INFORMATICA, 1998, Vol. 9, No. 2, 195–208

Absolute Stability and Hyperstability of a Class of Hereditary Systems

Manuel de la SEN, Josu JUGO
Departamento de Electricidad y Electronica, Facultad de Ciencias
Universidad del Pais Vasco, Leioa(Bizkaia). Aptdo. 644 de Bilbao. Spain
e-mail: msen@we.lc.ehu.es

Received: October 1997

Abstract. This paper considers the control problem of a class of linear hereditary systems subjected to a nonlinear (perhaps) time-varying controller. The absolute stability for a class of nonlinear time-varying controllers are investigated. Sufficient conditions for absolute stability and hyperstability are given.

Key words: hereditary systems, hyperstability, absolute stability.

1. Introduction

The stabilizability of linear hereditary systems independent of delay has been investigated in Kamen (1982, 1983) and Kamen *et al.* (1984). Also, an open-loop stabilizability problem was proposed in Olbrot (1978) for general linear autonomous systems with both discrete and distributed delays in state and control variables. It was proven that a simple algebraic rank condition, similar to the well known Hautus condition, is necessary for open-loop stabilizability. Such a condition was also shown to be sufficient by constructing a proper stabilizing state feedback. When the delays appear in control variables only the state-feedback spectrum assignability is equivalent to formal controllability of a certain pair of real matrices and, equivalently, to systems state controllability.

In the above context, hereditary systems with delayed inputs may be stabilized in some cases through controls whose decays to the equilibrium are slower than those of the state (trajectory stabilizable systems). They may be also stabilized with controls being proportional to the state (state stabilizable systems). Both properties were investigated in Tadmor (1988) through comparisons with the results in Olbrot (1978). On the other hand, some connections between controllability and observability and stability of such systems has been stated in De la Sen (1992). In particular, it has been found that under controllability and observability, a class of such systems is exponentially asymptotically stable if and only if it is bounded-input bounded-state stable.

The purpose of this paper is to extend some basic results of absolute stability and hyperstability from linear plants subject to a wide class of non-linear controllers to plants having delays in the state while being controlled by the same classes of no n-linear controllers.

2. Systems with Delays

Consider the linear and time-invariant single-input single-output system

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + bu(t) + b_1u(t-h'), \tag{1a}$$

$$\sigma(t) = c^T x(t), \quad x(0) = x_0, \tag{1b}$$

subjected to the nonlinear (and perhaps time-varying) controller

$$u(t) = -\varphi(\sigma, t), \tag{1c}$$

where $x \in R^n$, $u, \sigma \in R$ are, respectively, the state vector and scalar input and output and $h > 0$ and $h' > 0$ are, respectively, the internal and external delays. $\varphi(\sigma, \cdot)$ is a nonlinear characteristic to be specified later. All vectors and matrices in Eqs. 1 are constant and of appropriate dimensionalities. Positive definiteness (semidefiniteness) are denoted by > 0 ($\geqslant 0$).

Lemma 1. *The following propositions hold:*

(i) Lefschetz. *Given $A \in \{\bar{A}_1\}$, (A, b) completely controllable, a real vector k, scalars ψ and $\varepsilon > 0$ and an arbitrary real matrix $L = L^T > 0$, then a real vector q and a real matrix $P = P^T > 0$ satisfying*

$$A^T P + PA = -qq^T - \varepsilon L,$$

$$Pb - k = \sqrt{\psi} q$$

exist if and only if ε is sufficiently small and

$$G(s) \triangleq \frac{1}{2}\psi + k^T(sI - A)^{-1}b$$

belongs to $\{SPR\}$, namely, the class of strictly positive real functions.

(ii) Meyer. *Given $A \in \{\bar{A}_1\}$, (A, b) completely controllable, a real vector k and a real scalar ψ, then a real vector q and matrices $P = P^T > 0$, $M = M^T \geqslant 0$ satisfying*

$$A^T P + PA = -qq^T - M,$$

$$Pb - k = \sqrt{\psi} q,$$

with (A, q^T) being completely observable exist, if and only if $G(s)$ defined as in (i) *belongs to $\{PR\}$, namely, the class of positive real functions.*

(iii) Taylor and Narendra. *Given $\hat{A} = A + \mu I \in \{\bar{A}_1\}$, namely, the class of Hurwitzian matrices, $(\hat{A}, b)$ (or, equivalently (A, b)) completely controllable with μ being a* *real positive constant, a real vector k and a real positive scalar ψ, then a real vector q and matrices $P = P^T$, $N = N^T \geqslant 0$ satisfying*

$$A^T P + PA = -qq^T - N - 2\mu P,$$

$$Pb - k = \sqrt{\psi} q,$$

exist if and only if $\hat{G}(s) = G(s - \mu) = \frac{1}{2}\psi + k^T \left((s - \mu)I - A\right)^{-1} b$ *belongs to* $\{PR\}$.

