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Solvability and Asymptotic Behavior of a Pair Formation Model

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Abstract. The unique solvability and asymptotic behavior for large time of two cases of symmetric bisexual population model are presented. One of them includes the harmonic mean mating law, while in the other one pair formation occurs only within the same age class.

Key words: population dynamics, age structure, pair formation, demography.

1. Introduction

Pair formation models are of great importance for human demography and epidemiology, in particular for modeling sexually transmitted diseases (see, e.g., references in Hadeler (1993)).

It is well known the standard ordinary differential equations model for pair formation (see, e.g., Hadeler (1993), Prüss and Schappacher (1994) and references there). This type of bisexual population models consists of a system of three differential equations for S_m, S_f, S_p, the total number of single males, single females, and pairs, respectively, and reads

$$\begin{aligned} S_m' &= -\mu_m S_m + (\beta_m + \tilde{\mu}_f + \sigma)p - \varphi(S_m, S_f), \\ S_f' &= -\mu_f S_f + (\beta_f + \tilde{\mu}_m + \sigma)p - \varphi(S_m, S_f), \\ p' &= -(\tilde{\mu}_m + \tilde{\mu}_f + \sigma)p + \varphi(S_m, S_f). \end{aligned} \tag{1.1}$$

Here the prime indicates differentiation with respect to time t, $\mu_j > 0$ ($\tilde{\mu}_j > 0$) denotes the death rates of unmarried (married) males ($j = m$) or females ($j = f$), $\beta_j > 0$ the birth rates, $\sigma \geqslant 0$ the divorce rate, and φ the mating function. The model (1.1) was completely analyzed by Hadeler *et al.* (1988).

Ordinary differential equations model totally neglects the age structure of the population. Hadeler (1993) incorporated a maturation period τ by introducing a delay in the standard model and derived the system

$$S_m' = \beta_m \exp\{-\mu_m \tau\} p(t - \tau) - \mu_m S_m + (\tilde{\mu}_f + \sigma)p - \varphi(S_m, S_f),$$

$$S_f' = \beta_f \exp\{-\mu_f \tau\} p(t-\tau) - \mu_f S_f + (\tilde{\mu}_m + \sigma)p - \varphi(S_m, S_f), \tag{1.2}$$
$$p' = -(\tilde{\mu}_m + \tilde{\mu}_f + \sigma)p + \varphi(S_m, S_f),$$

for which in the case where $\tilde{\mu}_m = \mu_m$, $\tilde{\mu}_f = \mu_f$ he studied the persistent solutions.

There exist also few works devoted to pair formation models of sex-age-structured populations (see, e.g., Hadeler (1993), Prüss and Schappacher (1994) and references there). The most general sex-age-structured populations model have been proposed by Hoppensteadt (1975) and Staroverov (1977), and consists of a system of three itegro-differential equations for the density $x(t,a)$ of single females of age a, the density $y(t,b)$ of single males of age b, and the density $p(t,a,b,c)$ of pairs which are formed of females of age a, males of age b, and which have existed for a time c. Hadeler (1993) simplified this model by introducing a maturation period into the mating law. This simplified model reads

$$\begin{aligned}
&\partial_t x + \partial_a x + \mu_x x = 0, \quad 0 < a < \tau,\\
&\partial_t y + \partial_b y + \mu_y y = 0, \quad 0 < b < \tau,\\
&\partial_t x + \partial_a x + \mu_x x + \int_\tau^\infty p|_{c=0}\, db = \int_0^{a-\tau} dc \int_{\tau+c}^\infty (\tilde{\mu}_y + \sigma)p\, db, \quad a > \tau,\\
&\partial_t y + \partial_b y + \mu_y y + \int_\tau^\infty p|_{c=0}\, da = \int_0^{b-\tau} dc \int_{\tau+c}^\infty (\tilde{\mu}_x + \sigma)p\, da, \quad b > \tau,\\
&\partial_t p + \partial_a p + \partial_b p + \partial_c p + (\tilde{\mu}_x + \tilde{\mu}_y + \sigma) = 0,\\
&\quad a > \tau,\ b > \tau,\ 0 < c \leqslant \min(\mathrm{a} - \tau, \mathrm{b} - \tau),\\
&x|_{a=0} = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta_x p\, db, \quad [x|_{a=\tau}] = [y|_{b=\tau}] = 0,\\
&y|_{b=0} = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta_y p\, db,\\
&p|_{c=0} = \tilde{\varphi}(x,y)(t,a,b), \quad a > \tau,\ b > \tau,\\
&x|_{t=0} = x^0,\ y|_{t=0} = y^0,\ p|_{t=0} = p^0.
\end{aligned} \tag{1.3}$$

Here $\tau > 0$ is a maturation period, ∂_t, ∂_a, ∂_b and ∂_c indicate partial differentiation, $[x|_{a=\tau}]$ and $[y|_{b=\tau}]$ are jumps of x and y at the lines $a = \tau$ and $b = \tau$, respectively, x^0, y^0, p^0 denote the initial age distribution of single females, single males, and pairs, respectively, β_j the birth rates of males ($j = y$) or females ($j = x$), μ_y resp. μ_x the death rates of single males resp. single females, σ means the divorce rate of pairs, $\tilde{\mu}_y$ resp. $\tilde{\mu}_x$ the death rates of married males resp. females, and $\tilde{\varphi}$ the mating function. The harmonic mean mating law

$$\tilde{\varphi}(x,y) = mxy\Big(\int_\tau^\infty h_x x\,da + \int_\tau^\infty h_y y\,db\Big)^{-1} \tag{1.4}$$

generally is used. The non-negative demographic functions $\mu_x(t,a),\ \mu_y(t,b),\ m(t,a,b),\ h_x(t,a,b),\ h_y(t,a,b),\ \tilde{\mu}_x(t,a,b,c),\ \tilde{\mu}_y(t,a,b,c),\ \sigma(t,a,b,c),\ \beta_x(t,a,b,c),\ \beta_y(t,a,b,c)$, initial distributions $x^0(a),\ y^0(b),\ p^0(a,b,c)$ and maturation age τ are assumed to be prescribed. In addition we assume that $x^0,\ y^0$ and p^0 satisfy the following compatibility conditions

$$\begin{aligned}
&x^0(0) = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta_x|_{t=0}\, p^0\, db, \quad y(0) = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta_y|_{t=0}\, p^0\, db,\\
&p^0|_{c=0} = m|_{t=0} x^0 y^0 \Big(\int_\tau^\infty h_x|_{t=0} x^0\, da + \int_\tau^\infty h_y|_{t=0}\, y^0\, db\Big)^{-1},\\
&[x^0(\tau)] = [y^0(\tau)] = 0.
\end{aligned} \tag{1.5}$$

In the case where $\mu_x,\ \mu_y,\ \tilde{\mu}_x,\ \tilde{\mu}_y,\ \sigma,\ \beta_x,\ \beta_y$ and m are constants and $h_x = h_y = 1$ Hadeler (1993) integrating the system (1.3), (1.4) over age derived (1.2) and studied its persistent solutions. Prüss and Schappacher (1994) treated more general case $\mu_x(a),\ \mu_y(b), \tilde{\mu}_x(a),\ \tilde{\mu}_y(b), \sigma(a,b,c),\ \tau = 0,\ \beta_x(a,b,c),\ \beta_y(a,b,c), h_x = h_y = 1$ and obtained the conditions for existence and non-existence of persistent solutions of (1.3) in an L^1-setting.

In the case where population is symmetric in the sence that all vital rates are symmetric in a and b and independent of sex, i.e., $\beta_x = \beta_y = \beta,\ \mu_x = \mu_y = \mu,\ \tilde{\mu}_x = \tilde{\mu}_y = \tilde{\mu},\ h_x + h_y = 1$, the model (1.3)–(1.5) can be simplified and reads

$$\begin{aligned}
&\partial_t x + \partial_a x + \mu x = 0, \quad 0 < a < \tau,\\
&\partial_t x + \partial_a x + \mu x + \int_\tau^\infty p|_{c=0}\, db = \int_0^{a-\tau} dc \int_{\tau+c}^\infty (\tilde{\mu} + \sigma) p\, db, \quad a > \tau,\\
&\partial_t p + \partial_a p + \partial_b p + \partial_c p + (2\tilde{\mu} + \sigma) p = 0,\\
&\quad a > \tau,\ b > \tau,\ 0 < c \leqslant \min(\mathrm{a} - \tau, \mathrm{b} - \tau),\\
&x|_{a=0} = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta p\, db, \quad [x|_{a=\tau}] = 0,\\
&p|_{c=0} = mx(t,a)x(t,b)\Big(\int_\tau^\infty x(t,\xi)\, d\xi\Big)^{-1}, \quad a > \tau,\ b > \tau,\\
&x|_{t=0} = x^0,\ p|_{t=0} = p^0,
\end{aligned} \tag{1.6}$$

$$x^0(0) = \int_0^\infty dc \int_{\tau+c}^\infty da \int_{\tau+c}^\infty \beta|_{t=0} p^0 \, db, \quad [x^0(\tau)] = 0,$$

$$p^0|_{c=0} = m|_{t=0} x^0(a) x^0(b) \Big(\int_\tau^\infty x^0(\xi)\, d\xi \Big)^{-1}, \quad a > \tau,\ b > \tau.$$

The model (1.6) can be considerably simplified replacing the harmonic mean mating function by a law in which pair formation occurs only within the same age class:

$$\begin{aligned}
&\partial_t x + \partial_a x + \mu x = 0, \quad 0 < a < \tau,\\
&\partial_t x + \partial_a x + (\mu + m)x = \int_0^{a-\tau} (\tilde{\mu} + \sigma) p \, dc, \quad a > \tau,\\
&\partial_t p + \partial_a p + \partial_c p + (2\tilde{\mu} + \sigma) p = 0, \quad a > \tau,\ 0 < c \leqslant a - \tau, \qquad (1.7)\\
&x|_{a=0} = \int_0^\infty dc \int_{\tau+c}^\infty \beta p \, da, \quad [x|_{a=\tau}] = 0, \quad p|_{c=0} = mx, \quad a > \tau,\\
&x|_{t=0} = x^0, \quad p|_{t=0} = p^0,\\
&x^0(0) = \int_0^\infty dc \int_{\tau+c}^\infty \beta|_{t=0} p^0 da, \quad [x^0(\tau)] = 0,\\
&p^0|_{c=0} = m|_{t=0} x^0, \quad [x^0(\tau)] = 0.
\end{aligned}$$

Here $x(t,a)$, $p(t,a,c)$, $\beta(t,a,c)$, $\tilde{\mu}(t,a,c)$, $m(t,a)$, $\mu(t,a)$, $x^0(a)$, $p^0(a,c)$ depend on indicated variables only. In the case where $\tilde{\mu} = \mu(a)$, $m(a)$, $\sigma(a)$, $\beta(a)$ and $\tau = 0$ Hadeler (1989) analyzed the persistent solutions for this linear model.

