonempty set H. The symbol $\{0,1\}^H$ denotes the set of all mappings from H into $\{0,1\}$. Finally, the symbol $\oplus$ denotes coordinatewise addition defined on $\{0,1\}^H$. It means that if $i \in H$ and $M \subseteq \{0,1\}^H$, then

$$\left(\bigoplus_{S \in M} S\right)_i = \sum_{S \in M} S_i$$

reaches values from $\{0,1,2,\ldots\} \cup \{+\infty\}$.

Infinite binary sequences are considered in Kramosil (1991), i.e., the special case with $H = N = \{1,2,3,\ldots\}$ is investigated.

DEFINITION 9. A mapping P which takes $\mathcal{A}$ into $\{0,1\}^N$ is called a ***nonstandard B-valued probability measure*** if $P(\Omega) = \langle 1,1,1,\ldots\rangle$

and if for each countable system $\mathcal{A}_1 \subseteq \mathcal{A}$ of mutually disjoint sets it holds

$$P\left(\bigcup_{A\in\mathcal{A}_1} A\right) = \bigoplus_{A\in\mathcal{A}_1} P(A).$$

We shall represent nonstandard B-valued probability measures by means of SV-measures. Roughly speaking, whenever a sequence $S \in \{0,1\}^N$ is used we substitute it by a set $S' \in \exp N$ satisfying

$$\{i \in N \mid S_i = 1\} = S'.$$

Let M_H be the set of all mappings from $\mathcal{A}$ into $\exp H$, $M_H^{0,1}$ be the set of all mappings from $\mathcal{A}$ into $\{0,1\}^H$.

We introduce a mapping $\psi\colon M_H^{0,1} \longrightarrow M_H$ as follows. For every $P \in M_H^{0,1}$ we set

$$\psi(P)(A) = \{i \in H \mid P(A)_i = 1\}, \qquad \forall A \in \mathcal{A}. \tag{30}$$

Then, as could be easily seen, ψ is a bijection between $M_H^{0,1}$ and M_H. Moreover, the following proposition holds.

PROPOSITION 3. Let $P \in M_H^{0,1}$. Then $\psi(P)$ is an SV-measure iff

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigoplus_{n=1}^{\infty} P(A_n) \tag{31}$$

holds for any system $A_n \in \mathcal{A}$, $n \in N$ of mutually disjoint sets.

Proof. a) Let $\psi(P)$ be an SV-measure, denoted by μ. We consider mutually disjoint sets $A_n \in \mathcal{A}$, $n \in N$ and the set $A = \bigcup_{n=1}^{\infty} A_n$.

Let

$$M = \mu(A).$$

Then

$$M = \bigcup_{n=1}^{\infty} \mu(A_n)$$

and $\mu(A_n)$, $n \in N$ are mutually disjoint, as follows from (4) in Definition 1.

If $i \in H \setminus M$, then $P(A)_i = 0$ and $P(A_n)_i = 0$ hold for any $n \in N$. So that

$$P(A)_i = \bigoplus_{n=1}^{\infty} P(A_n)_i. \tag{32}$$

If $i \in M$, then there is just one $n(i) \in N$ such that $i \in \mu(A_{n(i)})$, because $\mu(A_n)$, $n \in N$ are mutually disjoint. Therefore $1 = P(A)_i = P(A_{n(i)})_i$ and $P(A_n)_i = 0$ for any $n \in N \setminus \{n(i)\}$. Thus (32) holds for $i \in M$ as well. So that P satisfies (31).

b) Let P fulfil (31) and μ denote $\psi(P)$. We shall prove that $P(\emptyset)_i = 0$ holds for any $i \in H$. We set $\emptyset = A_1 = A_2 = A_3 = \dots$ into (31). Then $P(\emptyset)_i = \bigoplus_{n=1}^{\infty} P(\emptyset)_i$, so that $P(\emptyset)_i = 0$.

Let $A, B \in \mathcal{A}$ be mutually disjoint. We set $A_1 = A$, $A_2 = B$ and $\emptyset = A_3 = A_4 = \dots$ into (31). Then $P(A \cup B) = P(A) \bigoplus P(B)$. Let $i \in H$. If $i \notin \mu(A)$, then $P(A)_i = 1$, so that $P(B)_i = 0$, therefore $i \notin \mu(B)$. Thus $\mu(A) \cap \mu(B) = \emptyset$ and (4) is satisfied.

If $A_n \in \mathcal{A}$, $n \in N$ are mutually disjoint, then (5) follows from (4) and (31).

It results from Proposition 3 and Definition 1 that if P is an n. B-v.p.m., then $\psi(P)$ is an SV-measure. Conversely if μ is an SV-measure and $\mu(\Omega) = N$, then $\psi^{-1}(\mu)$ is an n. B-v.p.m.

Appendix B. Compositions of a nonstandard B-valued probability measure with probability measures (called induced probability measures with respect to given n. B-v.p.m.) are studied in Kramosil (1991). There is an open question in Kramosil (1991) whether such a measure is discrete or not. We shall give an affirmative answer here.

To do that it suffices to prove that the composition of SV-measure μ ranging in a countable target set with a measure is a discrete measure.

REMARK. We call a probability measure ν defined on a measurable space $\langle X, \mathcal{X} \rangle$ *discrete* iff there is an at most countable collection

$\{A_i \mid i \in I\}$ of measurable sets such that for any $A \in \mathcal{X}$ it holds

$$\nu(A) = \sum_{i \in I} \nu(A \cap A_i).$$

Lemma 11. *Let $(\Omega, \mathcal{A}, H, \mu)$ be a space with SV-measure, ρ be a measure on $(\mu(\Omega), \mu(\mathcal{A}))$. Then $\mu \circ \rho$ is a measure on $(\Omega, \mathcal{A})$.*

Proof. Let $A_n \in \mathcal{A}$, $n \in N$ be mutually disjoint sets. Then $\mu(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} \mu(A_n)$ and $\mu(A_n)$, $n \in N$ are mutually disjoint sets again. Therefore

$$\mu \circ \rho\left(\bigcup_{n=1}^{\infty} A_n\right) = \rho\left(\bigcup_{n=1}^{\infty} \mu(A_n)\right) = \sum_{n=1}^{\infty} (\mu \circ \rho)(A_n).$$

Let the target H be countable. Then $\mu(\Omega)$ is the union of at most countable collection of atoms of $\mu(\mathcal{A})$, because μ is σ-quasiatomic by Lemma 9. Of course, any measure ρ on $(\mu(\Omega), \mu(\mathcal{A}))$ must be discrete. Therefore the composed measure $\mu \circ \rho$ is discrete as well.

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