(iv) Narendra and Taylor. *Given the marginally stable matrix $A \in \{\bar{A}_0\}$ (namely, the class of matrices having one single zero eigenvalue while all the remaining ones are strictly stable) of the form*

$$A = \begin{pmatrix} 0 & e_2^T \\ 0 & \hat{A} \end{pmatrix},$$

where e_2 is the second unitary vector in the Euclidean space, a symetric matrix $L_0 = L_0^T$ of the form

$$L_0 = \begin{pmatrix} 0 & 0 \\ 0 & \hat{L} \end{pmatrix},$$

$\hat{L} = \hat{L}^T > 0$ arbitrary, a real vector k, real scalars $\varepsilon > 0$ and ψ, then a real vector q and a matrix $P = P^T > 0$ satisfying

$$A^T P + PA = -qq^T - \varepsilon L_0,$$

$$Pb - k = \sqrt{\psi} q,$$

exist if and only if ε is sufficiently small and $G(s)$ of (i) *belongs to $\{SPR_0\}$, namely, the class of functions in SPR having a pole at the origin.*

3. Absolute Stability

In this section, system Eqs. 1 is investigated from a stability point of view with $h > 0$ (existence of internal delay) under two situations, namely

a $h' = 0$, $b_1 = 0$, i.e., absence of external delay,
b $h' > 0$, $b_1 \neq 0$, i.e., presence of external delay.

3.1. *Absolute Stability under Internal Delay in the Absence of External Delay*

Assume the following constraints for the nonlinear controller (1.c)

$$h_1 \sigma^2 \leqslant \sigma\varphi(\sigma) \leqslant h_2 \sigma^2, \quad h_2 > h_1 \geqslant 0. \tag{2}$$

The interval $[h_1, h_2]$ is called a Lur'e or a Popov sector as being inherent to (3a) below. The first inequality is strict for all $\sigma \neq 0$ and $\varphi(0) = 0$. See, for instance, Landau (1975), Narendra and Taylor (1973). Constraints (2) imply directly

$$h_1 \leqslant \frac{\varphi(\sigma)}{\sigma} \leqslant h_2, \tag{3a}$$

$$\psi(\sigma) = \int_0^{\sigma} \psi(\sigma')d\sigma = \int_{t_0}^{t} \psi(\sigma')\dot{\sigma}'d\tau \geqslant 0, \tag{3b}$$

$$\xi(0,t) = \int_0^{t} \varphi(\sigma)\sigma d\tau \geqslant 0, \tag{3c}$$

where $\dot{\sigma} = d\sigma/dt$ and $t_0 = \max\{\tau \in R^+ \colon \tau \leqslant t, \sigma(\tau) = 0\}$. The problem of absolute stability is that of the stability of system (1) for all nonlinear controller satisfying (2). For $h' = 0$ and $b_1 = 0$, define the nonnegative function

$$V(x, \varphi(\sigma), t) = x^T(t)Px(t) + \int_0^{t} x^T(\tau - h)P_1 x(\tau - h)d\tau + \beta \int_0^{\sigma} \varphi(\sigma')d\sigma', \tag{4}$$

with $P = P^T > 0,\ P_1 = P_1^T > 0,\ \beta > 0$. Stability is guaranteed if

$$\begin{aligned} \dot{V}(x, x(t-h), \varphi(t), t) &= \dot{x}^T Px + x^T P\dot{x} \\ &\quad + x^T(t-h)P_1 x(t-h) + \beta\varphi(\sigma)\dot{\sigma} \\ &= x^T(A^T P + PA)x - 2x^T Pb\varphi(\sigma) \\ &\quad + \beta c^T Ax\varphi(\sigma) - \beta c^T b\varphi^2(\sigma) \\ &\quad + x^T(t-h)P_1 x(t-h) + 2x^T PA_1 x(t-h) \\ &\quad + 2\beta c^T A_1 x(t-h)\varphi(\sigma) < 0, \end{aligned} \tag{5}$$

for all nonzero $z(t) = \left(x^T,\ x^T(t-h),\ \varphi(\sigma)\right)^T$. For notational simplicity, the argument t has been omitted although $x(t-h)$ is explicited. Eq. 5 may be rewritten as

$$\dot{V}(z) = -z^T Qz \leqslant 0, \ \text{for}\ \varphi(\sigma)/\sigma \in [h_1, h_2], \tag{6}$$

by defining the matrix

$$Q = Q^T = \begin{pmatrix} -(A^T P + PA) & -PA_1 & Pb - \frac{\beta}{2}A^T c \\ -A_1^T P & P_1 & -\frac{\beta}{2}A_1^T c \\ b^T P - \frac{\beta}{2}c^T A & -\frac{\beta}{2}c^T A_1 & \beta c^T b \end{pmatrix} \tag{7}$$

with equality to zero standing if and only if $z = 0$ provided that $Q > 0$. This is guaranteed under sufficient conditions by the next result.