In this paper we deal with (1.6) and obtain its solution in the case where $\tilde{\mu} = \mu$, m and σ are constant and β depends on all possible variables, and construct its asymptotic behavior when β is constant and x^0, p^0 are majorized by the initial value of the persistent solution for (1.6) (see Section 5). We treat also problem (1.7) with $\mu(t,a) = \tilde{\mu}(t,a)$, $\sigma(t,a)$, $m(t,a)$, $\beta(t,a)$ and obtain its solution. Moreover, we construct its asymptotic behavior when all vital rates are stationary and initial distributions are specialized (see Section 3). We also deal with the modified problem (1.7) with operators $\partial_t + \partial_a$, $\partial_t + \partial_a + \partial_c$ replaced by respective directional derivatives and prove its solvability when μ, $\tilde{\mu}$, σ, m are stationary, β depends on all possible variables and initial functions belong to special class. Moreover, for stationary β we construct the asymptotic behavior of this modified problem (see Section 4).

In Section 2 we analyze persistent solutions of (1.7). In all considered cases we found that product (persistent) solutions furnish the asymptotic behavior of general solution for large time ($t > a$). This result is analogues to that for the Lotka-Sharpe problem (see Busenberg and Iannelli (1985)).

2. Persistent Solutions of (1.7)

In this section we look for the functions

$$x = \exp\{\lambda(t-a)\}X(a), \quad p = \exp\{\lambda(t-a)\}P(a,c), \qquad (2.1)$$
$$x^0 = X(a)\exp\{-\lambda a\}, \quad p^0 = P(a,c)\exp\{-\lambda a\},$$

as the solution of (1.7) in the stationary case of $\mu,\ \tilde{\mu},\ \sigma,\ m,\ \beta$.

Substituting (2.1) into (1.7) yields

$$X' + \mu X = 0, \quad a < \tau, \qquad (2.2)$$

$$X' + (\mu + m)X = \int_0^{a-\tau} (\tilde{\mu} + \sigma)P\,dc, \quad a > \tau, \qquad (2.3)$$

$$\partial_a P + \partial_c P + (2\tilde{\mu} + \sigma)P = 0, \quad a > \tau,\ 0 < c \leqslant a - \tau, \qquad (2.4)$$

$$X(0) = \int_\tau^\infty \exp\{-\lambda a\}\,da \int_0^{a-\tau} \beta P\,dc, \quad P(a,0) = mX, \quad [X(\tau)] = 0, \qquad (2.5)$$

here the prime indicates differentiation with respect to age a. Eqs. (2.2) and (2.4), $(2.5)_2$ have the formal solution

$$X = X(0)\exp\Big\{-\int_0^a \mu ds\Big\}, \quad a \leqslant \tau, \qquad (2.6)$$

$$P = m(a-c)X(a-c)\exp\Big\{-\int_{a-c}^{a} (2\tilde{\mu} + \sigma)|_{(\eta,\eta-a+c)}\,d\eta\Big\}, \ a-c \geqslant \tau. \qquad (2.7)$$

Denoting

$$F(a) = \int_0^{a-\tau} P\,dc, \qquad (2.8)$$

and using (2.4) and $(2.5)_2$ yields

$$F' - mX = -\int_0^{a-\tau} (2\tilde{\mu} + \sigma)P\,dc, \quad F(\tau) = 0. \qquad (2.9)$$

Then combining (2.9), (2.3) and (2.6) one can obtain

$$(X + F)' + \mu X = -\int_0^{a-\tau} \tilde{\mu} P\,dc,$$

$$(X+F)|_{a=\tau} = X(\tau) = X(0)\exp\Big\{-\int_0^\tau \mu\, ds\Big\}. \tag{2.10}$$

In the case where $\tilde{\mu}$, σ and β are independent of c and $\tilde{\mu} = \mu$ Eq. 2.10 has the solution

$$X + F = G(a) = X(0)\exp\Big\{-\int_0^a \mu\, ds\Big\}, \tag{2.11}$$

which allows us to get the following equations

$$X' + (2\mu + \sigma + m)X = (\mu + \sigma)G, \quad X(\tau) = G(\tau),$$
$$F' + (2\mu + \sigma + m)F = mG, \quad F(\tau) = 0,$$

having the solutions

$$X = X(0)N_X(a)\exp\Big\{-\int_0^a \mu\, ds\Big\} = N_X(a)G(a), \tag{2.12}$$

$$F = X(0)N_F(a)\exp\Big\{-\int_0^a \mu\, ds\Big\} = N_F(a)G(a), \tag{2.13}$$

$$N_X = \exp\Big\{-\int_\tau^a (\mu + \sigma + m)\, ds\Big\} + \int_\tau^a (\mu+\sigma)\exp\Big\{-\int_\eta^a (\mu + \sigma + m)\, ds\Big\}d\eta,$$

$$N_F(a) = \int_\tau^a m(\eta)\exp\Big\{-\int_\eta^a (\mu + \sigma + m)\, ds\Big\}d\eta.$$

Substituting (2.13) into $(2.5)_1$ yields the following equation for λ

$$\kappa(\beta)(\lambda) = 1, \quad \kappa(\beta)(\lambda) = \int_\tau^\infty \beta N_F \exp\Big\{-a\lambda - \int_0^a \mu\, ds\Big\}da, \tag{2.14}$$

while $X(0) > 0$ is arbitrary.

If m, σ, μ and β are continuous and β is bounded, then $N_X \leqslant 1$, $N_F \leqslant 1$ and κ exists for $Re\lambda > \tilde{\lambda} = -\inf_{a \geqslant \tau}(1/a)\int_0^a \mu\, ds$, $\tilde{\lambda} < 0$. Assume $\lim_{\lambda \to \tilde{\lambda}+0} \kappa(\beta)(\lambda) > 1$. Let $\lambda_0 > \tilde{\lambda}$ be a unique real root of (2.14) and $\lambda_k = \alpha_k \pm i\beta_k$, $\alpha_k > \tilde{\lambda}$, $k = 1, 2, \dots$ its complex roots. Then $\alpha_k < \lambda_0$ for all k, $\text{sign}\lambda_0 = \text{sign}(\kappa(\beta)(0) - 1)$, and $\lambda_0 = 0$ if $\kappa(\beta)(0) = 1$.

Eq. 2.14, for $\tau = 0$, has been analyzed by Hadeler (1989). The qualitative dependence λ_0 on $\beta,\ \mu,\ \sigma,\ m$ for (2.14) is the same as that presented in Hadeler (1989) for $\tau = 0$. Clearly λ_0 is decreased as τ increased because $d\lambda_0/d\tau < 0$.

Now we consider the case $\tilde{\mu}(a,c),\ \sigma(a,c)$ and $\beta(a,c)$. Denoting $D_1 = \{a :\ a \geqslant \tau,\ \mu \leqslant \inf_c \tilde{\mu}\}$, $D_2 = \{a :\ a \geqslant \tau,\ \mu > \inf_c \tilde{\mu}\}$,

$$\mu_*(a) = \begin{cases} \mu & \text{in } \ D_1, \\ \inf_c \tilde{\mu} & \text{in } \ D_2, \\ \mu & \text{for } \ a \leqslant \tau, \end{cases}$$

from (2.10) one can obtain the estimate

$$X + F \leqslant G_*(a) = X(0) \exp\Big\{ - \int_0^a \mu_* \, ds \Big\}, \tag{2.15}$$

since $\mu_* \leqslant \mu,\ \tilde{\mu}$.

