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Controllability and Equilibrium Analysis of the Interaction between the Unemloyment Level and Related Government Expenditure

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Abstract. This paper addresses the study of the controllability and stability of the equilibrium in economic models which relate the unemployment level to the government expenditure. The interesting cases when the government expenditure is either bounded or a linear function of the national income are specifically considered. The relationships between both variables, namely, unemployment growth level and government expenditure is obtained by considering a Keynesian static model for the national income as well as a differential unemployment-inflation model of Phillips type. Both models are used to derive a new combined one by eliminating the common variable "taxes" which is driven by the investment and government expenditure.

Key words: controllability, economic models, equilibrium analysis.

1. Introduction

When dealing with economic models, an equilibrium analysis is normally performed (Chiang, 1984) while a dynamic analysis is very seldom considered (see, for instance, Medvegyev, 1984; Tabuchi, 1986; Popkov, 1988). The main reasons for this choice is that many economic variables require to have a precise model. These variables are often dependent on each other although some of them are considered endogenous for analysis purposes while others are stated as external inputs. The precise separation between the above two types is often difficult because of the interactions inherent to many economic models. The development of high-speed powerful computers has enabled econometricians to design high-complexity models with the largest feasible number of unknowns and equations. However, the associated high operating costs has led very often to the use of smaller size models which can be operated with comparatively less difficulties (Medvegyev, 1984; Balasko, 1984; Ivanov, 1992; Morishima, 1960; Tabuchi, 1986; Popkov, 1988). It can be pointed out that another reason for the pressure for small analytical models comes from the difficulty of seeing what happens in the large-scale models where the chain of events which produce the obtained results cannot be easily discussed or interpreted. This viewpoint is adopted in this paper when dealing with a static national income model and a dynamic unemployment-inflation one which are both related by eliminating the common input "taxes". The new model is driven by the investment and government expenditure which consists of two additive components, namely, an exogenous one to be designed for some specific purpose and a component being linearly dependent on the current level of unemployment. The free-design exogenous expenditure component is designed in particular to achieve the main objective in this paper, namely, the analytical study oriented to reach a predefined level of unemployment rate in a prescribed time. The overall government expenditure can be considered, if suited, as national income-dependent. The incorporation of more complexity in the models could be addressed by including the presence of unmodelled dynamics (Balasko, 1984; Di Benedetto *et al.*; De la Sen, 1993) with associate extra constraints in the production and consumption balances.

The paper is organized as follows. Section 2 presents both the national income and unemployment-expected inflation rate initial models which lead to a combined economic model which is obtained by relating the above two models through the taxes. It is assumed that the government expenditure is a combination of the two above mentioned additive components. The controllability, stability and equilibrium point are examined as being dependent on the proportional coefficient used in the government expenditure for the unemployment financial support. Some particular cases of interest are discussed in Section 3 concerning the statement of a constant expected inflation rate objective, and the use of phase state-variables (namely, the expected inflation time-derivative is proportional to the current unemployment level). In Section 4, the study of Sections 2 and 3 is extended to two more realistic situations, namely, (a) the overall government expenditure is proportional to the national income, and (b) the government expenditure is upper-bounded by a prefixed bound. Some numerical examples are given in Section 5 and, finally, conclusions end the paper.

2. The Rate of Unemployment Growth-Government Expenditure Differential Model

2.1. *Initial Models*

Consider the following two models:

$$(\mathrm{M1}):\ Y = C + I_0 + G_0, \tag{1a}$$

$$C = a + b(Y - T), \tag{1b}$$

$$T = d + t_T Y, \tag{1c}$$

which is a standard Keynes national income model. The exogenous inputs are I_0 (exogenously determined investment) and G_0 (government expenditure). The endogenous variables are Y (national income) and T (taxes), $a > 0$, $d > 0$, $0 < b < 1$, $0 < t_T < 1$ are fixed parameters related to the static situation and the rates of growth of the associated variables

$$
\begin{aligned}
\text{(M2)}: \quad & p = w - lT, && \text{(2a)}\\
& w = \alpha - \beta U + h\pi, && \text{(2b)}\\
& \frac{d\pi}{dt} = j(p - \pi), && \text{(2c)}\\
& \frac{dU}{dt} = k(p - m), && \text{(2d)}
\end{aligned}
$$

which is an unemployment-expected inflation model, where U is the level of the unemployment rate, p and π are the rates of inflation and expected inflation and w and m are the rates of growth of money wage and balance, respectively and $j \leqslant 1$, $h \leqslant 1$, k, β, l and α are positive constants. The l–constant regulates how the taxes increase makes to decrease the inflation rate. An empirical interpretation of (2.2) is that inflation is reduced by taxes because an increase in taxes can result in limiting prices since the economical possibilities of consumption of the population become reduced. Eq. 2.a could be corrected by using the rates of taxes but, in practice, the policies of taxes are not modified by governments during large periods of time so that the level of taxes assumed to be constant influences negatively in the inflation during short periods of time on which the taxes level can be assumed to be constant. Also, Eq. 2.b can be corrected by replacing U with $(U - U_N)$ with U_N being the natural rate of unemployment. Then, if the expectations are correct $p = \pi$ and $U = U_N$ (i.e., the current unemployment rate equals the given natural rate). It is assumed with no loos in generality that $U_N = 0$. Otherwise, all the subsequent results could be reformulated with the change of variable $U \to (U - U_N)$. Eqs. (2.a)–(2.b) are known as the expectation-augmented Phillips relations while Eqs. (2.c)–(2.d) are the adaptive expectations and growth of unemployment rule, respectively (Chiang, 1984). Eliminating T in (M1), one gets directly:

$$
\begin{aligned}
& T = a' + b'(I_0 + G_0);\\
& a' = \frac{1}{l}\left[\frac{(1-b)d + t_T a}{1 + (t_T - 1)b}\right]; \quad b' = \frac{1}{l}\left[\frac{t_T}{1 + (t_T - 1)b}\right]. \qquad (3)
\end{aligned}
$$

which reflects the fact that the taxes are endogenous variables dependent on the investment and government expenditure. *The following assumption is now introduced and then used or assumed to be violated in the some of the various obtained results. Also, related comments are given on its interpretation. It reflects the logic idea that the unemployment financial support by the government should be proportional to its current rate of growth.* However, the irrelevance of such an Assumption will be then seen since the government expenditure for this purpose over a predefined time period could be alternatively concentrated at a fixed decision time without alterating the final results of achieving a prefixed level of unemployment rate. This can be performed by using the exogenous component of the government expenditure to support the unemployment.

ASSUMPTION 1. (The attractive assumption but, in practive, irrelevant.) The government expenditure is a linear function of the unemployment growth rate having a free-design exogenous component, i. e., $G_0 = G_{0s} + \mu U(\mu \succ 0)$, in order to provide the economy of the country with unemployment-dependent financial support.