Lemma 2. *The following two propositions hold*

(i) *$Q > 0$ and $\dot{V} < 0$ ($\forall z \neq 0$) in Eqs. 6–7 if for some matrix $D_1 = D_1^T = \varepsilon_1 L_1$, $\varepsilon_1 > 0$, $L_1 = L_1^T > 0$ and scalar $\beta > 0$, there exist a scalar $\gamma > 0$, n-vectors q_1 and q_2, n-positive definite matrices $D = D^T > 0$, $P_1' = P_1' > 0$, $P_1 = D_1 + q_2 q_2^T$ in* (4) *and $n \times 2n$ matrices R and S fulfilling*

$$\hat{D} = \begin{pmatrix} D + q_1 q_1^T & -PA_1 \\ -A_1^T P & P_1 \end{pmatrix} = \begin{pmatrix} R \\ S \end{pmatrix} (R^T \; S^T) > 0, \tag{8a}$$

$$A^T P + PA = -q_1 q_1^T - D, \quad P = P^T > 0, \tag{8b}$$

$$A_1^T P_1' + P_1' A_1 = -P_1 = -q_2 q_2^T - D_1, \quad P_1' = P_1'^T > 0, \tag{8c}$$

$$\frac{\beta}{2} \bar{A}^T \bar{c} - \bar{P}\bar{b} = -\gamma \bar{q}, \tag{8d}$$

$$\beta \bar{c}^T \bar{b} = \gamma^2, \tag{8e}$$

where

$$\bar{A} = \begin{pmatrix} A & A_1 \\ 0 & A_1 \end{pmatrix},$$

$$\bar{q} = (q_1^T \; q_2^T)^T; \quad \bar{b} = (b_1^T \; b_2^T)^T, \quad \bar{c} = (c_1^T \; c_2^T)^T,$$

$$\bar{P} = \mathrm{Diag}(P,\, P_1'), \quad b_1 = \left(\frac{\beta}{2}\right) (P_1')^{-1} A_1^T c_1 \;\; (c_1 \textit{ arbitrary}) \tag{9}$$

c_2 and b_2 being arbitrary n-vectors satisfying the constraint

$$c_2^T b_2 = c^T b - c_1^T b_1 = \gamma^2/\beta - c_1^T b_1 = c^T b - \frac{\beta}{2} c_1^T (P_1')^{-1} A_1^T c_1.$$

(ii) *Eqs.* 8.b–8.c *and* 8.d–8.e *subjected to* (9) *may be rewritten, respectively, as*

$$\bar{A}^T \bar{P} + \bar{P}\bar{A} = -\bar{q}\bar{q}^T - \bar{D}, \tag{10a}$$

$$\bar{D} = \begin{pmatrix} D & PA_1 - q_1 q_2^T \\ A_1^T P - q_2 q_1^T & D_1 \end{pmatrix}, \tag{10b}$$

$$\bar{P}\bar{b} - \bar{k} = \sqrt{\psi}\bar{q}, \tag{10c}$$

$$\bar{k} = \frac{\beta}{2} \bar{A}^T \bar{c}; \; \psi = \beta \bar{c}^T \bar{b}. \tag{10d}$$

or if $b_1 = c_1 = 0$ in (12) *then, as particular case,*

$$Pb - k = \sqrt{\psi} q_1, \tag{11a}$$

$$k = \frac{\beta}{2} A^T c; \ \psi = \beta c^T b. \tag{11b}$$

The proof is omitted by space reasons.

Global stability results independent of the internal delay h are given below.

Theorem 1. *The following propositions hold for system Eqs.* 1 *and* 2 *with any nonlinear device in the sector* $[0, \infty)$ *and any bounded* $x(0) \in R^n$.

(i) *It is absolutely stable if Lemma* 2 (i) *stands.*

(ii) *If* $P_1 = 0$ *in* (4), *then* $G(s)$ *in Lemma* 1 (i), (ii), (iv) *(or* $\hat{G}(s)$ *in Lemma* 1 (iii) *satisfies the respective realness conditions as being obtained from* (8.b) *and* (11) *in Lemma* 2, *and matrix* $\hat{D}$ *in* (11.a) *is at least positive semidefinite, then* $\|x(t)\|$ *is uniformly bounded in time provided* $A \in \bar{A}_1$ *(*$A \in \bar{A}_0$ *for Lemma* 1 (iv)*).*

(iii) *If* $P_1 = P_1^T > 0$ *has sufficiently large spectral radius, and* $G(s) \in \{PR\}$ *in Lemma* 1 (ii) *(or* $\hat{G}(s)$ *in Lemma* 1 (iii)*) being defined by using equations* (8.b) *and* (11) *in Lemma* 2, *then* $\|x(t)\|$ *is uniformly bounded-in-time and converges asymptotically to zero provided* A, $A_1 \in \bar{A}_1$ *and* (8.a) *holds.*