Substituting (2.7) into (2.3) yields the problem

$$X^{*\prime} + (\mu + m) X^* = \int_\tau^a X^*(\eta) L(a,\eta) \, d\eta,$$

$$L(a,\eta) = (\tilde{\mu} + \sigma)|_{(a,a-\eta)} m(\eta) \exp\Big\{ - \int_\eta^a (2\tilde{\mu} + \sigma)|_{(\xi,\xi-\eta)} \, d\xi \Big\},$$

$$X^*(\tau) = \exp\Big\{ - \int_0^\tau \mu \, ds \Big\}, \quad X^*(a) = X(a)/X(0),$$

which can be rewritten as follows

$$\begin{aligned} X^* &= X^*(\tau) \exp\Big\{ - \int_\tau^a (\mu + m) \, ds \Big\} \\ &\quad + \int_\tau^a d\rho \exp\Big\{ - \int_\rho^a (\mu + m) \, ds \Big\} \int_\tau^\rho X^*(\eta) L(\rho,\eta) \, d\eta. \end{aligned} \tag{2.16}$$

If $m,\ \mu,\ \tilde{\mu}$ and σ are continuous then this Volterra type equation has a unique non-negative continuously differentiable local solution, which due to (2.7), (2.8) and (2.15) is global and has the estimate $X^*(a) \leqslant \exp\Big\{ - \int_0^a \mu_* \, ds \Big\}$. Combining $(2.5)_1$ and (2.7) we get the following equation for λ

$$\kappa_1(\beta)(\lambda) = 1, \quad \kappa_1(\beta)(\lambda) = \int_\tau^\infty da \exp\{-a\lambda\}$$

$$\times \int_{\tau}^{a} \beta(a, a-\xi) m(\xi) X^*(\xi) \exp\Big\{ -\int_{\xi}^{a} (2\tilde{\mu}+\sigma)|_{(\eta,\eta-\xi)}\, d\eta \Big\} d\xi, \qquad (2.17)$$

while $X(0) > 0$ is arbitrary. If $m,\ \beta,\ \tilde{\mu},\ \sigma$ are continuous and $\beta,\ m$ are bounded, then κ_1 exits for $Re\lambda > \lambda_* = -\inf_{a \geqslant \tau} (1/a) \int_0^a \mu_*\, ds,\ \lambda_* < 0$. If $\lambda_0 > \lambda_*$ and $\alpha_k \pm i\beta_k,\ \alpha_k > \lambda_*$ are unique real and complex roots of (2.17), then $\alpha_k < \lambda_0,\ \mathrm{sign}\lambda_0 = \mathrm{sign}(\kappa_1(\beta)(0) - 1)$. Clearly λ_0 is increased if in some age class β increased. Observe that if $\tilde{\mu},\ \sigma$ and β are independent of c and $\tilde{\mu} = \mu$, then $\kappa_1(\beta)(\lambda) = \kappa(\beta)(\lambda)$ and $\tilde{\lambda} = \lambda_*$.

As a result we have proved

Theorem 2.1. *Assume:*

(1) $\mu,\ \tilde{\mu},\ \sigma,\ m$ *and* β *are non-negative non-trivial functions, and* $\beta,\ m$ *are bounded,*

(2) $\mu \in C^0([0,\infty)),\ m \in C^1((0,\infty)) \cap C^0([0,\infty)),\ \beta \in C^0([\tau,\infty) \times [0, a-\tau]),$ $\tilde{\mu},\ \sigma \in C^{0,1}([\tau,\infty) \times (0, a-\tau]) \cap C^0([\tau,\infty) \times [0, a-\tau]),$

(3) $\lim_{\lambda \to \lambda_* + 0} \kappa_1(\beta)(\lambda) > 1,\ \lambda_* < 0$ *for real* λ.

Then (1.7) *has a unique non-negative normed solution of type* (2.1) *such that* $X \leqslant G_*$.

Notice that (2.6), (2.7) and (2.12) (or the solution of (2.16) in the case $\tilde{\mu}(a,c),\ \sigma(a,c),\ \beta(a,c)$) represent the stationary solution of (1.7) provided that $\kappa(0) = 1$ (or $\kappa_1(0) = 1$).

3. Model (1.7) in the Case $\tilde{\mu} = \mu(\mathrm{t}, \mathrm{a}),\ \sigma(\mathrm{t}, \mathrm{a}),\ \beta(\mathrm{t}, \mathrm{a})$

In this section we consider problem (1.7) with $\beta,\ \tilde{\mu},\ \sigma$ not depending on c and $\tilde{\mu} = \mu$, construct its solution, and obtain its asymptotic behavior for large time ($t > a$) and stationary $\mu,\ \sigma,\ m,\ \beta$.

Integrating $(1.7)_1$ and $(1.7)_3$ yields

$$x = \begin{cases} x^0(a-t) \exp\Big\{ -\int\limits_{a-t}^{a} \mu(\xi - a + t, \xi)\, d\xi \Big\}, & \leqslant t \leqslant a, \\ x(t-a, 0) \exp\Big\{ -\int\limits_{0}^{a} \mu(\xi - a + t, \xi)\, d\xi \Big\}, & 0 \leqslant a \leqslant \min(t, \tau), \end{cases} \qquad (3.1)$$

$$p = \begin{cases} p^0(a-t, c-t) \exp\Big\{ -\int\limits_{a-t}^{a} (2\mu+\sigma)|_{(\xi-a+t,\xi)}\, d\xi \Big\}, \\ \quad 0 \leqslant t \leqslant c \leqslant a-\tau, \\ (mx)|_{(t-c,a-c)} \exp\Big\{ -\int\limits_{a-c}^{a} (2\mu+\sigma)|_{(\xi-a+t,\xi)}\, d\xi \Big\}, \\ \quad 0 \leqslant c \leqslant \min(\mathrm{t}, \mathrm{a}-\tau). \end{cases} \qquad (3.2)$$

Denoting

$$f(t,a) = \int_0^{a-\tau} p\, dc, \quad f^0(a) = \int_0^{a-\tau} p^0\, dc, \quad a \geqslant \tau \qquad (3.3)$$

and using (1.7) we obtain

$$\partial_t x + \partial_a x + (\mu + m)x - (\mu + \sigma)f = 0, \tag{3.4}$$
$$x|_{t=0} = x^0, \quad x|_{a=\tau} = x(t,\tau),$$
$$\partial_t f + \partial_a f - mx + (2\mu + \sigma)f = 0, \tag{3.5}$$
$$f|_{t=0} = f^0, \quad f|_{a=\tau} = 0,$$

where $x(t,\tau)$ is defined by (3.1). In the stationary case of all vital rates Hadeler (1989) studied the exponential solutions for this system. Adding (3.4) and (3.5) we obtain

$$\partial_t(x+f) + \partial_a(x+f) + \mu(x+f) = 0,$$
$$(x+f)|_{t=0} = x^0 + f^0, \quad (x+f)|_{a=\tau} = x(t,\tau)$$

and consequently

$$x + f = g(t,a), \tag{3.6}$$
$$g = \begin{cases} (x^0(a-t) + f^0(a-t)) \exp\Big\{ -\int\limits_{a-t}^{a} \mu(\xi - a + t, \xi)\, d\xi \Big\}, & t \leqslant a - \tau, \\ x^0(a-t) \exp\Big\{ -\int\limits_{a-t}^{a} \mu(\xi - a + t, \xi)\, d\xi \Big\}, & a - \tau \leqslant t \leqslant a, \\ x(t-a, 0) \exp\Big\{ -\int\limits_{0}^{a} \mu(\xi - a + t, \xi)\, d\xi \Big\}, & t \geqslant a. \end{cases}$$

Eq. 3.6 allows us to separate (3.4), (3.5) into two following problems

$$\partial_t x + \partial_a x + (2\mu + \sigma + m)x = (\mu + \sigma)g, \quad x|_{t=0} = x^0, \quad x|_{a=\tau} = x(t,\tau),$$
$$\partial_t f + \partial_a f + (2\mu + \sigma + m)f = mg, \quad f|_{t=0} = f^0, \quad f|_{a=\tau} = 0, \tag{3.7}$$

having the formal solutions

$$x = x_1(t,a) = \exp\Big\{ -\int\limits_{a-t}^{a} \mu(\xi - a + t, \xi) d\xi \Big\} \Big\{ x^0(a-t)$$
$$\times \exp\Big\{ -\int\limits_{a-t}^{a} (\mu + \sigma + m)|_{(\xi - a + t, \xi)} d\xi \Big\}$$
$$+ \int\limits_{a-t}^{a} \exp\Big\{ -\int\limits_{\eta}^{a} (\mu + \sigma + m)|_{(\xi - a + t, \xi)}\, d\xi \Big\}$$
$$\times (\mu + \sigma)|_{(\eta - a + t, \eta)}\, d\eta \big(x^0(a-t) + f^0(a-t)\big) \Big\}, \quad a - t \geqslant \tau, \tag{3.8}$$

$$x = x_2(t,a) = x^0(a-t)\exp\Big\{-\int\limits_{a-t}^{a}\mu(\xi-a+t,\xi)\,d\xi\Big\}$$
$$\times\Bigg\{\exp\Big\{-\int\limits_{\tau}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}$$
$$+\int\limits_{\tau}^{a}\exp\Big\{-\int\limits_{\eta}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}$$
$$\times(\mu+\sigma)|_{(\eta-a+t,\eta)}\,d\eta\Bigg\},\quad 0\leqslant a-t\leqslant\tau, \tag{3.9}$$
$$x = x_3(t,a) = x(t-a,0)N_x(t,a),\quad t\geqslant a, \tag{3.10}$$
$$f = f_1(t,a) = \exp\Big\{-\int\limits_{a-t}^{a}\mu(\xi-a+t,\xi)d\xi\Big\}$$
$$\times\Bigg\{f^0(a-t)\exp\Big\{-\int\limits_{a-t}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}d\xi\Big\}$$
$$+\int\limits_{a-t}^{a}\exp\Big\{-\int\limits_{\eta}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}$$
$$\times m(\eta-a+t,\eta)\,d\eta\Big(x^0(a-t)+f^0(a-t)\Big)\Bigg\},\quad a-t\geqslant\tau, \tag{3.11}$$
$$f = f_2(t,a) = x^0(a-t)\exp\Big\{-\int\limits_{a-t}^{a}\mu(\xi-a+t,\xi)\,d\xi\Big\}$$
$$\times\int\limits_{\tau}^{a}\exp\Big\{-\int\limits_{\eta}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}m(\eta-a+t,\eta)\,d\eta,$$
$$0\leqslant a-t\leqslant\tau. \tag{3.12}$$
$$f = f_3(t,a) = x(t-a,0)N_f(t,a),\quad t\geqslant a, \tag{3.13}$$
$$N_x(t,a) = \exp\Big\{-\int\limits_{0}^{a}\mu(\xi-a+t,\xi)\,d\xi\Big\}$$
$$\times\Bigg\{\exp\Big\{-\int\limits_{\tau}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}$$
$$+\int\limits_{\tau}^{a}\exp\Big\{-\int\limits_{\eta}^{a}(\mu+\sigma+m)|_{(\xi-a+t,\xi)}\,d\xi\Big\}(\mu+\sigma)|_{(\eta-a+t,\eta)}\,d\eta\Bigg\},$$

$$N_f(t,a) = \exp\Big\{ -\int_0^a \mu(\xi - a + t, \xi)\, d\xi \Big\}$$
$$\times \int_\tau^a \exp\Big\{ -\int_\eta^a (\mu + \sigma + m)|_{(\xi-a+t,\xi)}\, d\xi \Big\}\, m(\eta - a + t, \eta)\, d\eta.$$