Since the growth rate of unemployment is normally neglectable related to the unemployment level, we can consider that the constant term G_{0s} includes an additive term which is proportional to that level and which is used over large periods, compared to the time in which an significant change of unemployment level is registered, to maintain the residual unemployment level in the case when $U = 0$, i. e., the unemployment stops growing. Under such an hypothesis, the term proportional to the unemployment level could be approximated by using the initial conditions for such a level during the observation time. The use of such an approach would not modify the theoretical results we obtain in the following developments. A nice discussion about when keeping or violating Assumption 1 will be later discussed in Corollary 2.2 and it will arrive together with the arguments in Remarks 2 to the practical irrelevance of Assumption 1 since the associated gradual support for unemployment can be equivalently substituted by a punctual but sufficient exogenous government expenditure, i.e., $G = G'_{0s}$ (with $G_{0s} \neq G_{0s}$) without modifying the expected results. *The questions to be now addressed in the rest of the paper are how long will the economy need to get to the equilibrium state and whether policy is needed to get there faster or to alter the initial conditions and hence determine what the natural rate is actually going to be.* Using Assumption 1 into (3) and introducing the obtained expression for T into (2.a)–(2.b) leads directly to the next two identities :

$$T = a' + b'(I_0 + G_{0s} + \mu U), \tag{4}$$

$$p = \alpha - a'' - b'G_{0s} - \beta' U + h\pi, \tag{5}$$

where

$$\begin{aligned} a'' &= a' + b' = \frac{(1-b)d + (1+a)t_T}{1+(t_T-1)b}, \\ \beta'' &= \beta' + b'\mu = \frac{(1-b)\beta + (b\beta + \mu)t_T}{1+(t_T-1)b}. \end{aligned} \tag{6}$$

Since $b, t \in (0, 1)$, then $1 + (t_T - 1)b > 0$ so that $\beta' > \beta > 0$. The substitution of (5) into (2.c) and (2.d) yields directly the next combined model from (M1) and (M2):

2.2. *Resulting Unemployment-Expected Inflation Model*

$$\text{(M3):} \quad \frac{d\pi}{dt} = j[\alpha - a''I_0 - b'G_{0s} - \beta' U + (h-1)\pi], \tag{7a}$$

$$\frac{dU}{dt} = k[\alpha - a''I_0 - b'G_{0s} - \beta' U + h\pi - m], \tag{7b}$$

which is driven by the static exogenous component of the government expenditure, the rate of growth of money balance and the investment.

2.3. *Controllability and Stability in the* (π, U) *– Plane of (M3)*

Eq. 7 can be rewritten more compactly as

$$\dot{x}(t) = Ax(t) + Bu + w = Ax(t) + B_e u_e, \tag{8}$$

where $x = (\pi, U)^T$, $u = (I_0, G_{0s}, m)^T$ and $w = (j, k)^T$ are the state vector, exogenous constant input and a constant deterministic disturbance vector with $u_e = (u^T, \alpha)^T$ being an extended input driven additionally by the α – constant, and

$$A = \begin{bmatrix} j(h-1) & -j\beta' \\ kh & -k\beta' \end{bmatrix}, \quad B = \begin{bmatrix} ja'' & -jb' & 0 \\ -ka'' & -kb' & -k \end{bmatrix}, \tag{9}$$

and $B_e = (B, (j,k)^T \alpha)$. Since $\text{rank}[B, AB] = \text{rank}[B] = \text{rank} \begin{bmatrix} jb' & 0 \\ -kb' & -k \end{bmatrix} = 2$ since j, b' and k are nonzero, the dynamic system is controllable and it can be then driven in finite time to any equilibrium by synthesizing the appropriate input. The unforced system (i.e., that arising for $u_e = 0$) is always stable since the characteristic equation $\text{Det}(sI - A) = s^2 + [(1-h)j + k\beta']s + jk\beta' = 0$ has stable roots

$$s_{1,2} = (1/2)[-(k\beta' + j(1-h)) \pm (k^2\beta'^2 + j^2(1-h)^2 - 2k\beta' j(1+h))^{\frac{1}{2}}]. \tag{10}$$

From (10), the next result holds under Assumption 1 or in the absence of Assumption 1.

PROPOSITION 1. Let $\beta'_{1,2} = (j/k)[1 + h \pm 2h^{1/2}] \succ 0$, since $h \in (0,1)$. Thus, the equilibrium is

(a) a stable node for $\beta' \in (0, \beta'_1] \cap [\beta'_2, \infty)$,

(b) a stable focus for $\beta' \in (\beta'_1, \beta'_2)$,

(c) a center for $k = 3$, $\beta' = j(h-1)/3 < 0 \Rightarrow \mu = \frac{1}{b}\left[\frac{j(h-1)}{3} - \beta\right] < 0$ from (6b) [Assumption 1 is violated].

Under Proposition 1(a)–(b), the system is asymptotically stable. Assume now the next technical assumption:

ASSUMPTION 2. The monetary policy is fixed a priori so that $m = m_0$ and $w = w_0$ in (8).

Under Assumption 2, the solution of (8) can be uniquely calculated as

$$x(t) = \psi(t)x_0 + \left(\int_0^t \psi(t-\tau)d\tau\right)(B_0 u_0 + w_0), \quad \text{all } t \geqslant 0, \tag{11}$$

where $u_0 = (I_0, G_{0s})^T$, $w_0 = [j\alpha, k(\alpha - m)]^T$ and for the case that $\beta' \in (0, \beta_1'] \cap [\beta_2', \infty)$ since the equilibrium point is a node,

$$\psi(t) = \begin{bmatrix} A_1 e^{s_1 t} + A_3^{s_2 t} & A_2 e^{s_1 t} + A_4^{s_2 t} \\ B_1 e^{s_1 t} + B_3^{s_2 t} & B_2 e^{s_1 t} + B_4^{s_2 t} \end{bmatrix}, \quad B_0 = \begin{bmatrix} -j_a'' & jb' \\ -ka'' & -kb' \end{bmatrix}, \tag{12}$$

where the constants are given by

$$A_1 = \frac{1}{2} - A_1'; \quad A_3 = \frac{1}{2} + A_1', \tag{13a}$$

$$A_2 = -A_4 = \frac{j\beta'}{K^{1/2}}, \tag{13b}$$

$$A_1' = \frac{1}{K^{1/2}}\left[\frac{3}{2}k\beta' + \frac{1}{2}j(1-h)\right], \tag{13c}$$

$$B_2 = \frac{1}{2} - B_2'; \quad B_4 = \frac{1}{2} + B_2', \tag{13d}$$

$$B_3 = -B_1 = \frac{kh}{K^{1/2}}, \tag{13e}$$

$$B_2' = \frac{1}{K^{1/2}}\left[\frac{1}{2}k\beta' + \frac{3}{2}j(1-h)\right], \tag{13f}$$

where $K = k^2\beta'^2 + j^2(1-h)^2 - 2k\beta' j(1+h)$.

The case of a node is now studied in detail. The constants $A_{(\cdot)}$ and $B_{(\cdot)}$ have to be recalculated for the focus case. In that situation, the alternative calculations would be direct although more involved. Assume now that the point to be reached in finite time is $\widehat{x}_e = (\widehat{\pi}_e, \widehat{U}_e)^T$ by generating some input u_0. This is possible since the system is controllable. From (12.b), B_0 is nonsingular with $\mathrm{Det}(B_0) = 2ja''kb' \neq 0$ and $\int_0^t \psi(t-\tau)\,d\tau$ is nonsingular for almost all finite t since $\psi(t)$ is a fundamental matrix of (8). Thus,

$$u_0 = \left[\left(\int_0^T \psi(T_0 - \tau)\,d\tau\right) B_0\right]^{-1} (\widehat{x}_e - \psi(T_0)x_0) - B_0^{-1} w_0 \tag{14}$$

can be used to reach $\widehat{x}_e$ in time T_0 from any initial condition $x_0 = (p_0, U_0)^T$ provided that $\int_0^{T_0} \psi(T_0 - \tau)\,d\tau = (\phi_{ij}(T_0))$ is nonsingular, where $\psi(T_0)$ is given by (12.a) subject to (13) and B_0 by (12.b). To calculate $\phi_{ij}(T_0)$, $i, j = 1, 2$, first define auxiliary constants $\delta_1, \delta_2 = \delta_1^q$, q being a positive real constant to be fixed with $\delta_1 \in (0, 1)$ such that

$$e^{s_1 T} = \delta_1, \quad e^{s_1 T} = \delta_2 \tag{15a}$$

so that $T_0 = |\ln \delta_1|/s_1 = q|\ln \delta_1|/s_2$. Denote $k\beta' + j(1-h) = R$, then

$$q = \frac{s_2}{s_1} = \frac{R - K^{1/2}}{R + K^{1/2}}; \quad R = k\beta' + j(1-h) \tag{15b}$$

is positive. Thus, (14) is evaluated after calculating $(\psi_{ij}(T))$ and $(\phi_{ij}(T))$ for T_0 fulfilling (15) for a given δ_1 as follows