(iv) *Proposition* (iii) *holds identically by using Lemma* 1 (i) *for* A, $A_1 \in \bar{A}_1$ *and* $G(s) \in \{SPR\}$. *By hypotheses of Lemma 1 (iv) and* $G(s) \in \{SPR_0\}$ *with* $A_1 \in \bar{A}_1$, $A \in \bar{A}_0$ *only state uniform boundedness may be guaranteed.*

(v) *If* $P_1 = 0$, *then* (iii)–(iv) *are modified so that only state boundedness is guaranteed.*

(vi) *A necessary condition for any proposition* (i) *and* (iv) *to hold is that Lemma* 2 (ii) *holds which may be tested under the respective realness or positive realness conditons in Lemma* 1.

The proof is omitted by space reasons.

3.2. *Absolutely Stability under Internal and External Delays*

Now, assume that $h > 0$, $h' > 0$. Assume that the following nonnegative function is defined for system Eqs. 1 with $b_1 \neq 0$:

$$V'(x, \varphi(\sigma), t) = V(x, \varphi(\sigma), t) + \beta_1 \int_0^t \varphi(\sigma'(\tau - h'))\, \dot{\sigma}'(\tau) d\tau, \tag{12}$$

with $\beta_1 > 0$ and V defined in (4) so that

$$\begin{aligned} \dot{V}(x(t), x(t\, -\, h), \varphi(\sigma(t)), \varphi(\sigma(t - h')), t, t - h') &= x^T(A^T P + PA)x \\ -\ 2x^T Pb\varphi(\sigma) - 2x^T Pb_1\varphi(\sigma(t - h')) &+ \beta c^T Ax\varphi(\sigma) \\ -\ \beta c^T b\varphi^2(\sigma) - \beta c^T b_1\varphi(\sigma)\varphi(\sigma(t - h')) &+ x^T(t - h)P_1 x(t - h) \\ +\ 2x^T PA_1 x(t - h) + 2\beta c^T A_1 x(t - h)\varphi(\sigma) & \\ +\ \beta_1 \varphi(\sigma(t - h'))\, \dot{\sigma}(t), & \end{aligned} \tag{13}$$

where, as in (5), only the time argument for $(t-h')$ has been explicited. Note in (13) that

$$\begin{aligned}\beta_1\varphi\left(\sigma(t-h')\right)\dot{\sigma}(t) &= \beta_1\varphi\left(\sigma(t-h')\right)c^T\dot{x}(t)\\ &= \beta_1 c^T[Ax\varphi(\sigma(t-h')) + A_1x(t-h)\varphi(\sigma(t-h'))\\ &\quad - b\varphi(\sigma)\varphi(\sigma(t-h')) - b_1\varphi^2(\sigma(t-h'))]. \end{aligned} \tag{14}$$

Now, define $\bar{z}(t) = \left(x^T,\ x^T(t-h),\ \varphi(\sigma)\ \varphi(\sigma(t-h'))\right)^T$ so that $\dot{V}(t) = -\bar{z}^T(t)\bar{Q}\bar{z}(t)$ where $\bar{Q}$ becomes as follows by substituting (14) into (13):

$$\bar{Q} = \begin{pmatrix} -(A^TP+PA) & -PA_1 & Pb-\frac{\beta}{2}A^Tc & Pb_1 - \frac{\beta}{2A}{}^T c \\ -A_1^TP & P_1 & -\frac{\beta}{2}A_1^Tc & -\frac{\beta_1}{2}A_1^Tc \\ b^TP-\frac{\beta}{2}c^TA & -\frac{\beta}{2}c^TA_1 & \beta c^Tb & \frac{\beta}{2}c^Tb_1 + \frac{\beta_1}{2}c^Tb \\ b_1^TP - \frac{\beta}{2}c^TA & -\frac{\beta_1}{2}c^TA_1 & \frac{\beta}{2}c^Tb_1 + \frac{\beta_1}{2}c^Tb & \beta_1c^Tb_1 \end{pmatrix}. \tag{15}$$

Thus, a generalization of Lemma 2 for this case becomes

Lemma 3. *The following two propositions hold*

(i) $\bar{Q} > 0$ *and* $\dot{V} < 0$ *(all* $\bar{z} \neq 0$*) if for some n-matrix* $D_1 = D_1^T = \varepsilon_1 L_1$, $\varepsilon_1 > 0$, $L_1 = L_1^T > 0$ *and positive scalars* β *and* β_1, *there exist a positive scalar* γ, *n-vectors* q_1 *and* q_2, *n-positive definite matrices* $D = D^T > 0$, $P_1' = P_1'^T > 0$, $P_1 = D_1 + q_2q_2^T$ *in* (4) *and* (12) *and* $n \times 2n$ *matrices R and S fulfilling Eqs.* 8–9 *in Lemma* 1 *and, furthermore, for some n-matrix* $D_1' = D_1'^T = \varepsilon_1' L_1'$, $\varepsilon_1' > 0$, $L_1' = L_1'^T > 0$, *there exists a positive scalar* γ' *and an n-vector* q_3 *such that for* $\bar{\bar{q}} = (\bar{q}^T,\ q_3^T)^T$,