Now we claim to justify these solutions. Eq. 3.6 and definition of N_x and N_f show that

$$x_1,\ f_1,\ x_2,\ f_2 \leqslant g, \quad \text{and} \quad N_f,\ N_x \leqslant \exp\{-\int_0^a \mu(\xi + t - a, \xi)\, d\xi\}, \tag{3.14}$$

provided that μ, σ, m are continuous.

For $x(t,0)$ we have the condition $x(t,0) = \int_\tau^\infty \beta f\, da$, i.e.,

$$x(t,0) = \int_\tau^{t+\tau} \beta f_2\, da + \int_{t+\tau}^\infty \beta f_1\, da, \quad 0 \leqslant t \leqslant \tau, \tag{3.15}$$

$$x(t,0) = \int_0^{t-\tau} (\beta N_f)|_{(t,t-\rho)} x(\rho,0)\, d\rho + \int_t^{t+\tau} \beta f_2\, da + \int_{t+\tau}^\infty \beta f_1\, da,$$
$$t \geqslant \tau. \tag{3.16}$$

Starting with (3.15) and going along axis t by step τ from (3.16) we can construct $x(t,0)$ for all $t > \tau$. Knowing $x(t,0)$ and using (3.8–3.10) and (3.2) allows us to construct $x(t,a)$ and $p(t,a,c)$ for all t. Thus we have

Theorem 3.1. *Assume:*

(1) x^0, p^0, μ, σ, m *and* β *are non-negative non-trivial functions,*

(2) $m \in C^1((0,\infty) \times [\tau,\infty)) \cap C^0([0,\infty) \times [\tau,\infty))$,
$\mu \in C^{1,0}((0,\infty) \times [0,\infty)) \cap C^0([0,\infty) \times [0,\infty))$,
β, $\sigma \in C^{1,0}((0,\infty) \times [\tau,\infty)) \cap C^0([0,\infty) \times [\tau,\infty))$ *and* m, β *are bounded,*

(3) $x^0 \in C^1((0,\infty)) \cap C^0([0,\infty)) \cap L^1(0,\infty)$,
$p^0 \in C^1((\tau,\infty) \times (0,a-\tau]) \cap C^0([\tau,\infty) \times [0,a-\tau]) \cap L^1((\tau,\infty) \times (0,a-\tau))$,

(4) $\int_{t+\tau}^\infty \partial_t(\beta(t,a) f_1(t,a))\, da$ *converges uniformly for* $0 < t < \infty$.

Then (1.7) *has a unique non-negative solution* x, p *such that* $x \in C^0([0,\infty) \times [0,\infty)) \cap C^1(((0,\infty) \times (0,\infty)) \setminus \{(t,a) : t = a,\ t = a - \tau,\ a = \tau\})$, $p \in C^0([0,\infty) \times [0,\infty) \times [0,a-\tau]) \cap C^1(((0,\infty) \times (\tau,\infty) \times (0,a-\tau)) \setminus \{(t,a,c) : t = a,\ t = a - \tau,\ t = c\})$.

Proof. Conditions (2) and (3) and estimates $(3.14)_1$ ensure the existence of $\int_{t+\tau}^\infty \beta f_1\, da$. The rest is obvious.

Our further purpose of this section is to construct the asymptotic behavior of x, f and p for stationary $\mu,\ m,\ \sigma,\ \beta$. We limit our attention to the case

$$x^0(a) \leqslant \tilde{X}(a,\lambda_0) = X(a)\exp\{-a\lambda_0\},$$
$$p^0(a,c) \leqslant \tilde{P}(a,c,\lambda_0) = P(a,c)\exp\{-a\lambda_0\}, \tag{3.17}$$

where X and P are defined by (2.6), (2.12) and (2.7), respectively, and λ_0 is a unique real root of (2.14). Of course, we assume that conditions of Theorem 1 for $\tilde{\mu} = \mu$ and Theorem 3.1 for the stationary case of $\mu,\ \sigma,\ m,\ \beta$ hold true.

We first prove that functions $x,\ f$ and p defined by (3.1), (3.8)–(3.13) and (3.2) are bounded by $\tilde{X}(a,\lambda_0)\exp\{t\lambda_0\}$, $\tilde{F}(a,\lambda_0)\exp\{t\lambda_0\}$ and $\tilde{P}(a,c,\lambda_0)\exp\{t\lambda_0\}$, respectively, where $\tilde{F}(a,\lambda_0) = F(a)\exp\{-a\lambda_0\}$. Eqs. (3.11), (3.12), (2.11), (2.6) and estimates (3.17) show that

$$\begin{aligned}
f_1 &\leqslant \exp\Big\{-\int_{a-t}^{a}\mu\,d\xi\Big\}\Big\{\tilde{F}(a-t,\lambda_0)\exp\Big\{-\int_{a-t}^{a}(\mu+\sigma+m)\,d\xi\Big\}\\
&\quad + G(a-t)\int_{a-t}^{a}\exp\Big\{-\int_{\eta}^{a}(\mu+\sigma+m)\,d\xi\Big\}m(\eta)\,d\eta\Big\}\\
&= G(a)\exp\{-\lambda_0(a-t)\}\Big\{N_F(a-t)\exp\Big\{-\int_{a-t}^{a}(\mu+\sigma+m)\,d\xi\Big\}\\
&\quad + \int_{a-t}^{a}\exp\Big\{-\int_{\eta}^{a}(\mu+\sigma+m)\,d\xi\Big\}\,d\eta\Big\}\\
&= G(a)N_F(a)\exp\{-\lambda_0(a-t)\} = \tilde{F}(a,\lambda_0)\exp\{t\lambda_0\},\\
f_2 &\leqslant \exp\Big\{-\int_{a-t}^{a}\mu\,d\xi\Big\}\int_{\tau}^{a}\exp\Big\{-\int_{\eta}^{a}(\mu+\sigma+m)\,d\xi\Big\}m(\eta)\,d\eta\,X(a-t)\\
&\quad \times\exp\{-\lambda_0(a-t)\} = N_F(a)G(a)\exp\{-\lambda_0(a-t)\}\\
&= \tilde{F}(a,\lambda_0)\exp\{t\lambda_0\},
\end{aligned}$$

which together with (3.15), (2.13) and (2.14) yield

$$x(t,0) \leqslant \exp\{t\lambda_0\}\int_{\tau}^{\infty}\beta F(a)\exp\{-a\lambda_0\}\,da = X(0)\exp\{t\lambda_0\}, \quad t \leqslant \tau.$$

Now from (3.13) we obtain $f_3 \leqslant X(0)\exp\Big\{\lambda_0(t-a) - \int_0^a \mu\,d\xi\Big\}N_F(a) = \tilde{F}(a,\lambda_0)\exp\{t\lambda_0\}$, and then, from (3.16), $x(t,0) \leqslant X(0)\exp\{t\lambda_0\}$ for $t > \tau$.

Similarly from (3.8)–(3.10), (3.1) and (3.2) we derive estimates

$$x \leqslant \tilde{X}(a, \lambda_0) \exp\{t\lambda_0\}, \quad p \leqslant \tilde{P}(a, c, \lambda_0) \exp\{t\lambda_0\} \quad \text{for all} \quad t > 0.$$

Hence there exist Laplace and its inverse transforms of x, f and p. Let $\widehat{x}(\lambda, a)$ and $\widehat{f}(\lambda, a)$ be the Laplace transforms of x and f. Then from $(1.7)_4$, (3.13) we find

$$\widehat{x}(\lambda, 0) = \int_{\tau}^{\infty} \beta \widehat{f}\, da, \; \widehat{f}(\lambda, a) = \int_{0}^{a-\tau} f_1 \exp\{-t\lambda\}\, dt + \int_{a-\tau}^{a} f_2 \exp\{-t\lambda\}\, dt$$
$$+ \int_{a}^{\infty} f_3 \exp\{-t\lambda\}\, dt,$$
$$\int_{a}^{\infty} f_3 \exp\{-t\lambda\}\, dt = \widehat{x}(\lambda, 0) N_F(a) \exp\{-a\lambda - \int_{0}^{a} \mu\, ds\}.$$

Hence

$$\widehat{x}(\lambda, 0) = S(\lambda)(1 - \kappa(\beta)(\lambda))^{-1},$$
$$S(\lambda) = \int_{\tau}^{\infty} \beta \Big\{ \int_{0}^{a-\tau} f_1 \exp\{-t\lambda\}\, dt + \int_{a-\tau}^{a} f_2 \exp\{-t\lambda\}\, dt \Big\}\, da,$$

where $\kappa(\beta)(\lambda)$ is defined by (2.14).