Finite – time final point algorithm

Step 1. Compute

$$T_0 = \frac{|2\ln\delta_1|}{|s_1|} = \frac{|\ln\delta_1|}{R - K^{1/2}}$$

for arbitrary $\delta_1 > 0$ and q from (16), which is a finite lead-time which exists from the rank conditions associated with controllability in Eq. 9.

Step 2. Calculate the real constants $A_{(.)}$ and $B_{(.)}$ from (13).

Step 3. By using the results of Steps 1–2 with $\delta_1 = e^{s_1 T_0}$, $\delta_2 = \delta_1^q$, calculate $(\psi_{ij}(T_0))$ leading after straight calculations to

$$\psi_{11}(T_0) = \frac{1}{2}\delta_1(1+\delta_1) + \frac{A_2}{j\beta'}\left[\frac{3}{2}k\beta' + \frac{1}{2}j(1-h)\right](\delta_1^{q-1} - 1), \tag{16a}$$

$$\psi_{12}(T_0) = \frac{1}{2}\delta_1(1-\delta_1^{q-1}), \quad \psi_{21}(T_0) = -B_3(1-\delta_1^{q-1}), \tag{16b}$$

$$\psi_{22}(T_0) = \frac{1}{2}\delta_1(1+\delta_1^{q-1}) + \frac{\psi_{21}(T_0)}{kh}\left[\frac{1}{2}k\beta' + \frac{3}{2}j(1-h)\right] \times (\delta_1^{q-1} - 1). \tag{16c}$$

Calculate also $\phi_{ij}(T_0) = \int_0^{T_0} \psi_{ij}(T_0 - \tau)\,d\tau$ by using the results of (16) leading to

$$\phi_{11}(T^0) = \frac{(\delta_1 - 1)T}{\delta_1 \ln(\delta_1)} + \left(\frac{1}{2} + \frac{A_2}{j\beta'}\left[\frac{3}{2}k\beta' + \frac{1}{2}j(1-h)\right]\right) \tag{17a}$$

$$\phi_{12}(T_0) = \gamma'_{22}A_2;$$
$$\phi_{21}(T_0) = -\gamma'_{21}B_3; \; \phi_{22}(T_0) = -\alpha'_{22} + \gamma'_{22}B_3, \tag{17b}$$

where the next auxiliary constants are used

$$\gamma'_{12} = \frac{1}{jk\beta'(1+h)}2\sqrt{k^2\beta'^2 + j^2(1-h)^2} - 2k\beta' j(1+h) - \delta_1[R + \delta_1^{q-1}\{R + \sqrt{K}\}], \tag{18a}$$

$$\gamma'_{21} = \frac{1}{4jk\beta'}\sqrt{K} + \delta_1\{\delta_1^{q-1}[R - \sqrt{K}] - [R + \sqrt{K}]\}, \tag{18b}$$

$$\gamma'_{22} = -\frac{1}{kh}\left(\frac{1}{2}k\beta' + \frac{3}{2}j(1-h)\right)\gamma''_{22}, \tag{18c}$$

$$\alpha'_{22} = \frac{1}{4jk\beta'}\left[R - \frac{1}{2}\left\{R + \sqrt{K} - [R + \sqrt{K}\delta_1^{q-1}]\delta_1\right\}\right] + \frac{1-\delta_1^q}{2}[R - \sqrt{K}], \tag{18d}$$

$$\gamma'_{22} = \frac{1}{4jk\beta'}\left[R - \frac{1}{2}\{R + \sqrt{K} - [R + \sqrt{K}\delta_1^{q-1}]\delta_1\}\right]. \tag{18e}$$

2.4. *Main Results of Section 2*

The first point of interest is searching for the equilibrium points x_e so that $\dot{x}_e = 0$. Thus, from (8), subject to (9),

$$x_e = (\pi_e, U_e)^T = -A^{-1}B_e u_e. \tag{19}$$

The next result stands directly from (19) as being a demonstration of the natural rate idea for the asymptotic achievement of $U_e = \dot{U}_e = 0$ since 1+ $1 + (t_T - 1)b > 0$ since bt and b are smaller than unity.

Theorem 1. *The zero-level growth of unemployment is asymptotically achieved under Assumptions* $1-2$, *irrespective of the choice of* b' *provided that*

$$d > \max\left(0, \frac{k-a-1}{t_T}\right); \quad m > \frac{\alpha}{1-h} \quad \textit{and}$$

$$\text{(a)} \quad G_{0s} = \frac{(1-b)d + (1+a-k)t_T}{1+(t_T-1)b},$$

$$I_0 \geqslant \frac{1+(t_T-1)b}{(1-b)d+(1+a-k)t_T}[\alpha + (h-1)m], \quad \textit{or}$$

$$\text{(b)} \quad I_0 = \frac{1+(t_T-1)b}{(1-b)d+(1+a-k)t_T}[G_{0s} + \alpha + (h-1)m]$$

with $G_{0s} \geqslant \max(0, (1-h)m - \alpha)$ if $k > a+1$.

The above result addresses the problem of reaching asymptotically (i.e., as $t \to \infty$) the equilibrium. The use of (14) is related to reach a given objective point $\widehat{x}_e$, not necessarily being equal to x_e, in the state trajectory in finite time (i.e., the minimum lead-time in controllable systems, see, for instance Kailath (1989)), from admissible (i.e., nonnegative) initial conditions U_0, π_0. The next result follows from (4) and it is related to the system's capability of reaching a prefixed level of unemployment $\widehat{U}_e$ in finite time what is intuitively obvious from the system's controllability.

Theorem 2. *Under Assumption* 2, *assume arbitrary initial conditions* $x_0 = (\pi_0, U_0)^T$. *Thus, a given final level of unemployment level* $\widehat{U}_e$ *is reached intime* T_0, *computed accordingly to Step* 1 *of the Algorithm, provided that the government expenditure for any given (positive, i.e., Assumption* 1 *holds, or nonpositive, i. e., Assumption* 1 *is violated)* μ *is*

$$\begin{aligned} G_{0s} = {} & \frac{1}{a''[j\phi_{21} + k\phi_{22}]}(\psi_{21}\pi_0 + \psi_{22}U_0 - \widehat{U}_e + b'(j\phi_{21} - k\phi_{22})I_0 \\ & + (j\phi_{21} - k\phi_{22})\alpha + k\phi_{22}m), \end{aligned} \tag{20}$$

subject to (ψ_{ij}) *and* (ϕ_{ij}) *been given by* (16)–(18) *provided that*

$$I_0 > \max\left(0, \frac{\widehat{U}_e - \psi_{21}\pi_0 - \psi_{22}U_0}{b'(j\phi_{21} - k\phi_{22})}\right) \quad \text{if} \quad j \neq k\phi_{22}/\phi_{21},$$
$$a''[j\phi_{21} + k\phi_{22}] > 0, \tag{21}$$
$$0 < I_0 < \frac{\widehat{U}_e - \psi_{21}\pi_0 - \psi_{22}U_0}{b'(j\phi_{21} - k\phi_{22})} \quad \text{if} \quad a''[j\phi_{21} + k\phi_{22}] < 0, \tag{22}$$

with $\widehat{U}_e = C(\widehat{U}_e)b'[j\phi_{21} - k\phi_{22}] + \psi_{21}\pi_0 + \psi_{22}U_0$ *for some final unemployment level-dependent) positive real constant* C.