$$\bar{\bar{D}} = \begin{pmatrix} \hat{D} & & Pb - \frac{\beta}{2}A^Tc \\ & & -\frac{\beta}{2}A_1^Tc \\ b^TP - \frac{\beta}{2}c^TA & -\frac{\beta}{2}c^TA_1 & \gamma^2 + d_2 \end{pmatrix} = \bar{\bar{D}}^T > 0, \tag{16}$$

$$\frac{\beta'}{2}\bar{\bar{A}}^T\bar{\bar{c}} - \bar{\bar{P}}\bar{\bar{b}} = -\gamma'\bar{\bar{q}},$$

$$\beta'\bar{\bar{c}}^T\bar{\bar{b}} = \gamma'^2,$$

$$\beta c^Tb = \gamma^2 + d_2, \quad \textit{some} \quad d_2 > -\gamma^2\ (\Rightarrow c^Tb > 0),$$

where $\hat{D}$ *is defined as in Lemma* 2 *and*

$$\bar{\bar{A}} = \begin{pmatrix} A & A_1 & \frac{\beta}{2}P^{-1}A^Tc - b \\ 0 & A_1 & \frac{\beta}{2}P_1'^{-1}A_1^Tc \\ 0 & 0 & -\frac{\beta}{2}(c^Tb) \end{pmatrix}; \quad \bar{\bar{P}} = \mathrm{Diag}(P,\ P_1',\ I), \tag{17a}$$

$$\bar{\bar{b}} = (\bar{b}^T,\ b_3^T), \quad \bar{\bar{c}} = (\bar{c}^T,\ c_3^T)^T, \tag{17b}$$

and b_3 *and* c_3 *are arbitrary n-vectors satisfying the constraint*

$$c_3^T b_3 = c^T b_1 - \frac{\gamma^2}{\beta} = \frac{\gamma'^2}{\beta_1} - \frac{\gamma^2}{\beta}. \tag{17c}$$

(ii) *Equations in proposition* (i) *may be compacted as*

$$\bar{\bar{A}}^T\bar{\bar{P}} + \bar{\bar{P}}\bar{\bar{A}} = -\bar{\bar{q}}\bar{\bar{q}}^T - \bar{\bar{D}}, \tag{18a}$$

$$\bar{\bar{D}} = \begin{pmatrix} D & PA_1 - q_1q_2^T & \frac{\beta}{2}A^Tc - Pb - q_1q_3^T \\ A_1^TP - q_2q_1^T & D_1 & \frac{\beta}{2}A_1^Tc - q_2q_3^T \\ \frac{\beta}{2}c^TA - b^TP - q_3q_1^T & \frac{\beta}{2}c^TA_1 - q_3q_2^T & -\beta c^Tb - q_3q_3^T \end{pmatrix} \tag{18b}$$

$$\bar{\bar{P}}\bar{\bar{b}} - \bar{\bar{k}} = \sqrt{\bar{\psi}}\bar{\bar{q}}, \tag{18c}$$

$$\bar{\bar{k}} = \frac{\beta'}{2}\bar{\bar{A}}^T\bar{\bar{c}},\ \bar{\psi} = \beta'\bar{\bar{c}}^T\bar{\bar{b}}. \tag{18d}$$

If $b_1 = b_2 = c_1 = c_2 = 0$, *then* (18.c)–(18.d) *may be substituted by* (11.a)–(11.b).

The proof is omitted by space reasons.

Lemma 4. *Assume that Lemma* 3 *stands. Thus,* $V'(t)$ *is uniformly bounded in time.*

Global asymptotic stability follows from Lemmas 1 to 4 as established in the next result.

Theorem 2. *Theorem* 1 *stands with the changes* $\hat{D} \rightarrow \bar{\bar{D}}$, $A_1 \rightarrow A_1' = \mathrm{Diag}(A_1, c^Tb)$, *except for the propositions referred to the case* $P_1 = 0$ *for the system with internal delay (i.e.,* $b_1 \neq 0$, $h' > 0$*), subjected to Lemma* 3 *instead of to Lemma* 2, *if* $V'(t) \geqslant 0\ \forall t \in R^+$ *with* $V'(\cdot) = 0$ *if and only if* $x(\cdot) = 0$. *Otherwise, (i.e., when nonnegativity of*

V' cannot be guaranteed for every time), the various propositions related to asymptotic stability/convergence in Theorem 1 *must be weakened to nonasymptotic stability. Proof follows from Lemmas* 1, 3 *and* 4.