Letting $Re\lambda = \alpha$ and using estimate $f_s \leqslant \tilde{F}(a, \lambda_0) \exp\{t\lambda_0\}$, $s = 1, 2$ we obtain

$$|S(\lambda)| \leqslant \int_{\tau}^{\infty} da\, \beta \tilde{F}(a, \lambda_0) \int_{0}^{a} \exp\{(\lambda_0 - \alpha)t\}\, dt$$
$$= X(0) \int_{\tau}^{\infty} da\, \beta N_F(a) \exp\Big\{ - a\lambda_0 - \int_{0}^{a} \mu\, ds \Big\} \int_{0}^{a} \exp\{(\lambda_0 - \alpha)t\}\, dt.$$

If β is bounded and α, $\lambda_0 > \tilde{\lambda} = -\inf_{a \geqslant \tau} (1/a) \int_0^a \mu\, ds$, $\tilde{\lambda} < 0$, then $|S(\lambda)|$ is bounded too. Let (2.14) has a unique real root λ_0 and assume that real part α_s of its complex roots verify $\tilde{\lambda} < \alpha_s$ for all s. Then $\alpha_s < \lambda_0$, and using the method of contour integral [1] for inverse Laplace transform yields

$$x(t, 0) \sim q \exp\{t\lambda_0\}, \quad q = S(\lambda_0)/(-\kappa'(\lambda_0)),$$

the prime indicates differentiation. At last from (3.1), (3.10), (3.13) and (3.2) one can obtain

$$\begin{aligned} &x \sim q \exp\Big\{\lambda_0(t-a) - \int_0^a \mu\, ds\Big\} \begin{cases} 1, & 0 \leqslant a \leqslant \tau, \\ N_X(a), & a \geqslant \tau, \end{cases} \\ &f \sim q N_F(a) \exp\Big\{\lambda_0(t-a) - \int_0^a \mu\, ds\Big\}, \\ &p \sim m(a-c) q N_X(a-c) \exp\Big\{\lambda_0(t-a) - \int_0^a \mu\, ds - \int_{a-c}^a (\mu+\sigma)\, ds\Big\} \end{aligned} \tag{3.18}$$

for large time ($t > a$). As we saw N_X and $N_F \leqslant 1$. Clearly the main term of x, f and p for large time ($t > a$) is the persistent solution with $X(0) = q$.

4. Problem (1.7) in the Case $\tilde{\mu}(\mathrm{a}, \mathrm{c})$, $\sigma(\mathrm{a}, \mathrm{c})$

In this section we replace terms $\partial_t x + \partial_a x$ and $\partial_t p + \partial_a p + \partial_c p$ of (1.7) by $\sqrt{2}\, \partial_{l_x} x$ and $\sqrt{3}\, \partial_{l_p} p$, where l_x and l_p are positive directions of the characteristics of operators $\partial_t + \partial_a$ and $\partial_t + \partial_a + \partial_c$, respectively. This new problem we will call the modified (1.7) problem. Our purpose is to prove the unique solvability of the modified (1.7) problem in the stationary case of μ, $\tilde{\mu}$, σ, m and general function β, and to construct the asymptotic behavior of its solution for stationary β. From (1.7) we have the formal expression

$$x = \begin{cases} x^0(a-t) \exp\Big\{ - \int_{a-t}^a \mu(\xi)\, d\xi\Big\}, & 0 \leqslant t \leqslant a \leqslant \tau, \\ x(t-a, 0) \exp\Big\{ - \int_0^a \mu(\xi)\, d\xi\Big\}, & 0 \leqslant a \leqslant \min(\mathrm{t}, \tau), \end{cases} \tag{4.1}$$

$$p = \begin{cases} p^0(a-t, c-t) \exp\Big\{ - \int_{a-t}^a (2\tilde{\mu} + \sigma)|_{(\xi, \xi-a+c)}\, d\xi\Big\}, \\ \qquad 0 \leqslant t \leqslant c \leqslant a - \tau, \\ m(a-c) x(t-c, a-c) \exp\Big\{ - \int_{a-c}^a (2\tilde{\mu} + \sigma)|_{(\xi, \xi-a+c)}\, d\xi\Big\}, \\ \qquad c \leqslant \min(\mathrm{t}, \mathrm{a} - \tau). \end{cases} \tag{4.2}$$

Denoting

$$\varphi(t, a) = \int_0^{a-\tau} (\tilde{\mu} + \sigma) p\, dc, \quad a \geqslant \tau \tag{4.3}$$

and taking into account (4.2) and $(1.7)_2$ we have

$$\varphi(x)(t,a) = \begin{cases} \varphi_1(t,a) + \int_0^t x(t-c, a-c) L_1(a,c)\, dc, \\ \qquad 0 \leqslant t \leqslant a-\tau, \\ \int_0^{a-\tau} x(t-c, a-c) L_1(a,c)\, dc, \quad t \geqslant a-\tau, \end{cases} \tag{4.4}$$

$$L_1 = (\tilde{\mu}+\sigma)|_{(a,c)} m(a-c) \exp\Big\{ -\int_{a-c}^{a} (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\, d\xi \Big\},$$

$$\varphi_1(t,a) = \int_t^{a-\tau} (\tilde{\mu}+\sigma)|_{(a,c)} p^0(a-t, c-t)$$

$$\times \exp\Big\{ -\int_{a-t}^{a} (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\, d\xi \Big\}\, dc, \quad t \geqslant a-\tau,$$

$$x = x_1(t,a) = x^0(a-t) \exp\Big\{ -\int_{a-t}^{a} (\mu+m)\, d\xi \Big\}$$

$$+ \int_{a-t}^{a} \exp\Big\{ -\int_{\rho}^{a} (\mu+m)\, d\xi \Big\}$$

$$\times \varphi(x)(\rho-a+t, \rho)\, d\rho, \quad 0 \leqslant t \leqslant a-\tau, \tag{4.5}$$

$$x = x_2(t,a) = x^0(a-t) \exp\Big\{ -\int_{a-t}^{a} \mu\, d\xi - \int_{\tau}^{a} m\, d\xi \Big\}$$

$$+ \int_{\tau}^{a} \exp\Big\{ -\int_{\rho}^{a} (\mu+m) ds \Big\}$$

$$\times \varphi(x)(\rho-a+t, \rho)\, d\rho, \quad a-\tau \leqslant t \leqslant a. \tag{4.6}$$

Substituting

$$x = x_3(t,a) = x(t-a,0) r(a), \quad p = x(t-a,0) Q(a,c),$$
$$\varphi = x(t-a,0) \psi(Q)(a), \quad t \geqslant a$$

into (4.1)–(4.3) and $(1.7)_2$ yields

$$\psi(Q)(a) = \int_0^{a-\tau} (\tilde{\mu}+\sigma)|_{(a,c)} Q(a,c)\, dc,$$

$$Q = m(a-c)r(a-c)\exp\Big\{-\int\limits_{a-c}^{a}(2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\},$$

$$r' + (\mu+m)r = \psi(Q)(a), \quad r(\tau) = \exp\Big\{-\int\limits_{0}^{\tau}\mu\,d\xi\Big\}.$$

Equation for r is the same as the differential form of (2.16) and therefore $r = X^*$.

Let us denote

$$l(t,\rho,\eta) = \exp\Big\{-\int\limits_{0}^{t-\rho}(2\tilde{\mu}+\sigma)|_{(\xi+\eta,\xi)}\,d\xi\Big\}\beta(t,t-\rho+\eta,t-\rho)\,m(\eta).$$

Then $x(t,0) = \int_0^\infty dc\int_{\tau+c}^\infty \beta\,p\,da = I_1 + I_2$,

$$I_1 = \int\limits_{t}^{\infty}dc\int\limits_{\tau+c}^{\infty}\beta(t,a,c)\,p^0(a-t,c-t)$$
$$\times\exp\Big\{-\int\limits_{a-t}^{a}(2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}\,da,$$

$$I_2 = \int\limits_{0}^{t}dc\int\limits_{\tau+c}^{\infty}\beta(t,a,c)m(a-c)x(t-c,a-c)$$
$$\times\exp\Big\{-\int\limits_{a-c}^{a}(2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}\,da = \int\limits_{0}^{t}d\rho\int\limits_{\tau}^{\infty}x(\rho,\eta)l(t,\rho,\eta)\,d\eta.$$

If $t \leqslant \tau$, then

$$I_2 = \int\limits_{0}^{t}d\rho\Big\{\int\limits_{\tau}^{\rho+\tau}x_2(\rho,\eta)l(t,\rho,\eta)\,d\eta + \int\limits_{\rho+\tau}^{\infty}x_1(\rho,\eta)l(t,\rho,\eta)\,d\eta\Big\},$$

and

$$x(t,0) = I_1 + \int\limits_{0}^{t}d\rho\Big\{\int\limits_{\tau}^{\rho+\tau}x_2(\rho,\eta)l(t,\rho,\eta)\,d\eta + \int\limits_{\rho+\tau}^{\infty}x_1(\rho,\eta)l(t,\rho,\eta)\,d\eta\Big\}. \quad (4.7)$$