Proof. For $\widehat{\pi}_e$ being arbitrary but $C = \frac{\widehat{U}_e - \psi_{21}\pi_0 - \psi_{22}U_0}{b'(j\phi_{21} - k\phi_{22})} > 0$ if $a''[j\phi_{21} + k\phi_{22}] < 0$, G_{0s} computed from (20) leads to $X_2(T_0) = \widehat{U}_e$ in (11) with T_0 chosen from Step 1 of the algorithm. $G_{0s} > 0$ is guaranteed under (21), if $a''[j\phi_{21} + k\phi_{22}] > 0$, or under (22) otherwise.

Some issues about stability and controllability addressed in Theorems 1 – 2 are discussed in the next remarks relating the current problem at hand to the general theory of dynamic systems.

REMARK 1. Note that while Theorem 1 applies for the asymptotic achievement of the equilibrium, Theorem 2 is related to the achievement of a given final state in finite time based on the controllability property, which is a well-known result in classical theory of dynamic systems (Sen, 1993; Kailath, 1989).

Such a property appears in this paper related to the behavior of two coupled economic models. Note that, in the general context, controllability, which implies stabilizability through feedback of an unstable system but not current stability (since a controllable system can be unstable for the current control policy) is a stronger property than stability since under controllability, the equilibrium can be reached after a minimum finite lead-time. However, the converse is not true since a system being currently stable or feedback-stabilizable is not necessarily controllable in the sense that not always the equilibrium can be achieved in a minimum finite lead-time. Mathematically, a system is controllable iff its closed-loop modes can be freely assigned through feedback while it is stabilizable iff there is a feedback control law which places the closed-loop modes anywhere within the stable complex plane region, Kailath (1989). In our current problem, both properties hold simultaneously. Thus, the system will tend to an equilibrium point unforced, i. e., without policy although the particular equilibrium coordinates will depend, in general, of the forcing terms established by the policy what is reflected by Eq. 19 linked with Eq. 8. Theorem 2 can be reformulated for a prefixed expected inflation on a given unemployment growth level or for the achievement in time T of a prefixed state $(\widehat{\pi}_e, \widehat{U}_e)^T$ by simultaneous calculation of G_{0s} and I_0. Note also that for a designed G_{0s}, the necessary values of β' should be computed numerically. Note that Assumption 1 is not required by Theorem 2 but, if $\beta' > \beta$ then $\mu > 0$ and then Assumption 1 holds. Otherwise, Assumption 1 would be violated without modifying the results, except in the fact that G_{0s} depends on μ, through (6), (15) and (16) since β and then ψ_{ij} depend on μ, as it is now more clearly discussed in the particular cases dealt with in the next section.

3. Particular Cases

Two particular tractable cases are now examined, namely :

Case 1: $j = 0$. Then $\pi(T_0) = \pi_0 = \widehat{\pi}_e(T_0)$ from (2.c) for all $T_0 \geqslant 0$. Also, from (9),

$$A = \begin{bmatrix} 0 & 0 \\ kh & -k\beta' \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 \\ ka'' & -kb' \end{bmatrix}, \quad w_0 = \begin{bmatrix} 0 \\ k(\alpha - m) \end{bmatrix}.$$

From (10), the eigenvalues are $s_1 = \frac{k\beta'}{2}$ and, $s_1 = \frac{-3k\beta'}{2}$ from (13) and (16)–(18),

$$B_3 = \frac{h}{\beta'}, \quad \phi_{21} = \frac{h(\delta_1^{q-1} - 1)}{\beta'}, \quad \psi_{22} = \frac{1}{2}\delta_1^{q-1}[\delta_1 + 1] - \frac{1}{2}.$$

From (13)

$$B_1 = \frac{h'}{\beta'}, \quad B_2' = \frac{1}{2},$$

and, from (17)–(18),

$$\phi_{22}(T_0) = \frac{2}{3k\beta'}(1 - \delta_1^q), \quad j\phi_{21}(T_0) = \frac{2jh}{k\beta'^2}\left(\frac{2}{3} + \frac{1}{3}\delta_1^q - \delta_1\right) = 0.$$

Applying now Theorem 2, one gets directly

$$G_{0s} = \bar{G}'_{0s} + \beta'\widetilde{G}'_{0s} = \bar{G}'_{0s} + \left[\beta + \mu\frac{t_T}{1 + (t_T - 1)b}\right]\widetilde{G}'_{0s} = \bar{G}_{0s} + \mu\widetilde{G}_{0s}, \quad (23)$$

where

$$\begin{aligned} \bar{G}_{0s} &= \frac{3}{2}h(\delta_1^{q-1} - 1)\frac{[1 + (t_T - 1)b]\pi_0}{2[(1 - b)d + (1 + a)t_T](1 - \delta_1^q)} \\ &\quad + \frac{[1 + (t_T - 1)b](m - \alpha - b'I_0)}{(1 - b)d + (1 + a)t_T}, \qquad (24a) \\ \widetilde{G}'_{0s} &= \frac{3}{2}\frac{1 + (t_T - 1)b}{[(1 - b)d + (1 + a)t_T](1 - \delta_1^q)} \\ &\quad \times \left\{\frac{1}{2}[(\delta_1 + 1)\delta_1^{q-1} - 1]U_0 - \widehat{U}_e\right\}, \qquad (24b) \end{aligned}$$

$$\bar{G}_{0s} = \bar{G}_{0s} + \beta\widetilde{G}'_{0s}; \quad \widetilde{G}_{0s} = \frac{t\widetilde{G}'_{0s}}{1 = (t_T - 1)b}$$
$$= \frac{3}{2}\frac{t_T}{[(1-b)d + (1+a)t_T](1-\delta_1^q)}\left\{\frac{1}{2}[(\delta_1 + 1)\right.$$
$$\left.\times\, \delta_1^{q-1} - 1]U_0 - \widehat{U}_e\right\}. \tag{24c}$$

The next result follows immediately from the above calculations :

COROLLARY 1. Assume $j = 0$ so that $\pi(t) = \widehat{\pi}_e = \pi_0$, all $t \geqslant 0$. Thus, for any given pair (G_{0s}, δ_1) of positive scalars and any initial conditions (π_0, U_0), it is possible to lead he unemployment growth rate to a prefixed value $\widehat{U}_e \geqslant 0$ if μ is chosen as $\mu = \widetilde{G}_{0s}^{-1}(G_{0s} - \bar{G}_{0s})$, subject to the definitions of (24.c) of $\bar{G}_{0s}$, perhaps violating Assumption 1 (i.e., $\mu < 0$). In the same way, if μ is prefixed then $G_{0s} = \bar{G}_{0s} + \mu\widetilde{G}_{0s}$.