4. Hyperstability

Now, the set of nonlinear static and dynamic characteristics verifying the so-called Popov's inequality:

$$\xi(t_0, t_1) = \int_{t_0}^{t_1} \varphi(\sigma, t)\sigma(t)st \geqslant -\gamma_0^2 \quad \forall t_1 \geqslant t_0 \tag{19}$$

is considered. Such a description includes as particular case the nonlinear characteristics considered in Section 3. First, the hyperstability is investigated in the time domain and subsequently in the frequency domain. This last study is organized as follows. First, conditions are derived from Parseval's inequalities which ensure that the input asymptotically converges to zero by using stability tools for composite blocks (De la Sen, 1986). Subsequently, conditions ensuring the global asymptotic stability of the solutions of the free system (De la Sen 1988a, 1988b) are obtained from some results of Section 3.

4.1. *Analysis in the Time Domain*

Consider the function

$$\begin{aligned} V(t) &= x^T(t)Px(t) + \int_{-\infty}^{t} x^T(\tau - h)P_1x(\tau - h)d\tau \\ &\quad + \beta \int_{-\infty}^{t} \varphi(\sigma(\tau))\sigma(\tau)d\tau + \beta_1 \int_{-\infty}^{t} \varphi(\sigma(\tau - h'))\sigma(\tau - h')d\tau \\ &\quad + \lambda \int_{0}^{t} \varphi^2(\tau)d\tau + \lambda_1 \int_{0}^{t} \varphi^2(\tau - h)d\tau, \end{aligned} \tag{20}$$

for some matrices $P = P^T > 0$, $P_1 = P_1^T > 0$ and nonnegative scalars β, β_1, λ and λ_1 which may be chosen freely to pursue the proof of stability. By omitting, the time argument for time t, one has directly from (20).

$$\begin{aligned} \dot{V}(t) &= \dot{x}^T Px + x^T P\dot{x} + \dot{x}^T(t-h)P_1x(t-h) + \beta\varphi(\sigma)\sigma \\ &\quad + \beta_1\varphi(\sigma(t-h))\sigma(t-h) + \lambda\varphi^2 + \lambda_1\varphi^2(t-h) \\ &= x^T(A^TP + PA)x - \varphi(\sigma)[2b^TP - \beta c^T]x + x^T(t-h)A_1^TPx \\ &\quad + x^TPA_1x(t-h) - b_1^T\varphi(\sigma(t-h'))Px - x^TPb_1\varphi(\sigma(t-h)) \\ &\quad + x^T(t-h)P_1x(t-h) + \beta_1\varphi(\sigma(t-h'))c^Tx(t-h) + \lambda\varphi^2(\sigma) \\ &\quad + \lambda_1\varphi^2(t-h) = v^T(t)Rv(t), \end{aligned} \tag{21}$$

where $v(t) = \left(\bar{v}^T(t),\ \varphi(\sigma)\right)^T = \left(x^T(t),\ x^T(t-h),\ \varphi(\sigma(t-h)), \varphi(\sigma)\right)^T$, and

$$R = \begin{pmatrix} R' & r \\ r^T & \lambda \end{pmatrix}, \quad r = \left(\frac{1}{2}\beta c^T - b^T P,\ 0^T,\ 0^T\right)^T, \tag{22a}$$

$$R' = \bar{A}^T \bar{P} + \bar{P}\bar{A} = \begin{pmatrix} A^T P + PA & PA_1 & Pb_1 \\ A_1^T P & P_1 & 0 \\ b_1^T P & \frac{\beta_1}{2} c^T & \lambda_1 \end{pmatrix} \tag{22b}$$

for some matrices $\bar{A}$ and $\bar{P}$ of appropriate orders which are nonunique. Assume $\bar{P} = \text{Diag}(P,\ I,\ 1)$ with $\bar{A}$ being redefined with respect to Section 3 as

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} & a_{13} \\ A_{21} & A_{22} & a_{23} \\ a_{31}^T & a_{32}^T & a_{33} \end{pmatrix}, \tag{23}$$

where A_{ij} are n-matrices, a_{i3} $(i = 1, 2)$ are n-vectors $(a_{ij} = a_{ji})$ and a_{33} is a scalar. Direct calculus with (22)–(23) yields that a solution to (22.b) is

$$\bar{A} = \begin{pmatrix} A & A_1 - P^{-1}A_{21}^T & (I + P^{-1})^{-1} b_1 \\ A_{21} & \frac{P_1}{2} & 0 \\ 0^T & \frac{\beta_1}{4} c^T & \frac{\lambda_1}{2} \end{pmatrix}, \tag{24}$$

for any arbitrary n-matrix A_{21}. A particular case is

$$\bar{A} = \begin{pmatrix} A & 0 & (I + P^{-1})^{-1} b_1 \\ A_1^T P & \frac{P_1}{2} & 0 \\ 0^T & \frac{\beta_1}{4} c^T & \frac{\lambda_1}{2} \end{pmatrix}. \tag{25}$$