If $t > \tau$, then

$$I_2 = \int\limits_{0}^{\tau}d\rho\int\limits_{\tau}^{\infty}x(\rho,\eta)l(t,\rho,\eta)\,d\eta + \int\limits_{\tau}^{t}d\rho\int\limits_{\tau}^{\infty}x(\rho,\eta)l(t,\rho,\eta)\,d\eta$$

$$= \int_0^\tau d\rho \int_\tau^\infty x(\rho,\eta) l(t,\rho,\eta)\, d\eta + \int_\tau^t d\rho \Big\{ \int_\tau^\rho x_3(\rho,\eta) l(t,\rho,\eta) d\eta$$

$$+ \int_\rho^{\rho+\tau} x_2(\rho,\eta) l(t,\rho,\eta) d\eta + \int_{\rho+\tau}^\infty x_1(\rho,\eta) l(t,\rho,\eta) d\eta \Big\}$$

$$= \int_\tau^t d\rho \int_\tau^\rho x(\rho-\eta,0) X^*(\eta) l(t,\rho,\eta)\, d\eta + I_3,$$

$$I_3 = \int_0^\tau d\rho \Big\{ \int_\tau^{\rho+\tau} x_2(\rho,\eta) l(t,\rho,\eta) d\eta + \int_{\rho+\tau}^\infty x_1(\rho,\eta) l(t,\rho,\eta) d\eta \Big\}$$

$$+ \int_\tau^t d\rho \Big\{ \int_\rho^{\rho+\tau} x_2(\rho,\eta) l(t,\rho,\eta) d\eta + \int_{\rho+\eta}^\infty x_1(\rho,\eta) l(t,\rho,\eta) d\eta \Big\},$$

and finally

$$x(t,0) = \int_0^{t-\tau} d\xi x(\xi,0) \int_{\xi+\tau}^t X^*(\rho-\xi) l(t,\rho,\rho-\xi)\, d\rho + I_1 + I_3, \quad t > \tau. \tag{4.8}$$

Starting with (4.7) and going along axis t by step τ we can construct $x(t,0)$ for all $t > \tau$ provided that x_1 and x_2 are known and all integrals exist.

It remains to prove the solvability of (4.5) and (4.6). We restrict ourselves to the case $\beta(t,a,c) \leqslant \beta^*(a,c)$, $x^0 \leqslant \tilde{X}(a,\lambda_0)$, $p^0 \leqslant \tilde{P}(a,c,\lambda_0)$, where $\beta^*(a,c)$ is a suitable function, $\tilde{X}$, $\tilde{P}$ are defined by (3.17), and λ_0 is a unique real root of equation $\kappa_1(\beta^*)(\lambda) = 1$ (see (2.17)). Denoting the right-hand side of (4.5) and (4.6) by $K_s(x)$, $s = 1,2$, respectively, we first will prove that $K_s(x) \leqslant \tilde{X}(a,\lambda_0)\exp\{t\lambda_0\}$, x being the same.

Using (2.16) and (2.6), (2.7) from (4.2), (4.4)–(4.6) we obtain the following estimates:

$$p \leqslant \tilde{P}(a-t,c-t,\lambda_0) \exp\Big\{ -\int_{a-t}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\, d\xi \Big\}$$

$$= m(a-c) X(a-c) \exp\Big\{ -\int_{a-c}^{a-t} (2\tilde{\mu}+\sigma)|_{(\eta,\eta-a+c)}\, d\eta - \lambda_0(a-t)$$

$$- \int_{a-t}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\, d\xi \Big\}$$

$$= m(a-c)X(a-c)\exp\Big\{-\int_{a-c}^{a}(2\tilde{\mu}_{+}\sigma)|_{(\eta,\eta-a+c)}\,d\eta-\lambda_0(a-t)\Big\}$$

$$= \exp\{\lambda_0 t\}\tilde{P}(a,c,\lambda_0),\quad 0\leqslant t\leqslant c,$$

$$p\leqslant m(a-c)X(a-c)\exp\{\lambda_0(t-c)\}\exp\Big\{-\lambda_0(a-c)$$

$$-\int_{a-c}^{a}(2\tilde{\mu}_{+}\sigma)|_{(\eta,\eta-a+c)}\,d\eta\Big\}=\exp\{\lambda_0 t\}\tilde{P}(a,c,\lambda_0),\quad t>c,$$

$$\varphi(t,a)\leqslant\exp\{\lambda_0(t-a)\}\psi(P)(a)\,dc,$$

$$K_1(x)\leqslant\tilde{X}(a-t,\lambda_0)\exp\Big\{-\int_{a-t}^{a}(\mu+m)\,ds\Big\}$$

$$+\int_{a-t}^{a}\exp\Big\{-\int_{\rho}^{a}(\mu+m)\,ds\Big\}\exp\{\lambda_0(t-a)\}\psi(P)(\rho)\,d\rho$$

$$=\exp\{-\lambda_0(a-t)\}\Bigg\{X(\tau)\exp\Big\{-\int_{\tau}^{a-t}(\mu+m)ds\Big\}$$

$$+\int_{\tau}^{a-t}\exp\Big\{-\int_{\eta}^{a-t}(\mu+m)ds\Big\}\psi(P)(\eta)\,d\eta\Bigg\}\exp\Big\{-\int_{a-t}^{a}(\mu+m)ds\Big\}$$

$$+\int_{a-t}^{a}\exp\Big\{-\int_{\rho}^{a}(\mu+m)ds+\lambda_0(t-a)\Big\}\psi(P)(\rho)\,d\rho$$

$$=\exp\{\lambda_0(t-a)\}\Bigg\{X(\tau)\exp\Big\{-\int_{\tau}^{a}(\mu+m)ds\Big\}$$

$$+\int_{\tau}^{a-t}\exp\Big\{-\int_{\eta}^{a}(\mu+m)ds\Big\}\psi(P)(\eta)d\eta$$

$$+\int_{a-t}^{a}\exp\Big\{-\int_{\rho}^{a}(\mu+m)ds\Big\}\psi(P)(\rho)d\rho\Bigg\}=\exp\{\lambda_0 t\}\tilde{X}(a,\lambda_0),$$

$$K_2\leqslant\tilde{X}(a-t,\lambda_0)\exp\Big\{-\int_{a-t}^{a}\mu\,ds-\int_{\tau}^{a}m ds\Big\}$$

$$+\int_{\tau}^{a}\exp\Big\{-\int_{\eta}^{a}(\mu+m)ds\Big\}\exp\{\lambda_0(t-a)\}\psi(P)(\eta)\,d\eta$$

$$= \exp\{\lambda_0(t-a)\}\left\{X(0)\exp\Big\{-\int\limits_0^a \mu\, ds - \int\limits_\tau^a m ds\Big\} + \int\limits_\tau^a \exp\Big\{-\int\limits_\eta^a (\mu+m)ds\Big\}\psi(P)(\eta)\, d\eta\right\} = \exp\{\lambda_0 t\}\tilde{X}(a,\lambda_0).$$

Similarly,

$$x(t,0) = \int\limits_\tau^\infty da \int\limits_0^{a-\tau} \beta p\, dc \leqslant \exp\{\lambda_0 t\}\int\limits_\tau^\infty da \int\limits_0^{a-\tau} \beta^* \tilde{P}\, dc$$
$$= \exp\{\lambda_0 t\}X(0), \quad t > 0,$$
$$x_3 \leqslant \exp\{\lambda_0 t\}\tilde{X}(a,0), \quad t \geqslant a.$$

Let $x = z(t,a)\exp\{\lambda_0(t-a)\}$, $z \leqslant X(a)$ and assume $z' \leqslant X(a)$, $z'' \leqslant X(a)$. Then from (4.4)-(4.6) it follows that

$$|\varphi(x'') - \varphi(x')| \leqslant \| z'' - z' \| \int\limits_\tau^a L_1(a, a-\eta)\, d\eta \exp\{\lambda_0(t-a)\},$$
$$|K_s(x'') - K_s(x')| \leqslant \| z'' - z' \| \,\epsilon \exp\{\lambda_0(t-a)\},$$
$$\epsilon = \sup_{a \geqslant \tau} \int\limits_\tau^a d\rho \exp\Big\{-\int\limits_\rho^a (\mu+m)ds\Big\}\int\limits_\tau^\rho (\tilde{\mu}+\sigma)|_{(\rho,\rho-\eta)} m(\eta) \times \exp\Big\{-\int\limits_\eta^\rho (2\tilde{\mu}+\sigma)|_{(\xi,\xi-\eta)}\Big\}\, d\eta.$$

Therefore if $\epsilon < 1$, then operator $\tilde{K}_s$ defined by equation

$$z = \tilde{K}_s(z) = \exp\{-\lambda_0(t-a)\}K_s(\exp\{\lambda_0(t-a)\}z), \quad s = 1, 2$$

is contractive, and Eqs. (4.5), (4.6) have a unique solution. Here $\| x'' - z' \|$ is the norm in space C^0. Observe that $\epsilon \leqslant m(\mu+\sigma)(\mu+m)^{-1}(2\mu+\sigma)^{-1} < 1$ for the constant $\tilde{\mu} = \mu$, σ, m.

Thus we have proved

Theorem 4.1. *Assume:*

(1) $\beta(t,a,c) \leqslant \beta^*(a,c)$, $\beta \in C^0([0,\infty) \times [\tau,\infty) \times [0, a-\tau])$ *and* μ, $\tilde{\mu}$, σ, β^*, m *satisfies the conditions of Theorem* 2.1,

(2) $x^0 \in C([0,\infty))$, $p^0 \in C^0([\tau,\infty) \times [0, a-\tau))$ *are such that* $x^0 \leqslant X(a)\exp\{-\lambda_0 a\}$, $p^0 \leqslant P(a,c)\exp\{-\lambda_0 a\}$, *where* X *and* P *are defined in Sec.* 2 *and* λ_0 *is a real root of Eq.* $\kappa_1(\beta^*)(\lambda) = 1$,

(3) $\epsilon < 1$.