The interpretation of Corollary 1 is that the final value of $\widehat{U}_e$, which is arbitrary, subject to admissibility only but which can be, in particular, fixed to the equilibrium U_e, can be reached in arbitrary finite time with an arbitrary prefixed exogenous government expenditure provided that μ is chosen as in Corollary 1 for taxes and, according to (4),

$$\beta' = \beta + \mu b' = \left[1 + (t_T - 1)b\right]^{-1}\left\{(1-b)\beta + (\mu + b\beta)t_T\right\}, \tag{25}$$

for the current inflation relationship in the combined model (M3) for the given β in (M2). If μ is negative (i.e., Assumption 1 is violated) then the taxes have to be decreased as the unemployment growths in order to reach the expected unemployment growth level with a given prefixed exogenous government expenditure.

Case 2: $h = 1$ so that

$$A = -j\beta'\begin{bmatrix} 0 & 1 \\ \frac{-kh}{j\beta'} & \frac{k}{j} \end{bmatrix}, \quad B_0 = \begin{bmatrix} -ja'' & -jb'' \\ -ka'' & -kb' \end{bmatrix},$$
$$w_0 = \begin{bmatrix} j\alpha \\ k(\alpha - m) \end{bmatrix}. \tag{26}$$

The negative real eigenvalues, provided that $\beta' > 0$ under Assumption 1, are $s_{1,2} = \frac{1}{2}\left[-k\beta' \pm (k^2\beta'^2 - 4\beta' j)^{1/2}\right]$ and $\delta_i = \exp(s_i T_0)$, $i = 1, 2$ with $\delta_2 = \delta_1^q$. It is well-known that $A = P^{-1}A_d P$ where $A_d = \mathrm{Diag}(s_1, s_2)$ and $P^{-1} = \begin{bmatrix} 1 & 1 \\ s_1 & s_2 \end{bmatrix}$ is a Vandermonde matrix since the state-variables are phase variables in this case; i. e., $\dot{\pi}(t) = -j\beta' U(t)$. Since the matrix transformation P that diagonalizes A is the same that diagonalizes its fundamental matrix $\psi(t) = \exp(At) = P^{-1}\exp(A_d t)P$, it follows by direct calculations using P^{-1} and P and s_i $(i = 1, 2)$ that

$$\psi_{21} = \frac{k\beta' - \left(k^2\beta'^2 - 4k\beta' j\right)^{1/2}}{2\left(k^2\beta'^2 - 4k\beta' j\right)^{1/2}} \delta_1 \left[1 - \delta_1^{q-1}\right],$$

$$\psi_{22} = \frac{\delta_1}{2\left(k^2\beta'^2 - 4k\beta' j\right)^{1/2}} \left[\left(k\beta' + (k^2\beta'^2 - 4k\beta' j)^{1/2}\right)(1 + \delta_1^{q-1})\right]. \tag{27}$$

Also, since $\int_0^{T_0} e^{st}\, dt = (e^{sT_0} - 1)/s$ for $s = s_1, s_2$ then $\phi_{2i}(T_0) = \int_0^{T_0} \psi_{2i}(T_0 - \tau)\, d\tau$ $(i = 1, 2)$ so that ϕ_{2i} have have the same expressions as ψ_{2i} by replacing $\delta_1 \to \delta_1'$, $\delta_2 = \delta_1^q \to \delta_2'$ where

$$\delta_1' = \frac{(\delta_1 - 1)}{s_1} = \frac{2(1 - \delta_1)}{k\beta' - (k^2\beta'^2 - 4k\beta' j)^{1/2}},$$

$$\delta_2' = \frac{(\delta_1^q - 1)}{s_2} = \frac{2(\delta_1^q - 1)}{k\beta' - (k^2\beta'^2 - 4k\beta' j)^{1/2}}.$$

To easy the subsequent developments, define the auxiliary variables

$$F_1 = k\beta', \quad F_2 = (k^2\beta'^2 - 4k\beta' j)^{1/2}, \tag{28}$$

so that Eq. 27 and the corresponding ones for ϕ_{2i} $(i = 1, 2)$ become

$$\psi_{21} = \frac{F_1 - F_2}{2F_2}[\delta_1 - \delta_2],$$

$$\psi_{22} = \frac{F_1}{2F_2}[\delta_2 - \delta_1] + \frac{1}{2}[\delta_1 + \delta_2], \tag{29}$$

with

$$\phi_{21} = \frac{F_1 - F_2}{2F_2}[\delta_1' - \delta_2'] = \frac{1}{F_2}[2 - \delta_1 - \delta_2], \tag{30a}$$

$$\phi_{22} = \frac{F_1}{2F_2}[\delta_2' - \delta_1'] + \frac{1}{2}[\delta_1' + \delta_2']$$

$$= \frac{1}{F_1 - F_2}\left\{\frac{F_1}{F_2}[\delta_1 + \delta_2 - 2] + (\delta_2 - \delta_1)\right\}, \tag{30b}$$

so that G_{0s} in Theorem 2 can be rewritten from (28)–(30) and the definition of a'' and b' as

$$G_{0s} = \frac{[1 + (t_T - 1)b]}{\{(1 - b)d + (1 + a)t_T\}}$$

$$
\begin{aligned}
&\times \frac{1}{[j(F_1 - F_2)(2 - \delta_1 - \delta_2) + k\{F_1(\delta_1 + \delta_2 - 2)F_2(\delta_2 - \delta_1)\}]} \\
&\times \left[\frac{(F_1 - F_2)^2}{2}(\delta_1 - \delta_2)\pi_0 + \Big(\frac{F_1}{2}(F_1 - F_2)(\delta_2 - \delta_1)\right. \\
&+ \frac{F_2}{2}(F_1 - F_2)(\delta_1 + \delta_2)\Big)U_0 - F_2(F_1 - F_2)\widehat{U}_e \\
&+ \Big\{j(F_1 - F_2)(2 - \delta_1 - \delta_2) - kF_1(\delta_1 + \delta_2 - 2) \\
&- kF_2(\delta_2 - \delta_1)\Big\}\left[\frac{tI_0}{1 + (t_T - 1)b} + \alpha\right] \\
&\left. + k\{F_1(\delta_1 + \delta_2 - 2) + F_1(\delta_2 - \delta_1)\}m\right]. \qquad (31)
\end{aligned}
$$

To solve (31) with less independent variables, define the auxiliary variables μ_i $(i = 1, 2, 3)$ from the next relationships:

$$
F_1 - F_2 = \mu_1 F_1, \quad F_1 + F_2 = \mu_2 F_1, \quad F_2 = \mu_3 F_1, \qquad (32)
$$

which lead to

$$
F_1 = \frac{F_2}{1 - \mu_1} = -\frac{F_2}{1 - \mu_2} = \frac{F_2}{\mu_3} \Rightarrow \mu_2 = 2 - \mu_1, \quad \mu_3 = 1 - \mu_1. \qquad (33)
$$