To reduce R in (22.a) to a convenient form in order to apply stability results, put for its last column, for some $(2n+1)$-vectors $\bar{c}$ and $\bar{b}$

$$r = \frac{1}{2}\beta\bar{c} - \bar{P}\bar{b} = \begin{pmatrix} \frac{1}{2}\beta c - Pb \\ \frac{\beta_1}{2} c \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta c \\ \frac{\beta}{2}\left(\frac{\beta_1}{\beta}\right) c \\ 0 \end{pmatrix} - \begin{pmatrix} Pb \\ 0 \\ 0 \end{pmatrix}, \tag{26}$$

which leads to $\bar{c} = \left(c^T, \frac{\beta_1}{\beta}c^T, 0\right)^T$, $\bar{b} = (b^T, 0^T, 0^T)^T$. Using (22.b) and (26) with $\lambda = 0$, it follows that

$$R = \begin{pmatrix} \bar{A}^T\bar{P} + \bar{P}\bar{A} & \frac{1}{2}\beta\bar{c} - \bar{P}\bar{b} \\ \frac{1}{2}\beta\bar{c}^T - \bar{b}^T\bar{P} & 0 \end{pmatrix}. \tag{27}$$

Substitution of (27) into (21) implies that this one may be rewritten as

$$\dot{V}(t) = \bar{v}^T(\bar{A}^T\bar{P} + \bar{P}\bar{A})\bar{v} - \varphi(\sigma)[2\bar{b}^T\bar{P} - \beta\bar{c}^T]\bar{v}, \tag{28}$$

so that it is sufficient for negative definiteness that

$$\left.\begin{aligned} &\bar{A}^TP + \bar{P}\bar{A} = -Q_0 = -L_oL_0^T > 0 \\ &2\bar{b}^T\bar{P} = \beta\bar{c}^T \end{aligned}\right\}. \tag{29}$$

Standard well-known results (see, for instance, Landau, 1975) prove that (29) define a strictly positive real transfer function. Furthermore, V in (20) is a Lyapunov function since $\dot{V} < 0$ for all nonzero $\bar{v}$, and $V > 0$ with $P > 0$, $P_1 > 0$ and β, β_1, λ, λ_1 being nonnegative since under the hypothesis $\int_{-\infty}^{0} \varphi(\sigma)\sigma d\tau = \gamma_0^2 \geqslant 0$, it follows that $\int_{-\infty}^{t} \varphi(\sigma)\sigma d\tau \geqslant 0$. Thus, by taking into account (25) and (29), the following result stands.

Theorem 3. *The system Eqs.* 1 *is globally asymptotically hyperstable (i.e., globally asymptotically stable for all nonlinear and eventually time-varying controller satisfying* $\int_0^t \varphi(\sigma)\sigma d\tau \geqslant -\int_{-\infty}^{0} \varphi(\sigma)\sigma d\tau = \gamma_0^2$*) provided that* $\hat{h}_0(s) = \bar{c}^T(sI - \bar{A})^{-1}\bar{b} = \bar{b}^T\bar{P}(sI - \bar{A})^{-1}\bar{b} \in \{SPR\}$*, where* $\beta = 2$ *in (29) has been used without loss in generality, for some matrices* $P > 0$*,* $P_1 > 0$ *and nonnegative scalars* β*,* β_1 *and* λ_1*. If* $b_1 = 0$*,* $h' = 0$ *(i.e., there are no external delays), then the above condition is simplified by neglecting the last row and column of* $\bar{A}$ *in (25), and the last (zero) components of vectors* $\bar{b}$ *and* $\bar{c}$ *(thus* $\bar{A}$ *becomes of order* $2n$ *and* $\bar{b}$ *and* $\bar{c}$ *are* $2n$*-vectors).*

The proof is omitted by space reasons.

The considerations $\hat{D} = \hat{D}^T > 0$, some $P = P^T > 0$, $P_1' = P_1'^T > 0$ (i.e., $A_1 \in \{\bar{A}_1\}$) lead to the next result.

Theorem 4. *System Eqs.* 1.a *to* 1.c *is globally asymptotically hyperstable for all (perhaps time-varying) nonlinearity satisfying the inequality* $\int_0^t \varphi(\sigma)\sigma d\tau \geqslant -\gamma_0^2$ *if Eqs.* 8.a *to* 8.c *in Lemma* 2 *hold and* $(h(s) + \hat{h}(s)) \in \{SPR\}$*, with* $h(s) = c^T(sI - A)^{-1}b$ *and* $\hat{h}(s) = c^T(sI - A - e^{-hs}A_1)^{-1}b_1$.

For global (nonasymptotically stability) the above conditions may be weakened in a direct way similarly to the procedure in Section 3.