Then the modified problem (1.7) *has a unique non-negative solution such that:* $x \in C^0([0,\infty)\times[0,\infty))$, $p \in C^0([0,\infty)\times[\tau,\infty)\times[0,a-\tau])$, $\partial_{l_x} x \in C^0((0,\infty)\times((0,\tau)\cup(\tau,\infty)))$, $\partial_{l_p} p \in C^0((0,\infty)\times(\tau,\infty)\times(0,a-\tau))$ *and* $x \leqslant X(a)\exp\{\lambda_0(t-a)\}$, $p \leqslant P(a,c)\exp\{\lambda_0(t-a)\}$ *for all* t, a, c.

At last in the case of stationary $\beta(a,c)$ we will construct the asymptotic behavior of x, p obtained above. The upper estimates of x and p ensure the existence of Laplace and its inverse transforms. Thus from $(1.7)_4$, (4.2) we have

$$
\begin{aligned}
\widehat{x}(\lambda,0) &= \int_0^\infty dc \int_{\tau+c}^\infty \beta\widehat{p}(\lambda,a,c)\,da,\\
\widehat{p}(\lambda,a,c) &= \int_c^\infty \exp\{-\lambda t\} m(a-c) x(t-c,a-c)\\
&\quad\times \exp\Big\{-\int_{a-c}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}dt\\
&\quad+\int_0^c p^0(a-t,c-t)\exp\Big\{-\lambda t-\int_{a-t}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}dt\\
&= m(a-c)\widehat{x}(\lambda,a-c)\exp\Big\{-\lambda c-\int_{a-c}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}\\
&\quad+\int_0^c p^0(a-t,c-t)\exp\Big\{-\lambda t-\int_{a-t}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}dt,\\
\widehat{x}(\lambda,a) &= \int_0^{a-\tau} x_1\exp\{-\lambda t\}dt+\int_{a-\tau}^a x_2\exp\{-\lambda t\}dt+\int_a^\infty x_3\exp\{-\lambda t\}dt\\
&= X^*(a)\exp\{-\lambda a\}\widehat{x}(\lambda,0)+\int_0^{a-\tau} x_1\exp\{-\lambda t\}dt+\int_{a-\tau}^a x_2\exp\{-\lambda t\}dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\widehat{p}(\lambda,a,c) &= \widehat{x}(\lambda,0)m(a-c)X^*(a-c)\\
&\quad\times\exp\Big\{-\lambda a-\int_{a-c}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)}\,d\xi\Big\}+\Gamma(\lambda,a,c),
\end{aligned}
$$

$$\Gamma(\lambda, a, c) = \int_0^c p^0(a-t, c-t) \exp\Big\{ -\lambda t - \int_{a-t}^{a} (2\tilde{\mu} + \sigma)|_{(\xi, \xi - a + c)}\, d\xi \Big\} dt$$
$$+ m(a-c) \exp\Big\{ -\lambda c - \int_{a-c}^{a} (2\tilde{\mu} + \sigma)|_{(\xi, \xi - a + c)}\, d\xi \Big\}$$
$$\times \Big\{ \int_0^{a-c-\tau} x_1(t, a-c) \exp\{-\lambda t\} dt$$
$$+ \int_{a-c-\tau}^{a-c} x_2(t, a-c) \exp\{-\lambda t\} dt \Big\}, \tag{4.9}$$

and

$$\widehat{x}(\lambda, 0) = S_1(\lambda)(1 - \kappa_1(\beta)(\lambda))^{-1},$$
$$S_1(\lambda) = \int_0^\infty dc \int_{\tau + c}^\infty \beta(a, c) \Gamma(\lambda, a, c)\, da. \tag{4.10}$$

Denoting $Re\lambda = \alpha$ and using upper estimates for x and p from (4.9), (4.2), (2.7) we obtain

$$|\Gamma(\lambda, a, c)| \leqslant \int_0^c \exp\{(\lambda_0 - \alpha)t\} \tilde{P}(a, c, \lambda_0)\, dt + m(a-c)$$
$$\times \exp\Big\{ - \int_{a-c}^{a} (2\tilde{\mu} + \sigma)|_{(\xi, \xi - a + c)}\, d\xi \Big\} \int_c^a x(t-c, a-c) \exp\{-\alpha t\}\, dt$$
$$\leqslant \int_0^c \exp\{(\lambda_0 - \alpha)t\} dt\, \tilde{P}(a, c, \lambda_0) + m(a-c)$$
$$\times \exp\Big\{ - \int_{a-c}^{a} (2\tilde{\mu} + \sigma)|_{(\xi, \xi - a + c)}\, d\xi \Big\} \tilde{X}(a-c, \lambda_0)$$
$$\times \int_c^a \exp\{\lambda_0(t-c) - \alpha t\}\, dt$$
$$= \tilde{P}(a, c, \lambda_0) \int_0^a \exp\{(\lambda_0 - \alpha)t\}\, dt. \tag{4.11}$$

Assume $\beta \leqslant \beta_0$, $m \leqslant m_0$, β_0 and m_0 being positive constants. Then taking into account (2.15) and (4.10), (4.11) we get

$$
\begin{aligned}
|S_1(\lambda)| &\leqslant \int_0^\infty dc \int_{\tau+c}^\infty da\, \beta(a,c)\tilde{P}(a,c,\lambda_0) \int_0^a \exp\{(\lambda_0-\alpha)t\}dt \\
&\leqslant \beta_0 X(0) \int_0^\infty dc \int_{\tau+c}^\infty da \exp\Big\{ -a\lambda_0 - \int_0^{a-c} \mu\, ds \\
&\quad - \int_{a-c}^a (\tilde{\mu}+\sigma)|_{(\eta,\eta-a+c)}\, d\eta \Big\} m(a-c) \int_0^a \exp\{(\lambda_0-\alpha)t\}\, dt \\
&= \beta_0 X(0) \int_\tau^\infty da \exp\Big\{ -a\lambda_0 - \int_0^a \mu_*\, ds \Big\} \int_0^a \exp\{t(\lambda_0-\alpha)\}\, dt \\
&\quad \times \int_0^{a-\tau} m(a-c) \exp\Big\{ -\int_{a-c}^a (\tilde{\mu}+\sigma)|_{(\eta,\eta-a+c)}\, d\eta \Big\} dc \\
&\leqslant m_0\beta_0 X(0) \int_\tau^\infty da \exp\Big\{ -a\lambda_0 - \int_0^a \mu_* ds \Big\}(a-\tau) \int_0^a \exp\{t(\lambda_0-\alpha)\}\, dt.
\end{aligned}
$$

If α, $\lambda_0 > \lambda_* = -\inf_{a\geqslant\tau} (1/a) \int_0^a \mu_*\, ds$, then $|S_1|$ is bounded. Let $\lambda_0 > \lambda_*$ and λ_k with $\alpha_k = Re\lambda_k > \lambda_*$, $k = 1,2\ldots$ be a unique real and complex roots of Eq. $\kappa_1(\beta)(\lambda) = 1$. Since $\alpha_k < \lambda_0$, the inverse Laplace transform yields

$$
x(t,0) \sim \exp\{\lambda_0 t\}q_1, \quad q_1 = S_1(\lambda_0)(-\kappa_1'(\lambda_0))^{-1},
$$

the prime indicates differentiation. Therefore

$$
\begin{aligned}
x &\sim \exp\{\lambda_0(t-a)\} \begin{cases} q_1 \exp\Big\{ -\int_0^a \mu ds \Big\}, & 0 \leqslant a \leqslant \tau \\ X(a), & a \geqslant \tau, \end{cases}, \\
p &\sim m(a-c) \exp\Big\{ \lambda_0(t-a) - \int_{a-c}^a (2\tilde{\mu}+\sigma)|_{(\xi,\xi-a+c)} \Big\}
\end{aligned}
$$

for large time ($t > a$). It is evident that the main term of x and p for large time is the persistent solution with $X(0) = q_1$.

5. Problem (1.6) in a Particular Case of the Vital Rates

In this section we obtain the solution of (1.6) in the case where $\tilde{\mu} = \mu$, σ and m are constants and β depends on all possible variables, and construct its asymptotic behavior for constant β and specialized x^0, p^0.

Substituting $z(t,a,c) = \int_{\tau+c}^{\infty} p\,db$, $z^0 = \int_{\tau+c}^{\infty} p^0 db$ into (1.6) we arrive at the problem

$$\partial_t x + \partial_a x + \mu x = 0, \quad 0 < a < \tau,$$
$$\partial_t x + \partial_a x + (\mu + m)x = (\mu + \sigma)\int_0^{a-\tau} z\,dc, \quad a > \tau,$$
$$\partial_t z + \partial_a z + \partial_c z + (2\mu + \sigma)z = 0, \quad a > \tau, \quad 0 < c \leqslant a - \tau,$$
$$x|_{a=0} = \int_0^{\infty} dc \int_{\tau+c}^{\infty} da \int_{\tau+c}^{\infty} \beta p\,db, \quad [x|_{a=\tau}] = 0,\ x|_{t=0} = x^0,$$
$$z|_{c=0} = mx, \quad z|_{t=0} = z^0, \quad a > \tau,$$
$$x^0(0) = \int_0^{\infty} dc \int_{\tau+c}^{\infty} da \int_{\tau+c}^{\infty} \beta|_{t=0} p^0 da, \quad [x^0(\tau)] = 0.$$

In addition to this system we must add the equation for p with respective conditions (see (1.6)).