To simplify the subsequent calculations, we introduce the constant $\mu_1 = F_1 = kb'$ without loss of generality since k and β' are of free design. Thus, the relationships $\mu_1 = \frac{F_1 - F_2}{F_1} = F_1 = k\beta'$ from (33) with F_2 satisfying (32) lead to an equivalent expression for (18) as follows:

$$
\begin{aligned}
G_{0s} = &\frac{[1 + (t_T - 1)b]}{(1 - b)d + (1 + a)t_T} \\
&\times \frac{1}{j\mu_1(2 - \delta_1 - \delta_2) + k\{(\delta_1 + \delta_2 - 2) + (1 - \mu_1)(\delta_2 - \delta_1)\}} \\
&\times \left[\frac{\mu_1}{2}(\delta_1 - \delta_2)\pi_0 + \frac{\mu_1^2}{2}(\delta_2 - \delta_1)U_0 + \Big\{j\mu_1(2 - \delta_1 - \delta_2)\right. \\
&- k(\delta_1 + \delta_2 - 2) + k(\mu_1 - 1)(\delta_2 - \delta_1)\Big\}\left[\frac{t_T I_0}{1 + (t_T - 1)b} + \alpha\right] \\
&+ k\{(\delta_1 + \delta_2 - 2) + (1 - \mu_1)(\delta_2 - \delta_1)\}m \\
&\left. + \mu_1^2(1 - \mu_1)\left[\frac{\delta_1 + \delta_1^q}{2}U_0 - \widehat{U}_e\right]\right]. \qquad (34)
\end{aligned}
$$

with $q = \frac{s_2}{s_1} = \frac{F_1 - F_2}{F_1 + F_2} = \frac{\mu F_1}{\mu_1 + (1-\mu_1)F_1} = \frac{\mu_1}{2-\mu_1} \leqslant 1 \Rightarrow \mu_1 \leqslant \frac{2}{3}$ from (16). Direct calculation with (34) and $\delta_2 = \delta_1^q$ yields

$$\begin{aligned} G_{0s1} &= G_{0s}(\mu_1 = 0) \\ &= \frac{[1 + (t_T - 1)b]}{(1-b)d + (1+a)t_T} \left[(m - \alpha)\left(1 + (t_T - 1)b\right) - t_T I_0\right], \end{aligned} \tag{35a}$$

$$\begin{aligned} G_{0s2} &= G_{0s}\left(\mu_1 = \frac{2}{3}\right) \\ &= \frac{[1 + (t_T - 1)b]}{\{(1-b)d + (1+a)t_T\}\left[\frac{2}{3}\delta_1(k-j) + \frac{2}{3}\delta_2(2k-j) + \frac{4}{3}j - 2k\right]} \\ &\quad \times \left\{ \delta_1 \left(\frac{1}{3}\pi_0 + \frac{2}{9}\left(\delta_1^{q-1} - \frac{2}{3}\right) U_0 + \frac{2}{3}km \right. \right. \\ &\quad \left. - \frac{2}{3}(j+k)\left[\frac{t_T I_0}{1 + (t_T - 1)b} + \alpha\right] \right) \\ &\quad + \delta_1^q \left(\frac{2}{9}U_0 - \frac{1}{3}\pi_0 + \frac{4}{3}km - \left(\frac{2}{3}j + \frac{4}{3}k\right)\left[\frac{t_T I_0}{1 + (t_T - 1)b} + \alpha\right] \right) \\ &\quad \left. + 2\left(\frac{2}{3}j + 2k\right)\left[\frac{t_T I_0}{1 + (t_T - 1)b} + \alpha\right] - 2km - \frac{4}{27}\widehat{U}_e \right\} \\ &= \frac{[1 + (t_T - 1)b]}{\{(1-b)d + (1+a)t_T\}\left[\frac{2}{3}\delta_1(k-j) + \frac{2}{3}\delta_2(2k-j) + \frac{4}{3}j - 2k\right]} \\ &\quad \times \left[\frac{t_T I_0}{1 + (t_T - 1)b} + \alpha\right] \\ &\quad \times \left[\frac{2j}{3}(2 - \delta_1 - \delta_1^q) + 2k\left(2 - \frac{1}{3}\delta_1 - \frac{2}{3}\delta_1^q\right)\right] \\ &\quad + \frac{1}{3}\delta_1\left(1 - \delta_1^q\right)\pi_0 + \frac{4}{9}\left(\delta_1^{q-1} - \frac{1}{3}\right)\delta_1 U_0 \\ &\quad + \frac{2}{3}km\delta_1\left(1 + 2\delta_1^{q-1}\right) - 2km - \frac{4}{27}\widehat{U}_e, \end{aligned} \tag{35b}$$

which is positive provided that

$$I_0 = \frac{1 + (t_T - 1)b}{t_T}\left[\frac{1}{\frac{2j}{3}(2 - \delta_1 - \delta_1^q) + 2k\left(2 - \frac{1}{3}\delta_1 - \frac{2}{3}\delta_1^q\right)}\right.$$

$$\times\left\{ \left(\frac{2}{3}\delta_1(k-j)+\frac{2}{3}\delta_2(2k-j)+\frac{4}{3}j-2k\right)\rho \right.$$

$$-\frac{1}{3}\delta_1(1-\delta_1^q)\pi_0+\frac{4}{9}\left(\delta_1^{q-1}-\frac{1}{3}\right)\delta_1 U_0$$

$$\left.\left. +\frac{2}{3}km\delta_1(1+2\delta_1^{q-1})-2km-\frac{4}{27}\widehat{U}_e\right\}-\alpha\right], \quad \text{any } \rho>0. \tag{36}$$

Thus, the following result follows:

COROLLARY 2. Assume Case 2 (i.e., $h=1$) with $\mu_1=2/3=F_1$ and arbitrary positive δ_1 and that (32)–(33) hold. Thus, Theorem 2 holds with positive exogenous government expenditure component given by (35.b) provided that the investment is given by (36) for any given admissible triple $(U_0,\pi_0,\widehat{U}_e)$ of nonnegative scalars. The given growth of unemployment equilibrium point $\widehat{U}_e$ is reached in time $T_0=\frac{|\ln\delta_1|}{k\beta'-(k^2\beta'^2-4k\beta' j)^{1/2}}$ with $\beta'=\frac{\mu_1}{k}=\frac{2}{3k}$ and $\mu=\frac{1+(t_T-1)b}{t_T}[\frac{2}{3k}-\beta]$ which satisfies Assumption 1 if $\beta<\frac{2}{3k}$. If $\beta\geqslant\frac{2}{3k}$ then Assumption 1 is violated.

Proof. (outline). If (35)–(36) hold with any positive ρ then $G_{0s}=\frac{1+(t_T-1)b}{(1-b)d+(1+a)t_T}\rho$ and I_0 are both positive. The remaining of the relations follow from (3), (6) and (28).

Note that the choice of δ_1 is crucial to choose T and to design G_{0s} in a trade-off context.

REMARK 2. The established models have a nice interpretation if Assumption 1 either holds or it is violated, namely, for positive or nonpositive unemployment support depending on its evolution rate . Assume $\mu\succ 0$ (i. e., Assumption 1 holds). Thus $\beta'>\beta$ ($\beta<\frac{2}{3k}$), that is, the expected inflation and the unemployment level become decreased for a given growth of money wage (see, (M3) – Eq. 7 and 2b) at the expense of increasing the taxes since $\mu\succ 0$ (Eq. 4). Alternatively, the growth of money wage can be decreased by increasing β over $2/3k$ (i.e., the wages become moderately increased) with negative μ (i. e., Assumption 1 is violated). This implies that the taxes do not increase (i. e., they remain either fixed or decreasing) since the government expenditure are either fixed to a constant G_{0s} (if μ is zero) or decreasing (if μ is negative) implying, from (1a) and (1c), that the national income (and then the taxes) do not increase. In the case, that expense is time-decreasing, a positive lower-bound for the overall government expenditure $G_{\min}$ has to be fixed what implies that the μ-coefficient for unemployment support must satisfy the constraint $|\mu|\leqslant\frac{G_{0s}-G_{\min}}{U}$ in order to keep the constraint of positive government expenditure. At the same time, the rates of growths of unemployment and expected inflation grow with β' in (M3). In the first case, there is a gradual-in-time government expenditure in unemployment support and Assumption 1 is satisfied. However, in the second one, (i.e., Assumption 1 is violated), there is less gradual expenditure in unemployment support while a more important punctual government expenditure arises through the choice of a fixed expenditure G_{0s} sufficiently larger than $G_{\min}$. It becomes apparent from the above comments that the better government strategy to keep people happy is the first one since a taxes readjustment can be decided with some delay related to the wages policy. Note also that the time needed to reach the suited unemployment growth rate objective can be regulated through the choice of δ_1 and the various associated constants which modify the necessary government exogenous expenditure and investment. Note also the following immediate consequence of Corollary 2.