4.2. *Analysis in the Frequency Domain*

The analysis of the absolute stability and hyperstability in the frequency domain can be addressed by taking Laplace transforms in (1.a)–(1.b) with the change $s = j\omega$ to analyze the frequency response. This technique also allows obtaining input/output energy balances by using the Parseval's theorem to evaluate the input-output time integrals equivalently in the frequency domain. The positive and strict positive realness of transfer functions can be also tested in the frequency domain when stablishing the stability results. Details are omitted by space reasons. System (1) can also be generalized by reformulating (1.b) as $\sigma(t) = c^T x(t) + \sigma_0 u(t)$ with σ_0 being a nonzero scalar quantifying a direct scalar interconnection from the input to the output. In particular, Theorem 3 still holds by replacing $\hat{h}(s) \rightarrow \hat{h}(s) + \sigma_0$.

5. Conclusions

This paper has dealt with the problems of absolute stability and hyperstability of plants with linear dynamics having both internal and external delays while being controlled by nonlinear devices satisfying standard sector or Popov inequalities. The various stability results have been interpreted in terms of positive realness conditions on transfer functions.

Acknowledgments

The authors are very grateful to DGICYT by its support of this work through Projects PB93-0005 and PB96-0257.

References

De la Sen, M. (1986). Stability of composite systems with an asymptotically hyperstable block. *Int. J. of Control*, **44**, 1769–1775.

De la Sen, M. (1988a). Fundamental properties of linear control systems with after-effect. Part I: The continuous Case. *Mathematical and Computer Modelling*, **10**, 473–489.

De la Sen, M. (1988b). Fundamental properties of linear control systems with after-effect. Part II: extensions including the discrete case. *Mathematical and Computer Modelling*, **10**, 491–502.

De la Sen, M. (1992). New results in stability of a class of hereditary linear systems. *Int. J. of Systems Sci.*, **26**, 915–933.

Kamen, E.W. (1982). Linear systems with commensurate time delays: stability and stabilization independent of delay. *IEEE Trans. Automat. Control*, **27**, 367–375.

Kamen, E.W. (1983). Correction to linear systems with commensurate time delays: stability and stabilization independent of delay. *IEEE Trans. Automat. Control*, **28**, 248–249.

Kamen, E.W. (1984). Pointwise stability and feedback control of linear systems with noncommensurate time delays. *Acta Math. Appl.*, **2**, 159–184.

Landau, I.D. (1975). *Systèmes non-linèaires. Notes de Cours*. Institut National Polytechnique de Grenoble (in French).

Narendra, K.S., and J.H. Taylor (1973).*Frequency Domain Criteria for Absolute Stability*. Academic Press, New York.

Olbrot, A.W. (1978). Stabilizability, detectability and spectrum assigment for linear autonomous systems with general time delays. *IEEE Trans. Automat. Control*, **23**, 887–890.

Tadmor, G. (1988). Trajectory stabilizing controls in hereditary linear systems. *SIAM J. Control and Optimization*, **26**, 138–154.

M. De la Sen was born in Arrigorriaga (Bizkaia) in the spanish Basque Country, in July, 1953. He obtained the Ms.Sci. degree with honors from the Basque Country University in applied physiscs in 1975, the Ph.D. degree in applied physics with high honors from the same university in 1979 and the degree of "Docteur d'Etat-ès-Sciences Physiques" (specialitè Automatique et Traitement du Signal) from the Universitè de Grenoble, France with "mention très honorable" in 1987. He has had several teaching positions in the Basque Country University in Bilbao, Spain, where he is currently Professor of systems and control engineering. He has also had the positions of visiting Professor in the University of Grenoble, France and the University of Newcastle, New South Wales, Australia. He has been author or coauthor of a number of papers in the fields of adaptive systems, mathematical systems, theory and ordinary differential equations, which are his research interest subjects. He acts or has acted as reviewer for several international journals of control theory and applied mathematics.

J. Jugo was born in Galdakao (Bizkaia) in the spanish Basque Country in 1968. He obtained the Ms. Sci. degree in applied physics by the University of Basque Country and the Ph.D. degree in applied physics (Cum Laude) from the same university in 1997. He is currently assistant professor of systems and control engineering in the Basque Country University. He has been coauthor of several papers in the fields of adaptive systems, control theory and applied mathematics.

Hereditarių sistemų klasės absoliutus stabilumas ir hiperstabilumas

Manuel de la SEN ir Josu JUGO

Straipsnyje nagrinėjamas tiesinių hereditarių sistemų su netiesinių ir, galimas dalykas, besikeičiančių laike, valdymo įrenginių klasės valdymo uždavinys. Tiriamas netiesinių, besikeičiančių laike valdymo įrenginių klasės absoliutus stabilumas. Čia pateiktos pakankamos sąlygos absoliučiam stabilumui ir hiperstabilumui.