If $\beta = const$, then $x|_{a=0} = \beta \int_{\tau}^{\infty} da \int_0^{a-\tau} z\,dc$ and we have the special case of linear problem (1.7) considered in Sec. 3. Knowing x we obtain p by formula

$$p = \begin{cases} p^0(a-t, b-t, c-t)\exp\{-(2\mu+\sigma)t\}, & 0 \leqslant t \leqslant c, \\ mx(t-c, a-c)x(t-c, b-c)\exp\{-(2\mu+\sigma)c\} & \\ \quad (\int_{\tau}^{\infty} x(t-c,\xi)\,d\xi)^{-1}, & c \leqslant c_*, \end{cases} \tag{5.1}$$
$$c_* = \min(\mathrm{t}, \mathrm{a}-\tau, \mathrm{b}-\tau).$$

If λ_0 and $Re\lambda_k > -\mu$ for all k, then $Re\lambda_k < \lambda_0$, and asymptotics of x can be expressed by (3.18), i.e.,

$$x \sim q\exp\{\lambda_0(t-a) - \mu a\}\begin{cases} 1, & a \leqslant \tau, \\ N_X(a), & a \geqslant \tau, \end{cases} \tag{5.2}$$

where

$$N_X(a) = (\mu+\sigma+m)^{-1}\{\mu+\sigma+m\exp\{-(\mu+\sigma+m)(a-\tau)\}\},$$

λ_0 and λ_k are a unique real and complex roots of equation

$$\kappa(\beta)(\lambda) = m\beta\exp\{-\tau(\lambda+\mu)\}(\lambda+\mu)^{-1}(\lambda+2\mu+\sigma+m)^{-1} = 1,$$

and $\text{sign}\lambda_0 = \text{sign}(\kappa(\beta)(0) - 1)$. For $\tau = 0$ this characteristic equation has been analyzed in Hadeler (1989)

Thus

$$p \sim mqN_X(a-c)N_X(b-c)\Big(\int_\tau^\infty N_X(\xi)\exp\{-\xi(\lambda_0+\mu)\}d\xi\Big)^{-1}$$
$$\times \exp\{\lambda_0(t+c-a-b) - \mu(a+b) - c\sigma\},$$
$$t > \min(\text{a}, \text{b}),\ \lambda_0 > -\mu. \tag{5.3}$$

Observe that the main term of x and p for large time ($t > a$) is the persistent solution of (1.6) with $X(0) = q$.

Now we will consider the general case of β. By using (5.1) one can obtain

$$x(t,0) = \int_0^\infty dc \int_{\tau+c}^\infty d\xi \int_{\tau+c}^\infty d\eta \beta(t, t+\xi, t+\eta, c)p^0(\xi,\eta,c)d\eta$$
$$\times \exp\{-t(2\mu+\sigma)\} + m\int_0^t H(t,\rho)\,d\rho,$$
$$H(t,\rho) = \exp\{-(2\mu+\sigma)(t-\rho)\}\Big(\int_\tau^\infty x(\rho,z)dz\Big)^{-1}\int_\tau^\infty d\xi\, x(\rho,\xi)$$
$$\times \int_\tau^\infty d\eta\, x(\rho,\eta)\beta(t,\xi+t-\rho,\eta+t-\rho,t-\rho), \tag{5.4}$$

where x is defined by (3.8)–(3.10), i.e.,

$$x = x_1 = x^0(a-t)\exp\{-t(2\mu+\sigma+m)\} + \big(x^0(a-t) + f(a-t)\big)$$
$$\times(\mu+\sigma)(\mu+\sigma+m)^{-1}$$
$$\times\big(\exp\{-t\mu\} - \exp\{-t(2\mu+\sigma+m)\}\big),$$
$$f^0(a) = \int_0^{a-\tau} dc \int_{\tau+c}^\infty p^0 db, \quad 0 \leqslant t \leqslant a-\tau,$$
$$x = x_2 = x^0(a-t)\exp\{-t\mu\}(\mu+\sigma+m)^{-1}$$
$$\times\{\mu+\sigma+m\exp\{-(\mu+\sigma+m)(a-\tau)\}\}, \quad a-\tau \leqslant t \leqslant a,$$
$$x = x_3 = x(t-a,0)N_X(a)\exp\{-\mu a\}, \quad t \geqslant a.$$

Let $\Lambda(t,\rho,\xi,\eta)$ be a function, which can be specified. If $t \leqslant \tau$, then $\int_\tau^\infty \Lambda\, x(\rho,\xi)\,d\xi = \int_\tau^{\tau+\rho} \Lambda x_2\,d\xi + \int_{\tau+\rho}^\infty \Lambda x_1\,d\xi$ and, applying this formula to integrals $\int_\tau^\infty x(\rho,z)\,dz$ and

$\int_\tau^\infty d\xi\, x(\rho,\xi) \int_\tau^\infty d\eta\, x(\rho,\eta)\beta(t,\xi+t-\rho,\eta+t-\rho,t-\rho)$, from (5.4) we obtain $x(t,0)$. Thus $\int_0^t H(t,\rho)\,d\rho$ is known for all $t \leqslant \tau$.

If $t > \tau$, then

$$\int_\tau^\infty \Lambda x(\rho,\xi)d\xi = \int_\tau^\rho \Lambda x_3 d\xi + \int_\rho^{\tau+\rho} \Lambda x_2 d\xi + \int_{\tau+\rho}^\infty \Lambda x_1 d\xi$$
$$= \int_\rho^{\tau+\rho} \Lambda x_2 d\xi + \int_{\tau+\rho}^\infty \Lambda x_1 d\xi$$
$$+ \int_0^{\rho-\tau} \Lambda(t,\rho,\rho-\gamma,\eta)x(\gamma,0)N_X(\rho-\gamma)\exp\{-\mu(\rho-\gamma)\}\,d\gamma. \quad (5.5)$$

If $\tau < t \leqslant 2\tau$, then $\rho - \tau \leqslant t - \tau \leqslant \tau$. Therefore by (5.5) we obtain $H(t,\rho)$ and then from (5.4), (5.5) we get $x(t,0)$ for $t \in (\tau, 2\tau]$ because $x(t,0)$ is known for $0 \leqslant t \leqslant \tau$.

Going along axis t by step τ we construct $x(t,0)$ for all $t > \tau$ and, consequently, x and p.

Theorem 5.1. *Assume x^0 and p^0 are non-negative non-trivial and $x^0 \in L^1(0,\infty) \cap C^1(0,\infty) \cap C^0([0,\infty))$, $p^0 \in L^1((\tau+c,\infty)\times(\tau+c,\infty)\times(0,\infty)) \cap C^0([\tau+c,\infty)\times[\tau+c,\infty)\times[0,\infty)) \cap C^1((\tau+c,\infty)\times(\tau+c,\infty)\times(0,\infty))$.*

If $\tilde\mu = \mu$, σ, m and β are positive constants, then (1.6) *has a unique non-negative solution such that*

$$\begin{aligned} &x \in C^0([0,\infty)\times[0,\infty)) \cap C^1(((0,\infty)\times((0,\tau)\cup(\tau,\infty))) \setminus \\ &\quad \{(t,a) : t = a,\ t = a-\tau\}), \\ &p \in C^0([0,\infty)\times[\tau,\infty)\times[\tau,\infty)\times[0,\ \min(\mathrm{a}-\tau,\mathrm{b}-\tau)]) \\ &\quad \cap C^1(((0,\infty)\times(\tau,\infty)\times(\tau,\infty)\times(0,\ \min(\mathrm{a}-\tau,\mathrm{b}-\tau))) \setminus \\ &\quad \{(t,a,b,c) : t=a,\ t=b,\ t=c,\ t=a-\tau,\ t=b-\tau\}). \end{aligned} \quad (5.6)$$

If, moreover, $x^0 \leqslant \tilde X(a,\lambda_0)$, $f^0(a) \leqslant \tilde F(a,\lambda_0)$, where $\tilde X$, $\tilde F$ are defined in Sec. 3, *$\lambda_0 > -\mu$ and $\alpha_k \pm i\beta_k$ with $\alpha_k \in (-\mu,\lambda_0)$ are real and complex roots of equation $\kappa(\beta)(\lambda) = 1$, then* (5.2), (5.3) *represent the asymptotic behavior of x and p for large time ($t > a$), and $x \leqslant \tilde X(a,\lambda_0)\exp\{\lambda_0 t\}$.*

If $\tilde\mu = \mu$, σ and m are positive constants, $\beta \in C^1([0,\infty)\times[\tau,\infty)\times[\tau,\infty)\times[0,\ \min(\mathrm{a}-\tau,\mathrm{b}-\tau)])$, and β, $\partial_t\beta$, $\partial_a\beta$, $\partial_b\beta$, $\partial_c\beta$ are bounded, then (1.6) *has a unique non-negative solution satisfying* (5.6).

NOTE. If $\tilde\mu$, σ and β are independent of a, b, then substitution $z(t,a,c) = \int_{\tau+c}^\infty p\,db$ reduces (1.6) to problem (1.7) too, with p replaced by z.

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Porų formavimo modelio išsprendžiamumas ir asimptotika

Vladas Skakauskas

Įrodytas simetrinio dvilytės populiacijos modelio išsprendžiamumas dviem atvejais ir gautas jo sprendinio asimptotinis elgesys laiko kintamojo atžvilgiu. Vienu atveju poros formuojamos pagal harmoninio vidurkio dėsnį, kitu atveju poros partneriai yra vienodo amžiaus.