COROLLARY 3. If the constraints of Case 2 are satisfied then $\mu = \frac{(1-b)d+(1+a)t_T}{t_T}(\frac{2}{3k} - \beta)G_{0s}$ so that it is proportional to the required exogenous expenditure required to reach a given $\widehat{U}_e$ in time T given in Corollary 2 for both situations, i. e., $\beta < \frac{2}{3k}$ (i.e., Assumption 1 holds) or $\beta \geqslant \frac{2}{3k}$.

Proof. It follows directly from Corollary 2 and its proof.

4. Limited Government Expenditure

Two more realistic situations are now considered, namely:

(a) the government expenditure is proportional to the national income;

(b) the government expenditure is bounded by a prefixed bound.

Situation (a) is reduced to situation (b) after some calculations.

4.1. *Government Expenditure Being Proportional to the National Income*

In this case, $G_0 = gY$, some positive real constant g. From (1) and (3), one gets

$$Y = \frac{a - bT + I_0 + G_0}{1-b}, \tag{37a}$$

$$\begin{aligned} G_0 &= gY = g\frac{a - bT + I_0 + G_0}{1-b} \\ &= \frac{g}{1-b}(1-bb')G_0 + \frac{g}{1-b}(1-bb')I_0 + \frac{g}{1-b}(a-ba'), \end{aligned} \tag{37b}$$

and, recalculating G_0 from (37b),

$$G_0 = \left[\frac{g}{1-b+g(b+b'-1)}\right][(1-bb')I_0 + a - ba'], \tag{38}$$

which has to be positive for formulation coherency. This is guaranteed if, from (38) and (3)

$$\begin{aligned} I_0 &= \frac{\rho[1-b+g(b+b'-1)+a'b-a]}{1-bb'} \\ &= \frac{\rho[1-b+g(b+b'-1)][1+(t_T-1)b]+a(1-b)(d-1)}{(1-bb')[1+(t_T-1)b]} \end{aligned} \tag{39}$$

for any real positive ρ. The substitution of (39) into (38) with the definitions of a' and b' in (3) leads directly to a positive bounded government expenditure $G_0 = \rho g$.

4.2. *Upper-Bounded Government Expenditure*

In this case, there is $\overline{G}_0 > 0$ such that $G_0 < \overline{G}_0$. If the expenditure is proportional to the national income (Section 4.1) $\overline{G}_0 = \rho^* g \geqslant \rho g = G_0$ some known positive real constant ρ^*, all positive $\rho \leqslant \rho^*$ provided that I_0 satisfies (39). Thus, both situations can be described with a unified analysis in terms of boundedness of the government expenditure. From Assumption 1 $G_0 = G_{0s} + \mu U(t) \leqslant \overline{G}_0$ for all $t \geqslant 0$. Since $s_{1,2}$ have negative real parts for each positive real constant ε, there exist positive real constants $\delta = \delta(\varepsilon)$ and $T_0 = T_0(\varepsilon, \delta)$ such that $\widehat{U}_e \leqslant U(t) \leqslant \varepsilon$ all $t \geqslant T_0$ provided that $U_0 \leqslant \delta$. Thus, it suffices that $G_{0s} + \mu\varepsilon \leqslant \overline{G}_0 \Rightarrow \mu \leqslant \mu^* = \frac{\overline{G}_0 - G_{0s}}{\varepsilon}$. (In the case of Section 4.1, $\mu \leqslant \mu^* = \frac{\rho^* g - G_{0s}}{\varepsilon}$) with $T_0 = |\ln \delta_1|/|s_1|$ being chosen sufficiently large in the Finite-time Final-point Algorithm. All the topics dealt with in Sections 2–3 are reformulated according to this issue. Thus, note in particular:

(1) Case 1 of Section 3. In Corollary 1, Eq. 25, β' is checked with

$$\mu \leqslant \mu^* = \frac{(\overline{G}_0 - G_{0s})}{\varepsilon}$$

for bounded expenditure. If the expenditure is proportional to the income then

$$\mu \leqslant \mu^* = \frac{\rho^* g - G_{0s}}{\varepsilon}.$$

(2) Case 2 of Section 3. Corollary 2 holds with $\mu = \frac{1+(t_T-1)b}{t_T}[\frac{2}{3k} - \beta] \leqslant \mu^*$ and $\beta = \frac{2}{3k} - \frac{t_T(\overline{G}_0 - G_{0s}(\mu_1))}{\varepsilon[1+(t_T-1)b]}$; $\mu_1 \varepsilon (0, \frac{2}{3})$ leads to the necessary value in the wages policy of model (M2). In the case of expenditure being proportional to the income $\overline{G}_0 = g\rho$ with ρ being used in the investment Eq. 39.

5. Numerical Simulations

The following parametrization has been chosen for the model: $a = 0.5$, $b = 0.3$, $t_T = 0.1$, $l = 0.2$, $\alpha = 0.4$, $\beta = 0.2$, $j = 0.8$, $k = 1.51$, $h = 0.7$, $\mu = 0.6$ (i.e., the parameter indicating the contribution of unemployment level to the government expenditure in Assumption 1), $d = 0.11$. That choice results in the intermediate parameters $a' = 0.173973$, $b' = 0.684932$, $a'' = 0.858904$, $\beta' = 0.610959$. The eigenvalues of the matrix of dynamics of the system Eq. 8 become $-0.637123 \pm 0.649208i$. The controllability to the origin for extended controls being $(3.87584, 1.101317, 24.6321, 2.4576)^T$ and $(2.4576, 1.101317, 3.87584, 24.6321)^T$ and, both having identical Euclidean norms being 25.0802 are shown in Fig. 1a and 1b versus time from 1 sec. to 2 secs., respectively. The vertical axis is the Euclidean norm of all the initial states of (8) which can

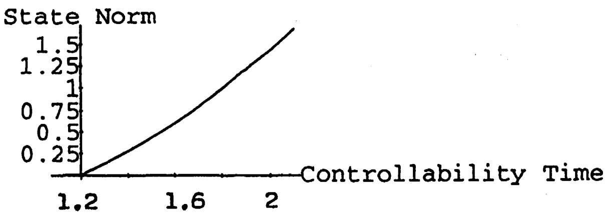

Fig. 1a. Controllability to the origin of several initial states versus controllability-time with extended control components (3.87584, 11. 01317, 24.6321, 2.4576) on the interval [1, 2] sec.

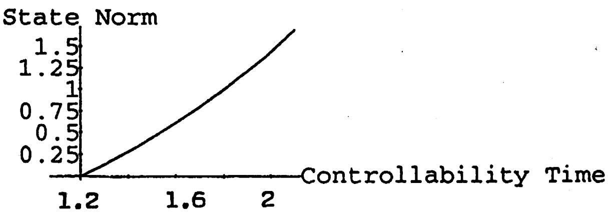

Fig. 1b. Ibid. with extended control components (2.4576, 1.101317, 3.875843, 24.6321).

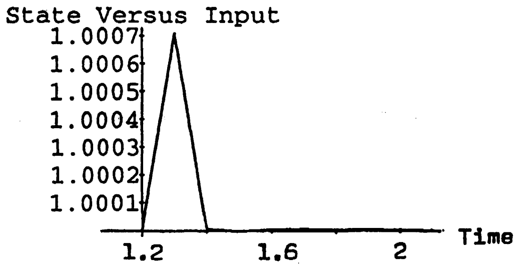

Fig. 1c. Relative values of the vertical axis shown in Figs. 1a–1b.

be driven to the origin with both given controls in different times belonging to such an interval. In particular, the initial states corresponding to $t = 1$ sec. for Figs. 1a–1b are, respectively, $(11.879, 25.4344)^T$ and $(9.07088, 21.2484)^T$ of respective Euclidean norms 28.0716 and 23.1036. It can be seen that both graphics are almost identical as emphasized by the corresponding relative values shown in Fig. 1c.

In Fig. 2, the three first components of the input are kept identical to the above experiment while the fourth one (i. e., the α – parameter of the initial model) is calculated so that the same initial state $(11.879, 25.4344)^T$ is driven to the origin in different times over the time interval [1, 2] (Fig. 2a) and [0.1, 0.2] (Fig. 2b). Note that the shapes of both figures are almost identical. The interpretation of those results relies on the influence of the constant term of the rate of growth of money wage in the Inflation-Unemployment model (2) through the α-parameter to drive an initial state to the origin faster or slower when the remaining input components in (8) are maintained constant. It is seen that α decreases as time increase so that a smaller constant term is allowed in such a model to drive the same initial state to the origin.

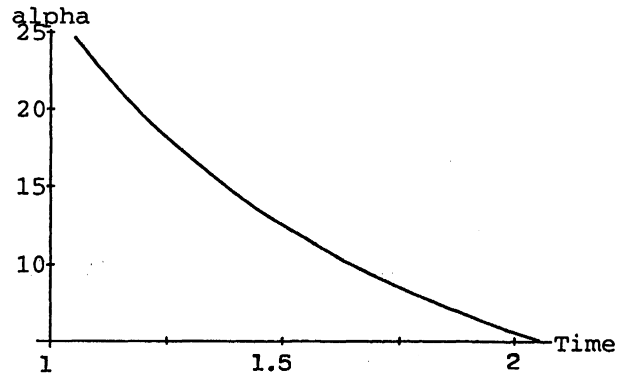

Fig. 2a. α – parameter versus controllability-time for the experiment of Fig. 1a. Time Interval [1, 2] sec. Initial state components (11.879, 25.3344).

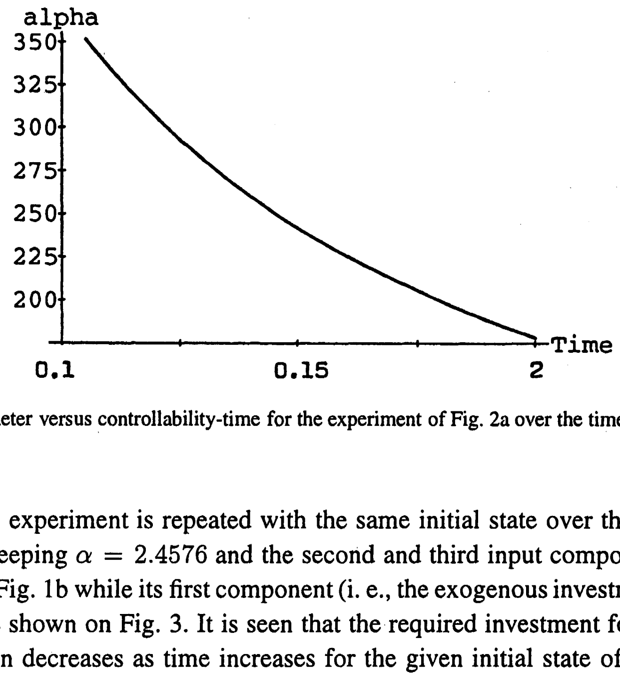

Fig. 2b. α – parameter versus controllability-time for the experiment of Fig. 2a over the time interval [0.1, 0.2] sec.

Finally, the experiment is repeated with the same initial state over the time interval [0.1, 0.3] by keeping $\alpha = 2.4576$ and the second and third input components as in the experiment of Fig. 1b while its first component (i. e., the exogenous investment) is varied. The results are shown on Fig. 3. It is seen that the required investment for controllability to the origin decreases as time increases for the given initial state of the combined Unemployment-Expected Inflation Dynamic Model Eq. 8.

6. Conclusions

This paper has dealt with the investigation of the dynamic relationships between level of unemployment and government expenditure. A stable and controllable inflation/unemployment model has been first obtained with respect to the exogenous input vector defined by investment government expenditure and money balance. The government expenditure has been decomposed into two (namely, exogenous and unemployment-

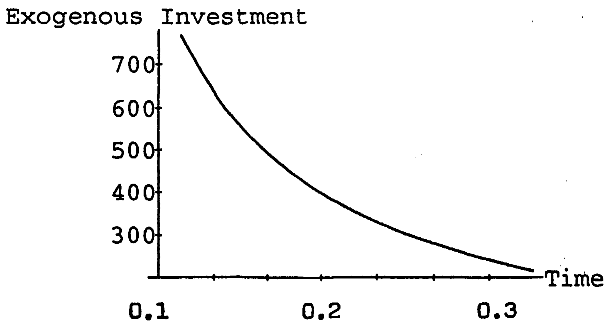

Fig. 3. Exogenous investment I_0 versus controllability-time for the experiment of Fig. 1b. Time interval [0.1, 0.3] sec.

dependent) additive components. The first component is associated with a fixed expenditure at a given time while the second one is associated with a gradual unemployment financial support. It is seen that, in fact, the design of the second expenditure component can be substituted by an appropriate redesign of the first one what means that the financial support for a prefixed level of unemployment support can be equivalently spent either at a fixed time or gradually during a suitable time interval. A trade-off between the final suitable value of the unemployment level and the time required to reach such a level has to be performed. Special emphasis is devoted to the cases when the government expenditure is bounded or proportional to the national income.

7. Acknowledgements

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Nedarbo lygio ir su juo susijusių vyriausybės išlaidų saveikos valdomumo ir pusiausvyros analizė

Manuel de la SEN

Šis darbas, tai – sąveikos, susiejančios nedarbo lygį su vyriausybės išlaidomis ekonominiuose modeliuose, valdomumo ir pusiausvyros studija. Čia yra nagrinėjami tie įdomūs atvejai, kai vyriausybės išlaidos yra arba apribotos, arba netiesinė nacionalinių pajamų funkcija. Analizuojant kaip Keinesio statinį nacionalinių pajamų, taip ir diferencijalinį Filipso tipo nedarbo-infliacijos modelius yra gauti savitarpio santykiai tarp abiejų kintamųjų, o būtent, tarp nedarbo augimo lygio ir vyriausybės išlaidų. Abu jie yra panaudoti naujo jungtinio modelio gavimui, eliminuojant bendrą kintamąjį – „mokesčius“, įvestą investicijoms ir vyriausybės išlaidoms.